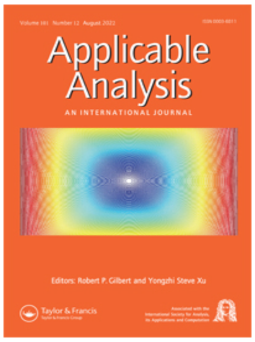

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Mathematical justification of a compressible bifluid system with different pressure laws: a continuous approach

Didier Bresch[a], Cosmin Burtea[b] and Frédéric Lagoutière[c]

[a]LAMA – UMR5127 CNRS, Le Bourget du Lac, France; [b]Université de Paris and IMJ-PRG-UMR7586, Paris, France; [c]Univ. Lyon, Université Claude Bernard Lyon 1, CNRS UMR5208, Institut Camille Jordan, Villeurbanne, France

ABSTRACT
This paper concerns the mathematical justification of a macroscopic Baer–Nunziato PDE bifluid system with a physical relaxation term that is linked to the two viscosities and the two pressure laws of the two compressible phases of the fluid which may be different. This is achieved using an homogenization approach in a periodic framework from a mesoscopic PDE description of two immiscible compressible viscous fluids with interfaces and no mass transfer. Our result extends the work in Bresch D, Hillairet M. [Note on the derivation of multi-component flow systems. Proc Am Math Soc. 2015;143:3429–3443] by allowing to consider different pressure laws for each component introducing an order parameter. This paper is complementary to the recent work [Bresch D, Burtea C, Lagoutière F. Mathematical justification of a compressible bi-fluid system with different pressure laws: a semi-discrete approach and numerical illustrations. Submitted 2021] which focuses on a semi-discretized approach and numerical illustrations. These two papers correspond to the extended versions of the document arXiv:2012.06497.

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1. Preface

Andro Mikelić was a world-renowned specialist in homogenization and flows in porous media. Andro was one of the first to invite D. Bresch to Lyon to discuss fluid mechanics and homogenization while sharing a good time for diners when D. Bresch was fresh out of his PhD thesis: these moments of sharing counted a lot in his way of seeing science and he never hesitated to interview Andro for his encyclopedic knowledge in homogenization and porous media. Andro Mikelic was full professor at ICJ Université Claude Bernard (Lyon) where this work has been done when C. Burtea was post-doc with F. Lagoutière (also at ICJ) and D. Bresch (CNRS, Univ. Savoie Mont Blanc). For the authors, this is an honor and a real pleasure to write a paper on homogenization for compressible viscous fluids in the memory of our friend and colleague Andro.

2. Introduction

In this paper, we propose to present the rigorous mathematical justification, in a one space dimension domain Ω, of a single velocity two-phase flow model with two different pressure laws. We consider 1-periodic initial data without loss of generality and denote the domain $\Omega = \mathbb{T}^1 = (0,1)$. This work follows a methodology proposed in a paper by the first author with M. Hillairet (see [1]) on the

CONTACT Didier Bresch didier.Bresch@univ-smb.fr
This paper is dedicated to the memory of our friend and colleague Andro Mikelić.

justification of a one velocity Baer–Nunziato type models, with a joint pressure to both phases. We recall that this classical model for multi-phase flows has been derived in [2]. A generalization to consider two possible different pressure laws asks for new existence results of solutions *à la Hoff* (intermediate regularity) on compressible Navier–Stokes with pressure depending on two quantities: one satisfying an advective equation and the other a transport equation. It is then necessary to introduce a homogenization parameter and to justify a two-scale asymptotic towards an underlying kinetic model. We obtain the two-phase limit model from the characterization of the defect measures satisfying this kinetic equation under initial hypothesis using a good control of the space derivative of the velocity even if the other unknowns are only bounded. Note that other possible formal derivation of multiphase flow models (including two-velocity models) exists for instance derivation by least action principle (see [3]), averaging approach in the spirit of Shii-Hibiki and Drew Passman (see [4,5]).

More precisely, we perform homogenization in one space dimension starting with two compressible fluids (each with its own pressure law and constant viscosity) governed by the Navier–Stokes equations separated by an interface. Assuming continuity of the velocity and continuity of the stress at the interface, we can write the system as a single system with a pressure depending on an order parameter function advected by the flow that is used to distinguish the different fluid phases and depending on the density. This provides a pressure law depending on two quantities. Note that our method presented here has been extended recently in [6,7] to cover physical situations taken into account surface tension quantities. We also refer the reader to our recent paper [8] where we tackle the question of obtaining a bi-fluid PDEs-model from an ODEs mesoscopic description.

By considering that the mixture of these fluids is the limit of situations where the fluids are separated by interfaces (multi-fluid approach at the mesoscopic scale) but at $\varepsilon = 1/n$ scale more and more fine, we obtain a system satisfied by the limit $n \to +\infty$, for which we have a formula to calculate the pressure of the mixture, as well as an equation for the volume fraction of each component with a completely justified relaxation term including the difference of phasic pressures. We focus in this note on the two-component case but the result generalizes to the multi-component case. These results render mathematically rigorous the formal computations that can be found for example in [9–12] by avoiding formal closure hypothesis for the relaxation term. In some sense, deriving one-velocity multi-fluid systems turns them into a homogenization problem for a mono-fluid equation with oscillating-concentrated initial data. The derivation of systems with highly oscillating-concentrated density has been first studied in the one dimension in space case by W. E. [13], D. Serre [14] in parallel with A.A. Amosov and A.A. Zlotinkov [15] for instance. Recently, P. Plotnikov and I. Sokolowski [16] on the one hand and D. Bresch and M. Hillairet [1,17] on the other hand have investigated the multi-dimension in space case. More precisely, the first authors consider compressible Navier–Stokes equations with constant viscosities with rapidly oscillating boundary data. Working on global weak solution in the spirit of Leray, using Young measures theory, it is possible to derive kinetic equations (see Lions et al. [18] for an introduction on kinetic equations) which encode the mixing dynamic. However, as explained in [1,17,19–21], multifluid systems are interpreted as reduced systems satisfied by particular Young measure (namely convex combinations of a finite number of Dirac masses) solutions of the homogenized compressible Navier–Stokes equation. Proving propagation of the number of Dirac masses in Young measure solutions to this homogenized equation is then the key point to derive the multi-fluid system with new relaxation terms. This requires to work with solutions with intermediate regularity (see D. Hoff [22] and B. Desjardins [23] for the definition) namely with initial density in $L^\infty(\Omega)$ and initial velocity in $H^1(\Omega)$. The macroscopic model is derived by sending the number n to infinity and computing a limit system. Letting n go to infinity, we mathematically justify the following system in the periodic setting:

$$\begin{cases} \partial_t \alpha_\pm + u\,\partial_x \alpha_\pm = \dfrac{\alpha_+\alpha_-}{\alpha_+\mu_- + \alpha_-\mu_+}(\sigma_\pm - \sigma_\mp), \\ \partial_t(\alpha_\pm\rho_\pm) + \partial_x(\alpha_\pm\rho_\pm u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) - \partial_x(\mu_{\text{eff}}\partial_x u) + \partial_x p_{\text{eff}} = 0, \\ \alpha_+ + \alpha_- = 1, \quad \rho = \alpha_+\rho_+ + \alpha_-\rho_-, \end{cases} \tag{1}$$

where

$$\mu_{\text{eff}} = \frac{\mu_+\mu_-}{\alpha_+\mu_- + \alpha_-\mu_+}, \quad p_{\text{eff}} = \frac{\alpha_+ p_+(\rho_+)\mu_- + \alpha_- p_-(\rho_-)\mu_+}{\alpha_+\mu_- + \alpha_-\mu_+} \tag{2}$$

with $s \mapsto p_+(x)$ and $s \mapsto p_-(x)$ two given monotone pressure laws that may be different for each component satisfying

$$p_\pm \in \mathcal{C}^1([0,+\infty)) \quad \text{such that} \quad p_\pm(0) = 0 \quad \text{and} \quad a_\pm s^{\gamma_\pm - 1} - b_\pm \le p_\pm'(s) \le \frac{1}{a_\pm} s^{\gamma_\pm - 1} + b_\pm \tag{3}$$

for some constant $\gamma_\pm > 1$ and $a_\pm > 0$, $b_\pm \geq 0$ and $\mu_\pm$ two positive given constant viscosities that may be different for each component and where the stress σ_+ and σ_- are given through the formula:

$$\sigma_\pm = -\mu_\pm \partial_x u + p_\pm(\rho_\pm). \tag{4}$$

This paper is divided into three main parts: (1) statement of the main result, (2) well-posedness of a bifluid system, (3) derivation of the Baer–Nunziato system (1)–(4) with relaxation term. In some sense, we rigorously derive a relaxation system of the usual inviscid Baer–Nunziato with the algebraic closure equilibrium by taking into account the viscosity effect, the novelty being to take care of the dependency with respect to two scalar unknowns. To derive (1), we follow the lines introduced by Bresch and Hillairet [17]: global well-posedness of the initial system and then homogenization process with characterization of the Young measures family.

Remark 2.1: Note that formally, if μ_+ and μ_- tend to zero in (1) we get the Kapila inviscid Baer–Nunziato system with the algebraic closure equilibrium

$$\begin{cases} p_+(\rho_+) = p_-(\rho_-), \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) + \partial_x p_+(\rho_+) = 0, \\ \alpha_+ + \alpha_- = 1, \quad \rho = \alpha_+\rho_+ + \alpha_-\rho_-. \end{cases} \tag{5}$$

See for instance the recent work [24].

This paper is organized into three main parts. In Section 2, we present the statement of the main result. In Section 3, we investigate a model which is a bifluid system to prove global existence and uniqueness of solution à la Hoff. Then in Section 4, we explain the link of a mixture system at mesoscale with the system studied on the second part and then we perform a homogenization process following the methods developed by D. Bresch and M. Hillairet to rigorously justify the bifluid system (1) with a relaxation term link to the two viscosities and the two pressure laws.

3. Statement of the main result

Let us consider two immiscible compressible fluids that are separated by an interface and denote the quantities with $+$ or $-$ depending on whether we are in the part of fluid $+$ or the part of fluid $-$. We introduce a parameter c which takes the value 1 in Ω_+ and 0 in Ω_-. We assume the velocity field to be continuous at the interface so that the velocity field in the part $+$ and the part $-$ are restrictions of a velocity field u which is defined on the whole space. Denoting $\Omega = \Omega_+ \cup \overline{\Omega}_-$, we assume that the color function c is transported by the velocity field of the fluid namely

$$\partial_t c + u\,\partial_x c = 0 \quad \text{in } \Omega \quad \text{with} \quad c\,(1-c) = 0 \quad a.e.$$

Then we assume the following equations with the same velocity u for each component:

$$\partial_t(\rho_+ u) + \partial_x(\rho_+ u^2) - \mu_+ \partial_x^2 u + \partial_x p_+(\rho_+) = 0, \quad \partial_t \rho_+ + \partial_x(\rho_+ u) = 0 \quad \text{in } \Omega_+$$

and

$$\partial_t(\rho_- u) + \partial_x(\rho_- u^2) - \mu_+ \partial_x^2 u + \partial_x p_-(\rho_-) = 0, \quad \partial_t \rho_- + \partial_x(\rho_- u) = 0 \quad \text{in } \Omega_-$$

where ρ_+ and ρ_- are respectively defined in Ω_+ and Ω_-. The densities ρ_+ and ρ_- are extended by 0 in Ω keeping the same notations ρ_+ and ρ_-. Writing the equations satisfied by $c\rho_+$ and $(1-c)\rho_-$ and assuming continuity of the stress $\sigma_\pm = -\mu_\pm \partial_x u + p_\pm(\rho_\pm)$ at interfaces, we can write the full system in the unified equations:

$$\partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x((c\mu_+ + (1-c)\mu_-)\partial_x u) + \partial_x(c p_+(\rho) + (1-c)p_-(\rho)) = 0,$$

with

$$\partial_t \rho + \partial_x(\rho u) = 0 \text{ in } \Omega, \quad \rho = c\rho_+ + (1-c)\rho_-.$$

In conclusion, we have the following system:

$$\begin{cases} \partial_t c + u\,\partial_x c = 0 \text{ with } c(1-c) = 0, \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x(\rho u^2) - \partial_x((c\mu_+ + (1-c)\mu_-)\partial_x u) + \partial_x(c p_+(\rho) + (1-c)p_-(\rho_-)) = 0, \end{cases} \tag{6}$$

with the initial condition

$$\rho|_{t:=0} = \rho_0, \quad c|_{t=0} = c_0, \quad u|_{t=0} = u_0. \tag{7}$$

Note that such a system is included in the one studied in Section 3 choosing

$$\mu(c) = c\mu_+ + (1-c)\mu_-, \quad p(\rho, c) = p_+(\rho)c + p_-(\rho)(1-c).$$

More precisely in Section 4, we prove the existence Theorem 4.1 with intermediate regularity. Using this existence result, a multi-phase fluid can be represented by solutions (ρ, c, u) with (ρ, c) that oscillate widely in space. Following the formalism introduced by the first author and M. Hillairet, we compute a macroscopic system for multiphase fluids by introducing a parameter n encoding the oscillation scale in the initial data. The scheme we have in mind is to set:

$$c_0^n(x) = c_0(nx), \quad \rho_0^n = c_0(nx)\rho_{+,0}(x) + (1 - c_0(nx))\rho_{-,0}(x), \quad u_0^n(x) = u_0(x) \tag{8}$$

where c_0 is a fixed 1-periodic profile and $\rho_{+,0}, \rho_{-,0}$ are bounded 1-periodic initial data such that

$$c_0 \in \{0, 1\}, \quad 0 < M^{-1} < \rho_0 < M, \quad \|u_0\|_{H^1(\mathbb{T}^1)} < C < +\infty \tag{9}$$

Denoting $(\dot{u^n}) = \partial_t u^n + u^n \partial_x u^n$, Theorem 4.1 provides a global existence result of a unique solution (ρ^n, c^n, u^n) satisfying uniformly with respect to n the following bounds on $[0, T]$:

- The Energy defined for $t \in (0, T)$

$$\int_0^1 \frac{\rho^n (u^n)^2}{2} + \int_0^1 \left(c^n H_+(\rho^n) + (1-c^n)H_-(\rho^n)\right) + \int_0^t \int_0^1 \mu(c^n)(\partial_x u^n)^2$$

$$\leq \int_0^1 \frac{\rho_0 u_0^2}{2} + \int_0^1 \left(c_0^n H_+(\rho_0^n) + (1-c_0^n) H_-(\rho_0^n)\right)$$

where, denoting $\overline{\rho}$ a reference density, we have

$$H_\pm(\rho^n) = \rho^n \int_{\overline{\rho}}^{\rho^n} p_\pm(s)/s^2 \,\mathrm{d}s.$$

- The bounds on the density ρ^n and the values on c^n

$$C(T)^{-1} \leq \rho^n(t,x) \leq C(T), \quad c^n(t,x) \in \{0,1\}.$$

- A first estimate in the spirit of Hoff solutions for $t \in (0,T)$

$$\frac{1}{2}\int_0^1 \mu(c^n)(\partial_x u^n)^2 + \int_0^t \int_0^1 \rho^n (\dot{u^n})^2 \leq C(T).$$

- A second estimate in the spirit of Hoff solutions for $t \in (0,T)$

$$\frac{1}{2}\int_0^1 \kappa(t)\, \rho (\dot{u^n})^2 + \frac{1}{2}\int_0^t \int_0^1 \kappa(t)\, \mu(c^n) \left(\partial_x \dot{u^n}\right)^2 \leq C(T)$$

and some bounds on the velocity

$$\|u^n\|_{L^\infty((0,1)\times\mathbb{T}^1)} + \|u^n\|_{H^1((0,1)\times\mathbb{T}^1)} \leq C(T)$$

and

$$\left\|\kappa^{\frac{1}{2}}(\partial_t u^n(t), \partial_x u^n(t))\right\|_{L^2(0,T;L^\infty(\mathbb{T}^1))} \leq C(T)$$

for all $t \in [0,T]$ where $\kappa(t) = \min\{t,1\}$. It remains to let n tend to $+\infty$ to show the convergence to the bifluid system (1)–(4) under assumptions on the initial data sequence. More precisely we prove the following theorem.

Theorem 3.1: *Let us consider p_+ and p_- two given monotone pressure laws satisfying* (3) *and assume the initial data satisfy* (8) *and* (9). *Then there exists a unique global solution $(\rho^n, c^n, u^n)_{n\in\mathbb{N}\setminus\{0\}}$ of the compressible Navier–Stokes equations* (6) *with the initial data* (8) *satisfying uniformly with respect to n the bounds given above. Let Θ_0^n be defined by*

$$\langle \Theta_0^n, b\rangle = \int_{\mathbb{T}^1} b(x, \rho_0^n(x), c_0^n(x))\,\mathrm{d}x$$

for all $b \in \mathcal{C}_c(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$ where the indices precise on which variable x, ξ, η the domains apply. Assume there exists $(\alpha_{+,0}, \alpha_{-,0}, \rho_{+,0}, \rho_{-,0}) \in L^\infty(\mathbb{T}^1)$ such that

$$\langle \Theta_0^n, b\rangle \to \langle \Theta^0, b\rangle = \int_{\mathbb{T}^1} \left(\alpha_{+,0}\, b(x, \rho_{+,0}(x), 1) + \alpha_{-,0} b(x, \rho_{-,0}(x), 0)\right)\,\mathrm{d}x$$

for all $b \in \mathcal{C}_c(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$. Then there exists $(\alpha_+, \alpha_-, \rho_+, \rho_-) \in L^\infty((0,T)\times\mathbb{T}^1)$ such that, up to a subsequence, for all b in $\mathcal{C}(\mathbb{T}_x^1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta)$

$$\langle \Theta^n, b\rangle = \int_{\mathbb{T}^1} b(x, \rho^n(t,x), c^n(t,x))\,\mathrm{d}x \to \langle \Theta, b\rangle$$

$$= \int_{\mathbb{T}^1} \left(\alpha_+ b(x, \rho_+(t,x), 1) + \alpha_{-,0} b(x, \rho_-(t,x), 0)\right) \, dx.$$

In particular, up to a subsequence,

$$\rho^n \rightharpoonup \alpha_+\rho_+ + \alpha_-\rho_- \text{ weakly star in } L^\infty((0,T)\times\mathbb{T}^1)$$
$$p(\rho^n, c^n) \rightharpoonup \alpha_+ p_+(\rho_+) + \alpha_- p_-(\rho_-) \text{ weakly star in } L^\infty((0,T)\times\mathbb{T}^1)$$

along with

$$u^n \rightharpoonup u \quad \text{in } H^1((0,T)\times\mathbb{T}^1)$$

and $(\alpha_+, \alpha_-, \rho_+, \rho_-, u)$ *satisfy* (1)–(4) *with the initial conditions*

$$\alpha_\pm|_{t=0} = \alpha_\pm^0, \quad \rho_\pm|_{t=0} = \rho_\pm^0, \quad u|_{t=0} = u^0.$$

The existence result concerning the sequence of solutions will be a direct consequence of Theorem 4.1 in Section 4. The homogenization procedure will be justified in Section 5.

Remark 3.1: Let us remark that

$$\mu_{\text{eff}} = \frac{1}{\left\langle \frac{1}{\mu} \right\rangle} = \frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}}$$

and

$$p_{\text{eff}} = \frac{1}{\left\langle \frac{1}{\mu} \right\rangle} \left\langle \frac{p}{\mu} \right\rangle = \frac{1}{\frac{\alpha_+}{\mu_+} + \frac{\alpha_-}{\mu_-}} \left(\frac{\alpha_+ p_+(\rho_+)}{\mu_+} + \frac{\alpha_- p_-(\rho_-)}{\mu_-} \right)$$

where $\langle f \rangle = \int_0^1 f(\tau) d\tau$ with α_+ the volume fraction of the phase $+$ and $\alpha_- = 1 - \alpha_+$ the volume fraction of the phase $-$. The first formula is similar to the effective coefficient obtained by F. Murat and L. Tartar in 1971 for oscillating elliptic equation problem in one space dimension namely $\partial_x(\mu^\varepsilon \partial_x u^\varepsilon) = F'(x)$ with $u^\varepsilon(0) = u^\varepsilon(1) = 0$. The second expression is similar to the effective term we will get if we study the equation $-\partial_x(\mu^\varepsilon \partial_x u^\varepsilon - p^\varepsilon) = F'(x)$ with F regular with $u^\varepsilon(0) = u^\varepsilon(1) = 0$ and with two oscillating functions μ^ε and p^ε. Namely we calculate

$$\partial_x u^\varepsilon = -F \frac{1}{\mu^\varepsilon} + \frac{p^\varepsilon}{\mu^\varepsilon}$$

and passing to the limit

$$\partial_x u = -F(x) \left\langle \frac{1}{\mu} \right\rangle + \left\langle \frac{p}{\mu} \right\rangle$$

which gives

$$\partial_x \left(\left\langle \frac{1}{\mu} \right\rangle \partial_x u + \left\langle \frac{1}{\mu} \right\rangle \left\langle \frac{p}{\mu} \right\rangle \right) = F'(x).$$

4. Global well posedness of a bifluid system

In this section, we study the following one-dimensional system:

$$\begin{cases} \partial_t c + u\,\partial_x c = 0, \\ \partial_t \rho + \partial_x(\rho u) = 0, \\ \partial_t(\rho u) + \partial_x\left(\rho u^2\right) - \partial_x(\mu(c)\partial_x u) + \partial_x p = 0, \end{cases} \tag{10}$$

where ρ is the density, u the velocity field and c is the mass fraction which denotes the relative weighting for each fluid component associated to a generalized pressure $p = p(\rho, c) \geq 0$. This unknown will be the order parameter tracking the mixing of the fluid components in the sequel of the paper. We supplement (10) with periodic initial data (c_0, ρ_0, u_0) on $(0, 1)$ for which

$$\begin{cases} c_0 \in L^\infty(\mathbb{T}^1) \text{ such that } 0 \le \inf c_0 \le c_0(x) \text{ a.e. on } (0,1)\,; \\ \rho_0 \in L^\infty(\mathbb{T}^1) \text{ such that } 0 < \inf \rho_0 \le \rho_0(x) \text{ a.e. on } (0,1)\,; \\ G(\rho_0, c_0) \in L^1(\mathbb{T}^1) \text{ with } G(\rho_0, c_0) = \rho_0 \displaystyle\int_0^{\rho_0} p(s, c_0)/s^2 \mathrm{d}s; \\ u_0 \in H^1\left(\mathbb{T}^1\right). \end{cases} \tag{11}$$

Such a system is considered in [25] with density/mass fraction-dependent viscosities. It is also considered recently in [26,27] with constant viscosity. Note that the mentioned works concern the existence of global weak solution à la Leray. As recalled in the introduction, we are more interested in solutions with intermediate regularity namely L^∞ bounds on the density/mass fraction and H^1 bound on the velocity field. So, we want to establish/generalize the Cauchy theory of Hoff–Desjardins (see for instance [22,23]) in the case of the classical compressible Navier–Stokes system with constant viscosity.

Let us start with the hypothesis on the pressure and the viscosity functions that we consider. We are interested in pressure functions $p \in \mathcal{C}^1([0,\infty) \times [m, M])$ and viscosity functions $\mu \in \mathcal{C}^1([m, M])$ where $M = \sup_{[0,1]} c_0 < +\infty$ and $m = \inf_{[0,1]} c_0 \ge 0$ that verify:

- Hypothesis 1:

$$p(\rho, c) \ge 0, \quad \mu(c) \ge \mu_{\min} > 0 \tag{12}$$

- Hypothesis 2: for any finite $\rho \ge 0$ we have

$$\rho \int_0^\rho \frac{p(s,c)}{s^2} \mathrm{d}s < \infty. \tag{13}$$

- Hypothesis 3: there exists a constant C_0 such that:

$$p(\rho, c) \le C_0(\rho + G(\rho, c)), \tag{14}$$

where

$$G(\rho, c) = \rho \int_0^\rho \frac{p(s,c)}{s^2} \mathrm{d}s.$$

- Hypothesis 4:

$$\rho\, \partial_1 p(\rho, c) \in L^\infty_{loc}([0,\infty) \times [0,1])\,. \tag{15}$$

We are in the position of stating the first result of this paper

Theorem 4.1: *Consider a function $p \in \mathcal{C}^1([0,\infty) \times [m, M])$ and $\mu \in \mathcal{C}^1([0, M])$ verifying the hypothesis* (12)–(15). *Also, consider (c_0, ρ_0, u_0) as in* (11). *Then, there exists a unique weak solution (c, ρ, u) with*

$$c, \rho \in C([0,\infty); L^p_x(\mathbb{T}^1)), \quad u \in L^\infty\left([0,\infty); H^1_x(\mathbb{T}^1)\right), \quad \partial_x u \in L^2\left([0,\infty) \times \mathbb{T}^1\right).$$

Moreover, for any $T > 0$, there exists a constant $C(T)$ that depends only on the norms of the initial data and T such that the following uniform bounds hold true:

$$\int_0^1 \frac{\rho u^2}{2} + \int_0^1 G(\rho, c) + \int_0^t \int_0^1 \mu(c)(\partial_x u)^2 \le \int_0^1 \frac{\rho_0 u_0^2}{2} + \int_0^1 G(\rho_0, c_0)\,, \tag{16}$$

$$\inf_{x\in[0,1]} c_0(x) \le c(t,x) \le \sup_{x\in[0,1]} c_0(x), \tag{17}$$

$$C(T)^{-1} \le \rho(t,x) \le C(T), \tag{18}$$

$$\frac{1}{2}\int_0^1 \mu(c)(\partial_x u)^2 + \int_0^t\int_0^1 \rho\dot{u}^2 \le C(T), \tag{19}$$

$$\frac{1}{2}\int_0^1 \kappa(t)\,\rho\dot{u}^2 + \frac{1}{2}\int_0^t\int_0^1 \kappa(t)\,\mu(c)\,(\partial_x\dot{u})^2 \le C(T), \tag{20}$$

$$\|u\|_{L^\infty((0,1)\times\mathbb{T}^1)} + \|u\|_{H^1((0,1)\times\mathbb{T}^1)} \le C(T), \tag{21}$$

$$\left\|\kappa^{1/2}(\partial_t u(t), \partial_x u(t))\right\|_{L^2(0,T;L^\infty(\mathbb{T}^1))} \le C(T) \tag{22}$$

for all $t \in [0,T]$ *where* $\kappa(t) = \min\{t,1\}$.

Local-in-time existence and uniqueness of classical solutions to (10), with a blow-up criterion, follow from a standard Lagrangian transformation. To extend these solutions into Hoff–Desjardins solutions, the most important part is to prove that they satisfy uniform estimates related to Hoff–Desjardins' regularity. This yields existence of Hoff–Desjardins solutions for smooth initial data. For initial data with bounded initial density/mass fraction and H^1 initial velocity field, a compactness argument then shows that a sequence of solutions for regularized initial data converges to the expected solutions. In this paper, we skip the proof of the classical solutions to the system with regular initial data. We focus on the uniform estimates related to Hoff–Desjardins regularity and to the stability result allowing to relax the regularity of the initial data.

4.1. The proof of Theorem thm4.1

This section is devoted to the proof of the existence result. It is divided into eight parts. From the first part to the sixth part, we focus on the uniform estimates on classical solutions in the sense of Hoff–Desjardins' regularity. In the remaining parts, we prove a stability result allowing to relax the regularity of the initial data and we also provide a uniqueness result of solutions. Throughout the section, the functions $p \in \mathcal{C}^1([0,\infty)\times[0,M])$ and $\mu \in \mathcal{C}^1([0,M])$ are given and verify Hypothesis (12)–(15).

4.1.1. A-priori estimates

In this section, we suppose that we are given a positive $T > 0$ and a triplet (c, ρ, u) regular enough such that the computations make sense.

Proposition 4.1: *Consider* (c,ρ,u) *a classical solution to the system* (10) *with initial data* (c_0,ρ_0,u_0) *verifying the lower bounds of* (11). *Then there exists a constant* $C(T)$ *which depends only on* $\|u^0\|_{H^1(\mathbb{T})}, \|\rho_0\|_{L^\infty(\mathbb{T})}, \|1/\rho_0\|_{L^\infty(\mathbb{T})}$, *and T such that the following uniform bounds hold true:*

$$\int_0^1 \frac{\rho u^2}{2} + \int_0^1 G(\rho,c) + \int_0^t\int_0^1 \mu(c)(\partial_x u)^2 \le \int_0^1 \frac{\rho_0 u_0^2}{2} + \int_0^1 G(\rho_0,c_0), \tag{23}$$

$$\inf_{x\in[0,1]} c_0(x) \le c(t,x) \le \sup_{x\in[0,1]} c_0(x), \tag{24}$$

$$C(T)^{-1} \le \rho(t,x) \le C(T), \tag{25}$$

$$\frac{1}{2}\int_0^1 \mu(c)(\partial_x u)^2 + \int_0^t\int_0^1 \rho\dot{u}^2 \le C(T), \tag{26}$$

$$\frac{1}{2}\int_0^1 \kappa(t)\,\rho\dot{u}^2 + \frac{1}{2}\int_0^t\int_0^1 \kappa(t)\,\mu(c)\,(\partial_x\dot{u})^2 \le C(T), \tag{27}$$

$$\|u\|_{L^\infty((0,1)\times\mathbb{T}^1)} + \|u\|_{H^1((0,1)\times\mathbb{T}^1)} \leq C(T), \tag{28}$$

$$\left\|\kappa^{1/2}(\partial_t u(t), \partial_x u(t))\right\|_{L^2(0,T;L^\infty(\mathbb{T}^1))} \leq C(T). \tag{29}$$

for all $t \in [0,T]$ *where* $\kappa(t) = \min\{t, 1\}$.

The objective of the next sections is to prove the above proposition. For conciseness, we introduce in the following computations the symbol $\overline{C}$ to denote a universal constant. It may vary from line to line.

4.1.2. Conservation of mass and energy estimate (23)

First of all, let us observe that integrating over $[0,1]$ the equation satisfied by ρ leads to

$$\int_0^1 \rho(t,x)\,\mathrm{d}x = \int_0^1 \rho_0(x)\,\mathrm{d}x.$$

We denote by M_0 the total mass of the fluid:

$$M_0 \overset{\text{notation}}{=} \int_0^1 \rho_0(x) = \int_0^1 \rho(t,x)\,\mathrm{d}x. \tag{30}$$

Next, let

$$H : [0,\infty) \times [m,M] \to \mathbb{R}$$

be a C^1-function. Multiplying the first equation of (10) with $\partial_2 H(\rho,c)$, the second equation of (10) with $\partial_1 H(\rho,c)$ and adding up the results we get that

$$\partial_t H + u\partial_x H + \rho\partial_1 H\partial_x u = 0, \tag{31}$$

which also writes

$$\partial_t H + \partial_x(uH) + (\rho\partial_1 H - H)\partial_x u = 0. \tag{32}$$

Next, multiplying the second equation of (10) with $u^2/2$, the last equation with u, adding up the resulting identities and using (32) we are lead to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \left(\frac{\rho|u|^2}{2} + G(\rho,c)\right) + \int_0^1 \mu(c)|\partial_x u|^2 = 0, \tag{33}$$

where

$$G(\rho,c) = \rho\int_0^\rho \frac{p(s,c)}{s^2}\mathrm{d}s.$$

Thus after time integration of (33) we get that

$$E(t) := \int_0^1 \frac{\rho|u|^2}{2} + \int_0^1 G(\rho,c) + \int_0^t\int_0^1 \mu(c)|\partial_x u|^2 = \int_0^1 \frac{\rho_0|u_0|^2}{2} + \int_0^1 G(\rho_0,c_0). \tag{34}$$

In what follows, we denote

$$E_0 \overset{\text{def.}}{=} \int_0^1 \frac{\rho_0 u_0^2}{2} + \int_0^1 G(\rho_0,c_0) \tag{35}$$

The above computations show the validity of estimate (23).

4.1.3. The $L^2(0,T;L^2(\mathbb{T}^1))$-control on the velocity

Following the arguments introduced by P.-L. Lions in [28] we may recover an $L^2(0,T;L^2(\mathbb{T}^1))$-control on the velocity with the help of the energy inequality (34). For the sake of completeness, let us reproduce here the arguments leading to such an inequality. First, we write that

$$u(t,x) - u(t,y) = \int_y^x \partial_x u\,(t,z)\,\mathrm{d}z,$$

so that integrating in y yields

$$u(t,x) - \int_0^1 u(t,y)\mathrm{d}y = \int_0^1 \int_y^x \partial_x u\,(t,z)\,\mathrm{d}z\mathrm{d}y.$$

Let us multiply the above relation with $\rho(t,x)$, integrate it with respect to x and recall the mass conservation relation (30) to conclude that:

$$\int_0^1 \rho u(t,x)\mathrm{d}x - \int_0^1 \rho_0(x)\mathrm{d}x \int_0^1 u(t,x)\mathrm{d}x = \int_0^1 \left(\rho\,(t,x) \int_0^1 \int_y^x \partial_x u\,(t,z)\,\mathrm{d}z\mathrm{d}y\right)\mathrm{d}x.$$

From the above relation, we get that

$$\int_0^1 u(t,x)\mathrm{d}x = \frac{1}{M_0}\int_0^1 \rho u(t,x)\mathrm{d}x + \frac{1}{M_0}\int_0^1 \left(\rho\,(t,x) \int_0^1 \int_y^x \partial_x u\,(t,z)\,\mathrm{d}z\mathrm{d}y\right)\mathrm{d}x. \tag{36}$$

With straightforward arguments to bound the right-hand side of this inequality with (23), we conclude that for all $t \in [0,T]$, it holds true that:

$$\int_0^t \left(\int_0^1 u(t,x)\mathrm{d}x\right)^2 \leq \overline{C}\left(\frac{t}{M_0} + \frac{1}{\mu_{\min}}\right)E_0.$$

Of course, as one has that

$$\|u(t,\cdot)\|_{H^1(\mathbb{T}^1)} \leq \overline{C}\left(\left|\int_0^1 u(t,x)\mathrm{d}x\right| + \|\partial_x u\|_{L^2(\mathbb{T}^1)}\right)$$

one obtains that

$$\|u\|^2_{L^2(0,T;H^1)} \leq \overline{C}\left(\frac{T}{M_0} + \frac{1}{\mu_{\min}}\right)E_0$$

and using the embedding $H^1(\mathbb{T}) \hookrightarrow L^\infty(\mathbb{R})$ we get that

$$\|u\|_{L^1(0,T;L^\infty(\mathbb{R}))} \leq \sqrt{T}\,\|u\|_{L^2(0,T;L^\infty(\mathbb{T}))} \leq \overline{C}\sqrt{TE_0}\left(\frac{T}{M_0} + \frac{1}{\mu_{\min}}\right)^{\frac{1}{2}}. \tag{37}$$

4.1.4. The Lagrangian variable

Let us consider the flow generated by u:

$$X_t(x) = x + \int_0^t u(\tau, X_\tau(x))\,\mathrm{d}\tau \quad \forall t \geq 0 \quad \forall x \in \mathbb{R} \tag{38}$$

and observe that

$$\frac{\partial X_t}{\partial x}(x) = 1 + \int_0^t \partial_x u(\tau, X_\tau(x)) \frac{\partial X_\tau}{\partial x}(x)\mathrm{d}\tau.$$

Hence, we get

$$\frac{\partial X_t}{\partial x}(x) = \exp\left(-\int_0^t \partial_x u(\tau, X_\tau(x))\,\mathrm{d}\tau\right) \geq \exp\left(-\int_0^t \|\partial_x u(\tau,\cdot)\|_{L^\infty}\,\mathrm{d}\tau\right) > 0.$$

Thus, X_t is a local C^1-diffeomorphism for any $t \in [0, T]$. Next, with the help of (37), we write that for any $t \in [0, T]$ we have

$$\begin{aligned} |X_t(x) - x| &\leq \|u\|_{L^1(0,T;L^\infty(\mathbb{R}))} \\ &\leq \overline{C}\sqrt{TE_0}\left(\frac{T}{M_0} + \frac{1}{\mu_{\min}}\right)^{\frac{1}{2}}. \end{aligned} \tag{39}$$

Consequently, one has

$$\lim_{x\to\pm\infty} X_t(x) = \pm\infty \quad \forall t \in [0, T].$$

Consequently the application X_t realizes a $\mathcal{C}^1$-diffeomorphism on $\mathbb{R}$ for arbitrary $t \in [0, T]$. We note that since u is 1-periodic, we also have

$$X_t(x + k) = X_t(x) + k, \quad \forall x \in \mathbb{R}, \quad \forall k \in \mathbb{Z}.$$

In the next section, we make use of the following variant of the Reynolds transport theorem which we leave as an exercise to the reader.

Remark 4.1: For $[0, T]$ consider X_t the $\mathcal{C}^1$-diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$ defined by (38) and fix a point $a \in \mathbb{R}$. Then, for any function f for which the integrals appearing below are defined, we have that

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\int_a^{X_t(x)} f(t,y)\,\mathrm{d}y &= \int_a^{X_t(x)} \partial_t f(t,y)\,\mathrm{d}y + f(t, X_t(x))\, u(t, X_t(x)) \\ &= \int_a^{X_t(x)} \partial_t f(t,y)\,\mathrm{d}y + f(t, X_t(x))\, u(t, X_t(x)) - f(t,a)u(t,a) + f(t,a)u(t,a) \\ &= \int_a^{X_t(x)} \left\{\partial_t f + \partial_x(fu)\right\}(t,y)\,\mathrm{d}y + f(t,a)u(t,a). \end{aligned}$$

4.1.5. L^∞-estimate for the density and concentration

First, let us write that the equation of the concentration function c is equivalent to

$$c\,(t, X_t(x)) = c_0(x),$$

which implies that

$$0 \le m = \inf_{x\in[0,1]} c_0(x) \le c(t,x) \le \sup_{x\in[0,1]} c_0(x) = M < +\infty, \tag{40}$$

and we conclude that (24) is also valid. Next, using the result of Remark 4.1, we write that for all $y \in [0,1]$:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_y^{X_t(x)} (\rho u)(t,z)\mathrm{d}z &= \int_y^{X_t(x)} \left\{\partial_t(\rho u) + \partial_x(\rho u^2)\right\}(t,z)\mathrm{d}z + \rho u^2(t,y)\\ &= \int_y^{X_t(x)} \left\{\partial_x(\mu(c)\partial_x u) - \partial_x p\right\}(t,z)\mathrm{d}z + \rho u^2(t,y)\\ &= \big(\mu(c_0(x))\partial_x u(t,X_t(x)) - p(\rho(t,X_t(x)), c_0(x))\big)\\ &\quad - \big(\mu(c(t,y))\partial_x u(t,y) - p(\rho(t,y), c(t,y))\big) + \rho u^2(t,y).\end{aligned}$$

Integrating with respect to y gives us

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_y^{X_t(x)} (\rho u)(t,z)\mathrm{d}z\mathrm{d}y &= \big(\mu(c_0(x))\partial_x u(t,X_t(x)) - p(\rho(t,X_t(x)), c_0(x))\big)\\ &\quad - \int_0^1 \left\{\big(\mu(c(t,y))\partial_x u(t,y) - p(\rho(t,y), c(t,y))\big) + \rho u^2(t,y)\right\}\mathrm{d}y\\ &= \big(\mu(c_0(x))\partial_x u(t,X_t(x)) - p(\rho(t,X_t(x)), c_0(x))\big)\\ &\quad - \int_0^1 \mu(c)(t,y)\partial_x u(t,y)\,\mathrm{d}y + \int_0^1 \left\{p(\rho(t,y), c(t,y)) + \rho u^2(t,y)\right\}\mathrm{d}y\end{aligned}$$

Using the hypothesis (12) along with the fact that

$$\frac{\mathrm{d}}{\mathrm{d}t}\log\rho(t,X_t(x)) = -\partial_x u(t,X_t(x))$$

we infer that

$$\begin{aligned}&\frac{1}{\mu(c_0(x))}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1\int_y^{X_t(x)} (\rho u)(t,z)\mathrm{d}z\mathrm{d}y + \frac{\mathrm{d}}{\mathrm{d}t}\log\rho(t,X_t(x))\\ &= -\frac{1}{\mu(c_0(x))}p(\rho(t,X_t(x)), c_0(x))\\ &\quad + \frac{1}{\mu(c_0(x))}\int_0^1 \left\{p(\rho(t,y), c(t,y)) + \rho u^2(t,y) - \partial_t\mu(c)(t,y)\right\}\mathrm{d}y.\end{aligned}$$

Note that the last term comes from the term $\int_0^1 \mu(c)(t,y)\partial_x u(t,y)\,\mathrm{d}y$, integrating by parts and using the equation $\partial_t\mu(c) + u\,\partial_x\mu(c) = 0$. Thus we have that

$$\rho(t,X_t(x))\exp\left(\frac{1}{\mu(c_0(x))}\int_0^1\int_y^{X_t(x)} (\rho u)(t,z)\mathrm{d}z\mathrm{d}y - \frac{1}{\mu(c_0(x))}\int_0^1\int_y^{x} (\rho_0 u_0)(t,z)\mathrm{d}z\mathrm{d}y\right)$$

$$= \rho_0(x) \exp\left(-\frac{1}{\mu(c_0(x))}\int_0^t p(\rho(\tau, X_\tau(x)), c_0(x))\right)$$
$$\cdots \exp\left(\frac{1}{\mu(c_0(x))}\int_0^t \int_0^1 \left\{p(\rho(\tau,y), c(\tau,y)) + \rho u^2(\tau,y) - \partial_t\mu(c)(t,y)\right\} \mathrm{d}y\mathrm{d}\tau\right). \quad (41)$$

Using (12) and (14), we get that

$$\begin{cases} \frac{1}{\mu(c_0(x))}\int_0^t \int_0^1 \left\{p(\rho(\tau,y), c(\tau,y)) + \rho u^2(\tau,y) - \partial_t\mu(c)(t,y)\right\} \mathrm{d}y\mathrm{d}\tau \\ \quad \leq (E_0 + M_0)\frac{(C_0+2)t}{\mu_{\min}} + \frac{1}{\mu(c_0(x))}\int_0^1 \mu(c_0(y))\,\mathrm{d}y, \\ \exp\left(-\frac{1}{\mu(c_0(x))}\int_0^t p(\rho(\tau, X_\tau(x)), c_0(x))\right) \leq 1. \end{cases} \quad (42)$$

Moreover, due to the periodicity of ρu, for any $t \geq 0$ we have that (with $[\cdot]$ denoting the integer part)

$$\left|\int_0^1 \int_y^{X_t(x)} (\rho u)(t,z)\mathrm{d}z\mathrm{d}y\right| (|[X_t]| + 1) \leq \int_0^1 |\rho u|\,(t,z)\mathrm{d}z \leq (|X_t(x)| + 2)\sqrt{2M_0E_0}. \quad (43)$$

Putting together (42) and (43) gives us

$$\rho(t, X_t(x))$$
$$\leq \rho_0(x) \exp\left(\frac{(|X_t(x)|+2)\sqrt{2M_0E_0}}{\mu_{\min}} + (E_0+M_0)\frac{(C_0+2)t}{\mu_{\min}} + \int_0^1 \mu(c_0(y))\mathrm{d}y/\mu(c_0(x))\right),$$

and consequently, for $x \in [0,1]$:

$$\rho(t,x) \leq \rho_0(X_t^{-1}(x)) \exp\left(\frac{3\sqrt{2M_0E_0}}{\mu_{\min}} + (E_0+M_0)\frac{C_0 t}{\mu_{\min}} + \int_0^1 \mu(c_0(y))\mathrm{d}y/\mu(c_0(x))\right). \quad (44)$$

Thus, for any $T > 0$ we have an upper bound C_T for the density ρ. It turns out that we may use it along with the identity (41) and estimates (42), (43) to derive a lower bound for ρ. Indeed we observe that

$$\int_0^t \int_0^1 \left\{p(\rho(\tau,y), c(\tau,y)) + \rho u^2(\tau,y)\right\} \mathrm{d}y\mathrm{d}\tau \geq 0$$

and that

$$\rho(t,x) \geq \rho_0(X_t^{-1}(x)) \exp\left(-\frac{3\sqrt{2M_0E_0}}{\mu_{\min}} - \frac{t}{\mu_{\min}} \sup_{[0,C_T]\times[m,M]} p(s_1,s_2) - \int_0^1 \mu(c)(t,y)\mathrm{d}y/\mu(c_0(x))\right), \quad (45)$$

where C_T is an upper bound for $\rho(t,x)$ on $[0,T]$. Thus we obtain that ρ is bounded and bounded away from vacuum from the estimates (44) and (45) and using the property on μ. From now on, we will just use that there exists a constant $C(T)$ such that

$$C(T)^{-1} \leq \rho\,(t,x) \leq C(T) \quad (46)$$

This shows the validity of estimate (25).

4.1.6. Energy estimates 'à la Hoff'

L^2-estimates of $\partial_x u$. In this section, we aim at deriving higher order estimates for the velocity. We take inspiration in the techniques introduced by Hoff in [22]. In the following, we will denote

$$\dot{u} := u_t + u\partial_x u,$$

which allows us to rewrite the first equation as

$$\rho\dot{u} - \partial_x(\mu(c)\partial_x u) + \partial_x p = 0. \tag{47}$$

Using the equation satisfied by $\mu(c)$, we observe that

$$\begin{aligned} -\int_0^1 \dot{u}\,\partial_x(\mu(c)\partial_x u) &= -\int_0^1 (u_t + u\partial_x u)\,\partial_x(\mu(c))\partial_x u) \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mu(c)(\partial_x u)^2 + \frac{1}{2}\int_0^1 \mu(c)\,(\partial_x u)^3 . \end{aligned} \tag{48}$$

Next, we see that

$$\begin{aligned} \int_0^1 \dot{u}\partial_x p &= \int_0^1 u_t\partial_x p + \int_0^1 u\partial_x u\partial_x p = -\int_0^1 p\partial_{xt}^2 u + \int_0^1 u\partial_x u\partial_x p \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p\partial_x u + \int_0^1 \partial_x u\partial_t p + \int_0^1 u\partial_x u\partial_x p \\ &= -\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p\partial_x u - \int_0^1 \rho\partial_1 p(\rho,c)(\partial_x u)^2, \end{aligned} \tag{49}$$

where we have used the identity (31) with p instead of H. Thus if we multiply Equation (47) with $\dot{u}$ and we integrate in space, then, taking in consideration the identities (48) and (49), we end up with

$$\int_0^1 \rho\dot{u}^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \mu(c)(\partial_x u)^2 - \frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 p\partial_x u = -\frac{1}{2}\int_0^1 \mu(c)\,(\partial_x u)^3 + \int_0^1 \rho\partial_1 p(\rho,c)(\partial_x u)^2,$$

which gives us after time integration

$$\begin{aligned} A_1(c,\rho,u)\,(t) &\overset{\text{def.}}{=} \frac{1}{2}\int_0^1 \mu(c)(\partial_x u)^2 + \int_0^t\int_0^1 \rho\dot{u}^2 \\ &= \frac{1}{2}\int_0^1 \mu(c_0)(\partial_x u_0)^2 - \int_0^1 p\,(\rho_0,c_0)\,\partial_x u_0 \\ &\quad + \int_0^1 p\partial_x u - \frac{1}{2}\int_0^t\int_0^1 \mu(c)\,(\partial_x u)^3 + \int_0^t\int_0^1 \rho\partial_1 p(\rho,c)(\partial_x u)^2. \end{aligned} \tag{50}$$

In the computations that follow, we will use the notation $A_1(t)$ instead of $A_1(c,\rho,u)(t)$. Let us observe that for all $t \geq 0$, we have that

$$\begin{aligned} \int_0^1 p\partial_x u &\leq \frac{1}{\mu_{\min}}\int_0^1 p^2 + \frac{1}{4}\int_0^1 \mu(c)(\partial_x u)^2 \\ &\leq \frac{1}{\mu_{\min}}\left(\sup_{[0,C_T]\times[m,M]} p\,(\tau,c)\right)^2 + \frac{1}{4}\int_0^1 \mu(c)(\partial_x u)^2 \end{aligned} \tag{51}$$

$$\leq \frac{1}{\mu_{\min}}\left(\sup_{[0,C_T]\times[m,M]} p(\tau,c)\right)^2 + \frac{1}{4}A_1(t). \tag{52}$$

Moreover, we have that

$$\int_0^t\int_0^1 \rho\partial_1 p(\rho,c)(\partial_x u)^2 \leq \frac{1}{\mu_{\min}} \sup_{[0,C_T]\times[m,M]} \tau\partial_1 p(\tau,c)\, E_0. \tag{53}$$

We write the remaining term as

$$\begin{aligned} -\frac{1}{2}\int_0^t\int_0^1 \mu(c)\,(\partial_x u)^3 &= \frac{1}{2}\int_0^t\int_0^1 (\partial_x u)^2\left(-\mu(c)\partial_x u + p\right) - \frac{1}{2}\int_0^t\int_0^1 p\,(\partial_x u)^2 \\ &\leq \frac{1}{2}\int_0^t\int_0^1 (\partial_x u)^2\left(-\mu(c)\partial_x u + p\right). \end{aligned} \tag{54}$$

To treat the remaining term of (54), we reproduce the computations at the beginning of Section 4.1.3 remarking that $\partial_x(\mu(c)\partial_x u - p) = \rho\dot{u}$. We obtain that, for any $x \in [0,1]$:

$$\left(\mu(c)\partial_x u - p\right)(t,x) - \int_0^1 (\mu(c)(t,y)\partial_x u(t,y) - p(t,y))\mathrm{d}y = \int_0^1\int_y^x \rho\dot{u}(t,z)\mathrm{d}z\mathrm{d}y,$$

and using Cauchy's inequality along with hypothesis (14), there holds:

$$\left\|(\mu(c)\partial_x u - p(\rho,c)) - \int_0^1 (\mu(c)\partial_x u - p(\rho,c))\right\|_{L^\infty(\mathbb{T})} \leq M_0^{\frac{1}{2}}\left(\int_0^1 \rho\dot{u}^2\right)^{\frac{1}{2}}. \tag{55}$$

Consequently, there holds

$$\left\|\mu(c)\partial_x u - p(\rho,c) - \int_0^1 (\mu(c)\partial_x u - p(\rho,c))\right\|_{L^\infty(\mathbb{T})} \leq M_0^{\frac{1}{2}} A_1^{\frac{1}{2}}(t). \tag{56}$$

On the other hand, we have (with an obvious meaning for the symbol $\mu_{\max}$)

$$\begin{aligned} &\int_0^t\int_0^1 |\partial_x u|^2 \left|\int_0^1 (\mu(c)\partial_x u - p(\rho,c))\right| \\ &\leq \int_0^t\left(\int_0^1 |\partial_x u|^2\right)^2 \\ &\quad + \int_0^t\left|\int_0^1 \mu(c)|\partial_x u|\right|^2 + C_0\int_0^t\int_0^1 |\partial_x u|^2\left(\int_0^1 \rho + \int_0^1 G(\rho,c)\right) \\ &\leq \int_0^t \|\partial_x u\|_{L^2(\mathbb{T})}^4 + \left(\mu_{\max} + \frac{C_0(M_0+E_0)}{\mu_{\min}}\right)E_0. \end{aligned}$$

From the last relations, we deduce that for any $\varepsilon > 0$, there holds:

$$\begin{aligned} &\frac{1}{2}\int_0^t\int_0^1 (\partial_x u)^2\left(-\mu(x)\partial_x u + p(\rho,c)\right) \\ &\leq \frac{1}{2}\int_0^t \|\partial_x u\|_{L^2(\mathbb{T})}^2 \left\|(\mu(c)\partial_x u - p(\rho,c)) - \int_0^1 (\mu(c)\partial_x u - p(\rho,c))\right\|_{L^\infty(\mathbb{T})} \end{aligned}$$

$$+\frac{1}{2}\int_0^t\int_0^1(\partial_x u)^2\left|\int_0^1(\mu(c)\partial_x u-p(\rho,c))\right|$$
$$\leq\left(\frac{1}{16\varepsilon}+1\right)\int_0^t\|\partial_x u\|_{L^2(\mathbb{T})}^4+\left(\mu_{\max}+\frac{C_0(M_0+E_0)}{\mu_{\min}}\right)E_0+\varepsilon M_0A_1(t).$$

Thus, taking $\varepsilon M_0=\frac{1}{4}$ we see that

$$\frac{1}{2}\int_0^t\int_0^1(\partial_x u)^2\left(-\mu\partial_x u+p\right)$$
$$\leq\left(\mu_{\max}+\frac{C_0(M_0+E_0)}{\mu_{\min}}\right)E_0+\left(\frac{M_0}{4}+1\right)\int_0^t\|\partial_x u\|_{L^2(\mathbb{T})}^4+\frac{1}{4}A_1(t). \tag{57}$$

Thus denoting by

$$C_{2,T}=\frac{1}{2}\int_0^1\mu(\partial_x u_0)^2-\int_0^1p(\rho_0,c_0)\,\partial_x u_0+\frac{1}{\mu_{\min}}\left(\sup_{[0,C_T]\times[m,M]}P(\tau,c)\right)^2$$
$$+\frac{1}{\mu_{\min}}\sup_{[0,C_T]\times[m,M]}\tau\partial_1 p(\tau,c)\,E_0+\left(\mu_{\max}+\frac{C_0(M_0+E_0)}{\mu_{\min}}\right)E_0$$

and using the identity (50) along with the estimates (52), (53), (56), (57), we gather that

$$A_1(t)\leq 2C_{2,T}+(M_0+2)\int_0^t\|\partial_x u\|_{L^2}^4$$
$$\leq 2C_{2,T}+\frac{(M_0+2)}{\mu_{\min}}\int_0^t\left\|\sqrt{\mu(c)}\,\partial_x u\right\|_{L^2(\mathbb{T})}^2A_1(\tau)\,\mathrm{d}\tau.$$

Owing to Gronwall's lemma, we then obtain that for all $t\in[0,T]$:

$$A_1(t)\leq 2C_{2,T}\exp\left(\frac{(M_0+2)}{\mu_{\min}}E_0\right) \tag{58}$$

This shows the validity of estimate (26). Using now (55), we have for all t

$$\|\mu(c)\partial_x u\|_{L^\infty(\mathbb{T})}\leq\|p\|_{L^\infty(\mathbb{T})}+\|\mu(c)\partial_x u-p\|_{L^1(\mathbb{T})}+\sqrt{M_0}\sqrt{A_1(t)}$$
$$\leq\sqrt{\mu_{\max}}\|\sqrt{\mu\circ c}\,\partial_x u\|_{L^2(\mathbb{T})}+2\|p\|_{L^\infty(\mathbb{T})}+\sqrt{M_0}\sqrt{A_1(t)}.$$

Integrating in time, and applying (58) this yields:

$$\int_0^T\|\partial_x u\|_{L^\infty(\mathbb{T}^2)}^2$$
$$\leq\frac{1}{\mu_{\min}^2}\left(\mu_{\max}E_0+2\sup_{[0,C_T]\times[m,M]}p(s_1,s_2)^2T+2M_0TC_{2,T}\exp\left(\frac{2(M_0+2)}{\mu_{\min}}E_0\right)\right). \tag{59}$$

Higher order estimates with time weights. In this part, we aim at obtaining estimate for the L^2-norm of $\partial_x\dot{u}$. This will be useful to recover regularity properties of u. The idea is to apply the operator $\partial_t+u\partial_x$ to the velocity's equation:

$$(\partial_t+u\partial_x)(\rho\dot{u})-(\partial_t+u\partial_x)\partial_x(\mu(c)\partial_x u)+\left(\partial_t\partial_x p+u\partial_{xx}p\right)=0$$

and to test it with $\min\{1,t\}\dot{u}$. We begin by observing that

$$\int_0^1(\rho\dot{u})_t\,\dot{u}=\int_0^1\rho_t\dot{u}^2+\frac{1}{2}\int_0^1\rho\frac{\mathrm{d}\dot{u}^2}{\mathrm{d}t}$$

$$= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho\dot{u}^2 + \frac{1}{2}\int_0^1 \rho_t\dot{u}^2.$$

Observe that

$$\int_0^1 u\partial_x(\rho\dot{u})\,\dot{u} = -\int_0^1 \rho\dot{u}\partial_x(u\dot{u}) = -\int_0^1 \partial_x u\rho\dot{u}^2 + \frac{1}{2}\int_0^1 (\rho u)_x\,\dot{u}^2.$$

Summing the above two relations yields

$$\int_0^1 (\partial_t + u\partial_x)(\rho\dot{u})\dot{u} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho\dot{u}^2 - \int_0^1 \partial_x u\rho\dot{u}^2. \tag{60}$$

Next, we take a look at the second term

$$\begin{aligned}
&-\int_0^1 (\partial_t + u\partial_x)\partial_x(\mu(c)\partial_x u))\,\dot{u} \\
&= \frac{1}{2}\int_0^1 \mu(c)(\partial_x\dot{u})^2 - \frac{3}{2}\int_0^1 \mu(c)(\partial_x u)^2\partial_x\dot{u} + \frac{1}{2}\int_0^1 \partial_x\mu(c)|\partial_x u|^2\dot{u}.
\end{aligned} \tag{61}$$

Let us observe that

$$\begin{aligned}
\int_0^1 (\partial_x p_t + u\partial_{xx}^2 p)\dot{u} &= -\int_0^1 p_t\partial_x u + \int_0^1 u\partial_{xx}^2 p\dot{u} \\
&= \int_0^1 u\partial_x p\partial_x\dot{u} + \int_0^1 \rho\partial_1 p(\rho,c)\partial_x u\partial_x\dot{u} + \int_0^1 u\partial_{xx}^2 p\dot{u} \\
&= -\int_0^1 \partial_x u\partial_x p\dot{u} + \int_0^1 \rho\partial_1 p(\rho,c)\partial_x u\partial_x\dot{u} \\
&= \int_0^1 \partial_x u\rho\dot{u}^2 + \frac{1}{2}\int_0^1 \mu(c)\partial_x\dot{u}\,(\partial_x u)^2 \\
&\quad - \frac{1}{2}\int_0^1 |\partial_x u|^2\dot{u}\partial_x\mu(c) + \int_0^1 \rho\partial_1 p(\rho,c)\partial_x u\partial_x\dot{u}.
\end{aligned} \tag{62}$$

where we have used the equation of the velocity to replace

$$-\partial_x p = \rho\dot{u} - \partial_x(\mu(c)\partial_x u).$$

We sum up the relations (60), (61) and (62) to obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho\dot{u}^2 + \frac{1}{2}\int_0^1 \mu(c)\,(\partial_x\dot{u})^2 = -\int_0^1 \rho\partial_1 p(\rho,c)\partial_x u\partial_x\dot{u} + \int_0^1 \mu(c)\partial_x\dot{u}\,(\partial_x u)^2.$$

Multiplying with $\kappa(t)$ and integrating in time lead to

$$\begin{aligned}
A_2(c,\rho,u)(t) &\overset{\text{def.}}{=} \frac{1}{2}\int_0^1 \kappa(t)\,\rho\dot{u}^2 + \frac{1}{2}\int_0^t\int_0^1 \kappa\mu(c)\,(\partial_x\dot{u})^2 \\
&= \int_0^{\min(1,t)}\int_0^1 \rho\dot{u}^2 - \int_0^t\int_0^1 \kappa\rho\partial_1 p(\rho,c)\partial_x u\partial_x\dot{u} + \int_0^t\int_0^1 \kappa\mu(c)\partial_x\dot{u}\,(\partial_x u)^2.
\end{aligned} \tag{63}$$

In the computations that follow, we will use the notation $A_2(t)$ instead of $A_2(c,\rho,u)(t)$. Obviously using (58) we have that

$$\int_0^{\min(1,t)}\int_0^1 \rho\dot{u}^2 \leq 2C_{2,T}\exp\left(\frac{(M_0+2)}{\mu_{\min}}E_0\right), \tag{64}$$

for all $t\in[0,T]$. Next, using Cauchy's inequality we have that

$$\begin{aligned}
&\int_0^t\int_0^1 \kappa\rho\partial_1 p(\rho,c)\partial_x u\partial_x\dot{u}\\
&\leq \sup_{[0,C_T]\times[m,M]} \tau\partial_1 p(\tau,c)\left(\int_0^t\int_0^1 \frac{1}{\mu_{\min}}(\partial_x u)^2\right)^{\frac{1}{2}}\left(\int_0^t\int_0^1 \kappa\mu(c)|\partial_x\dot{u}|^2\right)^{\frac{1}{2}}\\
&\leq \left\{\sup_{[0,C_T]\times[m,M]} \tau\partial_1 p(\tau,c)\right\}^2\frac{4E_0}{\mu_{\min}^2}+\frac{1}{4}A_2(t).
\end{aligned}\tag{65}$$

Finally, using again (58), we arrive at

$$\begin{aligned}
\int_0^t\int_0^1 \kappa\mu(c)\partial_x\dot{u}\,(\partial_x u)^2 &\leq \frac{1}{8}\int_0^t\int_0^1 \kappa\mu(c)(\partial_x\dot{u})^2+2\mu_{\max}\int_0^t\int_0^1(\partial_x u)^4\\
&\leq \frac{1}{4}A_2(t)+\frac{4\mu_{\max}}{\mu_{\min}}\int_0^t\|\partial_x u\|_{L^\infty(\mathbb{T})}^2\sup_{(0,T)}\frac{1}{2}\int_0^1\mu(c)|\partial_x u|^2\\
&\leq \frac{1}{4}A_2(t)+C_{3,T},
\end{aligned}\tag{66}$$

with $C_{3,T}$ depending on initial data and T only. In view of the three estimates established above, (64), (65) and (66) we gather that

$$A_2(t)\leq\left\{\sup_{(\tau,c)\in[0,C_T]\times[0,1]}\tau\partial_1 p(\tau,c)\right\}^2\frac{8E_0}{\mu_{\min}^2}+4C_{2,T}\exp\left(\frac{(M_0+2)E_0}{\mu_{\min}^2}\right)+2C_{3,T}. \tag{67}$$

Thus, estimate (27) holds true. Combining estimate (55) with the bound on $A_2(t)$ we get that

$$\begin{aligned}
\kappa(t)^{\frac{1}{2}}\left\|\mu\partial_x u-p\right\|_{L^\infty} &\leq M_0^{\frac{1}{2}}\left(\int_0^1\sigma\rho\dot{u}^2\right)^{\frac{1}{2}}+C_0(M_0+E_0)\\
&\leq\sqrt{M_0A_2(t)}+C_0(M_0+E_0)+\sigma\sqrt{\mu_{\max}A_1(t)}.
\end{aligned}$$

In view of the uniform bounds for the density and concentration function, there exists a constant $C(T)$ such that

$$\kappa(t)^{\frac{1}{2}}\|\mu\partial_x u(t)\|_{L^\infty}\leq C(T), \tag{68}$$

for any $t\in[0,T]$.

Uniform bounds for the time derivative of the solution. In this part, we show how to use the bounds of $A_1(c,\rho,u)$ and $A_2(c,\rho,u)$ to get information for the time derivative of the solution. Analyzing the estimates (34), (58), (67) and (46) we get that

$$\int_0^1\frac{\rho u^2}{2}+\int_0^1 G(\rho,c)+\int_0^t\int_0^1\mu(c)(\partial_x u)^2+\frac{1}{2}\int_0^1\mu(c)(\partial_x u)^2+\int_0^t\int_0^1\rho\dot{u}^2$$

$$+\frac{1}{2}\int_0^1 \kappa(t)\,\rho\dot{u}^2+\frac{1}{2}\int_0^t\int_0^1 \kappa(t)\,\mu(c)\,(\partial_x\dot{u})^2\le C(T)$$

for any $t\in[0,T]$ and

$$C(T)^{-1}\le\rho(t,x)\le C(T)$$

for some constant $C(T)$ that depends on the initial data and on T.

Thus we get that

$$u\in L_T^\infty(L^2(\mathbb{T}^1)) \tag{69}$$

along with

$$\partial_x u\in L_T^\infty(L^2(\mathbb{T}^1))\cap L_T^2(L^2(\mathbb{T}^1)). \tag{70}$$

Next, let us note that

$$(\partial_t u+u\partial_x u)^2\ge\frac{1}{2}(\partial_t u)^2-(u\partial_x u)^2$$

and consequently

$$\frac{1}{2}\int_0^1(\partial_t u)^2\le C(T)\int_0^1\rho\dot{u}^2+2\,\|u\|_{L_T^\infty(L^\infty(\mathbb{T}))}\int_0^1(\partial_x u)^2.$$

By time integration, we obtain that

$$\begin{aligned}\frac{1}{2}\int_0^t\int_0^1(\partial_t u)^2&\le C(T)\int_0^t\int_0^1\rho\dot{u}^2+2\int_0^t\int_0^1(u\partial_x u)^2\\&\le C(T)A_1(t)+2\frac{\|u\|^2_{L_T^\infty(L^\infty(\mathbb{T}))}}{\mu_{\min}}E_0.\end{aligned}$$

Thus we obtain that

$$\partial_t u\in L_T^2(L^2(\mathbb{T}^1)) \tag{71}$$

As a consequence of (69), (70) and of (71), we obtain the uniform bound

$$u\in H^1\left((0,1)\times\mathbb{T}^1\right). \tag{72}$$

Moreover, we also have that

$$\begin{aligned}\frac{\kappa(t)}{2}\int_0^1(\partial_t u)^2&\le C(T)\int_0^1\kappa\rho\dot{u}^2+2\,\|u\|^2_{L_T^\infty(L^\infty(\mathbb{T}))}\int_0^1\kappa(\partial_x u)^2\\&\le C(T)A_2(t)+\frac{2}{\mu_{\min}}\,\|u\|^2_{L_T^\infty(L^\infty(\mathbb{T}))}\,A_1(t).\end{aligned}$$

Thus for all $t>0$ we have that

$$\partial_t u\in L^\infty((t,T);(L^2(\mathbb{T}^1)).$$

Next, we write that

$$\begin{aligned}\|\dot{u}(t,\cdot)\|^2_{L^\infty(\mathbb{T})}&\le\|\dot{u}\|_{L^2(\mathbb{T})}\,\|\partial_x\dot{u}\|_{L^2(\mathbb{T})}\le\sqrt{C(T)}\left\|\rho^{\frac{1}{2}}\dot{u}\right\|_{L^2(\mathbb{T})}\|\partial_x\dot{u}\|_{L^2(\mathbb{T})}\\&\le C(T)\left\|\rho^{\frac{1}{2}}\dot{u}\right\|^2_{L^2(\mathbb{T})}+\|\partial_x\dot{u}\|^2_{L^2(\mathbb{T})}\end{aligned}$$

thus we get that

$$\int_0^T \kappa(t)\,\|\dot{u}(t,\cdot)\|^2_{L^\infty(\mathbb{T})} \le C(T)\int_0^T\int_0^1 \kappa(t)\rho\dot{u}^2 + \int_0^T\int_0^1 \kappa(t)(\partial_x\dot{u})^2$$
$$\le (T + C(T))A_2(c,\rho,u)(T) \le C(T).$$

Using the uniform bounds for $u \in L^\infty_T(L^2(\mathbb{T}^1))$ and $\partial_x u \in L^\infty_T(L^2(\mathbb{T}^1))$, we get a uniform bound for u in $L^\infty((0,1)\times\mathbb{T}^1)$. Using the fact that $\dot{u} = u_t + u\partial_x u$ along with the estimate (68) we get that

$$\int_0^T \kappa(t)\,\|\partial_t u(t)\|^2_{L^\infty(\mathbb{T})}\,\mathrm{d}t \le C(T).$$

Gathering the information of this section yields estimates (28) and (29).

4.1.7. Stability of a sequence of solutions

Using the estimates established in the preceding sections, we are in the position of proving the following stability result.

Theorem 4.2: *Let us consider (c^n,ρ^n,u^n) a sequence of weak solutions of* (10) *with the initial data*

$$\begin{cases} c_0^n \in L^\infty(\mathbb{T}^1), 0 \le \inf_n \inf_{x\in\mathbb{T}^1} c_0^n \le \sup_n \sup_{x\in\mathbb{T}^1} c_0^n(x) < \infty \\ \rho_0^n \in L^\infty(\mathbb{T}^1) \text{ such that } 0 < \inf_n \inf_{x\in\mathbb{T}^1} \rho_0^n \le \sup_n \sup_{x\in\mathbb{T}^1} \rho_0^n < \infty \\ G\left(\rho_0^n,c_0^n\right) \in L^1(\mathbb{T}^1);\ \sup_n \int_0^1 G\left(\rho_0^n,c_0^n\right) < \infty \\ u_0^n \in H^1\left(\mathbb{T}^1\right),\ \sup_n \left\|u_0^n\right\|_{H^1} < \infty \end{cases}$$

such that

$$\begin{cases} c_0^n \to c_0 \text{ strongly in } L^2(\mathbb{T}^1), \\ \rho_0^n \to \rho_0 \text{ strongly in } L^2(\mathbb{T}^1), \\ u_0^n \rightharpoonup u_0 \text{ weakly in } H^1\left(\mathbb{T}^1\right). \end{cases}$$

Furthermore, we suppose that there exists a constant $M \in (0,\infty)$ so that the sequence of solutions $(c^n,\rho^n,u^n)_n$ satisfies the following uniform bounds:

$$\int_0^1 \frac{\rho^n(u^n)^2}{2} + \int_0^1 G\left(\rho^n,c^n\right) + \int_0^t\int_0^1 \mu(c^n)(\partial_x u^n)^2 \le \int_0^1 \frac{\rho_0^n(u_0)^n}{2} + \int_0^1 G\left(\rho_0^n,c_0^n\right), \tag{73}$$

$$0 \le \inf_n \inf_{x\in\mathbb{T}^1} c_0^n(x) \le c^n(t,x) \le \sup_n \sup_{x\in\mathbb{T}^1} c_0^n(x) < +\infty, \tag{74}$$

$$M^{-1} \le \rho^n(t,x) \le M, \tag{75}$$

$$\frac{1}{2}\int_0^1 \mu(c^n)(\partial_x u^n)^2 + \int_0^t\int_0^1 \rho^n(\dot{u}^n)^2 \le M, \tag{76}$$

$$\frac{1}{2}\int_0^1 \kappa(t)\,\rho^n(\dot{u}^n)^2 + \frac{1}{2}\int_0^t\int_0^1 \kappa(t)\,\mu(c)\left(\partial_x\dot{u}^n\right)^2 \le M, \tag{77}$$

$$\left\|u^n\right\|_{L^\infty((0,1)\times\mathbb{T}^1)} + \left\|u^n\right\|_{H^1((0,1)\times\mathbb{T}^1)} \le M, \tag{78}$$

$$\left\|\kappa^{\frac{1}{2}}(\partial_t u^n(t),\partial_x u^n(t))\right\|_{L^2(0,T;L^\infty(\mathbb{T}^1))} \le M, \tag{79}$$

for all $t \in [0,T]$ *where* $\kappa(t) = \min\{t,1\}$. *Then up to an extraction* (c^n, ρ^n, u^n) *tends to a triplet* (c, ρ, u) *which is a solution of the system* (10) *on* $[0,T]$ *with initial data* (c_0, ρ_0, u_0). *Moreover* (c, ρ, u) *verifies the bounds* (23)–(29).

Proof: The proof of this statement is rather classical and as such we give just a few details. Using the Arzelà–Ascoli theorem, we have that

$$u^n \to u \text{ uniformly on } [0,T] \times [0,1].$$

Let us consider X_t^n the flow corresponding to u^n:

$$X_t^n(x) = x + \int_0^t u^n(\tau, X_\tau^n(x))\mathrm{d}\tau.$$

Of course, we have that

$$\frac{dX_t^n}{\mathrm{d}x}(x) = \exp\left(-\int_0^t \partial_x u^n(\tau, X_\tau^n(x))\mathrm{d}\tau\right) \geq \exp\left(-\sqrt{T}M\right).$$

We note that

$$\begin{aligned}\left|X_t^n(x) - X_t^m(x)\right| &\leq \left|\int_0^t u^n(\tau, X_\tau^n(x))\mathrm{d}\tau - \int_0^t u^n(\tau, X_\tau^m(x))\mathrm{d}\tau\right| \\ &\quad + \left|\int_0^t u^n(\tau, X_\tau^m(x))\mathrm{d}\tau - \int_0^t u^m(\tau, X_\tau^m(x))\mathrm{d}\tau\right| \\ &\leq T\left\|u^n - u^m\right\|_{L^\infty([0,T]\times\mathbb{T}^1)} + \int_0^t \left\|\partial_x u^n(\tau,\cdot)\right\|_{L^\infty(\mathbb{T})}\left|X_\tau^n(x) - X_\tau^m(x)\right|\end{aligned}$$

which implies that

$$\begin{aligned}\left|X_t^n(x) - X_t^m(x)\right| &\leq T\left\|u^n - u^m\right\|_{L^\infty([0,T]\times\mathbb{T}^1)} \exp\left(\int_0^T \left\|u^n(\tau,\cdot)\right\|_{L^\infty(\mathbb{T})}\right) \\ &\leq C(M)T\left\|u^n - u^m\right\|_{L^\infty([0,T]\times\mathbb{T}},\end{aligned}$$

for all $(t,x) \in [0,T] \times \mathbb{R}$. The conclusion is that

$$X_t^n(x) \to X(x) \text{ uniformly on } [0,T] \times [0,1].$$

Next, from the a-priori bounds on ρ_n, we get the existence of a $\rho \in L^\infty((0,T) \times \mathbb{T}^1)$ such that

$$\rho^n \rightharpoonup \rho \text{ in } L^2((0,T)\times\mathbb{T}^1) \cap L^\infty\left((0,T)\times\mathbb{T}^1\right).$$

Mixing the strong convergence of u with the weak convergence of ρ_n it is not hard to see that

$$\partial_t \rho + \partial_x(\rho u) = 0.$$

Let us observe that this entails

$$\begin{cases}(\rho^n)_t^2 + \partial_x((\rho^n)^2 u^n) + (\rho^n)^2\partial_x u^n = 0, \\ \rho_t^2 + \partial_x(\rho^2 u) + \rho^2\partial_x u = 0\end{cases}$$

and thus by subtracting the above equation and integrating over $(0,1)$ we get that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1((\rho^n)^2 - \rho^2) \leq \int_0^1 \rho^2(\partial_x u^n - \partial_x u) + \int_0^1((\rho^n)^2 - \rho^2)\partial_x u^n.$$

We integrate in time and we find that

$$\int_0^1 ((\rho^n)^2 - \rho^2)(t) \leq \int_0^1 ((\rho_0^n)^2 - \rho_0^2) + \int_0^T \int_0^1 \rho^2(\partial_x u^n - \partial_x u) + \int_0^T \int_0^1 ((\rho^n)^2 - \rho^2)\partial_x u^n$$

for all $t \in [0, T]$. We arbitrarily fix $\varepsilon > 0$ and using the uniform bounds on $(\partial_x u^n)_n \in L^2((0, T) \times \mathbb{T}^1)$ and the fact that $\rho^2 \in L^2((0, T) \times \mathbb{T}^1)$, we may find n_ε such that

$$\int_0^1 \rho^2(\partial_x u^n - \partial_x u) \leq \varepsilon/T$$

for all $n \geq n_\varepsilon$. Therefore, via Gronwall's lemma we get that

$$\begin{aligned}\int_0^1 ((\rho^n)^2 - \rho^2)(t) &\leq \left(\int_0^1 ((\rho_0^n)^2 - \rho_0^2) + \varepsilon\right) \exp\left(\int_0^T \|\partial_x u^n\|_{L^\infty}\right) \\ &\leq \left(\int_0^1 ((\rho_0^n)^2 - \rho_0^2) + \varepsilon\right) \exp\left(\sqrt{T}M\right).\end{aligned}$$

for all $n \geq n_\varepsilon$. We perform a time integration in the above relation to obtain that

$$\int_0^T \int_0^1 ((\rho^n)^2 - \rho^2) \leq T\left(\int_0^1 ((\rho_0^n)^2 - \rho_0^2) + \varepsilon\right) \exp\left(\sqrt{T}M\right).$$

After some obvious manipulation, we conclude that

$$\lim_{n\to\infty} \int_0^T \int_0^1 (\rho^n)^2 = \int_0^T \int_0^1 \rho^2$$

which combined with the weak convergence of ρ^n towards ρ in $L^2((0, T) \times \mathbb{T}^1)$ leads to the conclusion that

$$\rho^n \to \rho \text{ strongly in } L^2((0, T) \times \mathbb{T}^1)$$

and, modulo a subsequence

$$\rho^n \to \rho$$

almost everywhere on $(0, T) \times \mathbb{T}^1$. Moreover, the same type of argument allows to conclude that

$$\rho^n \to \rho \text{ strongly in } L_T^\infty(L^2(\mathbb{T}^1)).$$

A similar conclusion holds for the sequence $(c^n)_n$: there exists $c \in L^\infty((0, T) \times \mathbb{T}^1)$ such that up to a subsequence we have that

$$\begin{cases} c^n \rightharpoonup c \text{ weakly in } L^\infty\left((0, T) \times \mathbb{T}^1\right) \text{ and} \\ c^n \to c \text{ a.e. on } (0, T) \times \mathbb{T}^1. \end{cases}$$

The fact that $p(\rho^n, c^n)$ is uniformly bounded in $L^\infty((0, T) \times \mathbb{T}^1)$ along with the continuity of p and the convergence properties of $(\rho^n, c^n)_n$ enable us to conclude that

$$p\left(\rho^n, c^n\right) \to p\left(\rho, c\right) \text{ weakly in } L^\infty\left((0, T) \times \mathbb{T}^1\right).$$

The expected uniform bounds for (ρ, c, u) yield also from classical pointwise/weak convergence arguments. ■

4.1.8. Uniqueness of solutions

The aim of this section is to prove the uniqueness part of Theorem 4.1. To accomplish this task, we work in Lagrangian coordinates. This framework has the advantage of decoupling the hyperbolic part of the equation from the parabolic part. We are left with a more complicated equation but which, on short times, is not far from being parabolic. In what remains of this section, we consider two solutions (ρ_i, c_i, u_i) of (10) on $(0, T)$ satisfying estimates claimed in Theorem 4.1. We note that the involved constant $C(T)$ may be chosen independent of the solution.

The 1*D Navier–Stokes system in Lagrangian coordinates.* In this part, we derive the Lagrangian formulation of the system (10). To do so, let us recall that for arbitrary classical solution (ρ, c, u) to (10) (such as the one chosen above) the flow of u is defined by

$$\begin{cases} \dfrac{\partial X}{\partial t}(t,x) = u(t, X(t,x)), \\ X(0,x) = x. \end{cases} \tag{80}$$

Moreover, as was proved in Section 4.1.4, for each $t \in (0, T)$, $X(t, \cdot)$ is a diffeomorphism from $\mathbb{R}$ to $\mathbb{R}$. We emphasize that for the sake of clarity when dealing with its partial derivatives, in the rest of the section we will denote the flow of u by $X(t, x)$. As u is 1−periodic, it is easy to see that

$$X(t, x+1) = X(t,x) + 1$$

For any function $v : [0, \infty) \times \mathbb{T}^1 \to \mathbb{R}$, we denote by $\tilde{v}$ the function defined as

$$\tilde{v}(t,x) := v(t, X(t,x))$$

Then, we also have

$$\begin{aligned} \tilde{v}(t, x+1) &:= v(t, X(t, x+1)) = v(t, X(t,x)+1) \\ &= v(t, X(t,x)) = \tilde{v}(t,x). \end{aligned}$$

In the following lines, we aim at rewriting system (10) in the variables $(\tilde{c}, \tilde{\rho}, \tilde{u})$. First of all, as c is transported by the flow, we have that

$$\frac{\partial \tilde{c}}{\partial t}(t,x) = 0. \tag{81}$$

Next, we see that

$$X(t,x) = x + \int_0^t u(\tau, X(\tau, x)) = x + \int_0^t \tilde{u}(\tau, x)\, d\tau,$$

and thus

$$\frac{\partial X}{\partial x}(t,x) = 1 + \int_0^t \partial_x \tilde{u}(\tau, x)\, d\tau.$$

It follows that

$$\partial_x \tilde{v}(t,x) = \widetilde{\partial_x v}(t,x)\, \frac{\partial X}{\partial x}(t,x) \tag{82}$$

and that

$$\widetilde{\partial_x v}(t,x) = \frac{\partial X}{\partial x}(t,x)^{-1}\, \partial_x \tilde{v}(t,x) = \frac{1}{1 + \int_0^t \partial_x \tilde{u}(\tau,x)\, d\tau}\, \partial_x \tilde{v}(t,x). \tag{83}$$

Let us investigate the equation for ρ. We observe that

$$0 = \tilde{\rho}_t(t,x) + \tilde{\rho}(t,x)\, \widetilde{\partial_x u}(t,x) = \tilde{\rho}_t(t,x) + \tilde{\rho}(t,x) \frac{1}{1 + \int_0^t \partial_x \tilde{u}(\tau,x)\, d\tau}\, \partial_x \tilde{u}(t,x),$$

which leads to

$$0 = \left(1 + \int_0^t \partial_x \tilde{u}\,(\tau, x)\,\mathrm{d}\tau\right)\tilde{\rho}_t + \partial_x \tilde{u}\,(t,x)\,\tilde{\rho}(t,x)$$

and we obtain that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\left(1 + \int_0^t \partial_x \tilde{u}\,(\tau, x)\,\mathrm{d}\tau\right)\tilde{\rho}\right) = 0. \tag{84}$$

Finally, using (83) along with (84), we may rewrite the equation of the velocity as

$$\rho_0\,(x)\,\partial_t \tilde{u} - \partial_x \left(\left(\frac{\partial X}{\partial x}\right)^{-1} \mu(\tilde{c})\,\partial_x \tilde{u}\right) + \partial_x p\left(\tilde{\rho}, \tilde{c}\right) = 0. \tag{85}$$

Putting together Equations (81), (84) and (85) we deduce that the system (10) can be written in Lagrangian coordinates as

$$\begin{cases} \partial_t \tilde{c} = 0, \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\left(\dfrac{\partial X}{\partial x}\,(t,x)\,\tilde{\rho}\right) = 0, \\ \rho_0\,(x)\,\partial_t \tilde{u} - \partial_x \left(\left(\dfrac{\partial X}{\partial x}\right)^{-1} \mu(\tilde{c})\,\partial_x\,\tilde{u}\right) + \partial_x p\left(\tilde{\rho}, \tilde{c}\right) = 0, \\ X\,(t,x) = x + \displaystyle\int_0^t \tilde{u}\,(\tau, x)\,\mathrm{d}\tau. \end{cases} \tag{86}$$

Let us also derive some useful inequalities. Of course, we have that

$$C(T)^{-1} \leq \tilde{\rho}(t,x) \leq C\,(T)\,.$$

Thus one gets that

$$\frac{C(T)}{\inf \rho_0} \geq \left(\frac{\partial X}{\partial x}\,(t,x)\right)^{-1} = \frac{\tilde{\rho}(t,x)}{\rho_0(x)} \geq \frac{C(T)^{-1}}{\sup \rho_0}. \tag{87}$$

The proof of the uniqueness part of Theorem 4.1. Let us consider two solutions (c_i, ρ_i, u_i), $i \in \{1,2\}$ generated by the same initial data:

$$\begin{cases} \partial_t c_i + u_i \partial_x c_i = 0, \\ \partial_t \rho_i + \partial_x\,(\rho_i u_i) = 0, \\ \partial_t\,(\rho_i u_i) + \partial_x\left(\rho_i u_i^2\right) - \partial_x(\mu(c_i)\partial_x u_i) + \partial_x p_i = 0, \\ \left(c_{i|t=0}, \rho_{i|t=0}, u_{i|t=0}\right) = (c_0, \rho_0, u_0)\,. \end{cases} \tag{88}$$

Considering the flows generated by u_i

$$X_i(t,x) = x + \int_0^t u_i\,(\tau, X_\tau\,(x))\,\mathrm{d}\tau$$

and denoting with tilde

$$\tilde{v}_i\,(t,x) = v_i\,(t, X_i\,(t,x))$$

for $v \in \{c, \rho, u\}$ we get that

$$\begin{cases} \partial_t \tilde{c}_i = 0 \\ \partial_t \left(\dfrac{\partial X^i}{\partial x} \tilde{\rho}_i \right) = 0, \\ \rho_0 \partial_t \tilde{u}_i - \mu \partial_x \left(\left(\dfrac{\partial X^i}{\partial x} \right)^{-1} \mu(\tilde{c}_i)\, \partial_x \tilde{u}_i \right) + \partial_x p\left(\tilde{\rho}_i, \tilde{c}_i\right) = 0, \\ X_i(t,x) = x + \displaystyle\int_0^t \tilde{u}_i\left(\tau, x\right) \mathrm{d}\tau. \end{cases} \tag{89}$$

Remarking that $\mu(\tilde{c}_1) = \mu(\tilde{c}_2)$, we have that

$$\rho_0 \partial_t \delta\tilde{u} - \partial(\mu(\tilde{c}_1)\partial_x \delta\tilde{u}) = \partial_x \delta F_1 + \partial_x \delta F_2, \tag{90}$$

where

$$\begin{cases} \delta F_1 = p\left(K\left(\displaystyle\int_0^t \partial_x \tilde{u}_1\right)\rho_0, c_0\right) - p\left(K\left(\displaystyle\int_0^t \partial_x \tilde{u}_2\right)\rho_0, c_0\right), \\ \delta F_2 = \mu(\tilde{c}_1) L\left(\displaystyle\int_0^t \partial_x \tilde{u}_1\right)\partial_x \tilde{u}_1 - \mu(\tilde{c}_1) L\left(\displaystyle\int_0^t \partial_x \tilde{u}_2\right)\partial_x \tilde{u}_2. \end{cases}$$

where

$$K\left(s\right) = \frac{1}{1+s}, \quad L\left(s\right) = \frac{s}{1+s}.$$

We multiply (89) with $\delta\tilde{u}$ and integrate over $\mathbb{T}^1$. Using Cauchy's inequality, we obtain that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int_0^1 \rho_0(\delta\tilde{u})^2 + \frac{\mu_{\min}}{4}\int_0^1 \left(\partial_x(\delta\tilde{u})\right)^2 \leq \frac{1}{\mu_{\min}}\int_0^1 (\delta F_1)^2 + (\delta F_2)^2.$$

Let us estimate the L^2-norm of δF_1 and δF_2. We begin with by noticing that

$$K\left(\int_0^t \partial_x \tilde{u}_1\right) - K\left(\int_0^t \partial_x \tilde{u}_2\right) = \frac{\int_0^t \partial_x \delta\tilde{u}}{\partial_x X^1(t,x)\partial_x X^2(t,x)}$$

so that we may write

$$\begin{aligned} \delta F_1\left(t\right) &= p\left(K\left(\int_0^t \partial_x \tilde{u}_1\right)\rho_0, c_0\right) - p\left(K\left(\int_0^t \partial_x \tilde{u}_2\right)\rho_0, c_0\right) \\ &\leq \sup_{\tau \in [C(T)^{-1}, C(T)]} \partial_1 p(\tau, c_0) \frac{\left|\int_0^t \partial_x \delta\tilde{u}\right|}{\partial_x X^1(t,x)\partial_x X^2(t,x)}. \end{aligned}$$

Using inequality (87), we get that

$$\frac{1}{\mu_{\min}}\int_0^1 (\delta F_1)^2 \leq t\frac{C(T)}{\mu_{\min}}\int_0^t\int_0^1 (\partial_x \delta\tilde{u})^2. \tag{91}$$

Next, we have that

$$\delta F_2 = \mu(\tilde{c}_1) L\left(\int_0^t \partial_x \tilde{u}_1\right)\partial_x \tilde{u}_1 - \mu(\tilde{c}_1) L\left(\int_0^t \partial_x \tilde{u}_2\right)\partial_x \tilde{u}_2$$

$$= \mu(\tilde{c}_1)\left(K\left(\int_0^t \partial_x \tilde{u}_1\right) - K\left(\int_0^t \partial_x \tilde{u}_2\right)\right)\partial_x \tilde{u}_1 + \mu(\tilde{c}_1)L\left(\int_0^t \partial_x \tilde{u}_2\right)\left(\partial_x \delta \tilde{u}\right).$$

Thus, using again the inequality (87), we get that

$$\int_0^1 (\delta F_2)^2 \leq \mu_{\max} C(T)\left(t\int_0^1 \left\|\partial_x \tilde{u}_1\right\|^2_{L^\infty(\mathbb{T})}\int_0^t (\partial_x \delta\tilde{u})^2 + \mu_{\max} t\int_0^1 \left\|\partial_x \tilde{u}_2\right\|^2_{L^\infty(\mathbb{T})}\int_0^t (\partial_x \delta\tilde{u})^2\right). \tag{92}$$

Putting together the inequalities (91), (92) and integrating in time, we get that

$$\frac{1}{2}\int_0^1 \rho_0(\delta\tilde{u})^2 + \frac{\mu_{\min}}{8}\int_0^t\int_0^1 \left(\partial_x(\delta\tilde{u})\right)^2 \leq 0$$

for any $t \in [0, T_0]$ with T_0 sufficiently small. Thus we get a local uniqueness property. Reiterating this process gives us the two solutions coincide on their whole domain of definition.

5. Definition of the mesoscopic system and derivation of the homogenized compressible bifluid system

In this section, we justify mathematically the derivation of the Baer–Nunziato model with physical relaxation term at the macroscopic level from the mesoscopic description that we have proposed in Section 2.

5.1. Homogenization procedure and main result

From now on, we consider a sequence of initial data (ρ_0^n, c_0^n, u_0^n) that satisfy:

$$\begin{cases} \rho_0^n \in L^\infty\left(\mathbb{T}^1\right) \text{ with } 0 < \inf\limits_{n,x\in\mathbb{T}^1} \rho_0^n(x) \leq \rho_0^n(x) \leq \sup\limits_{n,x\in\mathbb{T}^1} \rho_0^n(x) \leq M < +\infty, \\ c_0^n(1-c_0^n) = 0 \quad \text{a.e. on } \mathbb{T}^1 \text{ with } \quad c_0^n \in [0,1], \\ \displaystyle\int_0^1 \left(c_0^n p_+(\rho_0^n) + (1-c_0^n)p_-(\rho_0^n)\right) \leq M, \\ u_0^n \in H^1\left(\mathbb{T}^1\right) \text{ such that } \left\|u_0^n\right\|_{H^1} \leq M \end{cases} \tag{93}$$

with $M > 0$ independent of n. We note that these assumptions are satisfied in particular for initial configurations as depicted in (8). The bounds (93) allow us to conclude that there exists $(\rho_0, c_0, u_0) \in L^\infty(\mathbb{T}^1) \times L^\infty(\mathbb{T}^1) \times H^1(\mathbb{T}^1)$ such that

$$\rho_0^n \rightharpoonup \rho_0 \text{ in } L^\infty\left(\mathbb{T}^1\right) - w\star, \quad c_0^n \rightharpoonup c_0 \text{ in } L^\infty\left(\mathbb{T}^1\right) - w\star, \quad u_0^n \rightharpoonup u_0 \text{ in } H^1\left(\mathbb{T}^1\right).$$

Furthermore, given $n \in \mathbb{N}^*$ the initial data (ρ_0^n, c_0^n, u_0^n) enters the scope of Theorem 4.1. So, we can associate to this initial data a solution (ρ^n, c^n, u^n) to (6). Moreover, this sequence satisfies the following uniform bounds on any interval $[0, T]$ independent of n:

$$\int_0^1 \frac{\rho^n(u^n)^2}{2} + \int_0^1 \left(G(\rho^n, c^n)\right) + \mu\int_0^t\int_0^1 (\partial_x u^n)^2 \leq C, \tag{94}$$

$$C(T)^{-1} \leq \rho^n(t,x) \leq C(T), \tag{95}$$

$$\frac{\mu}{2}\int_0^1 (\partial_x u^n)^2 + \int_0^t\int_0^1 \rho^n(\dot{u}^n)^2 + \int_0^t\int_0^1 \left|\partial_x(\mu\partial_x u^n - p^n)\right|^2 \leq C(T), \tag{96}$$

$$\frac{1}{2}\int_0^1 \kappa(t)\,\rho^n(\dot{u}^n)^2 + \frac{\mu}{2}\int_0^t\int_0^1 \kappa(t)\left(\partial_x \dot{u}^n\right)^2 \le C(T), \tag{97}$$

$$\left\|u^n\right\|_{L^\infty((0,1)\times\mathbb{T}^1)} + \left\|u^n\right\|_{H^1((0,1)\times\mathbb{T}^1)} \le C(T), \tag{98}$$

$$\kappa^{1/2}(t)\left\|(\partial_t u^n(t), \partial_x u^n(t))\right\|_{L^\infty} \le C(T). \tag{99}$$

Using the uniform bounds of (94)–(29), we conclude that

$$\begin{cases} \rho^n \rightharpoonup \rho, \quad p(\rho^n, c^n) \rightharpoonup \Pi \quad \text{in } L^\infty(\mathbb{R}_+; L^\infty(\mathbb{T}^1)) - w\star, \\ u^n \rightharpoonup u \text{ in } L^2(\mathbb{R}_+; H^1(\mathbb{T}^1)), \\ \sigma^n := \mu\partial_x u^n - p(\rho^n, c^n) \rightharpoonup \sigma := \mu\partial_x u - \Pi \text{ in } L^2(\mathbb{R}_+; H^1(\mathbb{T}^1)). \end{cases} \tag{100}$$

As explained previously, the density ρ^n and the parameter c^n are expected to oscillate widely in space. For this reason, it is hopeless to obtain stronger convergence on these sequences than in a weak L^p-setting. On the other hand, we need to recover some properties of the sequence $p(\rho^n, c^n)$ to compute a limit system satisfied by (ρ, u, σ). To this end, we associate to the sequence $(\rho^n, c^n)_{n\in\mathbb{N}}$ a sequence of measures on the space $\mathbb{T}^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta$ (here $\mathbb{R}_\xi$ must be understood as the range of the ρ^n while $\mathbb{R}_\eta$ is the range of the c^n). Namely, given $n \ge 0$ and $t \ge 0$, we consider the Young measure Θ^n on $\mathbb{T}^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta$ as defined by

$$\left\langle \Theta^n(t), b\right\rangle : \stackrel{\text{def.}}{=} \int_{\mathbb{T}^1} b\left(x, \rho^n(t,x), c^n(t,x)\right) \mathrm{d}x, \quad \forall\, b \in C_c\left(\mathbb{T}^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta\right) \tag{101}$$

We have the following proposition:

Proposition 5.1: *For fixed $n \in \mathbb{N}$ there holds*

$$\Theta^n \in C_w([0,\infty); \mathcal{M}_+(\mathbb{T}^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta)) \tag{102}$$

with

$$\mathrm{Supp}(\Theta^n(t)) \subset \mathbb{T}^1_x \times [C(t)^{-1}, C(t)] \times [0,1] \quad \langle \Theta^n, 1\rangle = 1. \quad \forall\, t \ge 0, \tag{103}$$

where $C(t)$ is given by (95).

Proof: The second identity (103) being obvious we only discuss (102). First, we note that, by definition Θ^n, is continuous in b for the topology of $L^1(\mathbb{T}^1_x; C(\mathbb{R}_\xi \times \mathbb{R}_\eta))$. Consequently, a standard density argument entails that we only need to prove that $t \mapsto \langle \Theta^n(t), b\rangle$ is continuous when $b \in C^1_c(\mathbb{T}^1_x \times \mathbb{R}_\xi)$. For this, we write that

$$\left|\left\langle \Theta^n(t), b\right\rangle - \left\langle \Theta^n(s), b\right\rangle\right| \le \|\partial_2 b\|_{L^\infty} \int_{\mathbb{T}^1} \left|\rho^n(t,x) - \rho^n(s,x)\right| \mathrm{d}x, \quad \forall\, (t,s) \in [0,\infty)^2,$$

and the fact that $\rho^n \in C([0,\infty), L^1(\mathbb{T}^1))$ allows to conclude. ∎

Once these Young measures are constructed, the rigorous justification of system (1)–(4) reduces to the following theorem:

Theorem 5.1: *Up to the extraction of a subsequence, we have* $\Theta^n \rightharpoonup \Theta$ *in* $C_w([0,\infty);\mathcal{M}_+(\mathbb{T}^1_x\times\mathbb{R}_\xi\times\mathbb{R}_\eta))$ *where* Θ *satisfies*

$$\partial_t\Theta+\partial_x(u\Theta)-\frac{1}{\mu(\eta)}\partial_\xi\left((\sigma\xi+\xi p(\xi,\eta))\Theta_\pm\right)-\frac{1}{\mu(\eta)}(\sigma+p(\xi,\eta))\Theta=0 \tag{104}$$

with (u,σ) *as defined in* (100). *Moreover, if there exists* $(\alpha_{+,0},\alpha_{-,0},\rho_{+,0},\rho_{-,0})\in L^\infty(\mathbb{T}^1)$ *such that*

$$\langle\Theta(0),b\rangle=\int_{\mathbb{T}^1}(\alpha_{+,0}(x)b\left(x,\rho_{+,0}(x),1\right)+\alpha_{-,0}(x)b\left(x,\rho_{-,0}(x),0\right))\mathrm{d}x,\quad\forall\, b\in C(\mathbb{T}^1_x\times\mathbb{R}_\xi\times\mathbb{R}_\eta) \tag{105}$$

then there exists $(\alpha_+,\alpha_-,\rho_+,\rho_-)\in[L^\infty_{\rm loc}([0,\infty);L^\infty(\mathbb{T}))\cap C([0,\infty);L^1(\mathbb{T}))]^4$ *such that, for any* $t\geq 0$ *we have*

$$\langle\Theta(t),b\rangle=\int_{\mathbb{T}^1}(\alpha_+(t,x)b\left(x,\rho_{+,0}(x),1\right)+\alpha_{-,0}(x)b\left(x,\rho_{-,0}(x),0\right))\mathrm{d}x,\quad\forall\, b\in C(\mathbb{T}^1_x\times\mathbb{R}_\xi\times\mathbb{R}_\eta). \tag{106}$$

Furthermore, $(\alpha_+,\alpha_-,\rho_+,\rho_-)$ *together with u satisfy* (1)–(4).

What remains of this section is devoted to the proof of this Theorem which will give the proof of the main theorem of the paper.

5.2. Proof of Theorem 5.1

We naturally divide the proof of Theorem 5.1 into two parts. First, we prove that the limiting Young measures verify the Equation (104) while in a second time we will prove that if at initial time Θ have the special structure (105), this structure propagates, i.e. (106) for all time $t>0$. This property will go along with the fact that the quantities $(\alpha_+,\alpha_-,\rho_+,\rho_-)$ satisfy (1) with u. We work on a truncated interval $[0,T]$. Since T is arbitrary, the result holds on $[0,\infty)$.

5.2.1. The equation verified by the limiting measure

Let us consider $b(x,\xi,\eta)\in C^1_c(\mathbb{T}^1_x\times\mathbb{R}_\xi\times\mathbb{R}_\eta))$. For all $N\geq 0$, we write

$$\rho^{n,N}(t):\stackrel{\text{def.}}{=}\omega_N*\rho^n(t)$$

where ω_N is a mollifier depending on the parameter N. For $t\in[0,T]$ and $p\in[1,\infty)$, we have that

$$\begin{cases}\lim_{N\to\infty}\left\|\rho^{n,N}(t)-\rho^n(t)\right\|_{L^p(\mathbb{T}^1)}=0.\\ \lim_{N\to\infty}\left\|\rho^{n,N}-\rho^n\right\|_{L^p([0,T]\times\mathbb{T}^1)}=0.\end{cases} \tag{107}$$

Let us apply ω_N to the second transport equation in (6) and write that

$$\partial_t\rho^{n,N}+\partial_x\left(\rho^{n,N}u^n\right)=r_N\left(\rho^n,u^n\right), \tag{108}$$

where $r_N(\rho^n,u^n):=\partial_x((\omega_N*\rho^n)u^n)-\partial_x(\omega_N*(\rho^nu^n))$ satisfies (see [29, Lemma II.1]):

$$\lim_{N\to\infty}\left\|r_N\left(\rho^n,u^n\right)\right\|_{L^2([0,T]\times\mathbb{T}^1)}=0 \tag{109}$$

Similarly, with the first transport equation of (6), we obtain

$$\partial_tc^{N,n}+u^n\partial_xc^{N,n}=r_N(c^n,u^n)-c^{N,n}\partial_xu^n+(c\partial_xu)^{N,n}, \tag{110}$$

with r_N satisfying also (109). We multiply (108) with $\partial_2b(x,\rho^{n,N},c^{n,N})$ and (110) with $\partial_3b(x,\rho^{n,N},c^{n,N})$ and we write that

$$\partial_tb(x,\rho^{n,N},c^{n,N})+\partial_x\left(u^nb(x,\rho^{n,N},c^{n,N})\right)-u^n\partial_1b\left(x,\rho^{n,N}_\pm,c^{n,N}\right)$$

$$
\begin{aligned}
&+\left(\rho^{n,N}\partial_2 b\left(x,\rho^{n,N},c^{n,N}\right)-b\left(x,\rho^{n,N},c^{n,N}\right)\right)\partial_x u^n\\
&=r_N\left(\rho^n,u^n\right)\partial_2 b(x,\rho^{n,N},c^{n,N})+r_N(c^n,u^n)\partial_3 b(x,\rho^{n,N},c^{n,N})\\
&-\left[c^{N,n}\partial_x u^n-(c\partial_x u)^{N,n}\right]\partial_3 b(x,\rho^{n,N},c^{n,N}).
\end{aligned}
$$

Remark 5.1: Let us mention that by $\partial_t b(x,\rho^{n,N},c^n)$, $\partial_x b(x,\rho^{n,N},c^n)$ we understand the derivative with respect to time/space of the function

$$(t,x)\to b(x,\rho^{n,N}(t,x),c^n(t,x))$$

while when using numbers $\partial_k b(t,x,\rho^{n,N})$, $k\in\{1,3\}$ represents the derivative of b with respect to its kth variable computed in $(x,\rho^{n,N}(t,x),c^n(t,x))$.

Moreover, in order to take advantage of the compactness properties of the effective flux, see (100),

$$\sigma^n=\mu(c^n)\partial_x u^n-p(\rho^n,c^n)$$

we rewrite the above equation as

$$
\begin{aligned}
&\partial_t b(x,\rho^{n,N},c^{n,N})+\partial_x\left(u^n b(x,\rho^{n,N},c^{n,N})\right)-u^n\partial_1 b(x,\rho^{n,N},c^{n,N})\\
&+\frac{1}{\mu(c^n)}\left(\rho^{n,N}\partial_2 b(x,\rho^{n,N},c^{n,N})-b(x,\rho^{n,N},c^{n,N})\right)\sigma^n\\
&+\frac{1}{\mu(c^n)}\left(\rho^{n,N}\partial_2 b(x,\rho^{n,N},c^n)-b(x,\rho^{n,N},c^{n,N})\right)p\left(\rho^n,c^n\right)\\
&=r_N\left(\rho^n,u^n\right)\partial_2 b(x,\rho^{n,N},c^{n,N})+r_N(c^n,u^n)\partial_3 b(x,\rho^{n,N},c^{n,N})\\
&-\left[c^{N,n}\partial_x u^n-(c\partial_x u)^{N,n}\right]\partial_3 b(x,\rho^{n,N},c^{n,N}).
\end{aligned}
\tag{111}
$$

Owing to (107), we get that up to the extraction of a subsequence, we have

$$
\begin{cases}
(\rho^{n,N},c^{n,N})\to(\rho^n,c^n)\text{ a.e. }[0,T]\times\mathbb{T}^1,\\
(\rho^{n,N}(T),c^{n,N}(T)\to(\rho^n(T),c^n(T))\text{ a.e. }\mathbb{T}^1,\\
(\rho^{n,N}(0),c^{N,n}(0))\to(\rho^n(0),c^n(0))\text{ a.e. }\mathbb{T}^1.
\end{cases}
\tag{112}
$$

Hence, by applying dominated convergence argument, we obtain that the left-hand side of (111) converges in $\mathcal{D}'((0,T)\times\mathbb{T}^1)$ to

$$
\begin{aligned}
&\partial_t b(x,\rho^n,c^n)+\partial_x\left(u^n b(x,\rho^n,c^n)\right)-u^n\partial_1 b(x,\rho^n,c^n)\\
&+\frac{1}{\mu(c^n)}\left(\rho^n\partial_2 b(x,\rho^n,c^n)-b(x,\rho^n,c^n)\right)\sigma^n\\
&+\frac{1}{\mu(c^n)}\left(\rho^n\partial_2 b(x,\rho^n,c^n)-b(x,\rho^n,c^n)\right)p\left(\rho^n,c^n\right)
\end{aligned}
$$

As for the right-hand side, we apply (109) together with the regularity $\partial_x u^n\in L^\infty_{loc}((0,T)\times\mathbb{T}^1)$ to yield that

$$\lim_{N\to\infty}\|c^{N,n}\partial_x u^n-(c\partial_x u)^{N,n}\|_{L^2_{loc}((0,T)\times\mathbb{T})}=0.$$

This entails that

$$\partial_t b(x,\rho^n,c^n)+\partial_x\left(u^n b(x,\rho^n,c^n)\right)-u^n\partial_1 b(x,\rho^n,c^n)$$

$$+\frac{1}{\mu(c^n)}\left(\rho^n\partial_2 b(x,\rho^n,c^n)-b(x,\rho^n,c^n)\right)\sigma^n=0,$$

or, using the definition of the sequences of measures Θ^n, i.e. (101)

$$\partial_t\Theta^n+\partial_x\left(u^n\Theta^n\right)-\partial_\xi\left(\left(\frac{\xi\sigma^n}{\mu(\eta)}+\frac{\xi p(\xi,\eta)}{\mu(\eta)}\right)\Theta^n\right)-\left(\frac{\sigma^n}{\mu(\eta)}+\frac{p(\xi,\eta)}{\mu(\eta)}\right)\Theta^n=0.$$

With the first statement, we obtain that, whatever $b\in C^1(\mathbb{T}^1\times\mathbb{R}_\xi\times\mathbb{R}_\eta)$, the quantity $\partial_t b(x,\rho^n,c^n)$ is bounded in $L^\infty(0,T;H^{-1}(\mathbb{T}^1))$. By a standard Arzela–Ascoli argument, applying that Θ^n have uniformly finite mass, we obtain that $\langle\Theta^n,b\rangle$ is precompact in $C([0,T])$. We can then use that Θ^n have compact support (uniformly in $\mathbb{N}$) to extract a limit for a denumerable set of b and combine with a density argument to obtain that Θ^n converge (up to the extraction of a subsequence) in $C_w([0,\infty);\mathcal{M}_+(\mathbb{T}^1_x\times\mathbb{R}_\xi\times\mathbb{R}_\eta))$.

We are now in position to pass to the limit $n\to\infty$ in this last equation. For this, we note that $\partial_t u_n$ is bounded in $L^2((0,T)\times\mathbb{T}^1)$ so that by a classical Ascoli–Arzela argument we have that (up to the extraction of a subsequence) u_n converges to u in $L^2((0,T);C(\mathbb{T}^1))$. Consequently

$$\Theta^n u^n\to\Theta u \text{ in } \mathcal{D}'((0,T)\times\mathbb{T}^1\times\mathbb{R}_\xi\times\mathbb{R}_\eta).$$

Concerning the remaining terms, the only difficulty lies in passing to the limit in the product $\sigma^n\Theta^N$. For this, we note that $\partial_t\rho^n$ is bounded in $L^\infty((0,T);H^{-1}(\mathbb{T}^1))$ while σ^n is bounded in $L^2((0,T);H^1(\mathbb{T}^1))$. By a classical compensated compactness argument (see also [1, Lemma 10]) we have then that

$$\lim_{n\to\infty}\int_0^T\zeta(t)\int_0^1\frac{1}{\mu(c^n(x))}\left(\rho^n(x),\partial_3 b(x,\rho^n(x),c^n(x))-b(x,\rho^n(x),c^n(x))\right)\sigma^n(t,x)\mathrm{d}x\mathrm{d}t$$
$$=\int_0^T\zeta(t)\int_0^1\frac{1}{\mu(c(x))}\left(\rho(x),\partial_3 b(x,\rho(x),c(x))-b(x,\rho(x),c(x))\right)\sigma^\infty(t,x)\mathrm{d}x\mathrm{d}t$$

whatever the test function $b\in C^1(\mathbb{T}^1_x\times\mathbb{R}_\xi\times\mathbb{R}_\eta)$ and $\zeta\in C_c^\infty(0,T)$. We obtain that Θ satisfies (104). This concludes the first part of Theorem 5.1.

5.2.2. Characterization of the limiting measure

The objective of this section is to prove the second part of Theorem 5.1. We follow the approach from [17] and construct explicit solutions to the limit system (1). Afterwards, using the uniqueness result we may identify the limit measure with the particular one we have constructed. At first, we note that the limiting velocity field has the regularity:

$$u\in C([0,T];L^2(\mathbb{T}^1))\cap L^2(0,T;W^{1,\infty}(\mathbb{T}^1)).$$

Consequently, classical arguments for semilinear hyperbolic problems yield that, given

$$(\alpha_{-,0},\alpha_{+,0},\rho_{-,0},\rho_{+,0})\in L^\infty(\mathbb{T}^1;\mathbb{R}^4)$$

such that

$$0\le\min(\alpha_{-,0},\alpha_{+,0},\rho_{-,0},\rho_{+,0})\quad \alpha_{-,0}+\alpha_{+,0}=1,$$

there exists a unique solution $(\alpha_-,\alpha_+,\rho_-,\rho_+)\in L^\infty((0,T)\times\mathbb{T}^1)\cap C([0,T];L^1(\mathbb{T}^1))$ to

$$\begin{cases}\partial_t\alpha_\pm+u\,\partial_x\alpha_\pm=\dfrac{\alpha_+\alpha_-}{\alpha_+\mu_-+\alpha_-\mu_+}(\sigma_\pm-\sigma_\mp),\\ \partial_t(\rho_\pm)+\partial_x(\rho_\pm u)=0,\end{cases}$$

where

$$\sigma_{\pm} = -\mu_{\pm}\partial_x u + p_{\pm}(\rho_{\pm}).$$

We note that the solution is a priori defined only locally. But, by uniqueness, the solution satisfies

$$0 \le \min(\alpha_-, \alpha_+, \rho_-, \rho_+) \quad \alpha_- + \alpha_+ = 1.$$

Furthermore, using the Lagrangian coordinates we see that

$$0 \le \rho_{\pm}(t, X_t(x)) = \rho_{\pm,0}(x) \exp\left(-\int_0^t \partial_x u\right). \tag{113}$$

And, since we have uniform bounds for $\|\partial_x u\|_{L^1(0,T;L^\infty(\mathbb{T}^1))}$ we may extend to global ones. At this point, we define an alternative measure on $\mathbb{T}^1 \times \mathbb{R}_\xi \times \mathbb{R}_\eta$ by the following formulae:

$$\left\langle \bar{\Theta}(t), b \right\rangle : \stackrel{\text{def.}}{=} \int_{\mathbb{T}^1} \alpha_-(t,x)\, b(x, \rho_-(t,x), 0) + \alpha_+(t,x)\, b(x, \rho_+(t,x), 1)\mathrm{d}x.$$

We observe that, for all $t \in [0, T]$, the measure $\bar{\Theta}(t)$ has compact support in $\mathbb{T}^1_x \times \mathbb{R}_\xi \times \mathbb{R}_\eta$ and that, given the system satisfied by $(\alpha_-, \alpha_+, \rho_-, \rho_+)$, the measure $\bar{\Theta}$ verify the following equations:

$$\partial_t \bar{\Theta} + \partial_x (u\,\Theta) - \partial_\xi \left(\left(\frac{\xi\sigma}{\mu(\eta)} + \frac{\xi p(\xi,\eta)}{\mu(\eta)}\right)\bar{\Theta}\right) - \left(\frac{\sigma}{\mu(\eta)} + \frac{p(\xi,\eta)}{\mu(\eta)}\right)\bar{\Theta} = 0. \tag{114}$$

Moreover, we have that

$$\lim_{t\to 0}\langle \bar{\Theta}(t), b\rangle = \int_{\mathbb{T}^1} (\alpha_{-,0}(x)\, b(x, \rho_{-,0}(x), 0) + \alpha_{+,0}(x)\, b(x, \rho_+(x), 1))\mathrm{d}x = \langle \Theta(0), b\rangle.$$

Let us fix $C(T) \ge 1$ such that Θ and $\bar{\Theta}$ have both support in $\mathbb{T}^1 \times [0, C(T)] \times [0, 1]$ on $[0, 1]$. Considering χ a compactly supported smooth function

$$\chi : \mathbb{R} \to [0,1] \quad \text{such that} \quad \chi = 1 \text{ on } [0, \max(C(T), 1)] \tag{115}$$

we can write that Θ and $\bar{\Theta}$ are both solutions to

$$\begin{cases} \partial_t \bar{\Phi} + \partial_x \left(u\,\bar{\Phi}\right) - \partial_\xi \left(\left(\frac{\xi\sigma}{\mu(\eta)} + \frac{\xi p(\xi,\eta)}{\mu(\eta)}\right)\chi(\xi)\,\chi(\eta)\,\bar{\Phi}\right) \\ \quad - \left(\frac{\sigma}{\mu(\eta)} + \frac{p(\xi,\eta)}{\mu(\eta)}\right)\left(\chi(\xi) - \xi\chi'(\xi)\right)\chi(\eta)\,\bar{\Phi} = 0, \\ \left\langle \bar{\Phi}_{|t=0}, b\right\rangle = \int_{\mathbb{T}^1} (\alpha_{-,0}(x)\, b(x, \rho_{-,0}(x), 0) + \alpha_{+,0}(x)\, b(x, \rho_+(x), 1))\mathrm{d}x. \end{cases} \tag{116}$$

Let us observe the equation is a transport equation with a velocity field $V = (V_1, V_2, V_3)$ with $V_1(t,x,\xi,\eta) = u(t,x)$, $V_2((t,x,\xi,\eta) = -[(\sigma(t,x) + p(\eta,\xi))/\mu(\eta)]\xi\chi(\eta)\chi(\xi)$ and $V_3(t,x,\xi,\eta) = 0$. Such velocity field is in $L^1(0,T;(L^\infty(\mathbb{T}^1))^3)$ and therefore the solution is unique namely $\Theta(t) = \bar{\Theta}(t)$. This concludes the proof of Theorem 3.

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