TURBULENCE FOR THE GENERALISED BURGERS EQUATION

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Abstract. In this survey, we review the rigorous results on turbulence for the generalised space-periodic Burgers equation:

$$u_t + f'(u)u_x = \nu u_{xx} + \eta,\ x \in S^1 = \mathbb{R}/\mathbb{Z},$$

studied by A.Biryuk and the author in [8, 10, 12, 13]. Here, f is smooth and strongly convex, whereas the constant $0 < \nu \ll 1$ corresponds to a viscosity coefficient.

We will consider both the unforced case ($\eta = 0$) and the randomly forced case, when η is smooth in x and irregular (kick or white noise) in t. In both cases, sharp bounds for Sobolev norms of u averaged in time and in ensemble of the type $C\nu^{-\delta},\ \delta \geq 0$, with the same value of δ for upper and lower bounds, are obtained. These results yield sharp bounds for small-scale quantities characterising turbulence, confirming the physical predictions [7].

Abbreviations

- 1d, 3d, multi-d: 1, 3, multi-dimensional
- a.e.: almost everywhere
- a.s.: almost surely
- (GN): the Gagliardo–Nirenberg inequality (Lemma 1.1)
- i.i.d.: independent identically distributed
- r.v.: random variable

Introduction

The generalised 1d space-periodic Burgers equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad \nu > 0,\ x \in S^1 = \mathbb{R}/\mathbb{Z} \tag{1}$$

(the classical Burgers equation [15] corresponds to $f(u) = u^2/2$) is a popular model for the Navier–Stokes equation. Indeed, both of them have similar nonlinearities and dissipative terms. Therefore, the physical arguments justifying various theories of hydrodynamical turbulence

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can usually be applied to describe the behaviour of solutions to the Burgers equation. Thus, this equation is often used as a benchmark for turbulence theories. It is also used as a benchmark for the numerical methods for turbulent flows. For more information, see [7].

For $\nu \ll 1$ and f strongly convex, i.e. satisfying:

$$f''(x) \geq \sigma > 0, \quad x \in \mathbb{R}, \tag{2}$$

solutions of (1) exhibit turbulent-like behaviour, called Burgers turbulence or "Burgulence" [6, 7]. To simplify the presentation, we restrict ourselves to solutions with zero mean value in space:

$$\int_{S^1} u(t,x)dx = 0, \quad \forall t \geq 0. \tag{3}$$

The space mean value does not change in time. Indeed, since u is 1-periodic in space, we have:

$$\frac{d}{dt}\int_{S^1} u(t,x)dx = -\int_{S^1} f'(u(t,x))u_x(t,x)dx + \nu\int_{S^1} u_{xx}(t,x)dx = 0.$$

If the mean value of the initial value u_0 on S^1 is equal to $b \neq 0$, we may consider the zero mean value function

$$v(t,x) = u(t,x+bt) - b,$$

which is a solution of (1) with $f(y)$ replaced by $g(y) = f(y+b) - by$. So the assumption (3) does not lead to a loss of generality.

In this survey, we consider both the unforced equation (1) and the generalised Burgers equation with an additive forcing term, smooth in space and irregular in time (see Subsection 1.2). We summarise the estimates obtained by Biryuk and the author [8, 10, 12, 13] for the Sobolev norms as well as for the dissipation length scale and the small-scale quantities relevant for the theory of hydrodynamical turbulence: the structure functions and the energy spectrum. This survey is partially based on the Ph.D. thesis of the author [11], where some technical points are covered in more detail.

The major difference between the unforced and the white-forced generalised Burgers equation is the energy picture. In the first case, we have a dissipative system: the L_2 norm is decreasing in time. Consequently, the regime where the energy $\int_{S^1} u^2/2$ dissipates fast enough (which yields a time-averaged lower bound on the Sobolev norms) is transient and depends on the initial condition through a certain quantity D (see (15) for its definition). On the contrary, in the second case, after a time needed either to dissipate energy if u_0 is large or to supply energy if u_0 is small, we are in a *quasi-stationary regime*, in the sense that in average on large enough time intervals, we have an approximate balance between the dissipation rate $-\nu \mathbf{E}\|u\|_1^2$ and the constant energy supply rate I_0.

For the *unforced* Burgers equation, some *upper* estimates for small-scale quantities are well-known. For example, Lemma 4.1 of our work is an analogue in the periodic setting of the one-sided Lipschitz estimate due to Oleinik, and the upper estimate for the structure function $S_1(\ell)$ follows from an estimate for the solution in the class of bounded variation functions BV. For references on these classical aspects of the theory of scalar conservation laws, see [20, 45, 49]. For some upper estimates for small-scale quantities, see [38, 53]. To our best knowledge, rigorous lower estimates were not known before Biryuk's and our work.

The research on the small-scale behaviour of solutions for the (forced) generalised Burgers equation is motivated by the problem of turbulence. It has been inspired by the pioneering works of Kuksin, who obtained lower and upper estimates for Sobolev norms by negative powers of the viscosity for a large class of equations (see [41, 42] and references in [42]). For more recent results obtained by Kuksin, Shirikyan and others for the 2D Navier–Stokes equation, see the book [43] and references therein.

The estimates for Sobolev norms and for small-scale quantities presented in our work are asymptotically sharp in the sense that viscosity enters lower and upper bounds at the same negative power. Such estimates are not available for the more complicated equations considered in [41, 42, 43].

This survey is also concerned with the problem of the invariant measure for the stochastic generalised Burgers equation. This problem has been treated prevously for the nonlinearity uu_x by Sinai [51, 52] and in the inviscid limit $\nu \to 0$ by E, Khanin, Mazel and Sinai [22]; see also [28, 31]. Here we present a simple approach to this problem, described in [13], which consists in using L_1-contractivity of the flow corresponding to the equation, and a coupling argument.

We do not consider other aspects of Burgulence, such as the inviscid limit, the behaviour of solutions for spatially rough forcing and the noncompact setting. We refer the reader to the survey by Bec and Khanin [7], which is concerned with physical aspects of the theory of Burgulence, and to the survey by Bakhtin [3], which discusses related probabilistic and ergodic results.

Organisation of the paper: We begin by introducing the notation and setup in Section 1. In Section 2, we present the K41 theory as well as the physical predictions for Burgulence. In Section 3, we formulate the main results.

In Section 4, we consider the solution $u(t,x)$ of the unforced equation (1). In Subsection 4.1, we begin by recalling the upper estimate for the quantity

$$\max_{s\in[t,t+1],\ x\in S^1} u_x(s,x),\ t\geq 1.$$

Using this bound, we get upper and lower estimates for the Sobolev norms of u. In Subsection 4.2 we study the implications of our results for Burgulence theory. Namely, we give sharp upper and lower bounds for the dissipation length scale, the increments and the spectral asymptotics for the flow $u(t,x)$, which hold uniformly for $\nu \leq \nu_0$. The quantity $\nu_0 > 0$ depends only on f and on the initial condition. These results justify rigorously the physical predictions for small-scale quantities which characterise Burgulence.

In the two last sections of the paper, we consider the randomly forced generalised Burgers equation. In Section 5, we obtain analogues of the results in Section 4, which also confirm the corresponding physical predictions [7]. Section 6 is concerned with the stationary measure.

1. Notation and setup

All functions which we consider in this paper are real-valued, except in Section 2, where vectors in $\mathbb{R}^3$ are written in bold script.

1.1. Functional spaces and Sobolev norms.

Consider a zero mean value integrable function v on S^1. For $p\in[1,\infty]$, we denote its L_p norm by $|v|_p$. The L_2 norm is denoted by $|v|$, and $\langle\cdot,\cdot\rangle$ stands for the L_2 scalar product. From now on $L_p,\ p\in[1,\infty]$, denotes the space of zero mean value functions in $L_p(S^1)$. Similarly, C^∞ is the space of C^∞-smooth zero mean value functions on S^1.

For a nonnegative integer m and $p\in[1,\infty]$, $W^{m,p}$ stands for the Sobolev space of zero mean value functions v on S^1 with finite homogeneous norm

$$|v|_{m,p} = \left|\frac{d^m v}{dx^m}\right|_p.$$

In particular, $W^{0,p} = L_p$ for $p\in[1,\infty]$. For $p=2$, we denote $W^{m,2}$ by H^m and abbreviate the corresponding norm as $\|v\|_m$.

Since the length of S^1 is 1, we have:

$$|v|_1 \leq |v|_\infty \leq |v|_{1,1} \leq |v|_{1,\infty} \leq \cdots \leq |v|_{m,1} \leq |v|_{m,\infty} \leq \dots$$

We recall a version of the classical Gagliardo–Nirenberg inequality (see [21, Appendix]):

Lemma 1.1. *For a smooth zero mean value function v on S^1,*

$$|v|_{\beta,r} \leq C\, |v|^{\theta}_{m,p}\, |v|^{1-\theta}_{q},$$

where $m > \beta \geq 0$, and r is defined by

$$\frac{1}{r} = \beta - \theta\Big(m - \frac{1}{p}\Big) + (1-\theta)\frac{1}{q},$$

under the assumption $\theta = \beta/m$ if $p = 1$ or $p = \infty$, and $\beta/m \leq \theta < 1$ otherwise. The constant C depends on m, p, q, β, θ.

From now on, we will refer to this inequality as (GN).

For any $s \geq 0$, H^s stands for the Sobolev space of zero mean value functions v on S^1 with finite norm

$$\|v\|_s = (2\pi)^s \Big(\sum_{k\in\mathbb{Z}} |k|^{2s} |\hat{v}(k)|^2\Big)^{1/2}, \tag{4}$$

where $\hat{v}(k)$ are the complex Fourier coefficients of $v(x)$. For an integer $s = m$, this norm coincides with the previously defined H^m norm. For $s \in (0,1)$, $\|v\|_s$ is equivalent to the norm

$$\|v\|'_s = \Bigg(\int_{S^1}\Big(\int_0^1 \frac{|v(x+\ell) - v(x)|^2}{\ell^{2s+1}} d\ell\Big) dx\Bigg)^{1/2} \tag{5}$$

(see [1, 54]).

Subindices t and x, which can be repeated, denote partial differentiation with respect to the corresponding variables. We denote by $v^{(m)}$ the m-th derivative of v in the variable x. For brevity, the function $v(t,\cdot)$ is denoted by $v(t)$.

1.2. **Well-posedness and different types of forcing.** In Section 4, we consider the unforced equation (1) with a C^∞-smooth initial condition u_0. This equation has a unique solution in C^∞: see for instance [39, Chapter 5].

In Section 5, we consider the generalised Burgers equation with two different types of additive forcing in the right-hand side, taking again a C^∞-smooth initial condition u_0. Since the forcing always has zero mean value in space and the initial condition satisfies (3), its solutions satisfy (3) for all time.

First, we consider the kick force. We begin by providing the space L_2 with the Borel σ-algebra $(\Omega, \mathcal{F})$. Then we consider an L_2-valued r.v. $\zeta = \zeta^\omega$ on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$. We suppose that ζ satisfies the following three properties.

(i) (Non-triviality)

$$\mathbf{P}(\zeta \equiv 0) < 1.$$

(ii) (Finiteness of moments for Sobolev norms) For every $m \geq 0$, we have:

$$I_m = \mathbf{E}\, \|\zeta\|_m^2 < +\infty.$$

(iii) (Vanishing of the expected value)

$$\mathbf{E}\zeta \equiv 0.$$

It is not difficult to construct explicitly ζ satisfying **(i)-(iii)**. One possibilty is to suppose that the real Fourier coefficients of ζ, defined for $k > 0$ by

$$a_k(\zeta) = \sqrt{2} \int_{S^1} \cos(2\pi kx) u(x);\ b_k(\zeta) = \sqrt{2} \int_{S^1} \sin(2\pi kx) u(x), \quad (6)$$

are independent r.v. with zero mean value and exponential moments tending to 1 fast enough as $k \to +\infty$.

Now let $\zeta_i = \zeta_i^\omega$, $i \in \mathbb{N}$ be i.i.d. r.v.'s having the same distribution as ζ. The sequence $(\zeta_i)_{i \geq 1}$ is a r.v. defined on a probability space which is a countable direct product of copies of Ω. From now on, this space will itself be called Ω. The meaning of $\mathcal{F}$ and $\mathbf{P}$ changes accordingly.

For $\omega \in \Omega$, the kick force ξ^ω is by definition the distribution defined by

$$\xi^\omega(t,x) = \sum_{i=1}^{+\infty} \delta_{t=i}\ \zeta_i^\omega(x),$$

where $\delta_{t=i}$ denotes the Dirac measure at the time moment i.

The kick-forced equation corresponds to the case where, in the right-hand side of (1), 0 is replaced by the kick force:

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = \xi^\omega. \quad (7)$$

This means that for integers $i \geq 1$, at the moments i the solution $u(x)$ instantly increases by the kick $\zeta_i^\omega(x)$, and that between these moments u solves (1). We make the additional assumption that the solution is a right-continuous function of time.

Existence and uniqueness of spatially smooth solutions to (7) follows directly from the corresponding fact for the unforced equation.

The other type of forcing considered here is the *white force*. Heuristically this force corresponds to a scaled limit of kick forces with more and more frequent kicks.

We provide each space $W^{m,p}$ with the Borel σ-algebra. Then we consider an L_2-valued Wiener process

$$w(t) = w^{\omega}(t),\ \omega \in \Omega,\ t \geq 0,$$

defined on a complete probability space $(\Omega,\ \mathcal{F},\ \mathbf{P})$, and an *adapted* filtration $\{\mathcal{F}_t,\ t \geq 0\}$ (i.e., for $t \geq 0$, $w(t)$ is $\mathcal{F}_t$-measurable, and $\mathcal{F}_t$ and the σ-algebra generated by the r.v.'s $w(t+s) - w(t),\ s \geq 0$ are independent).

We assume that for each m and each $t \geq 0$, $w(t) \in H^m$, almost surely. That is, for $\zeta, \chi \in L_2$,

$$\mathbf{E}(\langle w(s), \zeta\rangle \langle w(t), \chi\rangle) = \min(s,t) \langle Q\zeta, \chi\rangle ,$$

where Q is a symmetric operator which defines a continuous mapping $Q : L_2 \to H^m$ for every m. Thus, $w(t) \in C^\infty$ for every t, almost surely. From now on, we redefine the Wiener process so that this property holds for all $\omega \in \Omega$. We will denote $w(t)(x)$ by $w(t,x)$. For $m \geq 0$, we denote by I_m the quantity

$$I_m = Tr_{H^m}(Q) = \mathbf{E} \left\| w(1) \right\|_m^2 .$$

For more details on Wiener processes in Hilbert spaces, see [18, Chapter 4] and [44].

It is not difficult to construct $w(t)$ explicitly. For instance, we could consider the particular case of a "diagonal" noise:

$$w(t) = \sqrt{2} \sum_{k \geq 1} a_k w_k(t) \cos(2\pi k x) + \sqrt{2} \sum_{k \geq 1} b_k \tilde{w}_k(t) \sin(2\pi k x),$$

where $w_k(t),\ \tilde{w}_k(t),\ k > 0$, are standard independent Wiener processes and

$$I_m = \sum_{k \geq 1} (a_k^2 + b_k^2)(2\pi k)^{2m} < \infty$$

for each m. From now on, $dw(s)$ denotes the stochastic differential corresponding to the Wiener process $w(s)$ in the space L_2.

Now fix $m \geq 0$. By Fernique's Theorem [44, Theorem 3.3.1], there exist $\lambda_m, C_m > 0$ such that

$$\mathbf{E} \exp\Big(\lambda_m \left\| w(T) \right\|_m^2 / T\Big) \leq C_m, \quad T \geq 0. \tag{8}$$

Therefore by Doob's maximal inequality for infinite-dimensional submartingales [18, Theorem 3.8. (ii)] we have:

$$\mathbf{E} \sup_{t \in [0,T]} \left\| w(t) \right\|_m^p \leq \Big(\frac{p}{p-1}\Big)^p \mathbf{E} \left\| w(T) \right\|_m^p < +\infty, \tag{9}$$

for any $T > 0$ and $p \in (1, \infty)$.

The white-forced equation is obtained by replacing 0 by the weak derivative $\eta^\omega = \partial w^\omega / \partial t$ in the right-hand side of (1). Here, $w^\omega(t),\ t \geq 0$ is the Wiener process defined above.

DEFINITION 1.2. *For $T \geq 0$, we say that an H^1-valued process $u(t,x) = u^\omega(t,x)$ is a solution of the equation*

$$\frac{\partial u^\omega}{\partial t} + f'(u^\omega)\frac{\partial u^\omega}{\partial x} - \nu \frac{\partial^2 u^\omega}{\partial x^2} = \eta^\omega \tag{10}$$

for $t \geq T$ if:

(i) For every $t \geq T$, $\omega \mapsto u^\omega(t,\cdot)$ is $\mathcal{F}_t$-measurable.

(ii) For every ω, $t \mapsto u^\omega(t,\cdot)$ is continuous in H^1 for $t \geq T$ and satisfies

$$\begin{aligned} u^\omega(t) =& u^\omega(T) - \int_T^t \Big(\nu L u^\omega(s) + \frac{1}{2} B(u^\omega)(s)\Big) ds \\ &+ w^\omega(t) - w^\omega(T), \end{aligned} \tag{11}$$

where

$$B(u) = 2f'(u)u_x; \quad L = -\partial_{xx}.$$

For brevity, solutions for $t \geq 0$ will be referred to as solutions.

Existence and uniqueness of smooth solutions to (10) is proved by the mild solution technique (cf. [19, Chapter 14]). Since the forcing and the initial condition are smooth in space, the mapping $t \mapsto u(t)$ is time-continuous in H^m for every m, and $t \mapsto u(t) - w(t)$ has a space derivative in C^∞ for all t, a.s.

Now consider, for a solution $u(t,x)$ of (10), the functional $G_m(u(t)) = \|u(t)\|_m^2$ and apply Itô's formula [18, Theorem 4.17]:

$$\begin{aligned} \|u(t)\|_m^2 = \|u_0\|_m^2 - \int_0^t \left(2\nu \|u(s)\|_{m+1}^2 + \langle L^m u(s),\ B(u)(s)\rangle\right) ds + tI_m \\ + 2\int_0^t \langle L^m u(s),\ dw(s)\rangle \end{aligned}$$

(we recall that $I_m = Tr(Q_m)$.) Consequently,

$$\frac{d}{dt}\mathbf{E}\,\|u(t)\|_m^2 = -2\nu \mathbf{E}\,\|u(t)\|_{m+1}^2 - \mathbf{E}\,\langle L^m u(t),\ B(u)(t)\rangle + I_m.$$

As $\langle u,\ B(u)\rangle = 0$, for $m = 0$ this relation becomes

$$\frac{d}{dt}\mathbf{E}\,|u(t)|^2 = I_0 - 2\nu \mathbf{E}\,\|u(t)\|_1^2. \tag{12}$$

1.3. **Notation and agreements.** When considering a Sobolev norm in $W^{m,p}$, the quantity $\gamma = \gamma(m,p)$ denotes $\max(0,\ m-1/p)$.

In Subsection 2.1, $\mathbf{v}(t,\mathbf{x})$ denotes the velocity of a 3d flow with period 1 in each spatial coordinate. In the whole paper, $u(t,x)$ denotes a solution of the generalised Burgers equation with a given initial condition $u_0 = u(0,\cdot)$. In Section 4, we deal with the equation (1) under the assumptions (2-3). In Section 5 we deal with the equation (10) under the assumptions (2-3) and under the additional condition

$$\forall m \geq 0,\ \exists h \geq 0,\ C_m > 0 :\ |f^{(m)}(x)| \leq C_m(1+|x|)^h, \quad x \in \mathbb{R}, \quad (13)$$

where $h = h(m)$ is a function such that

$$1 \leq h(1) < 2 \quad (14)$$

(the lower bound on $h(1)$ follows from (2)). The results in that section also hold for the kicked equation (7), under the same assumptions as for (10), except (14), which is unnecessary.

When we consider the randomly forced generalised Burgers equation, $\mathbf{P}$ et $\mathbf{E}$ denote, respectively, the probability and the expected value with respect to the probability measure Ω (cf. Section 1.2).

All quantities denoted by C with sub- or superindices are positive and nonrandom. Unless otherwise stated, they depend only on the following parameters:

- When dealing with the K41 theory, the statistical properties of the forcing.

- When studying the unforced generalised Burgers equation, the function f determining the nonlinearity $f'(u)u_x$, as well as the parameter
$$D = \max(|u_0|_1^{-1},\ |u_0|_{1,\infty}) \quad (15)$$
which characterises how generic the initial condition is.

- When studying the randomly forced generalised Burgers equation, the function f determining the nonlinearity $f'(u)u_x$, as well as the statistical properties of the forcing. In the case of a kick force, by statistical properties we mean the distribution function of the i.i.d. r.v.'s ζ_i. In the case of a white force, we mean the correlation operator Q for the Wiener process w defining the random forcing.

In particular, these quantities never depend on the viscosity coefficient ν.

Constants which also depend on parameters $a_1, \dots, a_k$ are denoted by $C(a_1,\dots,a_k)$. By $X \overset{a_1,\dots,a_k}{\lesssim} Y$ we mean that $X \leq C(a_1,\dots,a_k)Y$. The notation $X \overset{a_1,\dots,a_k}{\sim} Y$ stands for

$$Y \overset{a_1,\dots,a_k}{\lesssim} X \overset{a_1,\dots,a_k}{\lesssim} Y.$$

In particular, $X \lesssim Y$ and $X \sim Y$ mean that $X \leq CY$ and $C^{-1}Y \leq X \leq CY$, respectively. Note that this notation is never used wit the parameter ν: in other words, dependence on the viscosity is always explicitly specified.

We use the notation $g^- = \max(-g,0)$ and $g^+ = \max(g,0)$.

In Subsection 2.1, the brackets $\langle\cdot\rangle$ denote the expected value. For the meaning of the brackets $\{\cdot\}$, see Subsection 4.1 in the deterministic case (where they correspond to averaging in time) and Subsection 5.3 in the random case (where they correspond to averaging in time and taking the expected value). The definitions of the relevant ranges and the length scales, as well as of the small-scale quantities, i.e. the structure functions $S_{p,\alpha}$ and $S_{p,1} = S_p$ and the spectrum $E(k)$ depend on the setting: see Subsections 2.1, 2.2, 4.2 and 5.3.

2. Turbulence and the Burgers equation

2.1. **Turbulence, K41 theory, intermittency.** It is well-known that giving a precise definition of turbulence is problematic. However, some features are generally recognised as characteristic of turbulence: many degrees of freedom, unpredictability/chaos, (small-scale) irregularity... For a more detailed discussion, see [27, 55]. Here, we will only present (in a slightly modified form) the vocabulary of the theory of turbulence which is relevant to the study of the Burgers model. In particular, we will proceed as if the flow $\mathbf{v}(t,\mathbf{x})$ under consideration is periodic in space, without concerning ourselves with the physical relevance of K41 in this setting. Without loss of generality, we may assume that $\mathbf{v}$ is 1-periodic in each coordinate x_1, x_2, x_3. Let us denote by ν the viscosity coefficient; we only consider the turbulent regime $0 < \nu \ll 1$.

We define the *space scale* as the inverse of the frequency under consideration. In particular, the Fourier coefficients $\hat{\mathbf{v}}(\mathbf{k})$ for large values of $\mathbf{k}$ or, in the physical space, the increments $\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})$ for small values of $\mathbf{r}$, are prototypical small-scale quantities.

The theory which may be considered as a starting point for the modern study of turbulence is essentially contained in three articles by Kolmogorov which have been published in 1941 [33, 34, 35]. Thus, it is referred to as the *K41 theory*.

The philosophy behind K41 is that although large-scale characteristics of a turbulent flow are clearly individual (depending on the forcing and/or on the boundary conditions), small-scale characteristics display some non-trivial universal features. To make this point clearer, we will introduce several definitions.

The *dissipation scale* ℓ_d is the smallest scale such that for all $|\mathbf{k}| \succeq \ell_d^{-1}$, the Fourier coefficients of the function $\mathbf{v}$ decrease super-polynomially in $|\mathbf{k}|$, uniformly in ν. The interval $\mathbf{J}_{diss} = (0, \ell_d]$ is called the *dissipation range*. The K41 theory claims that $\ell_d = C\nu^{3/4}$. The *energy range* $\mathbf{J}_{energ} = (\ell_e, 1]$ consists of the scales such that the corresponding Fourier modes support most of the L^2 norm of $\mathbf{v}$:

$$\sum_{|\mathbf{k}|<\ell_e^{-1}} \langle |\hat{\mathbf{v}}(\mathbf{k})|^2 \rangle \gg \sum_{|\mathbf{k}|\geq \ell_e^{-1}} \langle |\hat{\mathbf{v}}(\mathbf{k})|^2 \rangle.$$

K41 states that $\ell_e = C$.

Finally $\mathbf{J}_{inert} = (\ell_d, \ell_e]$ is the *inertial range*. K41 states that $\mathbf{J}_{inert} = (C\nu^{3/4}, C]$. This is the most interesting zone, where the flow exhibits non-trivial universal small-scale behaviour which will be described more precisely below. Heuristically, in the dissipation range the damping corresponds to an extinction of all relevant features of the flow, and in the energy range the flow is dominated by a heavy dependence on the large-scale features, i.e. the random forcing.

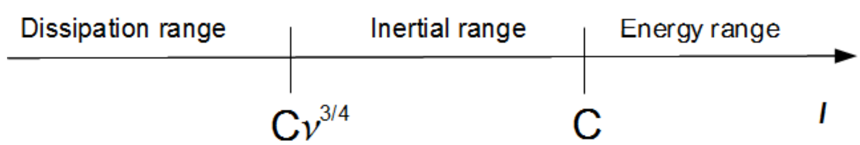

FIGURE 1. Kolmogorov scales

Two quantities used to describe the small-scale behaviour of a flow $\mathbf{v}(t, \mathbf{x})$ *at a fixed time moment* are:

- On one hand, the *longitudinal structure function*

$$S_p^{\|}(\mathbf{x}, \mathbf{r}) = \left\langle \left| \frac{(\mathbf{v}(\mathbf{x}+\mathbf{r}) - \mathbf{v}(\mathbf{x})) \cdot \mathbf{r}}{|\mathbf{r}|} \right|^p \right\rangle \tag{16}$$

- On the other hand, the *energy spectrum*

$$E(k) = \frac{\sum_{|\mathbf{n}| \in [M^{-1}k, Mk]} \langle |\hat{\mathbf{v}}(\mathbf{n})|^2 \rangle}{\sum_{|\mathbf{n}| \in [M^{-1}k, Mk]} 1}, \tag{17}$$

i.e. the average of $\langle |\hat{\mathbf{v}}(\mathbf{n})|^2 \rangle$ over a layer of $\mathbf{n}$ such that $|\mathbf{n}| \sim k$.

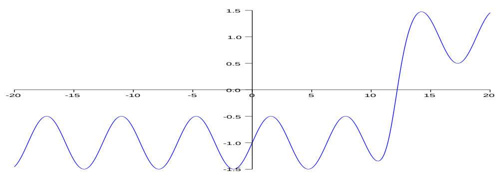

FIGURE 2. Example of a function exhibiting small-scale intermittency

The K41 theory predicts that under some conditions on the flow, for $\ell = |\mathbf{r}| \in \mathbf{J}_{inert}$ and for every $\mathbf{x}$, we have:

$$S_p^{||}(\mathbf{x},\mathbf{r}) \overset{p}{\sim} \ell^{p/3}, \quad p \geq 0. \tag{18}$$

On the other hand, for k such that $k^{-1} \in \mathbf{J}_{inert}$, K41 states that

$$E(k) \sim k^{-5/3} \tag{19}$$

(see [46, 47]).

The K41 predictions are in good agreement with experimental and numerical data for the energy spectrum and for the structure functions $S_p,\ p = 2, 3$. However, there are important discrepancies for the functions $S_p,\ p \geq 4$ [27, Chapter 8]. One of the possible explanations for these discrepancies involves the concept of *spatial intermittency*.

We say that a function is intermittent in space if at a given time moment, it is very strongly excited on a small subset of its domain of definition, as for the function whose graph is given in Figure 2.

Intermittency at the scale ℓ is quantified by *flatness*, defined as

$$F(\ell) = S_4^{||}(\ell)/S_2^{||}(\ell)^2 :$$

the larger the flatness, the more intermittent is the function. Therefore the K41 theory does not predict the intermittent features observed in the inertial range in turbulent flows, such as vortex stretching, which are clear manifestations of intermittency [50]. Indeed, for $\ell \in J_2$ the K41 predictions yield that

$$F(\ell) \sim \ell^{4/3}/(\ell^{2/3})^2 = 1.$$

Two parallel theories, due respectively to Kolmogorov himself [36] and to Frisch and Parisi [48] both give an explanation for the discrepancies between K41 and the experimental and numerical data which emphasises the role of spatial intermittency.

2.2. **Burgulence.** The 1d Burgers equation

$$u_t + uu_x = \nu u_{xx}, \tag{20}$$

where $\nu > 0$ is a viscosity coefficient, has first been considered by Forsyth [25] and Bateman [5] in the first decades of the XXth century. Here, we will only consider the space-periodic case, which after rescaling reduces to $\mathbf{x} \in S^1 = \mathbb{R}/\mathbb{Z}$.

Around 1950, the Burgers equation attracted considerable interest in the scientific community. In particular, it has been studied by the Dutch physicist whose name it bears ([14, 15]; see also [4]). His goal was to consider a simplified version of the incompressible Navier–Stokes equation

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla p; \quad \nabla \cdot \mathbf{u} = 0, \tag{21}$$

which would keep some of its features. This hope was shared by von Neumann [56, p. 437].

The Hopf-Cole-Florin transformation ([17, 24, 30]; see [9] for a historical account) reduces the Burgers equation to the heat equation. Indeed, if u is the solution of (20) corresponding to an initial condition u_0, then $u(t,x)$ is the space derivative of the function

$$-2\nu \ln(\phi(t,x)),$$

where ϕ is the solution of the heat equation

$$\phi_t = \nu \phi_{xx}$$

corresponding to the initial condition $\phi_0 = \exp(-H_0/2\nu)$. Here, H_0 is a primitive of u_0. This transformation can also be applied to the multi-d potential Burgers equation:

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u}; \quad \mathbf{u} = -\nabla \psi. \tag{22}$$

The fact that the Burgers equation can be reduced to the heat equation means that it is integrable and therefore its solutions do not exhibit chaotic behaviour. However, the Hopf-Cole-Florin transformation involves exponentials of a quantity divided by ν and does not provide information about the small-scale behaviour of solutions in the turbulent regime corresponding to $0 < \nu \ll 1$ in a transparent way. Moreover, this transformation cannot be applied to the generalised Burgers equation considered in our survey. Therefore we will not use this transformation, and our arguments will extend to the generalised Burgers equation (1) under the additional assumption (2).

The small-scale behaviour of solutions to the Burgers equation has been studied on a qualitative level by many physicists [2, 16, 26, 32, 37]. There is an agreement about the behaviour of the increments and of the energy spectrum in the inertial range, which corresponds to the interval $\mathbf{J}_{inert} = (C\nu, C]$. To explain the physical arguments in [2], we need to give more details on the structure of the solutions to (20). We assume that both the initial condition u_0 and its derivative have amplitude of the order 1.

First, consider the inviscid Hopf equation which is the limit case $\nu = 0$ of (20). Its solution is only smooth during a finite interval of time: it can be implicitly constructed using the method of characteristics (see for instance [20]). This method tells us that while the solution remains smooth, the value of u is constant along the lines $(t, x + tu_0(x))$ in the space-time. However, if u_0 is not constant, then lines corresponding to different values of u_0 cross after a finite time, forbidding the existence of smooth solutions. Nevertheless, a weak entropy solution can still be uniquely defined for all time in the class of bounded variation functions $BV(S^1)$. Such a solution is a limit in L_1 of classical solutions for the viscous equation as $\nu \to 0$. More precisely, this solution exhibits the N-wave behaviour [23], i.e. solutions are composed of waves similar to the Cyrillic capital letter И (the mirror image of N). In other words, at a fixed (large enough) time t the solution $u(t, \cdot)$ alternates between negative jump discontinuities and smooth regions where the derivative is positive and of the order 1. This is a clear manifestation of small-scale intermittency in space. Note that the solutions of (20) remain of order 1 during a time of order 1. On the other hand, for $t \to +\infty$ the solutions decay at least as Ct^{-1} in any Lebesgue space L_p, $1 \leq p \leq +\infty$, uniformly in ν (cf. for instance [40]).

Now let us give a more precise description of the N-wave behaviour. For a typical initial data u_0 (i.e. for $\max |u_0| \sim 1$ and $\max |(u_0)_x| \sim 1$) and for $t > 1/(\min(u_0)_x)$, $t \sim 1$, it is numerically observed [2] that a "typical" solution $u(t, \cdot)$ of the viscous equation has the following features (cf. Figure 3):

- Amplitude of the solution: ~ 1.
- Number of cliffs per period: ~ 1.
- "Vertical drop" at a cliff: ~ -1.
- "Width" of a cliff: $\sim \nu$.

Now we denote by $S_p(\ell)$ the structure function defined by

$$S_p(\ell) = \int_{S^1} |u(x+\ell) - u(x)|^p dx, \quad p \geq 1. \tag{23}$$

For $\ell \in \mathbf{J}_{inert}$, ℓ is typically smaller than the interval between two cliffs, but larger than the width of a cliff. Aurell, Frisch, Lutsko and

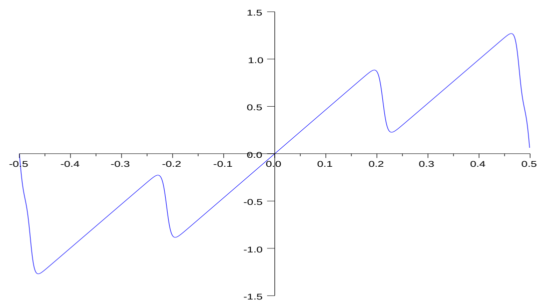

FIGURE 3. Typical solution of the Burgers equation

Vergassola [2] observe that there are three possibilities for the interval $[x, x+\ell]$.

- $[x, x+\ell]$ covers a large part of a cliff.
  Probability $= C\ell$. $u(x+\ell) - u(x) = \underbrace{-C}_{cliff} + \underbrace{C\ell}_{ramps} = -C$.

  $|u(x+\ell) - u(x)|^p \overset{p}{\sim} 1$.

- $[x, x+\ell]$ covers a small part of a cliff.
  The contribution of this term is negligible.

- $[x, x+\ell]$ does not intersect a cliff.
  Probability $= 1 - C\ell = C$. $u(x+\ell) - u(x) = \underbrace{C\ell}_{ramp}$.

  $|u(x+\ell) - u(x)|^p \overset{p}{\sim} \ell^p$.

Thus, $S_p(\ell) \overset{p}{\sim} \ell + \ell^p \overset{p}{\sim} \begin{cases} \ell^p, & 0 \leq p \leq 1, \\ \ell, & p \geq 1. \end{cases}$

In other words, for $p \geq 0$ the description above implies that for $\ell \in \mathbf{J}_{inert}$, the structure functions behave as follows:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, & 0 \leq p \leq 1, \\ \ell, & p \geq 1 \end{cases} \tag{24}$$

Consequently, for ℓ in the inertial range, the flatness $F(\ell)$ behaves as ℓ^{-1}.

Now we remark that asymptotically, the Fourier coefficients of an N-wave satisfy $|\hat{u}(k)| \sim k^{-1}$. Thus, it is natural to conjecture that for

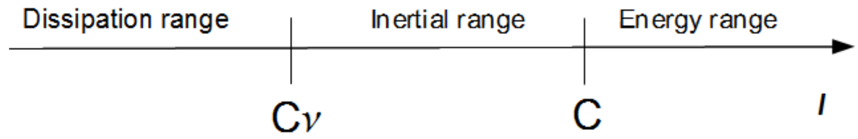

FIGURE 4. Space scales for the Burgers equation

ν small and for a certain range of k, energy-type quantities $\frac{1}{2}|\hat{u}(k)|^2$ behave, in average, as k^{-2}. Thus, for $k^{-1} \in \mathbf{J}_{inert}$ the physical predictions give $E(k) \sim k^{-2}$ with the same definition as above, up to the absence of the brackets $\langle \cdot \rangle$ for $E(k)$ [16, 26, 32, 37]. The restriction $k^{-1} \in \mathbf{J}_{inert}$ is due to a simple dimensional argument, ν being the natural small scale (width of a cliff) for the solutions.

Beginning from the 1980s, there has been an increasing interest in random versions of the Burgers equation. The most studied model has been the one with additive white in time noise, more or less smooth in space. Here, we will only consider the case where the noise is C^∞-smooth in space; for the general case, see the surveys [6, 7]. In this setting, numerical simulations and physical predictions give exactly the same results as in the deterministic case, up to the fact that we consider the expected values of the quantities [29]. Heuristically, this is due to the fact that forcing acts on large scales, in the energy range, and thus only influences smaller scales indirectly, as an energy source.

3. MAIN RESULTS

In Section 4, we are concerned with the deterministic Burgers equation. First, in Subsection 4.1, we prove sharp upper and lower bounds for some Sobolev norms of u. In Lemma 4.1, we recall the key estimate

$$u_x(t,x) \leq \min(D, \sigma^{-1}t^{-1}). \tag{25}$$

For the definition of D, see (15). The main results for the Sobolev norms of solutions are summed up in Theorem 4.8. Namely, for $m = 0, 1$ and $p \in [1,\infty]$ or for $m \geq 2$ and $p \in (1,\infty]$, we have:

$$\Big(\{|u(t)|^\alpha_{m,p}\}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \quad \alpha > 0, \tag{26}$$

where $\{\cdot\}$ denotes averaging in time over the interval $[T_1, T_2]$ defined by (38). We recall that $\gamma(m,p) = \max(0, m - 1/p)$.

In Subsection 4.2 we obtain sharp estimates for analogues of the quantities characterising the hydrodynamical turbulence. In what follows, we assume that $\nu \in (0, \nu_0]$, where $\nu_0 \in (0, 1]$ depends only on f

and on D. To begin with, we define the non-empty and non-intersecting intervals

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1].$$

For the definitions of ν_0, C_1 and C_2, see (51); these quantities depend only on f and on D. As a consequence of (25-26), in Theorem 4.17 we prove that for $\ell \in J_1$:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p,\ 0 \leq p \leq 1, \\ \ell^p \nu^{-(p-1)},\ p \geq 1, \end{cases}$$

and for $\ell \in J_2$:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p,\ 0 \leq p \leq 1, \\ \ell,\ p \geq 1. \end{cases}$$

Consequently, for $\ell \in J_2$ the flatness satisfies:

$$F(\ell) = S_4(\ell)/S_2^2(\ell) \sim \ell^{-1}.$$

Finally, we get estimates for the spectral asymptotics of Burgulence. On one hand, as a consequence of Theorem 4.8, for $m \geq 1$ we get:

$$\{|\hat{u}(k)|^2\} \overset{m}{\lesssim} k^{-2m} \|u\|_m^2 \overset{m}{\lesssim} (k\nu)^{-2m}\nu.$$

In particular, $\{|\hat{u}(k)|^2\}$ decreases at a faster-than-polynomial rate for $|k| \succeq \nu^{-1}$. On the other hand, by Theorem 4.21, for k such that $k^{-1} \in J_2$ the energy spectrum satisfies

$$E(k) = \frac{\sum_{|n| \in [M^{-1}k, Mk]} \langle |\hat{u}(n)|^2 \rangle}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \sim k^{-2},$$

where $M \geq 1$ depends only on f and on D.

Note that these results confirm rigorously the physical predictions exposed in Subsection 2.2. Moreover, averaging in the initial condition, as considered in [2], is actually not necessary. This is due to the particular structure of the deterministic generalised Burgers equation: an initial condition u_0 is as generic as the ratio between the orders of $(u_0)_x$ and of u_0 itself, which can be bounded from above using the quantity D.

The results in Section 5 can be formulated in exactly the same way, up to two modifications:

- The estimates hold uniformy in t (for t large enough) and in u_0. On the other hand, all estimated quantities should be replaced by their expected values. In particular, we modify the meaning of the brackets $\{\cdot\}$.

- Dependence on D should be replaced by dependence on the statistical properties of the forcing.

In Section 6, we expose results on existence and uniqueness of the stationary measure for the randomly forced generalised Burgers equation. These results yield that all estimates listed in Section 5 still hold if taking the expected value and averaging in time is replaced by averaging with respect to the stationary measure μ.

4. Estimates in the unforced case

The results in Subsection 4.1 have been obtained by Biryuk [8] for the Sobolev norms in $H^m,\ m \geq 1$, under a slightly different form. Our presentation follows the lines of [10], where we generalise Biryuk's results, obtaining estimates on the Sobolev norms $W^{m,p}$ for $m \in \{0,1\},\ p \in [1,\infty]$ or $m \geq 2,\ p \in (1,\infty]$, by Hölder's inequality and (GN). In [8], Biryuk also proved upper and lower spectral estimates, which allowed him to give the correct asymptotics for the dissipation length scale as $\nu \to 0$. These results have been sharpened in in [10], where moreover sharp bounds for the structure functions have been obtained. In Subsection 4.2 we will only give the results in [10], referring the reader to that paper for a comparison with the results in [8].

4.1. **Estimates for Sobolev norms.**

We begin by recalling the proof of a key upper estimate for u_x, which is a reformulation of the "Kruzhkov maximum principle" [40].

LEMMA 4.1. *We have:*

$$u_x(t,x) \leq \min(D, \sigma^{-1}t^{-1}).$$

Proof. Differentiating the equation (1) once in space we get:

$$(u_x)_t + f''(u)u_x^2 + f'(u)(u_x)_x = \nu (u_x)_{xx}. \tag{27}$$

Now consider a point (t_1, x_1) where u_x reaches its maximum on the cylinder $S = [0,t] \times S^1$. Suppose that $t_1 > 0$ and that this maximum is nonnegative. At such a point, Taylor's formula implies that we would have $(u_x)_t \geq 0$, $(u_x)_x = 0$ and $(u_x)_{xx} \leq 0$. Consequently, since by (2) $f''(u) \geq \sigma$, (27) yields that $\sigma u_x^2 \leq 0$, which is impossible. Thus u_x can only reach a nonnegative maximum on S for $t_1 = 0$. In other words, since $(u_0)_x$ has zero mean value, we have:

$$u_x(t,x) \leq \max_{x \in S^1} (u_0)_x(x) \leq D.$$

The inequality

$$u_x(t,x) \leq \sigma^{-1}t^{-1}$$

is proved in [40] by a similar maximum principle argument applied to the function $v = tu_x$. Indeed, this function can only reach a nonnegative maximum on S at a point (t_1, x_1) such that $t_1 > 0$. Multiplying (27) by t^2, we get:

$$t \underbrace{v_t}_{\geq 0} + tf'(u) \underbrace{v_x}_{0} + (-v + f''(u)v^2) = \nu t \underbrace{v_{xx}}_{\leq 0}.$$

Thus $v \leq \sigma^{-1}$ on S. In other words, $u_x \leq \sigma^{-1} t^{-1}$ for all $t > 0$. □

Since the space averages of $u(t)$ and $u_x(t)$ vanish for all t, we get the following upper estimates:

$$|u(t)|_p \leq |u(t)|_\infty \leq \int_{S^1} u_x^+(t) \leq \min(D, \sigma^{-1} t^{-1}), \quad 1 \leq p \leq +\infty. \quad (28)$$

$$|u(t)|_{1,1} = 2 \int_{S^1} u_x^+(t) \leq 2 \min(D, \sigma^{-1} t^{-1}). \quad (29)$$

Now we recall a standard estimate for the nonlinearity

$$\left\langle v^{(m+1)}, (f(v))^{(m)} \right\rangle,$$

which is proved in [13].

Lemma 4.2. *For $v \in C^\infty$ such that $|v|_\infty \leq N$, we have:*

$$N_m(v) = \left| \left\langle v^{(m+1)}, (f(v))^{(m)} \right\rangle \right| \overset{m,N}{\lesssim} \|v\|_m \|v\|_{m+1}, \quad m \geq 1.$$

Proof. Fix $m \geq 1$. In this proof, constants denoted by $\tilde{C}$ depend only on m, N. We have:

$$N_m(v) \leq \tilde{C} \sum_{k=1}^{m} \sum_{\substack{1 \leq a_1 \leq \cdots \leq a_k \leq m \\ a_1 + \cdots + a_k = m}} \int_{S^1} \left| v^{(m+1)} v^{(a_1)} \ldots v^{(a_k)} f^{(k)}(v) \right|$$

$$\leq \tilde{C} \max_{x \in [-N,N]} \max(f'(x), \ldots f^{(m)}(x))$$

$$\times \sum_{k=1}^{m} \sum_{\substack{1 \leq a_1 \leq \cdots \leq a_k \leq m \\ a_1 + \cdots + a_k = m}} \int_{S^1} |v^{(a_1)} \ldots v^{(a_k)} v^{(m+1)}|.$$

Using (13), Hölder's inequality and (GN), we get:

$$
\begin{aligned}
&N_m(v) \leq \tilde{C}(1+N)^{\max(h(1),\dots,h(m))} \\
&\times \sum_{k=1}^{m} \sum_{\substack{1\leq a_1\leq\dots\leq a_k\leq m\\ a_1+\dots+a_k=m}} \int_{S^1} |v^{(a_1)}\dots v^{(a_k)}v^{(m+1)}| \\
&\leq \tilde{C}\sum_{k=1}^{m} \sum_{\substack{1\leq a_1\leq\dots\leq a_k\leq m\\ a_1+\dots+a_k=m}} (\left|v^{(a_1)}\right|_{2m/a_1}\dots\left|v^{(a_k)}\right|_{2m/a_k}\left\|v\right\|_{m+1}) \\
&\leq \tilde{C}\left\|v\right\|_{m+1}\sum_{k=1}^{m} \sum_{\substack{1\leq a_1\leq\dots\leq a_k\leq m\\ a_1+\dots+a_k=m}} \Big((\left\|v\right\|_m^{a_1/m}\left|v\right|_\infty^{(m-a_1)/m})\times\dots \\
&\dots\times(\left\|v\right\|_m^{a_k/m}\left|v\right|_\infty^{(m-a_k)/m})\Big) \\
&\leq \tilde{C}(1+N)^{m-1}\left\|v\right\|_m\left\|v\right\|_{m+1} = \tilde{C}\left\|v\right\|_m\left\|v\right\|_{m+1}. \quad \square
\end{aligned}
$$

The following result shows the existence of a strong nonlinear damping which prevents the successive derivatives of u from becoming too large.

LEMMA 4.3. *We have:*

$$\left\|u(t)\right\|_1^2 \lesssim \nu^{-1}.$$

On the other hand, for $m\geq 2$,

$$\left\|u(t)\right\|_m^2 \overset{m}{\lesssim} \max(\nu^{-(2m-1)}, t^{-(2m-1)}).$$

Proof. Fix $m\geq 1$. Denote

$$x(t) = \left\|u(t)\right\|_m^2.$$

We claim that the following implication holds:

$$x(t)\geq C'\nu^{-(2m-1)} \Longrightarrow \frac{d}{dt}x(t)\leq -(2m-1)x(t)^{2m/(2m-1)}, \tag{30}$$

where C' is a fixed positive number, chosen later. Below, all constants denoted by C do not depend on C'.

Indeed, assume that $x(t)\geq C'\nu^{-(2m-1)}$. Integrating by parts in space and using (28) ($p=\infty$) and Lemma 4.2, we get the following energy dissipation relation:

$$
\begin{aligned}
\frac{d}{dt}x(t) &= -2\nu\left\|u(t)\right\|_{m+1}^2 + 2\left\langle u^{(m+1)}(t), (f(u(t)))^{(m)}\right\rangle \\
&\leq -2\nu\left\|u(t)\right\|_{m+1}^2 + C\left\|u(t)\right\|_m\left\|u(t)\right\|_{m+1}.
\end{aligned} \tag{31}
$$

Applying (GN) to u_x and then using (29), we get:

$$\begin{aligned}\|u(t)\|_m &\leq C\,\|u(t)\|_{m+1}^{(2m-1)/(2m+1)}\,|u(t)|_{1,1}^{2/(2m+1)} \\ &\leq C\,\|u(t)\|_{m+1}^{(2m-1)/(2m+1)}. \end{aligned} \tag{32}$$

Thus, we have the relation

$$\frac{d}{dt}x(t) \leq (-2\nu\,\|u(t)\|_{m+1}^{2/(2m+1)} + C)\,\|u(t)\|_{m+1}^{4m/(2m+1)}. \tag{33}$$

The inequality (32) yields that

$$\|u(t)\|_{m+1}^{2/(2m+1)} \geq Cx(t)^{1/(2m-1)}, \tag{34}$$

and then using the assumption $x(t) \geq C'\nu^{-(2m-1)}$ we get:

$$\|u(t)\|_{m+1}^{2/(2m+1)} \geq CC'^{1/(2m-1)}\nu^{-1}. \tag{35}$$

Combining the inequalities (33-35), for C' large enough we get:

$$\frac{d}{dt}x(t) \leq (-CC'^{1/(2m-1)} + C)x(t)^{2m/(2m-1)}.$$

Thus we can choose C' in such a way that implication (30) holds.

For $m = 1$, (15) and (30) yield that

$$x(t) \leq \max(C'\nu^{-1}, D^2) \leq \max(C', D^2)\nu^{-1},\ t \geq 0.$$

Now consider the case $m \geq 2$. We claim that

$$x(t) \leq \max(C'\nu^{-(2m-1)}, t^{-(2m-1)}). \tag{36}$$

Indeed, if $x(s) \leq C'\nu^{-(2m-1)}$ for some $s \in [0,t]$, then the assertion (30) ensures that $x(s)$ remains below this threshold up to time t.

Now, assume that $x(s) > C'\nu^{-(2m-1)}$ for all $s \in [0,t]$. Denote

$$\tilde{x}(s) = (x(s))^{-1/(2m-1)},\ s \in [0,t].$$

By (30) we have $d\tilde{x}(s)/ds \geq 1$. Therefore $\tilde{x}(t) \geq t$ and $x(t) \leq t^{-(2m-1)}$. Thus in this case, the inequality (36) still holds. This proves the lemma's assertion. □

By (GN) applied to $u^{(m)}$ we get the following inequality for $m \geq 1$:

$$|u(t)|_{m,\infty} \lesssim \|u(t)\|_m^{1/2}\,\|u(t)\|_{m+1}^{1/2} \overset{m}{\lesssim} \max(\nu^{-m}, t^{-m}).$$

Similarly, applying (GN) and interpolating between $|u|_{1,1}$ and $\|u\|_M$ for large values of M, we get the following result (we recall that $\gamma = \max(0, m-1/p)$):

THEOREM 4.4. *For $m \in \{0,1\}$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$,*

$$\Big(|u(t)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\lesssim} \max(t^{-\gamma},\nu^{-\gamma}), \quad \alpha > 0. \tag{37}$$

Now we define

$$T_1 = \frac{1}{4}D^{-2}\tilde{C}^{-1}; \quad T_2 = \max\Big(\frac{3}{2}T_1,\ 2D\sigma^{-1}\Big), \tag{38}$$

where $\tilde{C}$ is a constant such that for all t, $\|u(t)\|_1^2 \leq \tilde{C}\nu^{-1}$ (cf. Lemma 4.3). From now on, for any time-dependent Sobolev-space valued functional $A(\cdot)$, $\{A(t)\}$ is by definition the time average

$$\{A(t)\} = \frac{1}{T_2 - T_1}\int_{T_1}^{T_2} A(t).$$

LEMMA 4.5. *We have:*

$$\{\|u(t)\|_1^2\} \gtrsim \nu^{-1}.$$

Proof. Integrating by parts in space, we get the dissipation identity

$$\frac{d}{dt}|u(t)|^2 = \int_{S^1}(-2uf'(u)u_x + 2\nu u u_{xx}) = -2\nu\|u(t)\|_1^2. \tag{39}$$

Thus, integrating in time and using (15) and Lemma 4.3, we obtain that

$$|u(T_1)|^2 = |u_0|^2 - 2\nu\int_0^{T_1}\|u(t)\|_1^2 \geq D^{-2} - 2T_1\tilde{C} \geq \frac{1}{2}D^{-2}.$$

Consequently, integrating (39) in time and using (28) ($p = 2$) we get:

$$\begin{aligned}\{\|u(t)\|_1^2\} &= \frac{1}{2\nu(T_2 - T_1)}(|u(T_1)|^2 - |u(T_2)|^2)\\ &\geq \frac{1}{2\nu(T_2 - T_1)}\Big(\frac{1}{2}D^{-2} - \sigma^{-2}T_2^{-2}\Big)\\ &\geq \frac{D^{-2}}{8(T_2 - T_1)}\nu^{-1},\end{aligned}$$

which proves the lemma's assertion. □

This time-averaged lower bound yields similar bounds for other Sobolev norms.

LEMMA 4.6. *For $m \geq 1$,*

$$\{\|u(t)\|_m^2\} \overset{m}{\gtrsim} \nu^{-(2m-1)}.$$

Proof. Since the case $m = 1$ is covered by the previous lemma, we may assume that $m \geq 2$. By (29) and (GN), we have:

$$\{\|u(t)\|_m^2\} \overset{m}{\gtrsim} \{\|u(t)\|_m^2 \, |u(t)|_{1,1}^{(4m-4)}\} \overset{m}{\gtrsim} \{\|u(t)\|_1^{4m-2}\}.$$

Thus, using Hölder's inequality and Lemma 4.5, we get:

$$\{\|u(t)\|_m^2\} \overset{m}{\gtrsim} \{\|u(t)\|_1^{4m-2}\} \overset{m}{\gtrsim} \{\|u(t)\|_1^2\}^{(2m-1)} \overset{m}{\gtrsim} \nu^{-(2m-1)}. \ \square$$

The following lemma is proved similarly.

LEMMA 4.7. *For $m \geq 0$, $p \in [1,\infty]$, we have:*

$$\{|u(t)|_{m,p}^2\} \overset{m,p}{\gtrsim} \nu^{-2\gamma}.$$

The following theorem sums up the results of this section which will be used later, with the exception of Lemma 4.1.

THEOREM 4.8. *For $m \in \{0,1\}$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$, we have:*

$$\Big(\{|u(t)|_{m,p}^{\alpha}\}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \qquad \alpha > 0, \tag{40}$$

where $\{\cdot\}$ denotes time-averaging over $[T_1, T_2]$. The upper estimates in (40) hold without time-averaging, uniformly for t separated from 0. Namely, we have:

$$|u(t)|_{m,p} \overset{m,p}{\lesssim} \max(\nu^{-\gamma}, t^{-\gamma}).$$

On the other hand, the lower estimates hold for all $m \geq 0$, $p \in [1,\infty]$, $\alpha > 0$.

Proof. The upper estimates follow from Theorem 4.4. The lower estimates for $\alpha \geq 2$ follow from Lemma 4.7 by Hölder's inequality. For $m = 0$, $p \in [1,\infty]$ and for $m \geq 1$, $p \in (1,\infty]$ we obtain lower estimates for $\alpha \in (0,2)$ using lower estimates for $\alpha = 2$, upper estimates for $\alpha = 3$ and Hölder's inequality. Indeed,

$$\begin{aligned}\{|u(t)|_{m,p}^{\alpha}\} &\geq \Big(\{|u(t)|_{m,p}^2\}\Big)^{3-\alpha} \Big(\{|u(t)|_{m,p}^3\}\Big)^{-(2-\alpha)} \\ &\gtrsim \nu^{-(6-2\alpha)\gamma} \nu^{(6-3\alpha)\gamma} = \nu^{-\alpha\gamma}.\end{aligned}$$

For $|u|_{m,1}$, $m > 1$, the lower estimates follow from those for $|u|_{m-1,\infty}$. $\square$

For $p, \alpha = 2$, this theorem tells us that for integers $m \geq 1$, we have:

$$\{\|u\|_m^2\} \overset{m}{\sim} \nu^{-(2m-1)}. \tag{41}$$

By a standard interpolation argument (see (4)) the upper bound in (41) also holds for non-integer indices $s > 1$. Actually, the same is true for the lower bound, since for any integer $n > s$ we have:

$$\{\|u\|_s^2\} \geq \{\|u\|_n^2\}^{n-s+1}\{\|u\|_{n+1}^2\}^{-(n-s)} \overset{s}{\gtrsim} \nu^{-(2s-1)}.$$

4.2. **Estimates for small-scale quantities.** In this section, we study analogues of quantities which are important for the study of hydrodynamical turbulence. We consider quantities in the physical space (structure functions) as well as in the Fourier space (energy spectrum). We assume that $\nu \leq \nu_0$ and we define the intervals

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1].$$

The positive constants ν_0, C_1 and C_2 will be chosen in (50)-(51) in such a manner that $C_1\nu_0 < C_2 < 1$, which ensures that the intervals J_i are non-empty and non-intersecting.

By Theorem 4.8, we obtain that $\{|u|^2\} \sim 1$. On the other hand, by (29) we get (after integration by parts):

$$\begin{aligned} \{|\hat{u}(n)|^2\} &= (2\pi n)^{-2}\Big\{\Big|\int_{S^1} e^{2\pi i n x} u_x(x)\Big|^2\Big\} \\ &\leq (2\pi n)^{-2}\{|u|_{1,1}^2\} \leq C n^{-2}, \end{aligned} \tag{42}$$

and C_1 and C_2 can be made as small as we wish (cf. (52)). Consequently, the proportion of the sum $\{\sum |\hat{u}(n)|^2\}$ contained in the Fourier modes corresponding to J_3 can be made as large as we wish. For instance, we may assume that

$$\Bigg\{\sum_{|n|<C_2^{-1}} |\hat{u}(n)|^2\Bigg\} \geq \frac{99}{100}\Bigg\{\sum_{n\in\mathbb{Z}} |\hat{u}(n)|^2\Bigg\}.$$

For $p \geq 0$, we define the structure function of p-th order as:

$$S_p(\ell) = \Big\{\int_{S^1} |u(t,x+\ell) - u(t,x)|^p dx\Big\}.$$

The flatness $F(\ell)$, which measures spatial intermittency, is given by

$$F(\ell) = S_4(\ell)/S_2^2(\ell). \tag{43}$$

Finally, for $k \geq 1$, we define the (layer-averaged) energy spectrum by

$$E(k) = \Bigg\{\frac{\sum_{|n|\in[M^{-1}k,Mk]} |\hat{u}(n)|^2}{\sum_{|n|\in[M^{-1}k,Mk]} 1}\Bigg\}, \tag{44}$$

where $M \geq 1$ is a constant which will be specified later (see the proof of Theorem 4.21).

We begin by estimating the functions $S_p(\ell)$ from above.

LEMMA 4.9. *For* $\ell \in [0,1]$,

$$S_p(\ell) \overset{p}{\lesssim} \begin{cases} \ell^p, \ 0 \leq p \leq 1, \\ \ell^p \nu^{-(p-1)}, \ p \geq 1. \end{cases}$$

Proof. We begin by considering the case $p \geq 1$. We have:

$$\begin{aligned} S_p(\ell) &= \Big\{ \int_{S^1} |u(x+\ell) - u(x)|^p dx \Big\} \\ &\leq \Big\{ \Big(\int_{S^1} |u(x+\ell) - u(x)| dx \Big) \Big(\max_x |u(x+\ell) - u(x)|^{p-1} \Big) \Big\}. \end{aligned}$$

Using the fact that $\int_{S^1} u(\cdot + \ell) - u(\cdot) = 0$ and Hölder's inequality, we obtain that

$$\begin{aligned} S_p(\ell) \leq & \Big\{ \Big(2 \int_{S^1} (u(x+\ell) - u(x))^+ dx \Big)^p \Big\}^{1/p} \\ & \times \Big\{ \max_x |u(x+\ell) - u(x)|^p \Big\}^{(p-1)/p} \\ \leq & C\ell \Big\{ \max_x |u(x+\ell) - u(x)|^p \Big\}^{(p-1)/p}, \end{aligned} \tag{45}$$

where the second inequality follows from Lemma 4.1. Finally, by Theorem 4.8 we get:

$$S_p(\ell) \leq C\ell \Big\{ (\ell |u|_{1,\infty})^p \Big\}^{(p-1)/p} \leq C\ell^p \nu^{-(p-1)}.$$

The case $p < 1$ follows immediately from the case $p = 1$ since now $S_p(\ell) \leq (S_1(\ell))^p$, by Hölder's inequality. □

For $\ell \in J_2 \cup J_3$, we have a better upper bound if $p \geq 1$.

LEMMA 4.10. *For* $\ell \in J_2 \cup J_3$,

$$S_p(\ell) \overset{p}{\lesssim} \begin{cases} \ell^p, \ 0 \leq p \leq 1, \\ \ell, \ p \geq 1. \end{cases}$$

Proof. The calculations are almost the same as in the previous lemma. The only difference is that we use another bound for the right-hand side of (45). Namely, by Theorem 4.8 we have:

$$\begin{aligned} S_p(\ell) &\leq C\ell \Big\{ \max_x |u(x+\ell) - u(x)|^p \Big\}^{(p-1)/p} \\ &\leq C\ell \Big\{ (2|u|_\infty)^p \Big\}^{(p-1)/p} \leq C\ell. \ \square \end{aligned}$$

Remark 4.11. *Lemmas 4.9 and 4.10 actually hold even if we drop the time-averaging, since in deriving them we only use upper estimates which hold uniformly for $t \geq T_1$.*

To prove the lower estimates for $S_p(\ell)$, we need a lemma. Loosely speaking, this lemma states that there exists a large enough set $L_K \subset [T_1, T_2]$ such that for $t \in L_K$, several Sobolev norms are of the same order as their time averages. Thus for $t \in L_K$, we can prove the existence of a cliff of height at least C and width at least $C\nu$, using some of the arguments in [2] which we explained in Subsection 2.2.

Note that in the following definition, (46)-(47) contain lower and upper estimates, while (48) only contains an upper estimate. The inequality $|u(t)|_\infty \leq \max u_x(t)$ in (46) always holds, since $u(t)$ has zero mean value and the length of S^1 is 1.

Definition 4.12. *For $K > 1$, we denote by L_K the set of all $t \in [T_1, T_2]$ such that the assumptions*

$$K^{-1} \leq |u(t)|_\infty \leq \max u_x(t) \leq K \tag{46}$$

$$K^{-1}\nu^{-1} \leq |u(t)|_{1,\infty} \leq K\nu^{-1} \tag{47}$$

$$|u(t)|_{2,\infty} \leq K\nu^{-2} \tag{48}$$

hold.

Lemma 4.13. *There exist constants $C, K_1 > 0$ such that for $K \geq K_1$, the Lebesgue measure of L_K satisfies $\lambda(L_K) \geq C$.*

Proof. We begin by observing that if $K \leq K'$, then $L_K \subset L_{K'}$. By Lemma 4.1 and Theorem 4.8, for K large enough the upper estimates in (46)-(48) hold for all $t \geq T_1$. Therefore, if we denote by B_K the set of t such that

"The lower estimates in (46)-(47) hold for a given value of K",

then it suffices to prove the lemma's statement with B_K in place of L_K. Now denote by D_K the set of t such that

"The lower estimate in (47) holds for a given value of K".

By (GN) we have:

$$|u|_\infty \geq C|u|_{2,\infty}^{-1}|u|_{1,\infty}^2.$$

Thus if D_K holds, then $B_{K'}$ holds for K' large enough. Now it remains to show that there exists $C > 0$ such that for K large enough, we have the inequality $\lambda(D_K) \geq C$. We clearly have

$$\{|u|_{1,\infty}\mathbf{1}(|u|_{1,\infty} < K^{-1}\nu^{-1})\} < K^{-1}\nu^{-1}.$$

Here, $\mathbf{1}(A)$ denotes the indicator function of an event A. On the other hand, by the estimate for $\{|u|_{1,\infty}^2\}$ in Theorem 4.8 we get:

$$\{|u|_{1,\infty}\mathbf{1}(|u|_{1,\infty} > K\nu^{-1})\} < K^{-1}\nu\{|u|_{1,\infty}^2\} \leq CK^{-1}\nu^{-1}$$

Now denote by f the function

$$f = |u|_{1,\infty}\mathbf{1}(K_0^{-1}\nu^{-1} \leq |u|_{1,\infty} \leq K_0\nu^{-1}).$$

The inequalities above and the lower estimate for $\{|u|_{1,\infty}\}$ in Theorem 4.8 imply that

$$\{f\} \geq (C - K_0^{-1} - CK_0^{-1})\nu^{-1} \geq C_0\nu^{-1},$$

for some suitable constants C_0 and K_0. Since $f \leq K_0\nu^{-1}$, we get:

$$\lambda(f \geq C_0\nu^{-1}/2) \geq C_0K_0^{-1}(T_2 - T_1)/2.$$

Thus, since $|u|_{1,\infty} \geq f$, we have the inequality

$$\lambda(|u|_{1,\infty} \geq C_0\nu^{-1}/2) \geq C_0K_0^{-1}(T_2 - T_1)/2,$$

and therefore there exist $C, K_1 > 0$ such that $\lambda(D_K) \geq C$ for $K \geq K_1$. $\square$

Let us denote by $O_K \subset [T_1, T_2]$ the set defined as L_K, but with relation (47) replaced by

$$K^{-1}\nu^{-1} \leq -\min u_x \leq K\nu^{-1}. \tag{49}$$

Corollary 4.14. *For $K \geq K_1$ and $\nu < K_1^{-2}$, we have $\lambda(O_K) \geq C$. Here, C, K_1 are the same as in the formulation of Lemma 4.13.*

Proof. For $K = K_1$ and $\nu < K_1^{-2}$, the estimates (46)-(47) tell us that

$$\max u_x(t) \leq K_1 < K_1^{-1}\nu^{-1} \leq |u_x(t)|_\infty, \quad t \in L_K.$$

Thus, in this case we have $O_K = L_K$, which proves the corollary's assertion. Since increasing K while keeping ν constant increases the measure of O_K, it follows that for $K \geq K_1$ and $\nu < K_1^{-2}$ we still have $\lambda(O_K) \geq C$. $\square$

Now we fix

$$K = K_1, \tag{50}$$

and choose

$$\nu_0 = \frac{1}{6}K^{-2};\ C_1 = \frac{1}{4}K^{-2};\ C_2 = \frac{1}{20}K^{-4}. \tag{51}$$

In particular, we have $0 < C_1\nu_0 < C_2 < 1$: thus the intervals J_i are non-empty and non-intersecting for all $\nu \in (0, \nu_0]$. Everywhere below the constants depend on K.

Actually, we can choose any values of C_1, C_2 and ν_0, provided that:

$$C_1 \leq \frac{1}{4}K^{-2}; \quad 5K^2 \leq \frac{C_1}{C_2} < \frac{1}{\nu_0}. \tag{52}$$

LEMMA 4.15. *For $\ell \in J_1$,*

$$S_p(\ell) \overset{p}{\gtrsim} \begin{cases} \ell^p, \ 0 \leq p \leq 1, \\ \ell^p \nu^{-(p-1)}, \ p \geq 1. \end{cases}$$

Proof. By Corollary 4.14, it suffices to verify that the inequalities hold uniformly in t for $t \in O_K$, with $S_p(\ell)$ replaced by

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx.$$

Till the end of this proof, we assume that $t \in O_K$.

Denote by z the leftmost point on S^1 (considered as $[0,1)$) such that $u'(z) \leq -K^{-1}\nu^{-1}$. Since $|u|_{2,\infty} \leq K\nu^{-2}$, we have:

$$u'(y) \leq -\frac{1}{2}K^{-1}\nu^{-1}, \quad y \in [z - \frac{1}{2}K^{-2}\nu, z + \frac{1}{2}K^{-2}\nu]. \tag{53}$$

In other words, the interval

$$[z - \frac{1}{2}K^{-2}\nu, z + \frac{1}{2}K^{-2}\nu]$$

corresponds to (a part of) a cliff.

Case $\mathbf{p \geq 1}$. Since $\ell \leq C_1\nu = \frac{1}{4}K^{-2}\nu$, by Hölder's inequality we get:

$$\begin{aligned}
\int_{S^1} |u(x+\ell) - u(x)|^p dx &\geq \int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} |u(x+\ell) - u(x)|^p dx \\
&\geq (K^{-2}\nu/2)^{1-p} \Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} |u(x+\ell) - u(x)| dx \Big)^p \\
&= C(p)\nu^{1-p} \Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} \Big(\int_x^{x+\ell} -u'(y) dy \Big) dx \Big)^p \\
&\geq C(p)\nu^{1-p} \Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} \frac{1}{2}\ell K^{-1}\nu^{-1} \ dx \Big)^p = C(p)\nu^{1-p}\ell^p.
\end{aligned}$$

Case $\mathbf{p<1}$. By Hölder's inequality we obtain that

$$\int_{S^1}|u(x+\ell)-u(x)|^p dx \geq \int_{S^1}\Big((u(x+\ell)-u(x))^+\Big)^p dx$$
$$\geq \Big(\int_{S^1}\Big((u(x+\ell)-u(x))^+\Big)^2 dx\Big)^{p-1}\Big(\int_{S^1}(u(x+\ell)-u(x))^+ dx\Big)^{2-p}.$$

Using the upper estimate in (46) we get:

$$\int_{S^1}|u(x+\ell)-u(x)|^p dx$$
$$\geq \Big(\int_{S^1}\ell^2K^2 dx\Big)^{p-1}\Big(\int_{S^1}(u(x+\ell)-u(x))^+ dx\Big)^{2-p}.$$

Since $\int_{S^1}(u(\cdot+\ell)-u(\cdot))=0$, we obtain that

$$\int_{S^1}|u(x+\ell)-u(x)|^p dx$$
$$\geq C(p)\ell^{2(p-1)}\Big(\frac{1}{2}\int_{S^1}|u(x+\ell)-u(x)|dx\Big)^{2-p} \geq C(p)\ell^p.$$

The last inequality follows from the case $p=1$. □

The proof of the following lemma uses an argument from [2], which becomes quantitative if we restrict ourselves to the set O_K.

LEMMA 4.16. *For $m\geq 0$ and $\ell\in J_2$,*

$$S_p(\ell) \overset{p}{\gtrsim} \begin{cases}\ell^p, & 0\leq p\leq 1,\\ \ell, & p\geq 1.\end{cases}$$

Proof. In the same way as above, it suffices to verify that the inequalities hold uniformly in t for $t\in O_K$, with $S_p(\ell)$ replaced by

$$\int_{S^1}|u(x+\ell)-u(x)|^p dx,$$

and using Hölder's inequality we can restrict ourselves to the case $p\geq 1$. Again, till the end of this proof, we assume that $t\in O_K$.

Define z as in the proof of Lemma 4.15. We have:

$$\int_{S^1}|u(x+\ell)-u(x)|^p dx \geq$$
$$\int_{z-\frac{1}{2}\ell}^{z}\Big|\underbrace{\int_x^{x+\ell}u'^-(y)dy}_{cliffs}-\underbrace{\int_x^{x+\ell}u'^+(y)dy}_{ramps}\Big|^p dx.$$

Since $\ell \geq C_1\nu = \frac{1}{4}K^{-2}\nu$, by (53) for $x \in [z - \frac{1}{2}\ell, z]$ we get:

$$\int_x^{x+\ell} u'^-(y)dy \geq \int_z^{z+\frac{1}{8}K^{-2}\nu} u'^-(y)dy \geq \frac{1}{16}K^{-3}.$$

.

On the other hand, since $\ell \leq C_2$, by (46) and (51) we have:

$$\int_x^{x+\ell} u'^+(y)dy \leq C_2 K = \frac{1}{20}K^{-3}.$$

Thus,

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx \geq \frac{1}{2}\ell\left(\left(\frac{1}{16} - \frac{1}{20}\right)K^{-3}\right)^p \geq C(p)\ell. \ \square$$

Summing up the results above we obtain the following theorem.

THEOREM 4.17. *For $\ell \in J_1$,*

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, \ 0 \leq p \leq 1, \\ \ell^p \nu^{-(p-1)}, \ p \geq 1. \end{cases}$$

On the other hand, for $\ell \in J_2$,

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, \ 0 \leq p \leq 1, \\ \ell, \ p \geq 1. \end{cases}$$

The following result follows immediately from the definition (43).

COROLLARY 4.18. *For $\ell \in J_2$, the flatness satisfies $F(\ell) \sim \ell^{-1}$.*

By Theorem 4.8, for $m \geq 1$ we have:

$$\{|\hat{u}(k)|^2\} \leq (2\pi k)^{-2m}\{\|u\|_m^2\} \overset{m}{\sim} (k\nu)^{-2m}\nu.$$

Thus for $|k| \succeq \nu^{-1}$, $\{|\hat{u}(k)|^2\}$ decreases super-polynomially.

Now we want to estimate the H^s norms of u for $s \in (0,1)$.

LEMMA 4.19. *We have:*

$$\{\|u\|_{1/2}^2\} \sim |\log \nu|.$$

Proof. By (5) we have:

$$\|u\|_{1/2} \sim \left(\int_{S^1}\Big(\int_0^1 \frac{|u(x+\ell) - u(x)|^2}{\ell^2} d\ell\Big)dx\right)^{1/2}.$$

Consequently, by Fubini's theorem,

$$\begin{aligned}\{\|u\|_{1/2}^2\} &\sim \int_0^1 \frac{1}{\ell^2}\Big\{\int_{S^1} |u(x+\ell)-u(x)|^2 dx\Big\} d\ell \\ &= \int_0^1 \frac{S_2(\ell)}{\ell^2} d\ell = \int_{J_1} \frac{S_2(\ell)}{\ell^2} d\ell + \int_{J_2} \frac{S_2(\ell)}{\ell^2} d\ell + \int_{J_3} \frac{S_2(\ell)}{\ell^2} d\ell.\end{aligned}$$

By Theorem 4.17 we get:

$$\int_{J_1} \frac{S_2(\ell)}{\ell^2} d\ell \sim \int_0^{C_1\nu} \frac{\ell^2 \nu^{-1}}{\ell^2} d\ell \sim 1$$

and

$$\int_{J_2} \frac{S_2(\ell)}{\ell^2} d\ell \sim \int_{C_1\nu}^{C_2} \frac{\ell}{\ell^2} d\ell \sim |\log \nu|,$$

respectively. Finally, by Lemma 4.10 we get:

$$\int_{J_3} \frac{S_2(\ell)}{\ell^2} d\ell \leq C C_2^{-2} \leq C.$$

Thus,

$$\{\|u\|_{1/2}^2\} \sim |\log \nu|. \quad \square$$

The proof of the following result follows the same lines.

Lemma 4.20. *For* $s \in (0, 1/2)$,

$$\{\|u\|_s^2\} \overset{s}{\sim} 1.$$

On the other hand, for $s \in (1/2, 1)$,

$$\{\|u\|_s^2\} \overset{s}{\sim} \nu^{-(2s-1)}.$$

The results above tell us that $\{|\hat{u}(k)|^2\}$ decreases very fast for $|k| \gtrsim \nu^{-1}$ and that for $s \geq 0$ the sums $\sum |k|^{2s}\{|\hat{u}(k)|^2\}$ have exactly the same behaviour as the partial sums $\sum_{|k|\leq \nu^{-1}} |k|^{2s}|k|^{-2}$ in the limit $\nu \to 0^+$. Therefore we can conjecture that for $|k| \lesssim \nu^{-1}$, we have $\{|\hat{u}(k)|^2\} \sim |k|^{-2}$.

A result of this type actually holds (after layer-averaging), as long as $|k|$ is not too small. To prove it, we use that by Parseval's theorem, for a function $v \in L_2$ one has:

$$|v(\cdot + y) - v(\cdot)|^2 = 4 \sum_{n\in\mathbb{Z}} \sin^2(\pi n y)|\hat{v}(n)|^2. \tag{54}$$

Theorem 4.21. *For* k *such that* $k^{-1} \in J_2$, *we have* $E(k) \sim k^{-2}$.

Proof. We recall that by the definition (44),

$$E(k) = \left\{ \frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}(n)|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right\}.$$

Therefore proving the assertion of the theorem is the same as proving that

$$\sum_{|n| \in [M^{-1}k, Mk]} n^2 \{|\hat{u}(n)|^2\} \sim k. \tag{55}$$

From now on, we will indicate explicitly the dependence on M. The upper estimate in (55) holds without averaging over n such that

$$|n| \in [M^{-1}k, Mk].$$

Indeed, by (42) we know that

$$\{|\hat{u}(n)|^2\} \leq Cn^{-2}.$$

Also, this inequality implies that

$$\sum_{|n| < M^{-1}k} n^2 \{|\hat{u}(n)|^2\} \leq CM^{-1}k \tag{56}$$

and

$$\sum_{|n| > Mk} \{|\hat{u}(n)|^2\} \leq CM^{-1}k^{-1}. \tag{57}$$

To prove the lower estimate in (55) we note that

$$\begin{aligned}
\sum_{|n| \leq Mk} n^2 \{|\hat{u}(n)|^2\} &\geq \frac{k^2}{\pi^2} \sum_{|n| \leq Mk} \sin^2(\pi n k^{-1}) \{|\hat{u}(n)|^2\} \\
&\geq \frac{k^2}{\pi^2} \Big(\sum_{n \in \mathbb{Z}} \sin^2(\pi n k^{-1}) \{|\hat{u}(n)|^2\} - \sum_{|n| > Mk} \{|\hat{u}(n)|^2\} \Big).
\end{aligned}$$

Using (54) and (57) we get:

$$\begin{aligned}
\sum_{|n| \leq Mk} n^2 \{|\hat{u}(n)|^2\} &\geq \frac{k^2}{4\pi^2} \Big(\{|u(\cdot + k^{-1}) - u(\cdot)|^2\} - CM^{-1}k^{-1} \Big) \\
&\geq \frac{k^2}{4\pi^2} \Big(S_2(k^{-1}) - CM^{-1}k^{-1} \Big).
\end{aligned}$$

Finally, using Theorem 4.17 we obtain that

$$\sum_{|n| \leq Mk} n^2 \{|\hat{u}(n)|^2\} \geq (C - CM^{-1})k.$$

Now we use (56) and we choose $M \geq 1$ large enough to obtain (55). $\square$

5. Estimates in the randomly forced case

5.1. **Foreword.** The results in this section (estimates for Sobolev norms and for small-scale quantities) have been obtained in [13] for the equation forced by the white noise. For the simpler case of the kick force, estimates for Sobolev norms have been obtained in [12]. Since these estimates are used as a "black box" when studying small-scale quantities, generalisation of the small-scale estimates in [13] to the case of a kick force is immediate. Thus, in this section, we only consider the white-forced equation (10).

Some proofs in [13] are similar to the proofs in the unforced case. We will only give here the proofs of Theorem 5.1 and Lemma 5.6, as well as some comments on the proofs of small-scale results.

For simplicity, in the white-forced case we assume that the initial condition u_0 is deterministic. However, we can easily generalise all results to the case of a random initial condition u_0 independent of $w(t), t \geq 0$. Indeed, in this case for any measurable functional $\Phi(u(\cdot))$ we have:

$$\mathbf{E}\Phi(u(\cdot)) = \int \mathbf{E}\Big(\Phi(u(\cdot))|u(0) = u_0\Big) d\mu(u_0),$$

where $\mu(u_0)$ is the law of u_0, and all our estimates hold uniformly in u_0.

Moreover, for $\tau \geq 0$ and u_0 independent of $w(t) - w(\tau),\ t \geq \tau$, the Markov property, which can be proved in the same way as in [43], yields:

$$\mathbf{E}\Phi(u(\cdot)) = \int \mathbf{E}\Big(\Phi(u(\tau + \cdot))|u(\tau) = u_0\Big) d\mu(u_0).$$

Consequently, all estimates which hold for time t or a time interval $[t, t+T]$ actually hold for time $t+\tau$ or a time interval $[t+\tau, t+\tau+T]$, uniformly in $\tau \geq 0$.

The remarks above still hold for the kick-forced equation, up to some natural modifications due to the fact that the forcing is now discrete in time.

5.2. **Estimates for Sobolev norms.** The following key estimate is proved using a stochastic version of the Kruzhkov maximum principle (cf. [40]).

Theorem 5.1. *Denote by X_t the random variable*

$$X_t = \max_{s \in [t,t+1],\ x \in S^1} u_x(s,x).$$

For every $k \geq 1$, we have:

$$\mathbf{E}\ X_t^k \overset{k}{\lesssim} 1, \quad t \geq 1.$$

Proof. We take $t = 1$ and denote X_t by X.

Consider (10) on the time interval $[0, 2]$. Putting $v = u - w$ and differentiating once in space, we get:

$$\frac{\partial v_x}{\partial t} + f''(u)(v_x + w_x)^2 + f'(u)(v_x + w_x)_x = \nu(v_x + w_x)_{xx}. \tag{58}$$

Set $\tilde{v}(t, x) = tv_x(t, x)$ and multiply (58) by t^2. For $t > 0$, $\tilde{v}$ satisfies

$$\begin{aligned} &t\tilde{v}_t - \tilde{v} + f''(u)(\tilde{v} + tw_x)^2 + tf'(u)\tilde{v}_x + t^2 f'(u)w_{xx} \\ &= \nu t\tilde{v}_{xx} + \nu t^2 w_{xxx}. \end{aligned} \tag{59}$$

Now observe that if the zero mean function $\tilde{v}$ does not vanish identically on the domain $S = [0, 2] \times S^1$, then it attains its positive maximum N on S at a point (t_1, x_1) such that $t_1 > 0$. At (t_1, x_1) we have $\tilde{v}_t \geq 0$, $\tilde{v}_x = 0$ and $\tilde{v}_{xx} \leq 0$. By (59), at (t_1, x_1) we have the inequality

$$f''(u)(\tilde{v} + tw_x)^2 \leq \tilde{v} - t^2 f'(u)w_{xx} + \nu t^2 w_{xxx}. \tag{60}$$

Denote by A the random variable

$$A = \max_{t \in [0,2]} |w(t)|_{3,\infty}.$$

Since for every t, $tv(t)$ is the zero space average primitive of $\tilde{v}(t)$ on S^1, we get:

$$\begin{aligned} \max_{t \in [0,2],\ x \in S^1} |tu| &\leq \max_{t \in [0,2],\ x \in S^1} (|tv| + |tw|) \\ &\leq N + 2 \max_{t \in [0,2]} |w(t)|_\infty \leq N + 2A. \end{aligned}$$

Now denote by δ the quantity

$$\delta = 2 - h(1)$$

(cf. (13)). Since $\delta > 0$, we obtain that

$$\begin{aligned} \max_{t \in [0,2],\ x \in S^1} |t^2 f'(u)w_{xx}| &\leq A \max_{t \in [0,2],\ x \in S^1} t^\delta |t^{2-\delta} f'(u)| \\ &\leq CA \max_{t \in [0,2],\ x \in S^1} t^\delta (|tu| + t)^{2-\delta} \\ &\leq CA(N + 2A + 2)^{2-\delta}. \end{aligned}$$

From now on, we assume that $N \geq 2A$. Since $\nu \in (0, 1]$ and $f'' \geq \sigma$, the relation (60) yields

$$\sigma(N - 2A)^2 \leq N + CA(N + 2A + 2)^{2-\delta} + 4A.$$

Thus we have proved that if $N \geq 2A$, then $N \leq C(A+1)^{1/\delta}$. Since by (9), all moments of A are finite, all moments of N are also finite. By definition of $\tilde{v}$ and S, the same is true for X. This proves the theorem's assertion. $\square$

Corollary 5.2. *For $k \geq 1$,*

$$\mathbf{E} \max_{s\in[t,t+1]} |u(s)|_{1,1}^k \overset{k}{\lesssim} 1, \quad t \geq 1.$$

Corollary 5.3. *For $k \geq 1$,*

$$\mathbf{E} \max_{s\in[t,t+1]} |u(s)|_p^k \overset{k}{\lesssim} 1, \quad p \in [1,\infty],\ t \geq 1.$$

Lemma 5.4. *For $m \geq 1$,*

$$\mathbf{E} \max_{s\in[t,t+1]} \|u(s)\|_m^2 \overset{m}{\lesssim} \nu^{-(2m-1)}, \quad t \geq 2.$$

Theorem 5.5. *For $m \in \{0,1\}$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$,*

$$\Big(\mathbf{E} \max_{s\in[t,t+1]} |u(s)|_{m,p}^\alpha\Big)^{1/\alpha} \overset{m,p,\alpha}{\lesssim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq 2.$$

Now we give the key lower estmate.

Lemma 5.6. *There exists a constant $T_0 > 0$ such that we have:*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbf{E}\,\|u(s)\|_1^2\Big)^{1/2} \gtrsim \nu^{-1/2}, \qquad t \geq 1,\ T \geq T_0.$$

Proof. For $T > 0$, by (12) we get:

$$\mathbf{E}\,|u(t+T)|^2 \geq \mathbf{E}(|u(t+T)|^2 - |u(t)|^2) = TI_0 - 2\nu \int_t^{t+T} \mathbf{E}\,\|u(s)\|_1^2.$$

On the other hand, by Corollary 5.3 there exists a constant $C' > 0$ such that $\mathbf{E}\,|u(t+T)|^2 \leq C'$. Consequently, for $T \geq T_0 := (C'+1)/I_0$,

$$\frac{1}{T}\int_t^{t+T} \mathbf{E}\,\|u(s)\|_1^2 \geq \frac{TI_0 - C'}{2T}\nu^{-1} \geq \frac{I_0}{2(C'+1)}\nu^{-1},$$

which proves the lemma's assertion. $\square$

Theorem 5.7. *For $m \in \{0,1\}$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$, we have:*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbf{E}\,|u(s)|_{m,p}^\alpha\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma},\ \alpha > 0,$$
$$t \geq T_1 = T_0 + 2,\ T \geq T_0. \tag{61}$$

Moreover, the upper estimates hold with time-averaging replaced by maximising over $[t, t+1]$*, i.e.*

$$\Big(\mathbf{E} \max_{s \in [t,t+1]} |u(s)|^{\alpha}_{m,p}\Big)^{1/\alpha} \overset{m,p,\alpha}{\lesssim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq 2. \tag{62}$$

On the other hand, the lower estimates hold for all $m \geq 0$ *and* $p \in [1, \infty]$*. The asymptotics (61) hold without time-averaging if* m *and* p *are such that* $\gamma(m,p) = 0$*. Namely, in this case,*

$$\Big(\mathbf{E}\, |u(t)|^{\alpha}_{m,p}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} 1, \quad \alpha > 0,\ t \geq T_1. \tag{63}$$

5.3. **Estimates for small-scale quantities.** Consider an observable A, i.e. a real-valued functional on a Sobolev space H^m, which we evaluate on the solutions $u^{\omega}(s)$. We denote by $\{A\}$ the average of $A(u^{\omega}(s))$ in ensemble and in time over $[t, t+T_0]$:

$$\{A\} = \frac{1}{T_0} \int_t^{t+T_0} \mathbf{E}\ A(u^{\omega}(s)) ds,\ t \geq T_1.$$

The constant T_1 is the same as in Theorem 5.7. In this section, we assume that $\nu \leq \nu_0$, where ν_0 is a positive constant. The definitions and the choices for ν_0, the ranges and the small-scale quantities are word-to-word the same as in the unforced case, up to the changes in the meaning of the brackets $\{\cdot\}$.

Lemma 5.8. *For* $\alpha \geq 0$ *and* $\ell \in [0, 1]$*,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\lesssim} \begin{cases} \ell^{\alpha p},\ 0 \leq p \leq 1, \\ \ell^{\alpha p} \nu^{-\alpha(p-1)},\ p \geq 1. \end{cases}$$

Lemma 5.9. *For* $\alpha \geq 0$ *and* $\ell \in J_2 \cup J_3$*,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\lesssim} \begin{cases} \ell^{\alpha p},\ 0 \leq p \leq 1, \\ \ell^{\alpha},\ p \geq 1. \end{cases}$$

The following lemma states that with a not too small probability, during a not too small period of time, several Sobolev norms are of the same order as their expected values.

Definition 5.10. *For a given solution* $u(s) = u^{\omega}(s)$ *and* $K > 1$*, we denote by* L_K *the set of all* $(s, \omega) \in [t, t+T_0] \times \Omega$ *such that*

$$K^{-1} \leq |u(s)|_{\infty} \leq \max u_x(s) \leq K \tag{64}$$

$$K^{-1}\nu^{-1} \leq |u(s)|_{1,\infty} \leq K\nu^{-1} \tag{65}$$

$$|u(s)|_{2,\infty} \leq K\nu^{-2}. \tag{66}$$

LEMMA 5.11. *There exist constants $\tilde{C}, K_1 > 0$ such that for all $K \geq K_1$, $\rho(L_K) \geq \tilde{C}$. Here, ρ denotes the product measure of the Lebesgue measure and $\mathbf{P}$ on $[t, t+T_0] \times \Omega$.*

Proof. The proof is almost the same as in the deterministic case. One difference is that now we consider the product of the Lebesgue and the probability measures instead of only the Lebesgue measure. The other difference is that the upper estimates now hold with probability tending to 1 as $K \to +\infty$, and not with probability 1 for K large enough. $\square$

DEFINITION 5.12. *For a given solution $u(s) = u^\omega(s)$ and $K > 1$, we denote by O_K the set of all $(s, \omega) \in [t, t+T_0] \times \Omega$ such that the conditions (64), (66) and*

$$K^{-1}\nu^{-1} \leq -\min u_x \leq K\nu^{-1} \tag{67}$$

hold.

COROLLARY 5.13. *If $K \geq K_1$ and $\nu < K_1^{-2}$, then $\rho(O_K) \geq \tilde{C}$. Here, $\tilde{C}$ and K_1 are the same as in the statement of Lemma 5.11.*

THEOREM 5.14. *For $\alpha \geq 0$ and $\ell \in J_1$,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, & 0 \leq p \leq 1, \\ \ell^{\alpha p}\nu^{-\alpha(p-1)}, & p \geq 1. \end{cases}$$

On the other hand, for $\alpha \geq 0$ and $\ell \in J_2$,

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, & 0 \leq p \leq 1, \\ \ell^{\alpha}, & p \geq 1. \end{cases}$$

COROLLARY 5.15. *For $\ell \in J_2$, the flatness satisfies $F(\ell) \sim \ell^{-1}$.*

LEMMA 5.16. *We have:*

$$\begin{aligned} \{\|u\|_s^2\} &\overset{s}{\sim} 1, \quad s \in (0, 1/2), \\ \{\|u\|_{1/2}^2\} &\sim |\log \nu|, \\ \{\|u\|_s^2\} &\overset{s}{\sim} \nu^{-(2s-1)}, \quad s \in (1/2, 1). \end{aligned}$$

THEOREM 5.17. *If M in the definition of $E(k)$ is large enough, then for every k such that $k^{-1} \in J_2$, we have $E(k) \sim k^{-2}$. Moreover, we have:*

$$\left\{ \left(\frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}(n)|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right)^\alpha \right\} \overset{\alpha}{\sim} k^{-2\alpha}, \quad \alpha > 0.$$

6. Stationary measure and related issues

The results of this section are proved in [13] for the equation with white forcing. Up to some natural modifications due to the fact that the forcing is now discrete in time, they can be generalised to the kick force case. For more details, see [11]; see also [43], where a random forcing is introduced in a similar setup.

Theorem 6.1. *Consider two solutions u, $\overline{u}$ of (10), corresponding to the same random force but different initial conditions in C^∞. For all $t \geq 0$, we have:*

$$|u(t) - \overline{u}(t)|_1 \leq |u(0) - \overline{u}(0)|_1.$$

Since C^∞ is dense in L_1, Theorem 6.1 allows us to extend the flow of (10): now we can consider an initial condition in L_1. Note that since the Burgers flow is smoothing, the corresponding L_1-solutions become solutions to (10) for $t > 0$. This allows us to prove that the flow corresponding to (10) induces a time-continuous Markov process, and then we can define the corresponding semigroup S_t^* acting on Borel measures on L_1. For a more detailed account on the well-posedness in a similar setting, see [43].

A *stationary measure* is a Borel probability measure on L_1 invariant with respect to S_t^* for every t. A *stationary solution* of (10) is a random process v defined for $(t, \omega) \in [0, +\infty) \times \Omega$ and taking values in L_1, such that the distribution of $v(t)$ does not depend on t, satisfying (10). This distribution is automatically a stationary measure.

Now we consider the question of existence and uniqueness of a stationary measure, which implies existence and uniqueness (in the sense of distribution) of a stationary solution. Moreover, we obtain a bound for the rate of convergence to the stationary measure in an appropriate distance. This bound holds independently of the viscosity or of the initial condition.

Definition 6.2. *Fix $p \in [1, \infty)$. For a continuous function*

$$g: \ L_p \to \mathbb{R},$$

we define its Lipschitz norm as

$$|g|_L := \sup_{L_p} |g| + |g|_{Lip},$$

where $|g|_{Lip}$ is the Lipschitz constant of g. The set of continuous functions with finite Lipschitz norm will be denoted by $L(p) = L(L_p)$. We will abbreviate $L(1)$ as L.

DEFINITION 6.3. *For two Borel probability measures μ_1, μ_2 on L_p, we denote by $\|\mu_1 - \mu_2\|^*_{L(p)}$ the Lipschitz-dual distance:*

$$\|\mu_1 - \mu_2\|^*_{L(p)} := \sup_{g \in L(p),\ |g|_{L(p)} \leq 1} \Big| \int_{S^1} g d\mu_1 - \int_{S^1} g d\mu_2 \Big|.$$

Since we have u_0-uniform upper estimates, the existence of a stationary measure for the generalised Burgers equation is proved using the Bogolyubov-Krylov argument (see [43]).

Now we state the main result of this section. It immediately implies the uniqueness of a stationary measure μ for the equation (10), and an estimate on the speed of convergence to this measure which is algebraic in t, uniformly in the viscosity coefficient ν. Theorem 6.4 is proved using a simplified version of a coupling argument due to Kuksin and Shirikyan [43, Chapter 3]. The situation is actually simpler than for the stochastic 2D Navier Stokes equation. Indeed, in our setting the "damping time" needed to make the distance between two solutions corresponding to the same forcing small does not depend on the initial conditions, and moreover by Theorem 6.1 the flow of (10) is L_1-contracting.

THEOREM 6.4. *There exists a positive constant C' such that we have:*

$$\|S^*_t \mu_1 - S^*_t \mu_2\|^*_L \leq C' t^{-1/13}, \qquad t \geq 1, \tag{68}$$

for any probability measures μ_1, μ_2 on L_1.

COROLLARY 6.5. *For every $p \in (1, \infty)$, there exists a positive constant $C'(p)$ such that we have:*

$$\|S^*_t \mu_1 - S^*_t \mu_2\|^*_{L(p)} \leq C' t^{-1/13p}, \qquad t \geq 1, \tag{69}$$

for any probability measures μ_1, μ_2 on L_p.

Note that all the estimates in the previous sections still hold for a stationary solution, since they hold uniformly for any initial condition in L_1 for large times and a stationary solution has time-independent statistical properties. It follows that these estimates still hold when averaging in time and in ensemble (denoted by $\{\cdot\}$) is replaced by averaging solely in ensemble, i.e. by integrating with respect to μ. Namely, Theorem 5.7, Theorem 5.14 and Theorem 5.17 imply, respectively, the following results.

THEOREM 6.6. *For $m \in \{0, 1\}$ and $p \in [1, \infty]$, or for $m \geq 2$ and $p \in (1, \infty]$,*

$$\Big(\int |u|^\alpha_{m,p} d\mu(u) \Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \quad \alpha > 0.$$

THEOREM 6.7. *For $\alpha \geq 0$ and $\ell \in J_1$,*

$$\int \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^\alpha d\mu(u) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1, \\ \ell^{\alpha p} \nu^{-\alpha(p-1)}, \ p \geq 1. \end{cases}$$

On the other hand, for $\alpha \geq 0$ and $\ell \in J_2$,

$$\int \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^\alpha d\mu(u) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1, \\ \ell^{\alpha}, \ p \geq 1. \end{cases}$$

THEOREM 6.8. *For k such that $k^{-1} \in J_2$, we have:*

$$\int \frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}(n)|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} d\mu(u) \sim k^{-2}.$$

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