Noname manuscript No.
(will be inserted by the editor)

Exponential convergence to the stationary measure for a class of 1D Lagrangian systems with random forcing

Alexandre Boritchev

July 18, 2017

Abstract. We prove exponential convergence to the stationary measure for a class of 1d Lagrangian systems with random forcing in the space-periodic setting:

$$\phi_t + \phi_x^2/2 = F^{\omega},\ x \in S^1 = \mathbb{R}/\mathbb{Z}.$$

This is the first such result in a classical setting, i.e. in the dual-Lipschitz metric with respect to the Lebesgue space L_p for finite p. This partially answers the conjecture formulated in [10]. Our result is a consequence (and the natural stochastic PDE counterpart) of the results obtained in [6, 8]. It is also the natural analogue of the deterministic result [13] which holds in a generic setting.

University of Lyon

CNRS UMR 5208

University Claude Bernard Lyon 1

Institut Camille Jordan

43 Blvd. du 11 novembre 1918

69622 VILLEURBANNE CEDEX

FRANCE

E-mail: alexandre.boritchev@gmail.com

Keywords Burgers Equation, SPDEs, Lagrangian Systems, Weak KAM theory, Stationary Measure.

Mathematics Subject Classification (2000) 35Q53; 35R60, 35Q35, 37H10, 76M35

Contents

Abbreviations

- 1d: one-dimensional
- a.s.: almost surely
- i.i.d.: independent identically distributed
- r.v.: random variable

Introduction

We are concerned with convergence to the stationary measure for 1d random Lagrangian systems of the mechanical type, i.e. of the form:

$$L^{\omega}(x,v,t) = v^2/2 + F^{\omega}(x,t),\ x \in S^1 = \mathbb{R}/\mathbb{Z},$$

where $F^{\omega}(x,t)$ is a smooth function in x and a stationary random process in t (of the kick or white force type: see Section 1.1). The Legendre-Fenchel transform gives us the corresponding Hamiltonian:

$$H^{\omega}(x,p,t) = p^2/2 - F^{\omega}(x,t).$$

The corresponding Hamilton-Jacobi equation is:

$$\phi_t + \phi_x^2/2 = F^{\omega}. \tag{1}$$

Here, we consider only 1-periodic solutions ϕ. In this case the function $u = \phi_x$ satisfies the randomly forced inviscid Burgers equation:

$$u_t + uu_x = (F^{\omega})_x,\ x \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{2}$$

Note that it is equivalent to consider a solution of (2) and a solution of (1) defined up to an additive constant. Under the assumptions of Section 1.1, both of these equations are well-posed and their solutions define Markov processes. Therefore, we can consider the corresponding stationary measure. Its existence and uniqueness has been proved by E, Khanin, Mazel and Sinai in the white force case in the seminal work [8] using the Lagrangian representation of the solutions. For the case of a kick force, see for instance [11]. This result was clarified and generalised to the multi-d case by Khanin and his collaborators in [6, 10, 14] with more transparent assumptions on the forcing. In these papers, no explicit estimates on the speed of convergence to the stationary measure are given. However, an exponential bound locally in space away from the shocks has been obtained by Bec, Frisch and Khanin in [1]. In the papers mentioned above, the key object is the global minimiser and the key fact is its hyperbolicity.

We have previously obtained a bound on the speed of convergence to the stationary measure for solutions of (2) both in the 1d and in the multi-d case in [2, 3, 5] using stochastic PDE techniques. We proved that the speed of convergence is at least $C(p)t^{-\delta/p}$, $\delta > 0$, in the dual-Lipschitz metric corresponding to L_p, $p \in [1,\infty)$. Although this bound is given for the equation with an additional viscous term νu_{xx}, it is independent from the viscosity coefficient ν, and thus it still holds when we pass to the limit $\nu \to 0$. Note that for $\nu > 0$ there is exponential convergence to the stationary measure, but the speed of convergence is not a priori uniform in ν [16].

In our paper, we give an exponential bound for the speed of convergence to the stationary measure for solutions of (2) in the natural dual-Lipschitz metric mentioned above, which gives a partial answer in the 1d case to the conjecture stated in [10, Section 4]. This bound is important since it gives a natural SPDE analogue to the results on the exponential convergence of the minimising action curves obtained in [6, 8]. The part of the conjecture in [10] which remains open is proving that this exponential bound still holds if we add a positive viscosity coefficient ν, uniformly in ν. The main technical difficulty is that introducing this term destroys the well-understood structure of the minimisers.

It is very likely that the estimate we obtain is sharp since it coincides with the optimal one obtained in the generic nonrandom case by R. Iturriaga and H. Sanchez-Morgado [13]. Note that the metrics we use are also optimal since it is impossible to obtain such an estimate in the Lipschitz-dual space corresponding to L_∞ for solutions of (2) which are discontinuous with a strictly positive probability.

Finally, we would like to emphasize the link between our work and the corresponding deterministic results belonging to the realm of the weak KAM theory developed by A. Fathi and J. Mather [9]. In particular, there is a striking correspondence between the scheme of our proof and the one in [13], which follows a general rule: the results which hold in the random case under fairly weak assumptions are similar to the results which hold in the nonrandom case under more stringent genericity assumptions. For more on this subject and the link with the Aubry-Mather theory, see [11].

REMARK 0.1 *Our results extend to the case where ϕ, instead of being periodic in space, satisfies*

$$\phi(x+1) = \phi(x) + b, \ \ x \in \mathbb{R}.$$

The proofs are exactly the same since we use the results of [6, 8], which hold for all values of b.

Our results also extend to a class of non-mechanical convex in p Hamiltonians of the type $H(p) + F^\omega(t,x)$ with F^ω as above, under assumptions of the Tonelli type [9].

REMARK 0.2 *After the manuscript has been submitted, Iturriaga, Khanin and Zhang published a preprint containing more general results including also the multidimensional case [12]. However, their methods are more technically involved.*

1 Notation and setting

1.1 Random setting

We consider the mechanical Hamilton-Jacobi equation with two different types of additive forcing in the right-hand side and a C^∞-smooth initial condition ϕ^0.

We begin by formulating the assumptions on potentials, which are (except 1.1 (i) where we add an additional assumption for moments of the r.v.) the same as in the paper [6]:

Assumption 1.1 *In the "kicked" case, we assume the following.*

(i) The kicks at integer times j are of the form

$$F^{\omega}(j)(x) = \sum_{k=1}^{K} c_k^{\omega}(j) F^k(x),$$

where F^k are C^{∞}-smooth potentials on $S^1 = \mathbb{R}/\mathbb{Z}$. The random vectors $(c_k^{\omega}(j))_{1 \leq k \leq K}$ are i.i.d. $\mathbb{R}^K$-valued r.v.'s defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. Their distribution on $\mathbb{R}^K$, denoted by λ, is assumed to be absolutely continuous with respect to the Lebesgue measure, and all of its moments are assumed to be finite.

(ii) The potential 0 belongs to the support of λ.

(iii) The mapping from S^1 to $\mathbb{R}^K$ defined by

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding.

Assumption 1.2 *In the case of the white force potential, we assume the following.*

(i) The forcing has the form

$$F^{\omega}(x,t) = \sum_{k=1}^{K} (W_k^{\omega})_t(t) F^k(x),$$

where F^k are C^{∞}-smooth potentials on S^1, and $(W_k^{\omega})_t$ are independent white noises defined on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$, i.e. weak time derivatives of independent Wiener processes $W_k^{\omega}(t)$.

(ii) The mapping from S^1 to $\mathbb{R}^K$ defined by

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding.

Remark 1.1 *For both types of forcing, our results extend to the case of infinite-dimensional noise. The necessary restriction is that the noise remains smooth in space. For instance, we can put independent white noises on each Fourier mode in such a way that the amplitude of the noise decreases exponentially with the wavenumber.*

In the white noise case, we denote by G an antiderivative in time of the forcing:

$$G^\omega(x,t) = \sum_{k=1}^{K} W_k^\omega(t) F^k(x),$$

where $W_k^\omega(t)$ are independent standard Wiener processes with $W_k^\omega(0) = 0$. Since we will only consider time differences of G, the particular choice of antiderivative has no importance.

In both cases, F^ω will be abbreviated as F, and in the white force case $F(\cdot,t)$ will be abbreviated as $F(t)$, and similarly for G.

Note that since we have $G(t) \in C^\infty$ for every t, a.s., we can redefine the forcing F in the white force case so that $G(t) \in C^\infty$ for all $\omega \in \Omega$.

1.2 Functional spaces and Sobolev norms

Consider an integrable function v on S^1. For $p \in [1,\infty]$, we denote its L_p norm by $|v|_p$. The L_2 norm is denoted by $|v|$, and $\langle \cdot,\cdot \rangle$ stands for the L_2 scalar product.

For a nonnegative integer m and $p \in [1,\infty]$, $W^{m,p}$ stands for the Sobolev space of zero mean value functions v on S^1 with finite homogeneous norm

$$|v|_{m,p} = \left| \frac{d^m v}{dx^m} \right|_p .$$

In particular, $W^{0,p} = L_p$ for $p \in [1,\infty]$. We will never use Sobolev norms with $m \geq 1$ for non-zero mean functions: in particular, for solutions of (1) we will only consider the

Lebesgue norms. On the other hand, C^0 (resp. C^∞) will denote the space of C^0-smooth (resp. C^∞-smooth) (not necessarily zero mean value!) functions on S^1.

Since the length of S^1 is 1, we have:

$$|v|_1 \leq |v|_\infty \leq |v|_{1,1} \leq |v|_{1,\infty} \leq \cdots \leq |v|_{m,1} \leq |v|_{m,\infty} \leq \dots$$

We denote by $L_\infty/\mathbb{R}$ the space of functions in L_∞ defined modulo an additive constant endowed with the norm:

$$|u-v|_{L_\infty/\mathbb{R}} = \inf_{K\in\mathbb{R}} |u-v-K|_\infty$$

We recall a version of the classical Gagliardo–Nirenberg inequality (see [7, Appendix]):

Lemma 1.2 *For a smooth zero mean value function v on S^1,*

$$|v|_{\beta,r} \leq C\, |v|_{m,p}^{\theta}\, |v|_q^{1-\theta},$$

where $m > \beta \geq 0$, and r is defined by

$$\beta - \frac{1}{r} = \theta\Big(m - \frac{1}{p}\Big) + (1-\theta)\Big(0 - \frac{1}{q}\Big),$$

under the assumption $\theta = \beta/m$ if $p = 1$ or $p = \infty$, and $\beta/m \leq \theta < 1$ otherwise. The constant C depends on m, p, q, β, θ.

Subindices t and x, which can be repeated, denote partial differentiation with respect to the corresponding variables. We denote by $v^{(m)}$ the m-th derivative of v in the variable x. For brevity, the function $v(t,\cdot)$ is denoted by $v(t)$.

1.3 Agreements

All functions which we consider in this paper are real-valued.

The quantities denoted by K_i or M_i are positive constants which only depend on the general features of the system (i.e. the statistical distribution of the forcing): they are nonrandom and do not depend on the initial condition. Moreover the constants $K_1(p), \ldots, K_5(p)$ depend on the Lebesgue exponent $p \in [1, \infty)$.

There are two quantities, denoted respectively by $C_1(\omega)$ and $C_2(p)(\omega)$, which are time-independent r.v.'s with all moments finite, which do not depend on the initial condition, but only "pathwise" on the forcing; moreover the quantity $C_2(p)$ depends on the parameter p.

Quantities denoted by $C_i(s, \omega)$, $i \geq 3$ being a natural number, are time-dependent r.v.'s, which also have finite moments and do not depend on the initial condition, but only "pathwise" on the forcing ω. Moreover, these r.v.'s are stationary in the sense that $C_i(s, \omega)$ coincides with $C_i(s+t, \theta^t \omega)$ for every t, where θ^t denotes the time shift [8].

We will always denote by $\phi(t, x)$ a solution of (1) and by $u(t, x)$ its derivative, which solves (2), respectively for initial conditions ϕ^0 and $u^0 = \phi^0_x$. We will denote accordingly the solutions for two initial conditions $\phi^0, \overline{\phi^0}$. The assumptions on the forcing are the ones given in Section 1.1.

2 Dynamical objects and stationary measure

Here we introduce the Lagrangian dynamical objects in the setting described in the previous section. Note that all the results in Sections 2.2 hold under much more general assumptions: for instance, it is possible to drop (iii) in Assumption 1.1 or (ii) in

Assumption 1.2. However, these hypotheses will be extremely important for the results which will be given in Section 2.3. For more details on the definitions given below, see [10, 11].

2.1 Lagrangian formulation and minimisers

Definition 2.1 *For a time interval $[s,t]$ and $x,y \in S^1$, we say that a curve $\gamma_{s,t}^{y,x}(\tau)$ is a* ***minimiser*** *if it minimises the action*

$$A(\gamma) = \frac{1}{2}\int_s^t \gamma_t(\tau)^2 d\tau + \sum_{n\in(s,t]} \Big(F^n(\gamma(n))\Big)$$

in the "kicked" case and the action

$$\begin{aligned} A(\gamma) =& \frac{1}{2}\int_s^t \gamma_t(\tau)^2 d\tau + \int_s^t \left(\gamma_t(\tau)\Big(\frac{\partial G}{\partial x}(\gamma(\tau),s) - \frac{\partial G}{\partial x}(\gamma(\tau),\tau)\Big)\right) d\tau \\ &+ \Big(G(\gamma(t),t) - G(\gamma(t),s)\Big) \end{aligned}$$

in the white force case, respectively, over all absolutely continuous curves γ such that $\gamma(t) = x$ and $\gamma(s) = y$.

Remark 2.2 *In the kicked case, it is easy to see that minimising curves are linear on intervals $[n, n+1]$ for integer values of n.*

Definition 2.3 *For a time interval $[s,t]$, $x \in S^1$ and a continuous function $\phi : S^1 \to \mathbb{R}$, we say that a curve $\gamma_{s,t,\phi}^{x}(\tau) : [s,t] \to S^1$ is a* ***ϕ-minimiser*** *if it minimises $A(\gamma) + \phi(\gamma(s))$ over all absolutely continuous curves on $[s,t]$ such that $\gamma(t) = x$. In particular, all ϕ-minimisers are minimisers.*

Now we can define the (pathwise) solution to (1) for a given $\omega \in \Omega$ and a given continuous initial condition.

DEFINITION 2.4 *For a time interval $[s,t]$ and a continuous initial condition $\phi(s)$: $S^1 \to \mathbb{R}$, for every ω by definition the (pathwise) solution $\phi : [s,t] \times S^1 \to \mathbb{R}$ of (1) is defined by the ω-dependent action A:*

$$\phi(\tau, x) = A(\gamma) + \phi(s, \gamma(s)), \ \tau \in [s,t],$$

where $\gamma = \gamma^x_{s,\tau,\phi(s)}$ is an ω-dependent $\phi(s)$-minimiser defined on $[s,\tau]$ satisfying $\gamma(\tau) = x$.

REMARK 2.5 *Note that by a compactness argument, one can show that given an initial condition ϕ, for any given endpoint x, a ϕ-minimiser γ on $[s,t]$ such that $\gamma(t) = x$ exists. In the white force case, this minimiser gives a time-continuous solution in C^0, whereas in the "kicked" case the solution is a cadlag in time (right-continuous and with a limit to the left) C^0-valued function.*

REMARK 2.6 *It is easy to check that the solution ϕ verifies the semigroup property: in other words, one can define a solution operator*

$$S^{t_2}_{t_1} : \ \phi(t_1) \mapsto \phi(t_2), \ s \le t_1 \le t_2 \le t,$$

such that for $t_1 \le t_2 \le t_3$,

$$S^{t_3}_{t_2} \circ S^{t_2}_{t_1} = S^{t_3}_{t_1}.$$

In particular, the following holds:

LEMMA 2.7 *For any $\tau \in (s,t)$, the restriction of any $\phi(s)$-minimiser defined on $[s,t]$ on the time interval $[\tau,t]$ is a $S^{\tau}_{s}\phi(s)$-minimiser.*

REMARK 2.8 *Note that the solution ϕ is the limit in C^0 of the strong solutions to the equation obtained if we add a viscous term $\nu\phi_{xx}$ to (1) and then we make ν tend to* 0 *(see [10]).*

Definition 2.9 *For a time t and a point $x \in S^1$, we say that a curve $\gamma_t^{x,+}(\tau)$: $[t, +\infty) \mapsto S^1$ is a forward one-sided minimiser if it minimises $A(\gamma)$ over all absolutely continuous curves such that $\gamma(t) = x$ for compact in time perturbations.*

Namely, we require that if for a curve $\tilde{\gamma}$ such that $\tilde{\gamma}(t) = x$ there exists T such that $\tilde{\gamma}(s) \equiv \gamma(s)$ for $s \geq T$, then $A(\gamma) - A(\tilde{\gamma}) \leq 0$ (this difference is well-defined since it is equal to the difference of the actions on the finite interval $[t, T]$).

2.2 Stationary measure and related issues

Here we give a few results which hold under weak assumptions and are sufficient to ensure that the stationary measure corresponding to (2) exists and is unique. These results are not new and hold both in the one-dimensional and in the multi-dimensional setting: see [8, 11]. Estimates for the speed of convergence are given in [2, 3, 5], where all proofs are stated for $\nu > 0$, but still hold for $\nu = 0$ [8, 10]. Up to some natural modifications due to the fact that the forcing is now discrete in time, the convergence estimates can be generalised to the kick force case in 1d [2]. For more details, see also [15], where a random forcing is introduced in a similar setup.

The flow corresponding to (2) induces a Markov process, and then we can define the corresponding semigroup denoted by S_t^*, acting on Borel measures on any L_p, $1 \leq p < \infty$. A *stationary measure* for (2) is a Borel probability measure defined on L_p, invariant with respect to S_t^* for every t. A *stationary solution* of (2) is a random process v defined for $(t, \omega) \in [0, +\infty) \times \Omega$, satisfying (2) and taking values in L_p, such that the distribution of $v(t)$ does not depend on t. This distribution is automatically a stationary measure.

Existence of a stationary measure for (2) is obtained using uniform bounds for solutions in $W^{1,1}$, which is compactly injected into $L_p,\ p \in [1,\infty)$, and the Bogolyubov-Krylov argument. It is more delicate to obtain uniqueness of a stationary measure, which implies uniqueness for the distribution of a stationary solution.

REMARK 2.10 *The most natural space for our model would be the space $L_\infty/\mathbb{R}$, on which we could have treated directly the solutions to the equation (1). Moreover, this is the space in which exponential convergence to the unique stationary solution is proved in the deterministic generic setting in [13]. However, this space is not separable, which makes it delicate to deal with the stationary measure.*

DEFINITION 2.11 *Fix $p \in [1,\infty)$. For a continuous function*

$$g:\ L_p \to \mathbb{R},$$

we define its Lipschitz norm as

$$|g|_{L(p)} := |g|_{Lip} + \sup_{L_p} |g|,$$

where $|g|_{Lip}$ is the Lipschitz constant of g. The set of continuous functions with finite Lipschitz norm will be denoted by $L(p)$.

DEFINITION 2.12 *For two Borel probability measures μ_1, μ_2 on L_p, we denote by $\|\mu_1 - \mu_2\|^*_{L(p)}$ the Lipschitz-dual distance:*

$$\|\mu_1 - \mu_2\|^*_{L(p)} := \sup_{g \in L(p),\ |g|_{L(p)} \leq 1} \Big| \int_{S^1} g d\mu_1 - \int_{S^1} g d\mu_2 \Big|.$$

The following result proved in [2, 3, 5] is, as far as we are aware, the first explicit estimate for the speed of convergence to the stationary measure of (2) which is uniform with respect to the viscosity coefficient ν and is formulated in terms of Lebesgue spaces only. It holds both in the white force and in the kick-force setting in 1d, and only in the white force case in the multidimensional setting. However, a result which holds in the L_∞ norm locally in space away from the shocks in 1d has been obtained by Bec, Frisch and Khanin in [1]. The proof in [2, 3, 5] uses a version of the coupling argument due to Kuksin and Shirikyan [15, Chapter 3]. The situation is actually simpler than for the stochastic 2D Navier-Stokes equations. In particular, in our setting the "damping time" needed to make the distance between two solutions corresponding to the same forcing small does not depend on the initial conditions. Moreover, since the flow of (2) is L_1-contracting, the coupling argument is simplified.

THEOREM 2.13 *There exists $\delta > 0$ such that for every $p \in [1, \infty)$, there exists a positive constant $K_1(p)$ such that we have:*

$$\|S_t^* \mu_1 - S_t^* \mu_2\|_{L(p)}^* \leq K_1(p) t^{-\delta/p}, \qquad t \geq 1,$$

for any probability measures μ_1, μ_2 on L_p.

2.3 Main results and scheme of the proof

Now we are ready to state the main result of the paper.

THEOREM 2.14 *There exists $M_1 > 0$ such that for every $p \in [1, \infty)$, there is a positive constant $K_2(p)$ such that we have:*

$$\|S_t^* \mu_1 - S_t^* \mu_2\|_{L(p)}^* \leq K_2(p) exp(-M_1 t/p), \qquad t \geq 0, \tag{3}$$

for any probability measures μ_1, μ_2 on L_p.

The proof is, in the spirit, similar to the proof of [13, Theorem 1]. In that paper the authors use the objects of the weak KAM theory such as the Peierls barrier, which do not have any directly available counterparts in our setting. However, there is a straightforward dynamical interpretation of their method in the simplest case. Namely, consider a mechanical Lagrangian

$$v^2/2 - V(x)$$

such that the potential V is smooth and generic (i.e. it reaches its maximum at a unique point y with $V''(y) < 0$).

An energy-minimising curve on $[0, T]$ remains in a small neighbourhood of y on $[\tau, T - \tau]$ (with τ T-independent). Consequently, since y is a nondegenerate maximum for V, we obtain by linearising the Euler-Lagrange equation that at the time $T/2$, all minimisers (independently of the initial condition) are $C\exp(-CT)$-close to y, and then a standard argument allows us to conclude that for any initial conditions ϕ^0, $\overline{\phi^0}$, the solutions of (1) at time T are $C\exp(-CT)$-close up to an additive constant, i.e.:

$$\begin{aligned}
&\sup_{\phi^0,\ \overline{\phi^0} \in C^0} \inf_{\tilde{c} \in \mathbb{R}} \left| \phi(T,x) - \overline{\phi}(T,x) - \tilde{c} \right|_\infty \\
&\leq \sup_{\phi^0,\ \overline{\phi^0} \in C^0} \left| \phi(T,x) - \overline{\phi}(T,x) - (\phi(T/2,y) - \overline{\phi}(T/2,y)) \right|_\infty \\
&\leq C\exp(-CT).
\end{aligned}$$

There are two main ingredients in the proof. Roughly speaking, the first one is that for a given initial condition ϕ^0, the ϕ^0-minimisers concentrate exponentially. The second one is that the one-sided minimisers, which are limits of the ϕ^0_T-minimisers on $[0, T]$ as $T \to +\infty$ for *any* set of initial conditions $\{\phi^0_T\}$, also concentrate exponentially.

Now we introduce some definitions.

The diameter of a closed set Z can be thought of as the minimal length of a closed interval on S^1 containing Z.

Definition 2.15 *Consider a closed subset Z of S^1. Let $a(Z)$ denote the maximal length of a connected component of $S^1 - Z$. We define the diameter of Z as*

$$d(Z) = 1 - a(Z).$$

Definition 2.16 *For $-\infty < r < s \leq t < +\infty$ and for a fixed function $\phi^0 : S^1 \to \mathbb{R}$, let Ω_{r,s,t,ϕ^0} be the set of points reached, at the time s, by ϕ^0-minimisers on $[r,t]$:*

$$\Omega_{r,s,t,\phi^0} = \{\gamma^x_{r,t,\phi^0}(s),\ x \in S^1\}.$$

Now we give the formulations of the two key lemmas. The first of them is (up to notation) [6, Corollary 2.1.]; the only small difference is that we require that C_3 does not depend on ϕ_0 and is stationary in s (which follows immediately from the proof) and all its moments are finite (which follows without any problems using the Borel-Cantelli lemma in the same way as in that paper). The second one is a forward-in-time version of [8, Lemma 5.6.(a)]; the finiteness of the moments of C follows from the Borel-Cantelli lemma like in the proof of [8, Lemma 5.4.]. Time-reversal can be done without any measurability issues, since this is a "pathwise" result.

Remark 2.17 *Although the second lemma is only proved in the white force setting in [8], its proof in the kick force setting follows the same lines and is technically simpler.*

Lemma 2.18 *There exist a r.v. $C_3(s,\omega)$ and a constant K_6 such that we have the inequality:*

$$\sup_{\phi^0 \in C^0} d(\Omega_{0,s,s+s',\phi^0}) \leq C_3 \exp(-K_6 s').$$

Lemma 2.19 *There exists a r.v. $C_1(\omega)$ and a constant K_7 such that we have:*

$$\sup_{\tilde{\gamma}_1, \tilde{\gamma}_2 \in \Gamma} |\tilde{\gamma}_1(t) - \tilde{\gamma}_2(t)| \leq C_1 \exp(-K_7 t),\ t \geq 0. \tag{3}$$

where Γ is the set of all forward one-sided minimisers defined on the time interval $[0, +\infty)$.

Corollary 2.20 *Consider an initial condition ϕ^0 and a time $t > 0$. Then there exists a r.v. $C_4(t, \omega)$ and a constant K_8 such that for any ϕ^0-minimiser $\gamma : [0, 2t] \to S^1$ and any forward one-sided minimiser $\delta :\ [0, +\infty) \to S^1$ we have:*

$$|\gamma(t) - \delta(t)| \leq C_4 \exp(-K_8 t). \tag{4}$$

Proof of Corollary 2.20: As we know from [8, Section 5], extracting a subsequence of minimisers (and in particular of ϕ^0-minimisers) on $[0, s]$ and taking the limit while letting s go to $+\infty$, one obtains a forward one-sided minimiser. In particular, for every ϵ there exists $s(\epsilon) \geq 2t$, a ϕ^0-minimiser $\tilde{\gamma}$ defined on $[0, s]$ and a forward one-sided minimiser $\tilde{\delta}$ on $[0, +\infty)$ such that:

$$|\tilde{\gamma}(t) - \tilde{\delta}(t)| \leq \epsilon. \tag{5}$$

By Lemma 2.19 we have:

$$|\delta(t) - \tilde{\delta}(t)| \leq C_1(\omega) \exp(-K_7 t), \tag{6}$$

and by Lemma 2.18, since the restriction $\tilde{\gamma}|_{[0,2t]}$ is a ϕ^0-minimiser, we have:

$$|\gamma(t) - \tilde{\gamma}(t)| \leq C_3(t, \omega) \exp(-K_6 t), \tag{7}$$

Combining the inequalities (5)-(7) and using the triangular inequality, and then letting ϵ go to 0, we get (4) with $K_8 = \min(K_6,\ K_7)$ and $C_4(t, \omega) = C_1(\omega) + C_3(t, \omega)$.

□

3 Proof of Theorem 2.14

First we state two auxiliary lemmas. To prove Lemma 3.1 one can take the ν-uniform estimates in [2, 3] and consider the limit $\nu \to 0$. On the other hand, [8, Lemma 3.1] yields Lemma 3.3 after a few standard modifications (namely proving as previously that all moments of the stationary random variable are finite).

LEMMA 3.1 *There is a r.v.* $C_5(t,\omega)$ *such that for* $t \geq 1$, *we have:*

$$\sup_{\phi^0 \in C^0} |u(t)|_{1,1} \leq C_5.$$

COROLLARY 3.2 *For* $t \geq 1$, *we have:*

$$\sup_{\phi^0 \in C^0} |u(t)|_{\infty} \leq C_5,$$

where C_5 *is the same as above.*

LEMMA 3.3 *For* $t \geq 1$, *there is a r.v.* $C_6(t,\omega)$ *such that we have:*

$$\sup_{s \in [t,t+1], \gamma \in \Gamma} |\gamma_t(s)| \leq C_6,$$

where Γ *is the set of minimisers defined on* $[0,t+1]$.

Moreover, we will need the following lemma, analogous to [8, Section 3, Fact 1].

LEMMA 3.4 *Consider two minimisers* γ_1, γ_2, *both defined on* $[t,T]$, $T \geq t+1$, *and satisfying* $\gamma_1(T) = \gamma_2(T)$. *There is a r.v.* $C_7(t,\omega)$ *such that if for* $\epsilon > 0$ *we have:*

$$|\gamma_1(t) - \gamma_2(t)| \leq \epsilon,$$

then we have the following inequality for the actions of the minimisers:

$$|A(\gamma_1) - A(\gamma_2)| \leq C_7(t,\omega)(\epsilon + \epsilon^2).$$

Proof: By symmetry, it suffices to prove that:

$$A(\gamma_2) \leq A(\gamma_1) + C_7(t,\omega)(\epsilon + \epsilon^2); \tag{8}$$

C_7 will be fixed later. Consider the curve $\tilde{\gamma}_1 : [t,T] \to S^1$ defined by:

$$\tilde{\gamma}_1(s) = \gamma_1(s) + (t+1-s)(\gamma_2(t) - \gamma_1(t)),\ \ s \in [t, t+1].$$

$$\tilde{\gamma}_1(s) = \gamma_1(s),\ \ s \in [t+1, T].$$

Using Definition 2.1 and Lemma 3.3, we get:

$$A(\tilde{\gamma}_1) \leq A(\gamma_1) + C_7(t,\omega)(\epsilon + \epsilon^2).$$

On the other hand, since $\tilde{\gamma}_1$ has the same endpoints as the minimiser γ_2, we get:

$$A(\gamma_2) \leq A(\tilde{\gamma}_1).$$

Combining these two inequalities yields (8). □

The proof of the following lemma follows the lines of [13].

LEMMA 3.5 *Consider two solutions ϕ and $\overline{\phi}$ of (1) defined on the time interval $[0, +\infty)$. There there exist $M_2 > 0$ and a r.v. $C_8(t,\omega)$ such that we have:*

$$|\phi(t) - \overline{\phi}(t)|_{L_\infty/\mathbb{R}} \leq C_8 \exp(-M_2 t),\ \ t \geq 0.$$

Proof of Lemma 3.5: Consider two solutions ϕ and $\overline{\phi}$ to (1) corresponding to the same forcing and different initial conditions at time 0. Using Definition 2.4, we get for any $t \geq 1$ and $x \in S^1$:

$$\phi(2t,x) - \overline{\phi}(2t,x) \tag{9}$$

$$= \phi(t, \gamma_1(t)) + A(\gamma_1|_{[t,2t]}) - \overline{\phi}(t, \gamma_2(t)) - A(\gamma_2|_{[t,2t]}),$$

where γ_1 and γ_2 are respectively a ϕ^0- and a $\overline{\phi^0}$-minimiser on $[0, 2t]$ ending at x. By Corollary 2.20, we have:

$$|\gamma_i(t) - y| \leq C_4(t, \omega) \exp(-K_8 t), \; i = 1, 2, \tag{10}$$

where we fix *any* point y such that $y = \delta(t)$ for a one-sided minimiser δ defined on $[0, \infty)$. By Corollary 3.2, this inequality yields that:

$$|\phi(t, \gamma_1(t)) - \overline{\phi}(t, \gamma_2(t)) - R|$$

$$\leq (|\phi_x(t)|_\infty |\gamma_1(t) - y| + |\overline{\phi}_x(t)|_\infty |\gamma_2(t) - y|)$$

$$\leq 2C_4(t, \omega) C_5(t, \omega) \exp(-K_8 t),$$

where

$$R = \phi(t, y) - \overline{\phi}(t, y),$$

Note that R does not depend on x. On the other hand, using (10), by Lemma 3.4 we get that there is a r.v. $C_9(t, \omega)$ such that:

$$|A(\gamma_1|_{[t,2t]}) - A(\gamma_2|_{[t,2t]})| \leq C_9 \exp(-K_8 t).$$

Therefore, by (9), we get:

$$|\phi(2t) - \overline{\phi}(2t)|_{L_\infty/\mathbb{R}} \leq \sup_{x \in S^1} |\phi(2t, x) - \overline{\phi}(2t, x) - R|$$

$$\leq (C_9(t, \omega) + 2C_4(t, \omega) C_5(t, \omega)) \exp(-K_8 t).$$

This proves the lemma's statement. □

COROLLARY 3.6 *Consider two solutions u and $\overline{u}$ of (2) defined on the time interval $[0, +\infty)$. Then there exists a p-dependant r.v. $C_2(p)(\omega)$ such that for any $p > 0$ we have:*

$$|u(t) - \overline{u}(t)|_p \leq C_2(p) \exp(-M_2 t / 2p), \; t \geq 0.$$

Proof: This result follows from Lemma 3.5 using Lemma 1.2, Lemma 3.1 and Corollary 3.2. Indeed, it suffices to observe that for any $R' \in \mathbb{R}$:

$$|u(t)-\overline{u}(t)|_p \leq K_3(p)|\phi_x(t)-\overline{\phi}_x(t)|_1^{1/p}|\phi_x(t)-\overline{\phi}_x(t)|_\infty^{1-1/p}$$

$$\leq K_4(p)|\phi(t)-\overline{\phi}(t)-R'|_1^{1/2p}|u_x(t)-\overline{u}_x(t)|_1^{1/2p}|u(t)-\overline{u}(t)|_\infty^{1-1/p}.$$

$$\leq K_5(p)|\phi(t)-\overline{\phi}(t)-R'|_1^{1/2p}\max(|u_x(t)|_1,\ |\overline{u}_x(t)|_1,\ |u(t)|_\infty,\ |\overline{u}(t)|_\infty)^{1-1/p}$$

$$\leq C_2(p)|\phi(t)-\overline{\phi}(t)-R'|_\infty^{1/2p}.\Box$$

Proof of Theorem 2.14: By the Fubini theorem, it suffices to prove this result in the case when the measures μ_1 and μ_2 are two Dirac measures concentrated at the initial conditions $u^0, \overline{u^0} \in L_p$.

By a contradiction argument, it follows from Corollary 3.6 that if we denote by B the event

$$B = \{\omega \in \Omega \mid |u(t)-\overline{u}(t)|_{L(p)} \geq \exp(-M_2 t/4p)\},$$

then we have:

$$\mathbf{P}(B) \leq \exp(-M_2 t/4p)\mathbf{E}C_2(p),\ t \geq 0.$$

Now consider a function g defined on L_p which satisfies $|g|_L \leq 1$. We have for $t \geq 0$:

$$\mathbf{E}(|g(\phi(t))-g(\overline{\phi}(t))|_p)$$

$$\leq \mathbf{P}(B)\ \mathbf{E}(|g(\phi(t))-g(\overline{\phi}(t))|_p \mid B)$$

$$+\mathbf{P}(\Omega-B)\ \mathbf{E}(|g(\phi(t))-g(\overline{\phi}(t))|_p \mid \Omega-B)$$

$$\leq 2\mathbf{P}(B)+\mathbf{P}(\Omega-B)\exp(-M_2 t/4p)$$

$$\leq (2\mathbf{E}C_2(p)(\omega)+1)\exp(-M_2 t/4p).$$

This proves the theorem's statement with $K_2(p) = 2\mathbf{E}C_2(p) + 1$ and

$K_1(p) = M_2/4p$. $\square$

REMARK 3.7 *The estimate in Lemma 3.5 is uniform with respect to the initial conditions: in other words, there exists a constant $K_9 > 0$ such that we have*

$$\mathbf{E} \sup_{\phi^0, \overline{\phi^0} \in C^0} |\phi(t) - \overline{\phi}(t)|_{L_\infty/\mathbb{R}} \leq K_9 \exp(-M_2 t), \ t \geq 0.$$

A similar statement holds for the estimate in Corollary 3.6.

Acknowledgements

I am very grateful to P. Bernard, R. Iturriaga, R. Joly, S. Kuksin, L. Rifford, P. Thieullen, N. Vichery and K. Zhang for helpful discussions. I would also like to thank the anonymous referee, whose careful reading helped to improve the manuscript. Finally, I would like to thank separately A. Davini, who spent a lot of time helping me to understand the paper [13], A. Fathi, who introduced we to the weak KAM community through the ANR project WKBHJ ANR-12-BS01-0020 and K. Khanin, who gave me the opportunity to visit him in Toronto, where I started my work on random Lagrangian dynamics.

References

1. J. Bec, U. Frisch, and K.Khanin. Kicked Burgers turbulence. *Journal of Fluid Mechanics*, 416:239–267, 8 2000.
2. A. Boritchev. Estimates for solutions of a low-viscosity kick-forced generalised Burgers equation. *Proceedings of the Royal Society of Edinburgh A*, (143(2)):253–268, 2013.

3. A. Boritchev. Sharp estimates for turbulence in white-forced generalised Burgers equation. *Geometric and Functional Analysis*, (23(6)):1730–1771, 2013.
4. A. Boritchev. Erratum to: Multidimensional Potential Burgers Turbulence. *Communications in Mathematical Physics*, 344(1):369–370, 2016. See [5].
5. A. Boritchev. Multidimensional Potential Burgers Turbulence. *Communications in Mathematical Physics*, 342:441–489, 2016. With erratum: see [4].
6. A. Boritchev and K. Khanin. On the hyperbolicity of minimizers for 1D random Lagrangian systems. *Nonlinearity*, 26(1):65, 2013.
7. C. Doering and J. D. Gibbon. *Applied analysis of the Navier-Stokes equations*. Cambridge Texts in Applied Mathematics. Cambridge University Press, 1995.
8. Weinan E, K. Khanin, A. Mazel, and Ya. Sinai. Invariant measures for Burgers equation with stochastic forcing. *Annals of Mathematics*, (151):877–960, 2000.
9. A. Fathi. Weak KAM Theorem in Lagrangian Dynamics, preliminary version. 2005.
10. D. Gomes, R. Iturriaga, K. Khanin, and P. Padilla. Viscosity limit of stationary distributions for the random forced Burgers equation. *Moscow Mathematical Journal*, (5):613–631, 2005.
11. R. Iturriaga and K. Khanin. Burgers turbulence and random Lagrangian systems. *Communications in Mathematical Physics*, (232:3):377–428, 2003.
12. R. Iturriaga, K. Khanin, and K. Zhang. Exponential convergence of solutions for random Hamilton-Jacobi equation. Preprint, arxiv: 1703.10218, 2017.
13. R. Iturriaga and H. Sanchez-Morgado. Hyperbolicity and exponential convergence of the Lax-Oleinik semigroup. *Journal of Differential Equations*, 246(5):1744 – 1753, 2009.

14. K. Khanin and K. Zhang. Hyperbolicity of minimizers and regularity of viscosity solutions for random Hamilton-Jacobi equations. arXiv:1201.3381, 2011.

15. S. Kuksin and A. Shirikyan. *Mathematics of two-dimensional turbulence*, volume 194 of *Cambridge tracts in mathematics*. Cambridge University Press, 2012.

16. Y. Sinai. Two results concerning asymptotic behavior of solutions of the Burgers equation with force. *Journal of Statistical Physics*, (64):1–12, 1991.