1d, multi-d and fractional Burgers turbulence: Sobolev norms and small-scale behaviour.

Alexandre Boritchev, University of Lyon

Outline

Motivation: 3D Incompressible Navier-Stokes Equations

$$\frac{\partial \mathbf{u}}{\partial t} + (\mathbf{u} \cdot \nabla)\mathbf{u} = -\nabla p + \nu \Delta \mathbf{u} + \eta; \quad \text{div } \mathbf{u} = 0. \qquad \text{(NSE)}$$

Supplemented by boundary conditions.

$\mathbf{u}(t, \mathbf{x})$ velocity
$p(t, \mathbf{x})$ pressure

$\nu > 0$ constant viscosity coefficient
($\nu \ll 1$: turbulent regime)
$\eta(t, \mathbf{x})$ (random) forcing, smooth as a function of $\mathbf{x}$

The idea is to study the statistical behaviour of $\mathbf{u}$ as ν varies, all other parameters being fixed.

The K41 Theory

In Fourier space, a scale is, roughly speaking, the inverse of the Fourier frequency under consideration.

In a periodic setting, typical small-scale quantities are:
-$\hat{\mathbf{u}}(\mathbf{k})$ for large $\mathbf{k}$.
-$\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})$ for small $\mathbf{r}$.

In this talk, we only consider space scales, not time scales.

Small-scale (or rather 'moderately-small scale') behaviour for a velocity field of turbulent fluid is a very old problem (1930s-: Taylor, Onsager...; Kolmogorov 1941).

Length Scales

There are also estimates for small-scale quantities in K41 (of course non-rigorous: 3D Navier-Stokes is one of the Clay millenium problems!)
These estimates are related to estimates for Sobolev norms.

Some estimates have been found for 2D Navier-Stokes, non-linear Schrödiger, KdV (see Kuksin '97-'98 and the book of Kuksin-Shirikyan), with and without random forcing.

However, these estimates are not sharp in the limit when the small parameter ν goes to 0 (different powers for upper and lower ones).

1D Periodic Generalised Burgers Equation

$$u_t + (f(u))_x = \nu u_{xx}, \ t \geq 0, \ x \in S^1 = \mathbb{R}/\mathbb{Z}. \qquad \text{(1DB)}$$

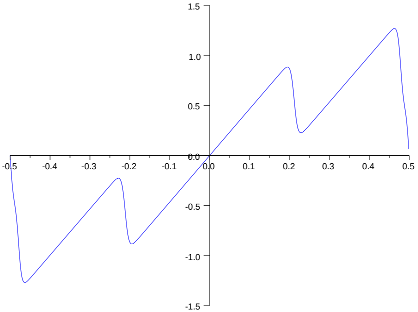
We assume that f is smooth, strongly convex.
Case $f(u) = u^2/2$: usual Burgers equation.

"Pressureless turbulence" considered by many physicists, for instance Polyakov '95 (and Zeldovich in the multi-d case '89).

We assume that $\nu > 0, \ \nu \ll 1$. Again, only ν varies.

For simplicity, we assume that $\int_{S^1} u(t, \cdot) = 0, \ \forall t$.

Typical Profile of a Burgers Solution



Amplitude of solution ~ 1. Cliffs (quasi-shocks): number of cliffs ~ 1, jump ~ -1, width $\sim \nu$.
Burgers turbulence or "Burgulence": see [Bec-Khanin 2007].
Ramp-cliff structure $\Rightarrow$ intermittency.

Estimates for the Norms of the Solution: Notation

$X \overset{a}{\sim} Y$: There exists $C > 0$ such that $C^{-1}X \leq Y \leq CX$.
C only depends on the parameter a, which is never the viscosity coefficient ν.

$|\cdot|_p$: the Lebesgue norm in the space $L_p(S^1)$.

$|\cdot|_{m,p}$: the Sobolev norm in the space $W^{m,p}(S^1)$.

$||\cdot||_\alpha$: the Sobolev norm in the space $H^\alpha(S^1) = W^{\alpha,2}(S^1)$.

$\{\dots\}$: averaging over a time period $[T_1, T_2]$, where T_1, T_2 only depend on $|u_0|_\infty$, $|u_0'|_\infty$.

Estimates for the Sobolev Norms of the Solution

In [Bor3], we obtain sharp estimates for the (1DB) solution.

Theorem 1

$$\{|u|_{m,p}^{\beta}\} \overset{m,p,\beta}{\sim} \nu^{-\beta\gamma}, \quad \forall m \geq 1,\ 1 < p \leq \infty, \quad \beta \geq 0.$$

Here $\gamma(m, p) = m - 1/p$.

- Upper and lower estimates the same up to a ν-independent constant.
- For all norms N, $\{N(u)^{\beta}\}$ behaves like $\{N(u)\}^{\beta}$.

Estimates for the Sobolev Norms of the Solution: Ideas of Proofs

Exact upper estimates are obtained by using Oleinik's estimate.

Exact lower estimates follow from the energy balance:

$$\frac{d}{dt}||u||_0^2 = -2\nu||u||_1^2.$$

combined with the 'inviscid energy dissipation'.

Propagation to higher order Sobolev norms follows from the Gagliardo-Nirenberg inequality and higher-order energy estimates.

Upper Bounds: Oleinik's Estimate

Consider unforced (1DB) on $S = (t, x) \in [0, T] \times S^1$:

$$u_t + uu_x = \nu u_{xx}.$$

Consider $v = tu_x$. The function v can only reach a str. positive maximum for $t > 0$. Then we would have:

$$\underbrace{v_t}_{\geq 0} + u \underbrace{v_x}_{0} + t^{-1}(-v + v^2) = \underbrace{\nu v_{xx}}_{\leq 0}.$$

Thus $v \leq 1$ on S. In other words, $u_x \leq t^{-1} \Rightarrow$ "damping".

Obtaining lower bounds

We have:

$$\frac{d}{dt}\int_{S^1} u^2 = \underbrace{-2\int_{S^1} uf'(u)u_x}_{0} + 2\nu \int_{S^1} uu_{xx} = -2\nu \int_{S^1} u_x^2.$$

Integrating in time, we get:

$$|u(T)|_2^2 - |u(0)|_2^2 = -2\nu T\{|u|_{1,2}^2\}.$$

Using the upper estimates, for $t \geq 1$ we have that:

$$|u(T)|_2^2 \leq (\max_x u_x(0,x))^2 \leq CT^{-2}.$$

Consequently, for T large enough:

$$\{|u|_{1,2}^2\} \geq C\nu^{-1}.$$

Small-Scale Results

$S_p(\ell)$: $\int_{S^1} \{|u(x+\ell) - u(x)|^p\} dx, \ p \geq 0.$

$E(k)$: average of $\{\frac{1}{2}|\hat{\mathbf{u}}(n)|^2\}$ over n such that $|n| \in [C^{-1}k, \ Ck]$, where $C > 0$ is a constant.

Theorem 2

For $\ell \in [C\nu, \ C]$, we have $S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, \ 0 \leq p \leq 1. \\ \ell, \ p \geq 1. \end{cases}$

Theorem 3

For $k^{-1} \in [C\nu, \ C]$, the energy spectrum satisfies $E(k) \sim k^{-2}$.

The fractional Burgers equation

We consider the equation with a fractional dissipation term ($\Lambda = \sqrt{-\Delta}$):

$$u_t + (f(u))_x = -\nu \Lambda^\alpha u, \ t \geq 0, \ x \in S^1 = \mathbb{R}/\mathbb{Z}. \qquad \text{(1DB)}$$

Periodic zero-average setting: fractional Laplacian well-defined as a Fourier multiplier by $(2\pi|k|)^\alpha$.

Physical motivation: study of combustion (Clavin-Denet).
Mathematical study: Alibaud-Imbert-Vovelle, Alfaro-Droniou, Kiselev-Nazarov-Shterenberg, Constantin-Vicol...

Scaling argument and technical difficulties (I)

Scaling analysis (ℓ caracteristic length scale):

$$u_t + \underbrace{u}_{1}\ \underbrace{u_x}_{\ell^{-1}} = -\nu \underbrace{\Lambda^\alpha u}_{\ell^{-\alpha}}.$$

$$\ell \sim \nu^\beta,\ \beta = \frac{1}{\alpha - 1}.$$

Supercritical case $0 < \alpha < 1$ ($\beta < 0$): if initial data large, solutions stop being smooth. Indeed:

$$||u||_0^2 \leq |u|_{1,1}^2 \leq t^{-2};\ \ ||u||_{\alpha/2}^2 \leq C(\alpha)|u|_{1,1}^2 \leq C,$$

On the other hand if solutions are smooth, no dissipation in the inviscid limit since:

$d||u||_0^2/dt = -2\nu||u||_{\alpha/2}^2 \implies$ contradiction.

Scaling argument and technical difficulties (II)

$$u_t + \underbrace{u}_{1} \underbrace{u_x}_{\ell^{-1}} = -\nu \underbrace{\Lambda^\alpha u}_{\ell^{-\alpha}}.$$

Critical case $\alpha = 1$ ($\beta = \infty$): we still have regularity. Exponentially small phenomenon (see Kiselev-Nazarov-Shterenberg).

Case $\alpha > 2$ problematic since Λ^α is not positive (therefore no good maximum principle).

Conclusion: The nicest case is the subcritical case $1 < \alpha < 2$.

Constantin-Vicol maximal principle

Assume $0 < \alpha < 2$. Let v be a smooth function on S^1 and consider a point x_1 where $v_x(\cdot)$ reaches its maximum on S^1. Then we have the following alternative (which corresponds to a nonlinear maximum principle:)

- Either $v_x(x_1) \lesssim \max(v(\cdot))$.
- Or $-\Lambda^\alpha v_x(t, x_1) \gtrsim (v_x(t, x_1))^{1+\alpha} / |v(t)|_\infty^\alpha$.

Proved in [Constantin-Vicol '12] using ideas form [Córdoba-Córdoba '04].

Results for Sobolev norms and small-scale quantities

Exactly the same as in the case $\alpha = 2$, except that ν has to be replaced with ν^β as expected.

The nonlinear maximum principle gives us optimal upper bounds; lower ones follow by Gagliardo-Nirenberg.

The random force

We only deal with additive random forcing.

$\eta(t,x) = \eta^{\omega}(t,x)$: smooth in space random force, white (or kicked) in time. Idea: independent white noises on each Fourier mode (decreasing amplitudes ensure smoothness).

$\{\dots\}$: averaging both over the time period $[t, t+T_0]$, where $t \geq T_0$ and T_0 is a constant, and in ensemble (taking the expected value).

The results and the methods

The results are exactly the same as in the deterministic case, up to the change of notation for the brackets (see previous slide).

The basic ideas of the methods are also the same, except we have to ensure that everything holds in the stochastic case.
In particular, we need Itô's formula, but also more refined tools such as the Burkholder-Davis-Gundy inequality.

Stationary Measure

The solutions u define a Markov process in $L_1(S^1)$.

Existence of a stationary measure: Bogolyubov-Krylov.

Uniqueness and rate of convergence to the stationary measure: coupling argument (cf. papers of Kuksin-Shirikyan). The Markov semigroup S_t is nonexpanding in L_1 in the 1d case (in multi-d argument OK with different norm):

$$|S_t^{\omega} u_0 - S_t^{\omega} \tilde{u}_0|_1 \leq |u_0 - \tilde{u}_0|_1, \ \forall \omega.$$

Thus, the distance between the solutions is made small since the solutions themselves become small during "small-noise intervals", and then this distance is nonincreasing.

Stationary Measure: the Speed of Convergence

The arguments above give us a convergence in $Ct^{-\delta}$, independently from the viscosity and the initial data.

In a recent preprint [Bor5], we gave an elementary proof of the fact that this speed is exponential in 1D in the inviscid case using Lagrangian methods (cf. [E-Khanin-Mazel Sinai '00]).

More involved statement in [Bec-Frisch-Khanin '00] (see also the recent preprint [Iturriaga-Khanin-Zhang] in the multi-d case).

Multi-d Burgers Turbulence: Setting

$$\mathbf{u}_t + (\nabla f(\mathbf{u}) \cdot \nabla) \cdot \mathbf{u} = \nu \Delta \mathbf{u} + \nabla \eta, \ t \geq 0, \ x \in \mathbb{T}^d = (\mathbb{R}/\mathbb{Z})^d.$$

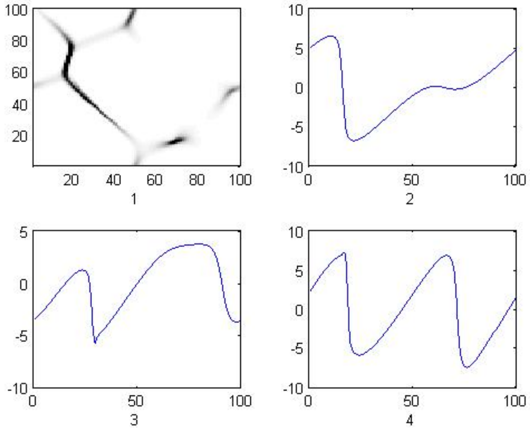
Key assumption: $\mathbf{u} = \nabla \psi$ (conserved by flow), where the potential ψ solves the viscous Hamilton-Jacobi equation:

$$\psi_t + f(\nabla \psi) = \nu \Delta \psi + \eta, \ t \geq 0, \ x \in \mathbb{T}^d.$$

As previously, f smooth, strongly convex, of moderate growth.
$\nu > 0, \ \nu \ll 1$, η smooth in space and white in time.

Multi-d Burgers turbulence: What is Expected

One expects a tesselation of smooth zones separated by shocks of codimension 1. In directions which are transverse to the shock, the longitudinal projection of the solution looks like a 1d solution [Gurbatov-Moshkov-Noullez 2010].



Notation: What Changes Compared to 1D

The Sobolev norms are defined as follows:

$$|\mathbf{v}|_{m,p} = \sum_{\alpha_1+\dots+\alpha_d=m,\ 1\le i\le d} \left| \frac{\partial^m v_i}{\partial x_1^{\alpha_1}\dots\partial x_d^{\alpha_d}} \right|_p .$$

The notation for small-scale quantities is:

$S_p(\mathbf{r})$: $\left\{ \int_{\mathbf{x}\in\mathbb{T}^d} |\mathbf{u}(\mathbf{x}+\mathbf{r})-\mathbf{u}(\mathbf{x})|^p d\mathbf{x} \right\}$.

$S_p(\ell)$: $c_d^{-1}\ell^{-(d-1)} \int_{\mathbf{r}\in\ell\mathcal{S}^{d-1}} S_p(\mathbf{r}) d\sigma(\mathbf{r}),\ p\ge 0$.

$E(k)$: Same definition as above: average of $\{\frac{1}{2}|\hat{\mathbf{u}}(\mathbf{n})|^2\}$ over $\mathbf{n}$ such that $|\mathbf{n}|\in[C^{-1}k,\ Ck]$, where $C>0$ is a constant.

Upper and Lower Estimates: What Changes Compared to 1D

The statements are almost exactly the same, up to the fact that we do not have estimates for the Sobolev norms $m \geq 1,\ p = \infty$.

In the deterministic case (or in the kicked case), we need mild anisotropy assumptions on the initial data.

It should be possible to combine multi-d and a fractional dissipation term.

Concluding Remarks

Our results give exact and rigorously proved small-scale estimates for a broad class of 1d and multid models which have a "good" maximum principle.

They confirm previous physical predictions under very general conditions on the initial data, for a physically reasonable class of forces. Our small-scale estimates also hold for solutions of the inviscid equation, and for the stationary solution.

First sharp estimates for Sobolev norms (and almost sharp estimates for small-scale quantities) in the unforced case are due to [Biryuk 2001].

Ongoing work on small-scale phenomena for more general equations with P.Biler and G.Karch (Wroclaw) and P.Laurençot (Toulouse).

Bibliography

[Bor1]: AB, Estimates for Solutions of a Low-Viscosity Kick-Forced Generalised Burgers Equation, Proc. Roy. Soc. Edinburgh A, 143(2), 2013, 253-268.

[Bor2]: AB, Sharp Estimates for Turbulence in White-Forced Generalised Burgers Equation, GAFA 23 (2013), 1730-1771

[Bor3]: AB, Note on Decaying Turbulence in a Generalised Burgers Equation, ARMA 214 (2014), 1, 331-357.

[Bor4]: AB, Multidimensional potential Burgers turbulence, CMP 342(2016): 441-489.

[Bor5]: AB, Exponential convergence to the stationary measure for a class of 1D Lagrangian systems with random forcing, SPDEs: Anal. and Comp (2017).

[Bor6]: AB, Decaying turbulence for the fractional subcritical Burgers equation, DCDS A, 2018, 38(5) : 2229-2249. .

[BK]: A.Boritchev, S.Kuksin, Introduction to SPDEs, Stochastic Burgers Equation and Burgulence, in preparation.