The Parabolic-Parabolic Keller-Segel equation and its toy models: well-posedness and blow-up.

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Outline

The Keller-Segel model

Chemotaxis=aggregation of bacteria, pollen, spermatozoids... through chemical signals.

Parabolic-parabolic Keller-Segel model:

$$u_t - \varepsilon\Delta u + \nabla\cdot(u\nabla\phi) = 0;$$
$$\tau\phi_t - \Delta\phi = u.$$

$u, \phi \geq 0$: resp. cell density and concentration of a chemical signal.

Mass preserved and solutions remain positive.

In the limit $\tau \to 0$ (instantaneously propagating information) we get the parabolic-elliptic aggregation-diffusion equation:

$$u_t - \varepsilon\Delta u + \nabla\cdot(u\nabla K * u) = 0,$$

with K the heat kernel.

Blow-up for aggregation-diffusion equations

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla K * u) = 0.$$

Whether we have blow-up or not depends on K. No singularities for the (attractive!) kernel $-|x|^a, \ a > 0$.

The logarithmic case (corresponding to the heat kernel K for $d = 2$) is 'borderline' ($a \to 0^+$). Everything depends on the mass.

The parabolic-elliptic Keller-Segel model.

From now on, we put $\varepsilon = 1$.

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \varphi), \\ \Delta \varphi + u = 0, & x \in \mathbb{R}^d, \ t > 0. \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{PE}$$

Relevant in astrophysics (dynamics of nebulae): Chandrasekhar stationary solutions...before the seminal paper Keller-Segel '70.

Under some mild hypotheses, sharp 8π critical mass threshold for the well-posedness in $2D$.

Jäger-Luckhaus '92, Herrero-Velázquez '97.
Many papers starting around 2004. Blanchet-Dolbeault-Perthame; Blanchet-Carlen-Carrillo; Mizoguchi, Senba...

For $d \geq 3$, mostly results with concentration/radiality assumptions: Biler, Naito, Biler-Zienkewicz...

The parabolic-parabolic Keller-Segel model

$$\begin{cases} u_t = \Delta u - \nabla \cdot (u \nabla \varphi), \\ \tau \varphi_t = \Delta \varphi + u, & x \in \mathbb{R}^d, \ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \tag{PP}$$

Same stationary solutions and same scale invariance as before:
$u(x,t) \mapsto \lambda^2 u(\lambda x, \lambda^2 t)$; $\phi(x,t) \mapsto \phi(\lambda x, \lambda^2 t)$.

No nice wellposedness threshold results; also few result about dependence on τ for well-posedness/explosion.

Biler-Guerra-Karch, Calvez-Corrias, Campos-Dolbeault (2D); Iwabuchi...; ill-posedness Winkler (2020)...

Some natural small-τ limit results: Raczynski, Biler-Brandolese, Lemarié-Rieusset, Corrias-Escobedo-Matos...

Two toy models

$$\begin{cases} u_t = \Delta u - u\Delta\varphi, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \qquad \text{(TM)}$$

Semilinear: well-posedness proved considering $u\Delta\phi$ as a perturbation of $\nabla(u\nabla\phi)$.

$$\begin{cases} u_t = \Delta u + (\Delta\varphi)^2, \\ \tau\varphi_t = \Delta\varphi + u, & x \in \mathbb{R}^d,\ t > 0, \\ u(0) = u_0, \quad \varphi(0) = \varphi_0, \end{cases} \qquad \text{(TM')}$$

'Less semilinear'. Besov-type spaces do not work.

Both models have the same scale invariance as (PP) but a different parabolic-elliptic limit (the nonlinear heat equation).

Functional spaces

Scale-invariance: $u(x,t) \mapsto \lambda^2 u(\lambda x, \lambda^2 t)$; $\phi(x,t) \mapsto \phi(\lambda x, \lambda^2 t)$.
For some parameters, the following spaces work:

Besov-type spaces for initial data:
well-posedness in $\operatorname{ess\,sup}_{t>0} t^a \|u(t,\cdot)\|_b$.

Pseudomeasure spaces for initial data:
well-posedness in $\operatorname{ess\,sup}_{t>0,\xi\in\mathbb{R}^d} t^a |\xi|^b |\hat{u}(t,\xi)|$.

Morrey spaces (Biler, Lemarié-Rieusset...);
Hardy spaces (Kozono, Sugiyama...)

Besov spaces: definitions

We consider the scale-invariant space

$$\mathcal{E}_p := \left\{ u \in L^\infty(0,\infty; L^p(\mathbb{R}^d)),\ \left|\!\left|\!\left| u \right|\!\right|\!\right|_p := \operatorname{ess\,sup} t^{1-d/2p} \|u(t)\|_p < \infty \right\}.$$

Lemma

Let $d \geq 2$, $p > d/2$. Then $u_0 \in \dot{B}^{-(2-d/p)}_{p,\infty}(\mathbb{R}^d)$ if and only if $(t \mapsto \mathrm{e}^{t\Delta} u_0) \in \mathcal{E}_p$. Moreover:

$$C(d,p)^{-1} \|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}} \leq \left|\!\left|\!\left| \mathrm{e}^{t\Delta} u_0 \right|\!\right|\!\right|_p \leq C(d,p) \|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}}.$$

.

Besov spaces: results

Theorem

For $d/2 < p < d$, provided

$$\|u_0\|_{\dot{B}^{-(2-d/p)}_{p,\infty}} < C_p\sqrt{\tau},$$

(PP) *has a global solution $u \in \mathcal{E}_p$, unique in a ball of size $C'_p\sqrt{\tau}$.*

A similar result holds for (TM) but not for (TM').

Besov spaces: proof

We define the operators L, B by

$$\widehat{Lu}(\xi,t) = (2\pi)^{-d}\int_0^t\int_{\mathbb{R}^d} e^{-(t-s)|\xi|^2}\xi\eta\,\widehat{u}(\xi-\eta,s)e^{-\tau^{-1}s|\eta|^2}\widehat{\varphi}_0(\eta)\,\mathrm{d}\eta\,\mathrm{d}s.$$

$$\widehat{B(u,v)}(\xi,t) =$$

$$(2\pi)^{-d}\int_0^t\int_0^s\int_{\mathbb{R}^d}\frac{\xi\eta}{\tau}\mathrm{e}^{-(t-s)|\xi|^2}\mathrm{e}^{-\frac{1}{\tau}(s-\sigma)|\eta|^2}\widehat{u}(\xi-\eta,s)\widehat{v}(\eta,\sigma)\,\mathrm{d}\eta\,\mathrm{d}\sigma\,\mathrm{d}s.$$

In this way, we see that u satisfies the integral equation

$$\widehat{u}(\xi,t) = e^{-t|\xi|^2}\widehat{u}_0(\xi) + \widehat{Lu}(\xi,t) + \widehat{B(u,u)}(\xi,t).$$

Besov spaces: proof (2)

For p, q in an appropriate range, we prove estimates of the form

$$\|\|\|Lz(t)\|\|\|_q \leq C\tau^{-1/2+d/2(1/p-1/q)}\|\|\|z\|\|\|_p;$$
$$\|\|\|B(u,z)\|\|\|_p \leq C\tau^{-1/2+d/2(1/p-1/q)}\|\|\|u\|\|\|_p\|\|\|z\|\|\|_p$$

and we obtain the uniqueness in a ball of size $C\sqrt{\tau}$ in the space $\mathcal{E}_p$ of a fixed point for

$$u \mapsto e^{t\Delta}u_0 + Lu + B(u,u).$$

Pseudomeasure spaces: definitions

Let $a \in \mathbb{R}$. We introduce the pseudomeasure spaces

$$\mathcal{PM}^b = \{f \in \mathscr{S}'(\mathbb{R}^d) \colon \|f\|_{\mathcal{PM}^b} = \operatorname{ess\,sup}_{\xi \in \mathbb{R}^d} |\xi|^b |\widehat{f}(\xi)| < \infty\},$$

We will construct our solutions in the scale-invariant space

$$\mathscr{Y}_a = \{u \in L^\infty_{\mathrm{loc}}(0,\infty; \mathscr{S}'(\mathbb{R}^d)) \colon$$
$$\|u\|_{\mathscr{Y}_a} = \operatorname{ess\,sup}_{t>0,\, \xi \in \mathbb{R}^d} t^{1+(a-d)/2} |\xi|^a |\widehat{u}(\xi, t)| < \infty\}.$$

Pseudomeasure spaces: results

Theorem

Let $d \geq 2$, $u_0 \in \mathcal{PM}^{d-2}$ and $\varphi_0 = 0$. Assume $\tau \geq \mathrm{e}^3$.
Then (PP) possesses a global solution, provided

$$\|u_0\|_{\mathcal{PM}^{d-2}} < C_d \tau / (\ln \tau)^3.$$

It is unique in a ball of $\mathscr{Y}_{d-4/\ln\tau}$, with radius $C'_d \tau/(\ln\tau)^3$.

Initial data of size 'almost τ' instead of $\sqrt{\tau}$ with Besov spaces.

We allow as initial data $C|x|^{-2}$ for $d \geq 3$ (multiples of the stationary Chandrasekhar solution).

Proof: again, a fixed point method with a mild formulation and linear/bilinear estimates.

A similar result holds for (TM) and (TM').

Explosion "a la Montgomery-Smith"

We define $w_0 \in \mathscr{S}(\mathbb{R}^d)$ given by $\widehat{w}_0(\xi) = A\ 1_{B(a,1/4)}(\xi)$, where $A(d) > 0$ and $a = (3/4, 0, \ldots, 0)$.

Theorem

Assume that $u_0 \in \mathscr{S}(\mathbb{R}^d)$ and $\tau \geq 1$ and we have

$$\widehat{u}_0(\xi) \geq \tau \widehat{w}_0(\xi).$$

Then the life span t^* of a solution to (TM) satisfies $t^* < 1$.

Only a logarithmic discrepancy w.r.t. the well-posedness result in pseudomeasure spaces.

An analogous result (but worse discrepancy with well-posedness since we require an initial data of size at least τ^2) holds for (TM').

Explosion "a la Montgomery-Smith": proof.

$$\begin{aligned}\widehat{u}(\xi,t) &= \mathrm{e}^{-t|\xi|^2}\widehat{u_0}(\xi) - \int_0^t \mathrm{e}^{-(t-s)|\xi|^2}\widehat{u\Delta\varphi}(\xi,s)\,\mathrm{d}s \\ &= \mathrm{e}^{-t|\xi|^2}\widehat{u_0}(\xi) + (2\pi)^{-d}\int_0^t\int_{\mathbb{R}^d} \mathrm{e}^{-(t-s)|\xi|^2}\widehat{u}(\xi-\eta,s)|\eta|^2\widehat{\varphi}(\eta,s)\,\mathrm{d}\eta\,\mathrm{d}s.\end{aligned}$$

Positivity of the Fourier transform is preserved.

This (and the fact that the nonlinearity is quadratic: compare with the ODE $v' = v^2$) allows us to build a sequence $t_k \to 1$ such that $|u(t_k)|_\infty \geq |u(0,t_k)| = |\hat{u}(t_k)|_1 \to +\infty$.

Moment method

Result on bounded domains.

Theorem

There exist positive solutions of system $(\mathrm{TM'})$ *for* $\tau \geq 2$ *with* $u_0 \geq 0$, $\varphi_0 \geq 0$ *of order* τ^2, τ, *with lifespan at most* $T_{max} = 1$.

Proof: For well-chosen initial data, the moment

$$J(t) = \int_\Omega \psi(x)\varphi(x,t)\,\mathrm{d}x,$$

where $\psi \geq 0$ is the normalized eigenfunction of Δ with the first eigenvalue $\lambda > 0$, cannot be continued past $T = 1$ (since it satisfies a differential inequality).

Perspectives

Blow-up for (PP) in the whole space (radiality assumptions?) for u large enough. The method needs to be changed.

Less specific assumptions for blow-up for (TM).

Better size for (TM'), for which unlike (TM) we have a gap between well-posedness and examples of ill-posedness.

Perspectives (2)

In the very beginning, in

$$u_t - \varepsilon \Delta u + \nabla \cdot (u \nabla \phi) = 0;$$
$$\tau \phi_t - \Delta \phi = u.$$

we fixed $\varepsilon = 1$. For some simpler (parabolic-elliptic) models, the limit $\varepsilon \to 0$ is non-trivial (beyond the limit, see ε-sharp estimates for small $\varepsilon > 0$ in [Biler-B.-Karch-Laurençot '21-'22]).

So we may look at parabolic-parabolic models for a different kernel for $\varepsilon \to 0$ (interaction with $\tau \to \infty$)?

Part of a more general program to look at small-parameter asymptotics for nonlinear PDEs, motivated by turbulence.

Bibliography

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THANK YOU!!!

For those who want to use this beamer theme, the name is 'Ann Arbor'.