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Sharp Sobolev estimates for concentration of solutions to an aggregation–diffusion equation

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In memory of Geneviève Raugel, colleague and mentor for many mathematicians in dynamics, PDEs and numerical analysis

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Abstract We consider the drift-diffusion equation

$$u_t - \varepsilon \Delta u + \nabla \cdot (u\ \nabla K * u) = 0$$

in the whole space with global-in-time solutions bounded in all Sobolev spaces; for simplicity, we restrict ourselves to the model case $K(x) = -|x|$.
We quantify the mass concentration phenomenon, a genuinely nonlinear effect, for radially symmetric solutions of this equation for small diffusivity ε studied in our previous paper [3], obtaining sharp upper and lower bounds for Sobolev norms.

Keywords nonlocal drift-diffusion equation; small diffusivity; concentration of solutions; Sobolev norms

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1 Introduction

We study the nonlinear nonlocal equation

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \, \nabla K * u) = 0, \qquad x \in \mathbb{R}^N, \; t > 0, \tag{1}$$

where $\varepsilon > 0$ is the diffusivity. We consider the simplest case of a *pointy potential.* In other words, to clarify the presentation we restrict ourselves to the radially symmetric kernel $K(x) = -|x|$ which has a mild singularity at the origin. Equation (1) belongs to a class of models describing numerous phenomena from biology and astrophysics; see the review [16] and [2, Introduction] for further references.

We make the following assumptions on the initial condition $u(\cdot, 0) \equiv u_0$:

(A) The function u_0 is C^∞-smooth, bounded and integrable along with all its derivatives. In other words,

$$u(\cdot, 0) \equiv u_0 \in \bigcap_{k \geq 0, \; 1 \leq p \leq \infty} W^{k,p}(\mathbb{R}^N),$$

where $W^{k,p}(\mathbb{R}^N)$ are the usual Sobolev spaces (see Section 2).

(B) The function u_0 is non-negative and radially symmetric.

(C) The mass of u_0 is sufficiently concentrated:

$$\int_{\mathbb{R}^N} |x| u_0(x) dx < \infty. \tag{2}$$

Since (1) is globally well-posed in any space $W^{k,1}(\mathbb{R}^N)$ for $k \in \mathbb{N}$, see [22], using Sobolev embeddings, it follows that the solutions u to (1) belong to $C([0, \infty), W^{k,p}(\mathbb{R}^N))$ for all $k \geq 0$ and $p \in [1, \infty]$. Also, $u(\cdot, t)$ remains non-negative and radially symmetric for all $t \geq 0$, and moreover we also have the mass conservation property, see (6) below. For more details on the well-posedness and regularity issues for (1), we refer to [18, 22].

In the limit case $\varepsilon = 0$, the solution to (1) blows up after a finite time, provided the initial condition is sufficiently concentrated in a neighbourhood of the origin [4]. For more results about blow-up depending on the choice of the kernel K, see [1],[5],[13],[14],[18],[21]. For a more comprehensive review of the results and open problems, see the very recent book of the first author [2], especially Chapter 5, Section 4.

In our work, we are concerned with the behaviour of solutions to (1) for $0 < \varepsilon \ll 1$. In our previous paper [3], we obtained optimal estimates for Lebesgue norms of u. Heuristically, after the solution is allowed enough time to concentrate in a neighbourhood of zero, the behaviour of the (time-averaged) Lebesgue norms of u is given by $\|u\|_p \sim \varepsilon^{-N(1-1/p)}$.

More rigorously, we proved that there are constants ε_*, $T_* > 0$ and $C_1(p) > 0$, $1 \leq p < \infty$, such that provided $0 < \varepsilon \leq \varepsilon_*$,

$$C_1^{-1} \varepsilon^{-N(1-1/p)} \leq \int_0^{T_*} \Big(\int_{\mathbb{R}^N} u^p(x,t) \, dx \Big)^{1/p} dt \leq C_1 \varepsilon^{-N(1-1/p)}. \tag{3}$$

Moreover, this result remains true if we integrate over x in a ball of radius $C\varepsilon$ instead of the whole space. For the precise formulation, see Theorem 2.3, Corollary 2.4 and Lemma 4.1 in [3].

To understand better small-scale behaviour of the solutions, it is relevant to look at norms beyond the Lebesgue setting. The Sobolev norms - which are natural candidates - have attracted much attention in models with physical motivation. Namely, in the pioneering works of Kuksin [19] and [20], upper and lower estimates of these norms for solutions of the nonlinear Schrödinger equation (with or without a random term) in a small dispersion regime have been obtained. After these seminal papers, study of the Sobolev norms in dispersive equations has become a very important field (see for example the paper [15] and the references therein).

Denoting by $\langle\cdot\rangle$ a time-average, dimensional analysis tells us that quantities of the type

$$\frac{\langle \|u\|_{\dot{H}^m}\rangle}{\langle \|u\|_{\dot{H}^{m+1}}\rangle}, \qquad m \geq 0, \tag{4}$$

(see Section 2 for the notation) provide a *characteristic length scale* of the solution. For a discussion, see the already mentioned papers of Kuksin [19] and [20], as well as [11, Chapter 6].

The main results of our paper, Theorem 1 and Theorem 2, state that, for the same ε_* and T_* as in the statement of (3), provided $0 \leq \varepsilon \leq \varepsilon_*$, for every $m \in \mathbb{N}$, there exists a ε-independent constant $C_1(m) > 0$ such that we have

$$C_1^{-1}\varepsilon^{-(2m+N)/2} \leq \int_0^{T_*} \|u(t)\|_{\dot{H}^m}\, dt \leq C_1 \varepsilon^{-(2m+N)/2}. \tag{5}$$

Consequently, up to averaging in time, all the quantities given by (4) are of order ε, as is the radius of the balls on which at least an ε-independent proportion of mass is concentrated. To the best of our knowledge, our paper is the first one which studies systematically models from mathematical biology using all-order Sobolev norms.

Moreover, our results for these norms - and therefore for the length scale - are sharp. Indeed, the upper and lower estimates only differ by a multiplicative constant which only depends on the initial condition through a finite number of parameters. This is a remarkable phenomenon, only previously observed in the Burgers equation and its generalisations. For more complex PDEs such as the 2D Navier–Stokes or the nonlinear Schrödinger equation, such results are beyond the reach of today's mathematics.

Our results are indeed similar to those obtained for the simpler Burgers equation and its fractional-dissipation and multidimensional analogues by the second author [7,8,9,10]. These papers were themselves inspired by the ideas and first results due to Biryuk [6]. Indeed, for Burgers-type equations the length scale is again the small parameter ε, and we have sharp Sobolev norm estimates. More precisely, up to a rescaling factor corresponding to the dimension N, $-u$ has the same behaviour with respect to Lebesgue and Sobolev norms as the derivative of a Burgers solution. In particular, the positivity of u seems to play a role analogous to that of Oleinik's upper bound on the positive part of the gradient for a solution of the Burgers equation. Heuristically, it seems that the rescaling N-dependent factor in the power of ε is due to a difference of geometry of the singular zones in the limit $\varepsilon \to 0$. Indeed, for (1) regions where the inviscid solution is not regular are of dimension zero (only the origin) and not shocks of codimension one as for the generalised Burgers equation.

Our methodology is essentially a combination of the approach used by the second author to study Sobolev norms in the papers cited above and of the arguments used by the three other authors to prove explosion under the concentration assumption in the paper [4] (see also [17]). The most delicate issue is to estimate the contribution of the nonlinearity in the energy estimates, which requires a subtle analysis of the convolution term using the classical Hardy–Littlewood–Sobolev inequality along with the Gagliardo–Nirenberg inequality within the admissible ranges for the exponents.

2 Notation, functional spaces and inequalities

We denote by M the total mass and recall that it is conserved by the flow of the equation (1) (and the non-negativity of its solution u is also conserved), so that

$$\|u(t)\|_1 = \int_{\mathbb{R}^N} u(x,t)dx = M := \int_{\mathbb{R}^N} u_0(x)dx, \qquad t \geq 0. \tag{6}$$

For multiindices $\mathbf{i}, \mathbf{j} \in \mathbb{Z}_+^N$, provided $i_k \leq j_k,\ 1 \leq k \leq N$ (which we denote as $\mathbf{i} \leq \mathbf{j}$), we use the generalised binomial coefficient notation

$$\binom{\mathbf{j}}{\mathbf{i}} = \prod_{k=1}^{N} \binom{j_k}{i_k}.$$

For $N = 1$ and a positive integer k, $u^{(k)}$ denotes the k-th spatial derivative of u, while we use the notation $\partial_{\mathbf{i}} u := \partial_{x_1}^{i_1} \ldots \partial_{x_N}^{i_N} u$ when $N > 1$ and $\mathbf{i} = (i_k)_{1\leq k\leq N}$ is a multiindex.

For $m \geq 0$ and $p \in [1,\infty]$, we will consider Lebesgue spaces $L^p(\mathbb{R}^N)$ and Sobolev spaces $W^{m,p}(\mathbb{R}^N)$. The Lebesgue norms will be denoted $\|\cdot\|_p$. As usual, we set $H^m(\mathbb{R}^N) = W^{m,2}(\mathbb{R}^N)$, $m \in \mathbb{N}$. For $m \in \mathbb{N}$ and $p \in [1,\infty]$, we denote the homogeneous seminorm in $W^{m,p}(\mathbb{R}^N)$ by

$$\|u\|_{\dot{W}^{m,p}} := \sum_{|\mathbf{i}|=m} \|\partial_{\mathbf{i}} u\|_p \quad \text{with} \quad \|\cdot\|_{\dot{H}^m} = \|\cdot\|_{\dot{W}^{m,2}}.$$

Throughout the paper, the notation C and C_i, $i \geq 1$, is used for various positive numbers which may vary from line to line. These numbers depend only on the dimension N, and on the initial condition u_0 through the total mass M and the quantity Λ (see the beginning of the proof of Theorem 2). The dependence upon additional parameters will be indicated explicitly.

Now we recall two classical inequalities.

Lemma 21 *(The Gagliardo–Nirenberg Inequality, [12]) For a C^∞-smooth function v on $\mathbb{R}^N$, we have*

$$\|v\|_{\dot{W}^{\beta,r}} \le C\|v\|^{\theta}_{\dot{W}^{m,p}}\|v\|_q^{1-\theta},$$

where $m > \beta \ge 0$, and r is defined by

$$\frac{N}{r} = \beta - \theta\Big(m - \frac{N}{p}\Big) + (1-\theta)\frac{N}{q},$$

under the assumption $\beta/m \le \theta < 1$ and with the exception of the case when $\beta = 0$, $r = q = \infty$ and $m - N/p$ is a nonnegative integer.
The constant C depends also on m, p, q, β, N.

Lemma 22 *(The Hardy–Littlewood–Sobolev Inequality.)*
[23, Theorem 4.3];[24, V.1.3.]
For a C^∞-smooth function v on $\mathbb{R}^N$, provided

$$1 < p, q < \infty,\ 1/p + \lambda/N = 1/q + 1,\ 0 < \lambda < N,$$

we have

$$\Big\|\ |x|^{-\lambda} * v\Big\|_q \le C\|v\|_p.$$

where $*$ denotes the convolution. The constant C depends on p, λ, N.

3 Upper estimates

The results proved in this section still hold without the radial symmetry assumption on the initial condition, and also without the concentration assumption (C). Nevertheless, in that case we do not have corresponding lower estimates with the same power of the parameter ε proved in the next section.

The scheme of the proof is very similar to that of the particular case $N = 1,\ m = 1$ already treated in [3].

Theorem 1 *For $m \in \mathbb{N}$ and $t \ge 0$, we have*

$$\|u(t)\|_{\dot{H}^m} \le \max\Big\{\|u_0\|_{\dot{H}^m},\ C(m)M^{(N+2m+2)/2}\varepsilon^{-(N+2m)/2}\Big\}.$$

Proof The case $m = 0$ is dealt with in [3], to which we refer, see [3, Lemma 4.1] for $p = 2$. From now on, we assume that $m \geq 1$.

The case $N = 1$. Integrating by parts and using that for any $p \in [1, \infty]$, $(K' * v)_x = -2v$ for $v \in L^p(\mathbb{R}^N)$, we obtain

$$
\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}&\|u\|_{\dot{H}^m}^2 \\
&= -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 - \int_{\mathbb{R}} u^{(m)} (u\ (K' * u))^{(m+1)} dx \\
&= -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 - \int_{\mathbb{R}} u^{(m)} u^{(m+1)} (K' * u) dx \\
&\quad - \sum_{k=0}^{m} \int_{\mathbb{R}} \binom{m+1}{k} u^{(m)} u^{(k)} (K' * u^{(m-k)})_x dx \\
&= -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 + \frac{1}{2} \int_{\mathbb{R}} (u^{(m)})^2 (K' * u)_x dx \\
&\quad - \sum_{k=0}^{m} \int_{\mathbb{R}} \binom{m+1}{k} u^{(m)} u^{(k)} (K' * u^{(m-k)})_x dx \\
&= -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 - \underbrace{\int_{\mathbb{R}} (u^{(m)})^2\ u\ dx}_{A_m} \\
&\quad + \sum_{k=0}^{m} \underbrace{\int_{\mathbb{R}} 2\binom{m+1}{k} u^{(m)} u^{(k)} u^{(m-k)} dx}_{B_{km}} .
\end{aligned}
$$

We first get, using the Hölder and then the Gagliardo–Nirenberg inequality (for $(\beta, r, m, p, q) = (m, \infty, m+1, 2, 1)$), as well as (6),

$$
\begin{aligned}
|A_m| &\leq \|u\|_{\dot{W}^{m,\infty}}^2 \|u\|_1 \leq C(m) (\|u\|_1^{1/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(2m+2)/(2m+3)})^2 \|u\|_1 \\
&= C(m) M^{(2m+5)/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(4m+4)/(2m+3)} .
\end{aligned}
$$

Similarly (with $(\beta, r, m, p, q) = (k, 2, m+1, 2, 1)$ and $(\beta, r, m, p, q) = (m - k, 2, m+1, 2, 1)$ for the Gagliardo–Nirenberg inequality), we obtain

$$
\begin{aligned}
|B_{km}| &\leq C(k, m) \|u\|_{\dot{W}^{m,\infty}} \|u\|_{\dot{H}^k} \|u\|_{\dot{H}^{m-k}} \\
&\leq C(k, m) \left(\|u\|_1^{1/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(2m+2)/(2m+3)} \right) \\
&\qquad \times \left(\|u\|_1^{(2m+2-2k)/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(2k+1)/(2m+3)} \right) \\
&\qquad \times \left(\|u\|_1^{(2k+2)/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(2m-2k+1)/(2m+3)} \right) \\
&= C(k, m) M^{(2m+5)/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(4m+4)/(2m+3)} .
\end{aligned}
$$

Consequently,

$$
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^m}^2 \leq -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 + C(m) M^{(2m+5)/(2m+3)} \|u\|_{\dot{H}^{m+1}}^{(4m+4)/(2m+3)} . \qquad (7)
$$

Now we observe that, interpolating $\|u(t)\|_{\dot H^m}$ between $\|u(t)\|_{\dot H^{m+1}}$ and $\|u(t)\|_1$ using the Gagliardo–Nirenberg inequality, we get, thanks to (6),

$$\|u(t)\|_{\dot H^{m+1}}^{2/(2m+3)} \geq C_{GN}(m) M^{-4/(2m+1)(2m+3)} \|u(t)\|_{\dot H^m}^{2/(2m+1)}. \tag{8}$$

Our goal is now to show that the inequality (7) implies that, for all $t \geq 0$,

$$\begin{aligned} &\|u(t)\|_{\dot H^m} \leq U_m \\ &\equiv \max\left\{\|u_0\|_{\dot H^m},\ C_{GN}(m)^{-(2m+1)/2} C(m)^{(2m+1)/2} M^{(2m+3)/2} \varepsilon^{-(2m+1)/2}\right\}, \end{aligned} \tag{9}$$

with $C(m)$ is the same as in (7) and $C_{GN}(m)$ the same as in (8). Indeed, for $\delta > 0$, consider the set

$$A_\delta := \left\{t \geq 0 :\ \|u(t)\|_{\dot H^m} \leq U_m + \delta\right\}.$$

Clearly, $0 \in A_\delta$ and the time continuity of u in $H^m(\mathbb{R}^N)$ ensures that

$$\tau_\delta := \sup\{t \geq 0 :\ [0,t] \subset A_\delta\} \in (0,\infty].$$

Assume now for contradiction that $\tau_\delta < \infty$. The definition of τ_δ implies that

$$\|u(\tau_\delta)\|_{\dot H^m}^2 = (U_m + \delta)^2 \geq \|u(t)\|_{\dot H^m}^2 \quad \text{for all} \ \ t \in (0,\tau_\delta).$$

Hence,

$$\frac{\mathrm{d}}{\mathrm{d}t}\|u(\tau_\delta)\|_{\dot H^m}^2 \geq 0. \tag{10}$$

We next infer from (7), (8) and the definition of U_m that

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u(\tau_\delta)\|_{\dot H^m}^2 \\ \leq& \varepsilon \|u(\tau_\delta)\|_{\dot H^{m+1}}^{(4m+4)/(2m+3)} \left(-\|u(\tau_\delta)\|_{\dot H^{m+1}}^{2/(2m+3)} + C(m) M^{(2m+5)/(2m+3)} \varepsilon^{-1}\right) \\ \leq& \varepsilon \|u(\tau_\delta)\|_{\dot H^{m+1}}^{(4m+4)/(2m+3)} \big(- C_{GN}(m) M^{-4/(2m+1)(2m+3)} \|u(\tau_\delta)\|_{\dot H^m}^{2/(2m+1)} \\ &+ C(m) M^{(2m+5)/(2m+3)} \varepsilon^{-1}\big) < 0, \end{aligned}$$

which contradicts (10). Consequently, $\tau_\delta = \infty$ and $A_\delta = [0,\infty)$ for all $\delta > 0$. Letting $\delta \to 0$ completes the proof of (9).

The case $N \geq 2$**.** Let $\mathbf{i} = (i_k)_{1 \leq k \leq N} \in \mathbb{N}^N$ be a multiindex with $|\mathbf{i}| = m$.

By applying Leibniz' formula, it follows from (1) that $\partial_{\mathbf{i}} u$ solves

$$(\partial_{\mathbf{i}} u)_t = \varepsilon \Delta (\partial_{\mathbf{i}} u) - \nabla \cdot \Big[\sum_{0 \leq \mathbf{j} \leq \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \partial_{\mathbf{j}} u \ (\nabla K * \partial_{\mathbf{i}-\mathbf{j}} u)\Big].$$

Multiplying the above equation by $\partial_{\mathbf{i}} u$, integrating over $\mathbb{R}^N$, summing over all multiindices $\mathbf{i}$ of length m and then integrating by parts, we get

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^m}^2 & \\
= & -\varepsilon \sum_{|\mathbf{i}|=m} \|\nabla(\partial_{\mathbf{i}} u)\|_2^2 \\
& + \sum_{|\mathbf{i}|=m} \sum_{0\le \mathbf{j}\le \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \int_{\mathbb{R}^N} \partial_{\mathbf{j}} u \,(\nabla K * \partial_{\mathbf{i}-\mathbf{j}} u)\cdot \nabla \partial_{\mathbf{i}} u \,\mathrm{d}x \\
= & -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 + \sum_{|\mathbf{i}|=m} \int_{\mathbb{R}^N} \partial_{\mathbf{i}} u \,(\nabla K * u)\cdot \nabla \partial_{\mathbf{i}} u \,\mathrm{d}x \\
& - \sum_{|\mathbf{i}|=m} \sum_{0\le \mathbf{j}< \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \int_{\mathbb{R}^N} \nabla\partial_{\mathbf{j}} u \cdot (\nabla K * \partial_{\mathbf{i}-\mathbf{j}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x \\
& - \sum_{|\mathbf{i}|=m} \sum_{0\le \mathbf{j}< \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \int_{\mathbb{R}^N} \partial_{\mathbf{j}} u \,(\Delta K * \partial_{\mathbf{i}-\mathbf{j}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x,
\end{aligned}$$

using that $\operatorname{div}(\nabla K * \partial_{\mathbf{i}-\mathbf{j}} u) = \Delta K * \partial_{\mathbf{i}-\mathbf{j}} u$. Since we can write

$$\begin{aligned}
& \sum_{0\le \mathbf{j}< \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \int_{\mathbb{R}^N} \nabla\partial_{\mathbf{j}} u \cdot (\nabla K * \partial_{\mathbf{i}-\mathbf{j}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x \\
= & \sum_{0\le \mathbf{j}< \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \sum_{k=1}^{N} \int_{\mathbb{R}^N} \partial_{x_k}\partial_{\mathbf{j}} u \,(\partial_{x_k} K * \partial_{\mathbf{i}-\mathbf{j}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x \\
= & \sum_{|\mathbf{l}|\le m} \sum_{r=1}^{N} \sum_{s=1}^{N} C_1(\mathbf{i},\mathbf{l},r,s) \int_{\mathbb{R}^N} \partial_{\mathbf{l}} u \,(\partial_{x_r}\partial_{x_s} K * \partial_{\mathbf{i}-\mathbf{l}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x
\end{aligned}$$

for some constants $C_1(\mathbf{i},\mathbf{l},r,s) \in \mathbb{R}$ and

$$\begin{aligned}
& \sum_{0\le \mathbf{j}< \mathbf{i}} \binom{\mathbf{i}}{\mathbf{j}} \int_{\mathbb{R}^N} \partial_{\mathbf{j}} u \,(\Delta K * \partial_{\mathbf{i}-\mathbf{j}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x \\
= & \sum_{|\mathbf{l}|\le m} \sum_{r=1}^{N} C_2(\mathbf{i},\mathbf{j},r) \int_{\mathbb{R}^N} \partial_{\mathbf{l}} u \left(\partial_{x_r}^2 K * \partial_{\mathbf{i}-\mathbf{l}} u\right) \partial_{\mathbf{i}} u \,\mathrm{d}x
\end{aligned}$$

for some constants $C_2(\mathbf{i},\mathbf{j},r) \in \mathbb{R}$, we obtain, after another integration by parts,

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot{H}^m}^2 = & -\varepsilon \|u\|_{\dot{H}^{m+1}}^2 - \frac{1}{2} \sum_{|\mathbf{i}|=m} \int_{\mathbb{R}^N} (\partial_{\mathbf{i}} u)^2 \,(\Delta K * u)\,\mathrm{d}x \\
& + \sum_{|\mathbf{i}|=m} \sum_{|\mathbf{l}|\le m} \int_{\mathbb{R}^N} \partial_{\mathbf{l}} u \,(P(\mathbf{i},\mathbf{l}) K * \partial_{\mathbf{i}-\mathbf{l}} u)\, \partial_{\mathbf{i}} u \,\mathrm{d}x,
\end{aligned} \tag{11}$$

where $P(\mathbf{i},\mathbf{l})$ are constant-coefficient differential operators of second order. We now split the last term of (11) to find, after moving partial derivatives inside the convolutions in an appropriate way,

$$\begin{aligned}
&\sum_{|\mathbf{i}|=m}\sum_{|\mathbf{l}|\le m}\int_{\mathbb{R}^N}\partial_{\mathbf{l}}u\,(P(\mathbf{i},\mathbf{l})K*\partial_{\mathbf{i}-\mathbf{l}}u)\,\partial_{\mathbf{i}}u\,\mathrm{d}x\\
&=\sum_{|\mathbf{i}|=m}\sum_{m-N+2\le|\mathbf{l}|\le m}\int_{\mathbb{R}^N}\partial_{\mathbf{l}}u\,(\partial_{\mathbf{i}-\mathbf{l}}P(\mathbf{i},\mathbf{l})K*u)\,\partial_{\mathbf{i}}u\,\mathrm{d}x\\
&\quad+\sum_{|\mathbf{i}|=m}\sum_{|\mathbf{l}|\le m-N+1}\int_{\mathbb{R}^N}\partial_{\mathbf{l}}u\,(P(\mathbf{i},\mathbf{l})K*\partial_{\mathbf{i}-\mathbf{l}}u)\,\partial_{\mathbf{i}}u\,\mathrm{d}x.
\end{aligned}$$

For the second term on the right hand side of the above identity, we observe that the conditions $|\mathbf{i}|=m$ and $|\mathbf{l}|\le m-N+1$ guarantee that $|\mathbf{i}-\mathbf{l}|\ge N-1$ and we can move $N-2$ partial derivatives from $\partial_{\mathbf{i}-\mathbf{l}}u$ on $P(\mathbf{i},\mathbf{l})K$ in the convolution $P(\mathbf{i},\mathbf{l})K*\partial_{\mathbf{i}-\mathbf{l}}u$ to find

$$\begin{aligned}
&\sum_{|\mathbf{i}|=m}\sum_{|\mathbf{l}|\le m}\int_{\mathbb{R}^N}\partial_{\mathbf{l}}u\,(P(\mathbf{i},\mathbf{l})K*\partial_{\mathbf{i}-\mathbf{l}})\,\partial_{\mathbf{i}}u\,\mathrm{d}x\\
&=\sum_{|\mathbf{i}|=m}\sum_{m-N+2\le|\mathbf{l}|\le m}\int_{\mathbb{R}^N}\partial_{\mathbf{l}}u\,(\partial_{\mathbf{i}-\mathbf{l}}P(\mathbf{i},\mathbf{l})K*u)\,\partial_{\mathbf{i}}u\,\mathrm{d}x\\
&\quad+\sum_{|\mathbf{i}|=m}\sum_{|\mathbf{l}|\le m-N+1}\sum_{|\mathbf{j}|=m-|\mathbf{l}|-N+2}\int_{\mathbb{R}^N}\partial_{\mathbf{l}}u\,(Q(\mathbf{i},\mathbf{l},\mathbf{j})K*\partial_{\mathbf{j}}u)\,\partial_{\mathbf{i}}u\,\mathrm{d}x,
\end{aligned}$$

where $Q(\mathbf{i},\mathbf{l},\mathbf{j})$ are constant-coefficient differential operators of order N. Inserting the above identity in (11) and computing the partial derivatives of K leads to

$$\begin{aligned}
\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|u\|_{\dot H^m}^2 &\le -\varepsilon\|u\|_{\dot H^{m+1}}^2+\frac{N-1}{2}\sum_{|\mathbf{i}|=m}\underbrace{\int_{\mathbb{R}^N}(\partial_{\mathbf{i}}u)^2\left(|x|^{-1}*u\right)\,\mathrm{d}x}_{A_{\mathbf{i}}}\\
&+\sum_{|\mathbf{i}|=m}\sum_{m-N+2\le|\mathbf{l}|\le m}C(\mathbf{i},\mathbf{l})\underbrace{\int_{\mathbb{R}^N}|\partial_{\mathbf{l}}u|\left(|x|^{-(m-|\mathbf{l}|+1)}*u\right)|\partial_{\mathbf{i}}u|\,\mathrm{d}x}_{D_{\mathbf{i},\mathbf{l}}}\\
&+\sum_{|\mathbf{i}|=m}\sum_{|\mathbf{l}|\le m-N+1}C(\mathbf{i},\mathbf{l})\underbrace{\sum_{|\mathbf{j}|=m-|\mathbf{l}|-N+2}\int_{\mathbb{R}^N}|\partial_{\mathbf{l}}u|\left(|x|^{-(N-1)}*|\partial_{\mathbf{j}}u|\right)|\partial_{\mathbf{i}}u|\,\mathrm{d}x}_{E_{\mathbf{i},\mathbf{l}}}.
\end{aligned}$$

Now it remains to estimate all the terms using first the Hölder, and then the Gagliardo–Nirenberg and the Hardy–Littlewood–Sobolev inequalities, along with the mass conservation (6). First,

$$
\begin{aligned}
|A_{\mathbf{i}}| \le & \|(\partial_{\mathbf{i}} u)^2\|_{2N/(2N-1)} \left\| |x|^{-1} * u \right\|_{2N} \le C(m) \|\partial_{\mathbf{i}} u\|^2_{4N/(2N-1)} \|u\|_{2N/(2N-1)} \\
\le & C(m) \|u\|^2_{\dot{W}^{m,4N/(2N-1)}} \|u\|_{2N/(2N-1)} \\
\le & C(m) \left(\|u\|_1^{3/(2m+N+2)} \|u\|_{\dot{H}^{m+1}}^{(4m+2N+1)/(2m+N+2)} \right) \\
& \times \left(\|u\|_1^{(2m+N+1)/(2m+N+2)} \|u\|_{\dot{H}^{m+1}}^{1/(2m+N+2)} \right) \\
\le & C(m) M^{(2m+N+4)/(2m+N+2)} \|u\|_{\dot{H}^{m+1}}^{(4m+2N+2)/(2m+N+2)}.
\end{aligned}
$$

Next, when $\mathbf{l}$ satisfies $0 \le m - |\mathbf{l}| \le N-2$,

$$
\begin{aligned}
|D_{\mathbf{i},\mathbf{l}}| \le & \|\partial_{\mathbf{l}} u\|_{2N/(N-m+|\mathbf{l}|)} \left\| |x|^{-(m-|\mathbf{l}|+1)} * u \right\|_{4N/(2m-2|\mathbf{l}|+1)} \|\partial_{\mathbf{i}} u\|_{4N/(2N-1)} \\
\le & C(\mathbf{i},\mathbf{l}) \|u\|_{\dot{W}^{|\mathbf{l}|,\ 2N/(N-m+L)}} \|u\|_{4N/(4N-2(m-|\mathbf{l}|)-3)} \|u\|_{\dot{W}^{m,4N/(2N-1)}} \\
\le & C(\mathbf{i},\mathbf{l}) (\|u\|_1^{1-\alpha} \|u\|_{\dot{H}^{m+1}}^{\alpha}) (\|u\|_1^{1-\beta} \|u\|_{\dot{H}^{m+1}}^{\beta}) (\|u\|_1^{1-\gamma} \|u\|_{\dot{H}^{m+1}}^{\gamma}) \\
\le & C(\mathbf{i},\mathbf{l}) M^{(2m+N+4)/(2m+N+2)} \|u\|_{\dot{H}^{m+1}}^{(4m+2N+2)/(2m+N+2)},
\end{aligned}
$$

where

$$
\alpha = \frac{m+|\mathbf{l}|+N}{2m+N+2}, \quad \beta = \frac{2(m-|\mathbf{l}|)+3}{2(2m+N+2)}, \quad \gamma = \frac{4m+2N+1}{2(2m+N+2)}.
$$

Finally, when $(\mathbf{l},\mathbf{j})$ satisfies $m - |\mathbf{l}| \ge N-1$ and $|\mathbf{j}| = m - |\mathbf{l}| - N + 2$,

$$
\begin{aligned}
|E_{\mathbf{i},\mathbf{l}}| \le & \|\partial_{\mathbf{l}} u\|_{(4m+4)/(2m-1)} \left\| |x|^{-(N-1)} * |\partial_{\mathbf{j}} u| \right\|_{4m+4} \|\partial_{\mathbf{i}} u\|_{(2m+2)/(m+2)} \\
\le & C(\mathbf{i},\mathbf{l}) \|u\|_{\dot{W}^{|\mathbf{l}|,(4m+4)/(2m-1)}} \|u\|_{\dot{W}^{m-|\mathbf{l}|-N+2,\ N(4m+4)/(N+4m+4)}} \\
& \times \|u\|_{\dot{W}^{m,(2m+2)/(m+2)}} \\
\le & C(\mathbf{i},\mathbf{l}) (\|u\|_1^{1-\delta} \|u\|_{\dot{H}^{m+1}}^{\delta}) (\|u\|_1^{(1-\delta')} \|u\|_{\dot{H}^{m+1}}^{\delta'}) \\
& \times (\|u\|_1^{1/(m+1)} \|u\|_{\dot{H}^{m+1}}^{m/(m+1)}) \\
\le & C(\mathbf{i},\mathbf{l}) M^{(2m+N+4)/(2m+N+2)} \|u\|_{\dot{H}^{m+1}}^{(4m+2N+2)/(2m+N+2)},
\end{aligned}
$$

where

$$
\delta = \frac{2(m+1)(2L+N)+3N}{2(2m+N+2)(m+1)}, \quad \delta' = \frac{4(m+1)(m-L+1)-N}{2(2m+N+2)(m+1)}.
$$

Summing up all the above estimates, we get

$$
\begin{aligned}
& \frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \|u\|^2_{\dot{H}^m} \\
\le & -\varepsilon \|u\|^2_{\dot{H}^{m+1}} + C(m) M^{(2m+N+4)/(2m+N+2)} \|u\|_{\dot{H}^{m+1}}^{(4m+2N+2)/(2m+N+2)} \\
= & \|u\|_{\dot{H}^{m+1}}^{(4m+2N+2)/(2m+N+2)} \\
& \times \left(C(m) M^{(2m+N+4)/(2m+N+2)} - \varepsilon \|u\|_{\dot{H}^{m+1}}^{2/(2m+N+2)} \right).
\end{aligned}
$$

From this energy inequality combined with the following consequence of the Gagliardo–Nirenberg inequality

$$\|u\|_{\dot{H}^{m+1}}^{2/(2m+N+2)} \ge C(m)\ M^{-4/(2m+N)(2m+N+2)} \|u\|_{\dot{H}^m}^{2/(2m+N)}$$

we deduce, arguing as in the case $N=1$, that

$$\|u(t)\|_{\dot{H}^m} \le \max\left\{ \|u_0\|_{\dot{H}^m}, C(m) M^{(N+2m+2)/2} \varepsilon^{-(N+2m)/2} \right\}, \qquad t \ge 0,$$

as announced.

4 Lower estimates for Sobolev norms

Here — unlike in the previous section — the positivity and radial symmetry assumptions (B) as well as the concentration assumption (C) on the initial condition u_0 play a crucial role.

Theorem 2 *Let $m \in \mathbb{N}$. For some explicit numbers $\varepsilon_* > 0$, $T_* > 0$ and $C_*(m) > 0$, independent of ε, the following inequality holds true:*

$$\int_0^{T_*} \|u\|_{\dot{H}^m}\,\mathrm{d}t \ge C_*(m) \varepsilon^{-(2m+N)/2}, \qquad \text{for all} \quad \varepsilon \in (0, \varepsilon_*). \tag{12}$$

Proof For $m=0$, see [3, Corollary 2.4] for $p=2$. Indeed, if (2) is true, then there exists $\Lambda > 0$ such that the assumption on the initial condition [3, Eq. (2.8)] holds true; see also [3, Remark 2.7].
For $m \ge 1$, we infer from the Gagliardo–Nirenberg inequality that

$$\|u(t)\|_2 \le C_{GN}(m) \|u(t)\|_{\dot{H}^m}^{N/(N+2m)} \|u(t)\|_1^{2m/(N+2m)}.$$

Hence, by (6),

$$\|u(t)\|_{\dot{H}^m} \ge C_{GN}(m)^{-(N+2m)/N} M^{-2m/N}\ \|u(t)\|_2^{(N+2m)/N}, \qquad t \ge 0,$$

and it follows from Hölder's inequality and the already established lower bound (12) for $m=0$ that

$$\begin{aligned}
\left(\int_0^{T_*} \|u(t)\|_{\dot{H}^m}^2\,\mathrm{d}t \right)^{1/2} &\ge C(m) M^{-2m/N}\ \left(\int_0^{T_*} \|u(t)\|_2^{2(N+2m)/N}\,\mathrm{d}t \right)^{1/2} \\
&\ge C(m) M^{-2m/N}\ T_*^{-(N+4m)/2N} \left(\int_0^{T_*} \|u(t)\|_2\,\mathrm{d}t \right)^{(N+2m)/N} \\
&\ge C_*^{(N+2m)/N} C(m) M^{-2m/N}\ T_*^{-(N+4m)/2N} \varepsilon^{-(N+2m)/2},
\end{aligned}$$

which completes the proof.

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References

1. J. Bedrossian, N. Rodríguez, and A.L. Bertozzi. Local and global well-posedness for aggregation equations and Patlak-Keller-Segel models with degenerate diffusion. *Nonlinearity*, 24(6):1683–1714, 2011.
2. P. Biler. *Singularities of solutions to chemotaxis systems.*, volume 6, De Gruyter Series in Mathematics and Life Sciences. Berlin: De Gruyter, 2020.
3. P. Biler, A. Boritchev, G. Karch, and Ph. Laurençot. Concentration phenomena in a diffusive aggregation model. *Journal of Differential Equations*, 271:1092–1108, 2021.
4. P. Biler, G. Karch, and Ph. Laurençot. Blowup of solutions to a diffusive aggregation model. *Nonlinearity*, 22(7):1559–1568, 2009.
5. P. Biler and W.A. Woyczynski. Global and exploding solutions for nonlocal quadratic evolution problems. *SIAM J. Appl. Math.*, 59(3):845–869, 1999.
6. A. Biryuk. Spectral properties of solutions of the Burgers equation with small dissipation. *Functional Analysis and its Applications*, 35:1:1–12, 2001.
7. A. Boritchev. Decaying turbulence in the generalised Burgers equation. *Arch. Ration. Mech. Anal.*, 214(1):331–357, 2014.
8. A. Boritchev. Turbulence in the generalised Burgers equation. *Uspekhi Mat. Nauk*, 69:6(6(420)):3–44, 2014.
9. A. Boritchev. Multidimensional potential Burgers turbulence, and Erratum. *Comm. Math. Phys.*, 342, 346(2):441–489, 369–370, 2016.
10. A. Boritchev. Decaying turbulence for the fractional subcritical Burgers equation. *Discrete Contin. Dyn. Syst.*, 38(5):2229–2249, 2018.
11. A. Boritchev and S. Kuksin. *One-dimensional turbulence and the stochastic Burgers equation.* Submitted.
12. H. Brezis and P. Mironescu. Where Sobolev interacts with Gagliardo–Nirenberg. *Journal of Functional Analysis*, 277:2839–2864, 2019.
13. J. A. Carrillo, M. DiFrancesco, A. Figalli, T. Laurent, and D. Slepčev. Global-in-time weak measure solutions and finite-time aggregation for nonlocal interaction equations. *Duke Math. J.*, 156(2):229–271, 2011.
14. J.A. Carrillo, F. James, F. Lagoutière, and N. Vauchelet. The Filippov characteristic flow for the aggregation equation with mildly singular potentials. *Journal of Differential Equations*, 260(1):304–338, 2016. 33 pages.
15. J. Colliander, M. Keel, G. Staffilani, H. Takaoka, and T. Tao. Transfer of energy to high frequencies in the cubic defocusing nonlinear Schrödinger equation. *Invent. Math.*, 181(1):39–113, 2010.
16. T. Hillen and K. J. Painter. A user's guide to PDE models for chemotaxis. *J. Math. Biol.*, 58(1-2):183–217, 2009.
17. G. Karch and K. Suzuki. Spikes and diffusion waves in a one-dimensional model of chemotaxis. *Nonlinearity*, 23(12):3119–3137, 2010.
18. G. Karch and K. Suzuki. Blow-up versus global existence of solutions to aggregation equations. *Appl. Math. (Warsaw)*, 38(3):243–258, 2011.
19. S. Kuksin. On turbulence in nonlinear Schrödinger equations. *Geometric and Functional Analysis*, (7):783–822, 1997.
20. S. Kuksin. Spectral properties of solutions for nonlinear PDEs in the turbulent regime. *Geometric and Functional Analysis*, 9:141–184, 1999.
21. L. Lafleche and S. Salem. Fractional Keller-Segel equation: global well-posedness and finite time blow-up. *Comm. Math. Sci.*, 17(8):2055–2087, 2019.

22. D. Li and J.L. Rodrigo. Wellposedness and regularity of solutions of an aggregation equation. *Rev. Mat. Iberoam.*, 26(1):261–294, 2010.
23. E.H. Lieb and M. Loss. *Analysis*, volume 14 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, second edition, 2001.
24. E.M. Stein. *Singular integrals and differentiability properties of functions.* Princeton: Princeton University Press, 1970.