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Equation de Burgers généralisée à force aléatoire et à viscosité petite
Generalised Burgers equation with random force and small viscosity

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Résumé

Cette thèse traite du comportement des solutions u de l'équation de Burgers généralisée sur le cercle :

$$u_t + f'(u)u_x = \nu u_{xx} + \eta,\ x \in S^1 = \mathbb{R}/\mathbb{Z}.$$

Ici, f est lisse, fortement convexe et satisfait certaines conditions de croissance. La constante $0 < \nu \ll 1$ correspond à un coefficient de viscosité. Nous considérons le cas où $\eta = 0$, ainsi que le cas où η est une force aléatoire, lisse en x et peu régulière (de type « kick » ou bruit blanc) en t. Nous obtenons des estimations sur les normes de Sobolev de u moyennées en temps et en probabilité de la forme $C\nu^{-\delta},\ \delta \geq 0$, avec les mêmes valeurs de δ pour les bornes supérieures et inférieures. On en déduit des estimations précises pour les quantités à petite échelle caractérisant la turbulence qui confirment exactement les prédictions physiques.

Nous nous intéressons également au comportement asymptotique des solutions. Nous obtenons un résultat d'hyperbolicité des minimiseurs pour l'action correspondant à l'équation de Hamilton-Jacobi stochastique, dont la dérivée en espace est l'équation de Burgers stochastique avec $\nu = 0$.

Abstract

This Ph.D. thesis is concerned with studying solutions u of a generalised Burgers equation on the circle:

$$u_t + f'(u)u_x = \nu u_{xx} + \eta,\ x \in S^1 = \mathbb{R}/\mathbb{Z}.$$

Here, f is smooth, strongly convex, and satisfies some growth conditions. The constant $0 < \nu \ll 1$ corresponds to a viscosity coefficient. We will consider both the case $\eta = 0$ and the case when η is a random force which is smooth in x and irregular ("kick" or white noise) in t. We obtain sharp bounds for Sobolev norms of u averaged in time and in ensemble of the type $C\nu^{-\delta},\ \delta \geq 0$, with the same value of δ for upper and lower bounds. These results yield sharp bounds for small-scale quantities characterising turbulence, which confirm physical predictions.

We are also concerned with the asymptotic behaviour of solutions: we prove hyperbolicity of minimizers for the action corresponding to the stochastic Hamilton-Jacobi equation, whose space derivative is the stochastic Burgers equation with $\nu = 0$.

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Organisation de la thèse

Dans le Chapitre 1, nous commencerons par introduire les notations, notamment pour les espaces fonctionnels. Ensuite, nous donnerons quelques éléments de théorie pour la turbulence hydrodynamique et les EDP stochastiques « markoviennes », avant de faire quelques commentaires historiques sur l'équation de Burgers. Enfin, nous présenterons un résumé des résultats contenus dans cette thèse et des méthodes utilisées, avant d'aborder les perspectives de recherche.

Les Chapitres 2-5 correspondent au contenu de mes 4 articles. Ceux-ci ne sont pas classés par ordre chronologique de l'écriture/publication, mais plutôt, pour les trois premiers, par complexité croissante. Dans chacun des Chapitres 2-4, on suit le même schéma. D'abord on obtient des estimations pour les solutions de l'équation de Burgers généralisée

$$u_t + f'(u)u_x = \nu u_{xx} + \eta,\ x \in S^1 = \mathbb{R}/\mathbb{Z}, \tag{1}$$

puis on en déduit des informations sur la « Burgulence » ou turbulence de Burgers.

Dans le Chapitre 2, on considère l'équation (1) avec $\eta = 0$. Dans le Chapitre 3, on introduit une force η de type « kick », lisse en espace. Finalement, dans le Chapitre 4, la force considérée est de type bruit blanc en temps (mais toujours lisse en espace). Ce cas correspond à une limite du cas précédent lorsqu'on considère des impulsions de plus en plus petites et rapprochées, avec un « scaling » approprié.

Le Chapitre 5 traite de l'équation inviscide, c'est-à-dire sans le terme νu_{xx}, aussi appelée équation de Hopf. Plus précisément, nous regardons l'équation de Hamilton-Jacobi stochastique, dont la dérivée spatiale est l'équation de Hopf stochastique. Ensuite, nous considérons les minimiseurs de l'action associée aux solutions de cette équation. Il s'agit alors de prouver un résultat d'hyperbolicité de ces minimiseurs.

Dans l'appendice A, on prouve que l'équation (1) avec η de type bruit blanc définit un problème de Cauchy bien posé dans L_1. Dans l'appendice B, on donne une esquisse de la preuve pour le caractère markovien des solutions de l'équation (1).

Chapitre 1

Introduction

1.1 Espaces fonctionnels et conventions de notation

1.1.1 Abbréviations

- p.s. (a.s.) : presque sûrement (almost surely)
- p.p. (a.e.) : presque partout (almost everywhere)
- v.a. (r.v.) : variable aléatoire (random variable)
- i.i.d. (i.i.d.) : indépendantes identiquement distribuées (independent identically distributed)
- SDA (RDS) : système dynamique aléatoire (random dynamical system)

1.1.2 Espaces de Lebesgue et de Sobolev

Soit v une fonction de moyenne nulle intégrable sur $S^1 = \mathbb{R}/\mathbb{Z}$. Pour $p \in [1, \infty]$, on note $|v|_p$ sa norme dans l'espace de Lebesgue L_p. La norme L_2 est notée $|v|$ et le produit scalaire dans L_2 $\langle \cdot, \cdot \rangle$.
Sauf si cela est explicitement précisé, L_p, $p \in [1, \infty]$ désigne l'espace des fonctions de moyenne nulle dans $L_p(S^1)$. De même, on note C^∞ l'espace des

fonctions C^∞-lisses de moyenne nulle sur S^1.

Pour un entier $m \geq 0$ et $p \in [1, \infty]$, $W^{m,p}$ désigne l'espace de Sobolev des fonctions v de moyenne nulle sur S^1 satisfaisant

$$|v|_{m,p} = \left| \frac{d^m v}{dx^m} \right|_p < \infty.$$

En particulier, $W^{0,p} = L_p$ pour $p \in [1, \infty]$. Pour $p = 2$, on utilise la notation $W^{m,2} = H^m$. La norme correspondante est notée $\|v\|_m$. Dans l'ensemble de la thèse, lorsqu'on considère une norme $|\cdot|_{m,p}$, on note γ la quantité

$$\gamma(m,p) = \max\Big(0, m - \frac{1}{p}\Big).$$

Comme S^1 est de longueur 1 et la valeur moyenne de v est égale à 0, nous avons

$$|v|_1 \leq |v|_\infty \leq |v|_{1,1} \leq |v|_{1,\infty} \leq \cdots \leq |v|_{m,1} \leq |v|_{m,\infty} \leq \ldots$$

Rappelons une version de l'inégalité de Gagliardo-Nirenberg (cf. [21, Appendice]) :

Lemme 1.1.1. *Pour une fonction v suffisamment lisse de moyenne nulle sur S^1,*

$$|v|_{\beta,r} \leq C\, |v|_{m,p}^{\theta}\, |v|_q^{1-\theta},$$

où $m > \beta$, et r est donné par

$$\frac{1}{r} = \beta - \theta\Big(m - \frac{1}{p}\Big) + (1-\theta)\frac{1}{q},$$

sous la condition $\theta = \beta/m$ si $p = 1$ ou $p = \infty$ et $\beta/m \leq \theta < 1$ sinon. La constante C dépend de m, p, q, β, θ.

Pour tout $s \geq 0$, H^s désigne l'espace de Sobolev des fonctions v de moyenne nulle sur S^1 de norme

$$\|v\|_s = (2\pi)^s \Big(\sum_{k \in \mathbb{Z}} |k|^{2s} |\hat{v}_k|^2 \Big)^{1/2} \tag{1.1}$$

finie. Ici, $\hat{v}_k$ est le k-ième coefficient de Fourier complexe de $v(x)$:

$$\hat{v}_k = \int_{S^1} e^{-2\pi i k x} v(x) dx.$$

Pour les valeurs entières de $s = m$, cette norme coïncide avec la norme H^m définie précédemment. Pour $s \in (0,1)$, $\|v\|_s$ est équivalente à la norme

$$\|v\|'_s = \left(\int_{S^1} \Big(\int_0^1 \frac{|v(x+\ell) - v(x)|^2}{\ell^{2s+1}} d\ell \Big) dx \right)^{1/2} \tag{1.2}$$

(voir [1, 58]). On a l'inégalité classique suivante :

$$|v|_\infty \leq C(s)\, \|v\|_s\,, \quad s > 1/2. \tag{1.3}$$

Soit maintenant v une fonction des variables (t, x). Les sous-indices (répétés) t et x désignent la dérivation par rapport aux variables correspondantes. On note $v^{(m)}$ la m-ième derivée de v en x. La fonction $v(t, \cdot)$ est notée $v(t)$.

1.1.3 Autres conventions de notation

Les lettres écrites en gras correspondent à des quantités vectorielles.

Lorsqu'on considère l'équation de Burgers avec un terme aléatoire, $\mathbb{P}$ et $\mathbb{E}$ désignent, respectivement, la probabilité et l'espérance par rapport à la mesure de probabilité considérée (cf. Section 1.4).

Toutes les constantes notées C avec des sous- or superindices sont strictement positives et non aléatoires. Sauf si cela est explicitement précisé, elles ne dépendent que des paramètres suivants, en fonction de l'équation considérée :

- Pour l'équation de Navier-Stokes avec une force aléatoire, les propriétés statistiques de la force η.

- Pour l'équation de Burgers généralisée sans force, la fonction f déterminant la nonlinéarité $f'(u)u_x$, ainsi que le paramètre D qui décrit la « typicité » de la condition initiale (voir (1.15) pour sa définition).

- Pour l'équation de Burgers généralisée avec une force aléatoire, la fonction f déterminant la nonlinéarité, ainsi que les propriétés statistiques de la force η. Dans le cas d'une force de type « kick », il s'agit de la fonction de distribution commune des v.a. i.i.d. η_k. Dans le cas d'une force de type bruit blanc, il s'agit de l'opérateur de corrélation pour le processus de Wiener w dont le bruit blanc est la dérivée au sens des distributions.

Les constantes qui dépendent aussi des paramètres $a_1, \dots, a_k$ sont notées $C(a_1, \dots, a_k)$. La notation $X \overset{a_1,\dots,a_k}{\lesssim} Y$ signifie que $X \leq C(a_1, \dots, a_k)Y$. La notation $X \overset{a_1,\dots,a_k}{\sim} Y$ est un raccourci pour dire qu'on a

$$Y \overset{a_1,\dots,a_k}{\lesssim} X \overset{a_1,\dots,a_k}{\lesssim} Y.$$

En particulier, $X \lesssim Y$ et $X \sim Y$ signifient que $X \leq CY$ et $C^{-1}Y \leq X \leq CY$, respectivement.

Toutes les constantes ne dépendent pas de la viscosité ν.

La condition initiale $u(0, \cdot)$ est notée u_0.

On utilise les notations $g^- = \max(-g, 0)$ et $g^+ = \max(g, 0)$.

1.2 Turbulence, théorie K41, intermittence

La turbulence est l'un des phénomènes physiques les plus difficiles à étudier. La définition de ce terme est déja extrêmement problématique. Les premiers mots qui viennent à l'esprit sont « très grand nombre de degrés de liberté », « imprévisibilité/chaos » ou encore « irrégularité ». S'il existe de nombreux types de turbulence (turbulence bidimensionnelle, turbulence magnétohydrodynamique...), les premiers exemples considérés étaient des écoulements de fluides tridimensionnels, souvent dans des contextes géométriques simples (cylindre...)

Pour décrire la turbulence, il est utile d'introduire la notion d'*échelle*. Pour cela, le plus simple est de considérer un flot $\mathbf{u}(t, \mathbf{x})$ périodique en espace. Sans perte de généralité, on peut supposer que la période vaut 1 suivant chaque coordonnée.

L'échelle spatiale correspond à l'inverse de la fréquence considérée. En d'autres mots, les grandes échelles correspondent aux basses fréquences et les petites échelles aux hautes fréquences.

Les coefficients de Fourier $\hat{\mathbf{u}}_\mathbf{k}$ pour $\mathbf{k}$ grand ou, dans l'espace physique, les incréments $\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})$ pour $\mathbf{r}$ petit sont des quantités à petite échelle.

La théorie qui fut en quelque sorte le point de départ de l'étude moderne de la turbulence est due à Kolmogorov. Elle est essentiellement contenue dans trois de ses articles publiés en 1941 [33, 34, 35]. On parle donc de *théorie K41*.

L'idée de départ est que, si le comportement d'un flot à grande échelle dépend de ses caractéristiques individuelles (forçage, conditions aux limites), le comportement à petite échelle est au contraire, dans un certain sens, universel. Pour donner un sens précis à cette affirmation, on commence par introduire quelques définitions.

L'*échelle dissipative* ℓ_d est l'échelle la plus petite telle que, pour $|\mathbf{k}| \succeq \ell_d^{-1}$, les coefficients de Fourier d'une fonction $\mathbf{u}$ décroissent très vite (typiquement, plus vite que l'inverse de n'importe quel polynôme en $|\mathbf{k}|$), uniformément en ν. On dit que $\mathbb{J}_{diss} = [0, \ell_d]$ est la *zone dissipative*. Pour K41, $\ell_d = C\nu^{3/4}$. La *zone énergétique* $\mathbb{J}_{energ} = [\ell_e, 1]$ est l'ensemble d'échelles telles que les modes de Fourier correspondants contiennent la plus grande partie de la norme L^2 de $\mathbf{u}$:

$$\sum_{|\mathbf{k}| \leq \ell_e^{-1}} \langle |\hat{\mathbf{u}}_\mathbf{k}|^2 \rangle \gg \sum_{|\mathbf{k}| > \ell_e^{-1}} \langle |\hat{\mathbf{u}}_\mathbf{k}|^2 \rangle.$$

Pour K41, $\ell_e = C$.

Finalement, on appelle $\mathbb{J}_{inert} = [\ell_d, \ell_e]$ la *zone inertielle.* Pour K41, $\mathbb{J}_{inert} = [C\nu^{3/4}, C]$.

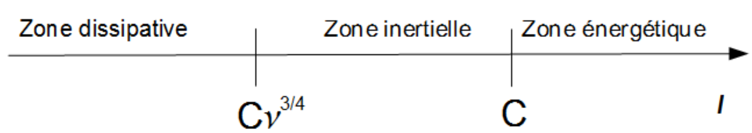

FIGURE 1.1 – Echelles de Kolmogorov

Pour étudier le comportement à petite échelle d'un flot $\mathbf{u}(t, \mathbf{x})$, les deux quantités essentielles sont :

- D'une part, la *fonction de structure*

$$S_p^{\|}(\mathbf{x}, \mathbf{r}) = \left\langle \frac{(\mathbf{u}(\mathbf{x} + \mathbf{r}) - \mathbf{u}(\mathbf{x})) \cdot \mathbf{r}}{|\mathbf{r}|} \right\rangle^p \tag{1.4}$$

- D'autre part, le *spectre d'énergie*

$$E(k) = \frac{\sum_{|\mathbf{n}| \in [M^{-1}k, Mk]} \langle |\hat{\mathbf{u}}_\mathbf{n}|^2 \rangle}{\sum_{|\mathbf{n}| \in [M^{-1}k, Mk]} 1}, \tag{1.5}$$

  en d'autres mots la moyenne de $\langle |\hat{\mathbf{u}}_\mathbf{n}|^2 \rangle$ sur l'ensemble des $\mathbf{n}$ qui sont du même ordre de grandeur que k.

La théorie K41 affirme que sous certaines conditions nous avons, pour $\ell = |\mathbf{r}| \in \mathbb{J}_{inert}$ et pour tout $\mathbf{x}$,

$$S_p^{\|}(\mathbf{x}, \mathbf{r}) \overset{p}{\sim} \ell^{p/3}, \quad p \geq 0. \tag{1.6}$$

D'autre part, pour tout k tel que $k^{-1} \in \mathbb{J}_{inert}$, la théorie prédit que

$$E(k) \sim k^{-5/3} \tag{1.7}$$

(voir [52, 53]).

Il s'est révélé que les prédictions de K41 étaient vérifiées expérimentalement et numériquement pour le spectre d'énergie et pour les fonctions de structure $S_p,\ p = 2, 3$. Cependant, il n'en était pas autant pour les fonctions de structure S_p avec $p \geq 4$ [27, Chapitre 8]. Il y a deux théories parallèles expliquant cette différence, dues respectivement à Kolmogorov lui-même [36] et à Frisch et Parisi [54], qui mettent en cause l'*intermittence* spatiale. En d'autres mots, en un temps donné, le flot est très fortement excité sur un ensemble de petite taille, comme pour la fonction représentée sur le graphe ci-dessous :

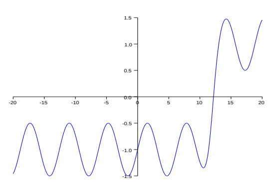

FIGURE 1.2 – Exemple d'une fonction intermittente à petite échelle

Ce type de comportement peut être quantifié par le *facteur d'aplatissement* (flatness)

$$F(\ell) = S_4^{\|}(\ell)/S_2^{\|}(\ell)^2 :$$

la fonction est d'autant plus intermittente à l'échelle ℓ que F est grand. Notons que cette notion ne doit pas être confondue avec celle de l'intermittence au sens de Pomeau et Manneville [55]

Il existe de nombreux ouvrages sur la turbulence. On peut notamment citer le texte historique de Batchelor [3], les monographies de Frisch, Lesieur et Tsinober [27, 47, 59] ainsi que le cours de Manneville [48].

1.3 EDP stochastiques : estimations et mesure stationnaire

Les définitions des termes utilisés dans cette section, ainsi que les résultats pour l'équation de Burgers stochastique, se trouvent dans l'appendice B. La terminologie, la présentation et les méthodes de preuve sont largement inspirées de celles du livre de Kuksin et Shirikyan [45].

Considérons tout d'abord une EDP d'évolution déterministe telle que le problème de Cauchy correspondant soit bien posé. Une telle EDP induit naturellement un flot S_s^{s+t} qui fait correspondre à une condition initiale au temps s la solution au temps $s+t$. Si l'EDP en question est autonome, ce flot ne dépend pas de s : on peut alors le noter S_t. Par dualité, un tel flot induit un flot S_t^* agissant sur les mesures dans l'espace fonctionnel dans lequel l'EDP est posée. On dit qu'une mesure de probabilité μ satisfaisant $S_t^*\mu = \mu$ pour tout $t > 0$ est une *mesure invariante*. L'existence et l'unicité d'une telle mesure, ainsi que la vitesse de convergence vers celle-ci, sont des questions cruciales, notamment lorsqu'on étudie les équations de Navier-Stokes. Vu la compléxité du système, même en 2D, il s'agit alors souvent de donner une réponse partielle en étudiant des objets tels que les attracteurs : voir à ce sujet les monographies [16, 21, 25]

Considérons maintenant une EDP d'évolution stochastique qui définit un processus de Markov. Typiquement, une équation bien posée où le seul terme non autonome est un bruit additif stationnaire en temps (comme le bruit blanc) ou admettant une invariance discrète (comme une force de type « kick ») satisfait cette condition. Il n'est alors pas difficile de définir une *mesure stationnaire* qui a les mêmes propriétés qu'une mesure invariante dans le cas déterministe. Lorsque le terme stochastique admet une invariance discrète, on demande seulement que la mesure admette une invariance discrète par rapport au flot correspondant.

Lorsque l'on a de bonnes estimations uniformes en temps pour les solutions, l'existence d'une mesure stationnaire est en général facile à prouver, en utilisant l'*argument de Krylov-Bogolyubov*. On prend une suite de mesures

$$\Big(\frac{1}{N}\int_0^N S_t^*\mu \; dt\Big)_N$$

où μ est une mesure de probabilité satisfaisant certaines conditions. Après avoir montré que cette suite est tendue, on utilise le théorème de Prokhorov pour en extraire une sous-suite qui converge vers une mesure stationnaire.

Les questions d'unicité d'une telle mesure sont bien plus délicates. Pour l'équation de Navier-Stokes 2D, il n'y a par exemple que des résultats partiels avec des hypothèses plus ou moins restrictives sur la forme de la force (voir [45] et les références qui y sont citées).

1.4 Les différents types de force considérés

Cette thèse traite surtout de l'équation de Burgers périodique généralisée avec un terme aléatoire additif. Il est à noter que, par souci de simplicité, on suppose partout que la condition initiale et le forçage sont de moyenne nulle sur une période spatiale $S^1 = \mathbb{R}/\mathbb{Z}$.

Une première possibilité est de considérer un terme aléatoire η de type « kick », lisse en espace. Concrètement, cela correspond au fait qu'aux moments entiers $t = k$ nous ajoutons une variable aléatoire η_k (k-ième « kick ») à la solution et qu'entre les moments entiers la solution vérifie l'équation sans le terme de force. On suppose que les η_k sont des variables aléatoires i.i.d. dans L_2, à valeurs presque sûrement dans C^∞. De plus, on suppose que leur distribution vérifie des hypothèses de non-trivalité et que leurs moments exponentiels dans les espaces H^m sont finis. Pour plus de précisions, voir Section 3.3.

Lorsqu'on considère des « kicks » de plus en plus rapprochés et de plus en plus petits avec un « scaling » approprié, en utilisant un argument heuristique de type théorème de Donsker on peut voir qu'on tend vers une force de type bruit blanc.

Une force η de type bruit blanc lisse en espace est par définition la dérivée faible d'un processus de Wiener de dimension infinie w. Il s'agit d'un processus de Wiener par rapport à une filtration $\mathcal{F}_t,\ t \geq 0$ défini sur un espace probabiliste complet $(\Omega, \mathcal{F}, \mathbb{P})$ et à valeurs dans H^m pour tout $m \geq 0$. En particulier, pour $\zeta, \chi \in L_2$,

$$\mathbb{E}(\langle w(s), \zeta \rangle \langle w(t), \chi \rangle) = \min(s,t) \langle Q\zeta, \chi \rangle ,$$

où Q est un opérateur symétrique tel que $Q : L_2 \to H^m$ est continue pour tout m. Ainsi, $w(t) \in C^\infty$ pour tout t, p.s. Pour plus de précisions sur les processus de Wiener dans les espaces de Banach, voir [19, Chapitre 4].

1.5 Equation de Burgers

L'équation de Burgers en une dimension :

$$u_t + uu_x = \nu u_{xx}, \tag{1.8}$$

où $\nu > 0$ est un coefficient de viscosité, a été considérée par Forsyth [26] et Bateman [4] dès les premières décennies du XXème siècle. Ici, nous nous limiterons toujours au cas périodique en espace : par un changement d'échelle, on pourra se ramener au cas où x se trouve sur le cercle $S^1 = \mathbb{R}/\mathbb{Z}$. Cette équation admet alors, au sens des distributions, une solution unique dès lors que la condition initiale est dans $L_1(S^1)$. Cela se prouve par des méthodes standard. En effet, nous avons affaire à une équation parabolique dont les solutions vérifient le principe du maximum : on peut par exemple se référer à [39, Chapitre 5].

L'équation de Burgers a acquis une certaine notoriété dans la communauté scientifique autour de 1950. A cette époque, elle a été étudiée par le physicien néerlandais auquel elle doit son nom ([12, 13] ; voir aussi [3]). Son objectif était de considérer une version simplifiée de l'équation de Navier-Stokes incompressible

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u} - \nabla p; \quad \nabla \cdot \mathbf{u} = 0, \tag{1.9}$$

qui garderait quelques-unes des propriétés essentielles de celle-ci. Cet espoir était partagé, notamment, par von Neumann [50, p. 437]. En effet, (1.8) et (1.9) possèdent des termes nonlinéaires et dissipatifs similaires ; la seule différence est l'absence du terme de pression (et de la condition d'incompressibilité, qui n'est pas adaptée au cas unidimensionnel).

En utilisant la transformation de Hopf-Cole [15, 30], on peut ramener l'équation de Burgers à l'équation de la chaleur. En effet, si u est la solution de (1.8) correspondant à une condition initiale u_0, alors $u(t, x)$ est la dérivée en espace de la fonction

$$-2\nu \ln(\phi(t, x)),$$

où ϕ est une solution de l'équation de la chaleur

$$\phi_t = \nu \phi_{xx},$$

correspondant à une condition initiale $\phi_0 = \exp(-H_0/2\nu)$. Ici, H_0 est une primitive de u_0. Pour une présentation plus complète de cette transformation, notamment d'un point de vue historique, voir [8].

Le fait que l'équation de Burgers puisse se ramener ainsi à l'équation de la chaleur signifie qu'elle est intégrable et que ses solutions n'ont pas de comportement chaotique. En effet, une petite perturbation de la condition initiale n'entraîne pas une grande perturbation au niveau du comportement asymptotique. Notons par ailleurs que la transformation de Hopf-Cole fonctionne également pour l'équation de Burgers multidimensionnelle dans le cas potentiel :

$$\mathbf{u}_t + (\mathbf{u} \cdot \nabla)\mathbf{u} = \nu \Delta \mathbf{u}; \quad \mathbf{u} = -\nabla \psi. \tag{1.10}$$

Cependant, le comportement à petite échelle des solutions pour $0 \leq \nu \ll 1$ ne découle pas immédiatement de la transformation de Hopf-Cole. Ainsi, des années 1960 jusqu'à aujourd'hui il y eut de nombreux articles qui en ont traité. On peut notamment citer les travaux de Kraichnan [38], Kida [32], Aurell, Frisch, Lutsko et Vergassola [2] ; voir aussi le livre de Chorin [14]. Dans tous ces travaux, il y a les mêmes conjectures pour le comportement des incréments et du spectre d'énergie dans la zone inertielle, qui correspond à l'intervalle $\mathbb{J}_{inert} = [C\nu, C]$.

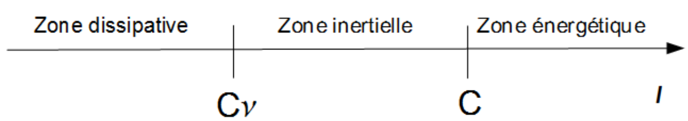

FIGURE 1.3 – Echelles pour l'équation de Burgers

Tout d'abord, si l'on note $S_p(\ell)$ la *fonction de structure* moyennée en espace définie par :

$$S_p(\ell) = \int_{S^1} |u(x+\ell) - u(x)|^p dx,$$

alors pour $\ell \in \mathbb{J}_{inert}$ on a

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, & 0 \leq p \leq 1. \\ \ell, & p \geq 1. \end{cases}$$

Notons que le facteur d'aplatissement $F(\ell)$ est de l'ordre de ℓ^{-1}, ce qui reflète bien le comportement intermittent à petite échelle d'une solution typique.

D'autre part, pour $k^{-1} \in \mathbb{J}_{inert}$, on a $E(k) \sim k^{-2}$.

Détaillons un peu les arguments de l'article [2]. Pour cela, précisons la structure d'une solution de (1.8) pour une condition initiale u_0 lisse « typique » d'amplitude 1 pour la fonction et sa dérivée spatiale.

Considérons d'abord l'équation de Burgers inviscide, appelée aussi équation de Hopf, correspondant au cas $\nu = 0$. La solution est alors lisse seulement pendant un temps fini : on peut la construire implicitement en utilisant la méthode des caractéristiques (voir par exemple [18]). Cette méthode nous dit que tant que la solution est lisse, la valeur de u est constante le long des droites d'équation $(t, x + tu_0(x))$ dans l'espace-temps. Or, lorsque u_0 est non constante, des droites correspondant à des valeurs de u_0 différentes vont se couper après un temps fini. Une solution lisse ne peut alors plus exister. Cependant, on peut définir une solution faible de manière unique dans la classe $BV(S^1)$ (fonctions à variation bornée). L'unicité est assurée par l'introduction de *conditions d'entropie.* Il est à noter que ces solutions sont des limites dans L_1 des solutions classiques de l'équation visqueuse lorsque $\nu \to 0$. Elles présentent des discontinuités (*chocs*) qui sont des sauts négatifs.

Regardons maintenant ce qui se passe pour un temps où la solution de l'équation inviscide n'est plus lisse, lorsque $\nu > 0$. Les régions qui correspondent à des chocs lorsque $\nu = 0$ sont alors des falaises (cliffs). Après un temps fini, l'amplitude de la solution, le nombre de falaises et le dénivelé au niveau de chaque falaise sont de l'ordre de 1. La largeur d'une falaise est de l'ordre de ν.

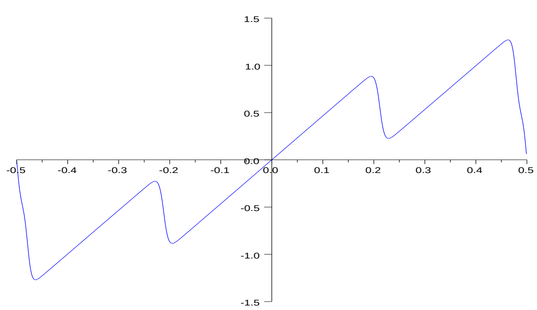

FIGURE 1.4 – Solution « typique » de l'équation de Burgers

Aurell, Frisch, Lutsko et Vergassola remarquent qu'on a 3 possibilités pour $\ell \in \mathbb{J}_{inert}$. Notons que dans ce cas, ℓ est au moins du même ordre que la largeur d'une falaise.

- $[x, x+\ell]$ coupe une grande partie d'une falaise.
  Probabilité $\simeq C\ell$. $|u(x+\ell)-u(x)|^p \overset{p}{\sim} 1$.

- $[x, x+\ell]$ coupe une petite partie d'une falaise.
  Contribution négligeable.

- $[x, x+\ell]$ ne coupe pas une falaise.
  Probabilité $\simeq 1 - C\ell$. $|u(x+\ell)-u(x)|^p \overset{p}{\sim} \ell^p$.

Ainsi, $S_p(\ell) = C(p)(\ell + \ell^p)$. On a bien

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, \ 0 \leq p \leq 1. \\ \ell, \ p \geq 1. \end{cases}$$

Pour finir, il faut mentionner le travail de Biryuk [7]. En utilisant les méthodes employées pour Kuksin pour étudier des équation plus complexes [41, 42], il a obtenu des estimations précises pour les solutions de l'équation de Burgers :

$$\|u(t)\|_m^2 \leq C\nu^{-(2m-1)}, \ \frac{1}{T}\int_0^T \|u(t)\|_m^2 \geq c\nu^{-(2m-1)}, m \geq 1, \nu \leq \nu_0. \quad (1.11)$$

Il est remarquable que les puissances de ν coïncident pour les bornes supérieures et inférieures. Les constantes ν_0, C, c et T dépendent ici non seulement de m et de f, mais aussi de la condition initiale u_0. Les bornes (1.11) impliquent des estimations pour les coefficients de Fourier de u qui confirment que la zone inertielle est bien l'intervalle $[C\nu, C]$.

1.6 Equation de Burgers stochastique

Dès le milieu des années 1980, de nombreux physiciens ont commencé à s'intéresser à des versions aléatoires de l'équation de Burgers, en espérant y trouver un modèle pour la turbulence réelle meilleur que l'équation déterministe (1.8). La première possibilité est de considérer une condition initiale aléatoire. La deuxième serait de rajouter un terme de forçage de type bruit blanc en temps et plus ou moins lisse en espace dans la partie droite. Pour une bibliographie très complète sur le sujet, voir les articles de synthèse [5, 6].

Ici, nous nous intéresserons uniquement au cas où l'on rajoute une force de type bruit blanc en temps et lisse en espace. Dans ce cas, les simulations numériques et les prédictions physiques indiquent qu'on a le même comportement que pour la turbulence de Burgers ou « Burgulence » non forcée, à ceci près qu'il faut considérer les espérances des quantités [29]. Intuitivement, cela s'explique par le fait que le forçage agit à grande échelle, donc dans la zone énergétique, et n'a ainsi qu'une influence indirecte, en tant que source d'énergie, sur les autres zones.

Dans l'article [23], E, Khanin, Mazel et Sinai s'intéressent aux propriétés de la mesure stationnaire dans le cas inviscide. Ils prouvent qu'une telle mesure existe et est unique. Pour cela, ils commencent par regarder l'équation de Hamilton-Jacobi stochastique

$$\phi_t + \frac{1}{2}\phi_x^2 = F \qquad (1.12)$$

qui s'obtient formellement en prenant une primitive en espace de l'équation de Burgers stochastique. Maintenant, notons b la constante

$$b = \int_{S^1} \phi_x(x,t)dx. \qquad (1.13)$$

et définissons pour une courbe $\gamma : [s,t] \mapsto S^1$ l'action associée à celle-ci par :

$$A(\gamma) = \tfrac{1}{2}\int_s^t (\dot{\gamma}(\tau) - b)^2 d\tau - \textstyle\sum_{n\in[s,t)} \tilde{\eta}_n(\gamma(n)),$$

dans le cas « kicked » et

$$\begin{aligned} A(\gamma) = \; & \tfrac{1}{2}\int_s^t (\dot{\gamma}(\tau) - b)^2 d\tau + \int_s^t \dot{\gamma}(\tau)\Big(w(\gamma(\tau),\tau) - w(\gamma(\tau),t)\Big)d\tau \\ & + \tilde{w}(\gamma(s),s) - \tilde{w}(\gamma(s),t) \end{aligned}$$

dans le cas du bruit blanc. Ici, $\tilde{\eta}_n$ et $\tilde{w}$ sont des primitives en espace de η_n et de w, respectivement. On rappelle que w est un processus de Wiener dont η est la dérivée faible en temps. Pour une fonction $\psi : S^1 \to \mathbb{R}$ donnée, un *ψ-minimiseur* $\gamma^x_{s,t,\psi}(\tau)$ minimise $A(\gamma) + \psi(\gamma(s))$ parmi toutes les courbes définies sur $[s,t]$ telles que $\gamma(t) = x$.

Definition 1.6.1. *Pour* $-\infty < r < s \leq t < +\infty$ *et une fonction donnée* $\psi(\cdot, r) : S^1 \to \mathbb{R}$*, on note* $\Omega_{r,s,t,\psi}$ *l'ensemble des points atteints, au temps* s*, par des* ψ*-minimiseurs sur* $[r, t]$ *:*

$$\Omega_{r,s,t,\psi} = \{\gamma^x_{r,t,\psi}(s),\ x \in S^1\}.$$

A partir de maintenant, on fixe ψ, et $\Omega_{s-1,s,t,\psi}$ sera noté $\Omega_{s,t}$.

Pour mesurer la taille d'un sous-ensemble de S^1, on utilise la quantité suivante.

Définition 1.6.2. *Soit* Z *un sous-ensemble fermé de* S^1*. Le diamètre de* Z *est alors par définition la quantité*

$$d(Z) = 1 - m(Z),$$

où $m(Z)$ *est la longueur maximale d'une composante connexe de* $S^1 - Z$*.*

On peut également voir $d(Z)$ comme la longueur minimale d'un intervalle de S^1 contenant Z. On a alors, sous des conditions supplémentaires sur le forçage, les résultats suivants (formulés de façon un peu différente par rapport à l'article original). Les constantes ci-dessous ne dépendent pas de s, t, ψ, mais dépendent de b.

Theorème 1.6.3. *Il existe des constantes* $\lambda, \tilde{C} > 0$ *telles que si* $-\infty < s \leq t < +\infty$*, alors*

$$\mathbb{E}(d(\Omega_{s,t})) \leq \tilde{C} \exp(-\lambda(t-s)).$$

Corollaire 1.6.4. *Il existe,* ω*-p.s., une constante aléatoire* $\tilde{C}(s, \omega) > 0$ *telle que*

$$d(\Omega_{s,t}) \leq \tilde{C}(s, \omega) \exp(-\lambda(t-s)/2), \quad t \geq s.$$

Ici, λ *est le même que dans le Theorème 1.6.3.*

Cette propriété de contraction exponentielle implique l'existence et l'unicité de la mesure stationnaire. De plus, en utilisant la théorie de Pesin, on en déduit l'hyperbolicité des minimiseurs.

E, Khanin, Mazel et Sinai en tirent des informations sur les quantités à petite échelle qui sont cohérentes avec la limite $\nu \to 0$ pour les estimations dans le cas visqueux [22].

Le cas multidimensionnel potentiel a été étudié par Gomes, Iturriaga, Khanin et Padilla [28, 31], qui ont notamment prouvé l'existence et l'unicité d'une mesure stationnaire, ainsi que la convergence des cette mesure pour $\nu \to 0$ vers la mesure stationnaire pour $\nu = 0$ en utilisant un argument conceptuel simple.

1.7 Résultats contenus dans la thèse

1.7.1 Estimations pour les solutions et Burgulence

Dans les Chapitres 2-4, nous considérons l'équation de Burgers généralisée

$$u_t + f'(u)u_x = \nu u_{xx} + \eta, \quad x \in S^1 = \mathbb{R}/\mathbb{Z}. \tag{1.14}$$

Le terme η correspond à un forçage aléatoire, toujours supposé lisse en espace. Le coefficient de viscosité ν est une constante telle que $0 < \nu \ll 1$. La fonction f est supposée fortement convexe (en d'autres mots, $f'' \geq \sigma > 0$) et, dans un certain sens, à croissance modérée. De plus, on suppose que la condition intiale u_0 et le forçage η sont de moyenne nulle sur S^1 : il en est donc de même pour $u(t)$ pour tout temps.

Dans le Chapitre 2, on considère le cas $\eta = 0$: on parle de turbulence de Burgers non forcée ou *en déclin (decaying Burgers turbulence)*. On pose

$$D = \max(|u_0|_1^{-1},\ |u_0|_{1,\infty}). \tag{1.15}$$

Dans les résultats suivants, T_1, T_2 et ν_0 ne dépendent que de f et de D. On note $\{\cdot\}$ la moyenne en temps sur $[T_1, T_2]$.

On commence par obtenir des estimations des normes de Sobolev analogues à celles de Biryuk (voir (1.11)). On rappelle qu'on note $\gamma(m,p) = \max(0, m - 1/p)$.

Theorème 1.7.1. *Pour* $0 \leq m \leq 1$ *et* $p \in [1,\infty]$*, ou pour* $m \geq 2$ *et* $p \in (1,\infty]$*, on a*

$$\Big(\{|u(t)|^{\alpha}_{m,p}\}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \qquad \alpha > 0. \tag{1.16}$$

La borne supérieure dans (1.16) est encore valable lorsqu'on enlève la moyenne en temps, et ceci uniformément pour t separé de 0. Nous avons :

$$|u(t)|_{m,p} \overset{m,p,\alpha}{\lesssim} \max(\nu^{-\gamma}, t^{-\gamma}).$$

D'autre part, la borne inférieure est valable pour tout m, p, α.

Encore une fois, les puissances de ν sont les mêmes pour les bornes supérieures et inférieures des normes de Sobolev. La différence entre ce théorème et les estimations de Biryuk est qu'ici, la période de moyennisation en temps $[T_1, T_2]$ et les constantes implicitement contenues dans le symbole $\sim$ ne dépendent de la condition initiale u_0 qu'à travers la quantité D. De plus, nos estimations sont valables pour $\nu \in (0, 1]$.

Dans un deuxième temps, on obtient des estimations pour les quantités à petite échelle qui confirment exactement les prédictions physiques (voir Section 1.5). Ces estimations sont valables pour $\nu \leq \nu_0(f, D)$. Si l'on pose

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1],$$

on obtient les résultats suivants. Notons que la constante M dans la définition (1.5) ne dépend que de f et de D.

Theorème 1.7.2. *Pour $\ell \in J_1$,*

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p,\ 0 \leq p \leq 1. \\ \ell^p \nu^{-(p-1)},\ p \geq 1. \end{cases}$$

D'autre part, pour $\ell \in J_2$,

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p,\ 0 \leq p \leq 1. \\ \ell,\ p \geq 1. \end{cases}$$

.

Theorème 1.7.3. *Pour $\ell \in J_2$, $E(k) \sim k^{-2}$.*

Par ailleurs, on prouve que J_1, J_2 et J_3 correspondent bien à la zone dissipative, à la zone d'inertie et à la zone énergétique, respectivement.

Dans le Chapitre 3, on considère un terme aléatoire η de type « kick ». On obtient des estimations precises pour les normes de Sobolev et les quantités à petite échelle du même type que dans le chapitre précédent, à ceci près qu'il s'agit d'estimations pour des espérances moyennées en temps. Le temps T_0, le coefficient de viscosité maximal ν_0 et la constante M dans la définition (1.5) ne dépendent que de f et des propriétés statistiques de η. On note toujours

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1].$$

THEORÈME 1.7.4. *Pour $m \in \{0,1\}$ et $p \in [1,\infty]$, ou pour $m \geq 2$ et $p \in (1,\infty]$,*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq T_0 + 2,\ T \geq T_0. \qquad (1.17)$$

De plus, dans les bornes supérieures on peut remplacer la moyenne en temps par un maximum en temps sur $[t, t+1]$ pour $t \geq 2$, c-à-d.

$$\Big(\mathbb{E} \max_{s\in[t,t+1]} |u(s)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\lesssim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq 2.$$

D'autre part, les bornes inférieures sont valables pour toutes les valeurs de m, p, α.

Pour les quantités à petite échelle, on peut prouver les résultats suivants :

THEORÈME 1.7.5. *Pour $\ell \in J_1$,*

$$S_p(\ell) \overset{p}{\gtrsim} \begin{cases} \ell^p,\ 0 \leq p \leq 1. \\ \ell^p \nu^{-(p-1)},\ p \geq 1. \end{cases}$$

D'autre part, pour $\ell \in J_2$,

$$S_p(\ell) \overset{p}{\gtrsim} \begin{cases} \ell^p,\ 0 \leq p \leq 1. \\ \ell,\ p \geq 1. \end{cases}$$

THEORÈME 1.7.6. *Pour $\ell \in J_2$, $E(k) \sim k^{-2}$.*

Ces deux derniers résultats ne sont pas démontrés dans le Chapitre 3. Cependant, leurs preuves seraient mot à mot les mêmes que celles des résultats correspondants dans le Chapitre 4. Notons que dans ce cadre, on peut toujours montrer que les zones pour les échelles de la théorie de la Kolmogorov correspondent bien à nos zones J_1, J_2 et J_3. Notons également que n'avons maintenant aucune dépendance de la condition initiale pour les estimations.

Une force de type « kick » est analogue à une force de type bruit blanc, à ceci près que l'apport et la dissipation pour l'énergie $\frac{1}{2}\int_{S^1} u^2$ ont lieu à des moments de temps distincts et non en parallèle. Ainsi, il n'est pas surprenant de voir que la valeur des quantités à petite échelle pour une force de type « kick » correspond bien aux prédictions physiques pour le cas du bruit blanc (voir Section 1.6).

Dans le Chapitre 4, on considère le cas où η est une force de type bruit blanc en temps. Les estimations pour les normes de Sobolev des solutions et pour les quantités statistiques à petite échelle sont exactement les mêmes que pour une force de type « kick » et sont donc conformes aux prédictions physiques.

1.7.2 Mesure stationnaire et hyperbolicité

Dans les deux dernières sections du Chapitre 4, nous donnons une preuve alternative de l'unicité d'une mesure stationnaire μ_{stat} dans l'espace des mesures de probabilité $\mathcal{P}(L_1)$ pour l'équation (1.14). Il est à noter que ce résultat, bien qu'il soit énoncé uniquement pour le cas du bruit blanc, est tout aussi valable lorsque la force est de type « kick ». L'avantage de notre méthode est que nous obtenons des résultats sur la vitesse de convergence vers μ_{stat} dans la distance Lipschitz-duale $\|\cdot\|_L^*$ correspondant à L_1 (voir Sous-section 4.8.2 pour sa définition).

THEORÈME 1.7.7. *Il existe $\mu_{stat} \in \mathcal{P}(L_1)$ telle que pour toute mesure de probabilité $\mu_0 \in \mathcal{P}(L_1)$ nous avons :*

$$\|S_t^* \mu_0 - \mu_{stat}\|_L^* \leq C t^{-\delta}, \quad t \geq 1.$$

Ici, C et δ ne dépendent ni de ν, ni de μ_0.

Ce résultat d'unicité de la mesure stationnaire implique que toutes les estimations énoncées précédemment sont également valables lorsqu'on remplace la moyenne en temps de l'espérance pour une condition initiale donnée par une intégrale par rapport à $d\mu_{stat}$.

Dans le Chapitre 5, nous considérons l'équation (1.14) dans le cas inviscide ($\nu = 0$) et avec une force de type « kick » ou bruit blanc en temps. Par souci de simplicité, nous nous restreignons au cas de la nonlinéarité classique ($f'(u) = u$). Nous nous restreignons également au cas où la force n'admet qu'un nombre fini de degrés de liberté.

On suppose qu'on a des conditions de non-dégénérescence sur les modes de forçage $(F^k)_{1 \leq k \leq K}$ de même nature que dans [23]. Elles sont satisfaites par exemple par le couple $(\sin x, \cos x)$, mais ne sont jamais satisfaites par un seul potentiel. De façon heuristique, on peut dire qu'elles servent entre autres à garantir une absence de symétrie suffisante pour que 1 soit la période spatiale minimale des solutions.

CONDITIONS 1.7.8. *Pour une force de type « kick », on suppose que :*
(i) le j-ième « kick » s'écrit

$$\eta^\omega(j) = \sum_{k=1}^{K} c_k^\omega(j) F^k,$$

où les F^k sont des fonctions lisses sur S^1 et $(c_k^\omega(j))_{1 \leq k \leq K}$ des variables aléatoires i.i.d. sur $\mathbb{R}^K$. De plus, leur distribution commune sur $\mathbb{R}^K$, notée μ, est absolument continue par rapport à la mesure de Lebesgue.
(ii) On a $0 \in Supp\ \mu$.
(iii) L'application de S^1 dans $\mathbb{R}^K$ donnée par

$$x \mapsto (F^1(x), ..., F^K(x))$$

est un plongement.

CONDITIONS 1.7.9. *Pour une force de type bruit blanc, on suppose que :*
(i) La force s'écrit

$$\sum_{k=1}^{K} \dot{W}_k^\omega(t) F^k(x),$$

où les F^k sont des fonctions lisses sur S^1, et pour $1 \leq k \leq K$ les $\dot{W}_k^\omega(t)$ sont des bruits blancs indépendants (dérivées faibles de processus de Wiener indépendants).
(ii) L'application de S^1 dans $\mathbb{R}^K$ donnée par

$$x \mapsto (F^1(x), ..., F^K(x))$$

est un plongement.

Nous donnons alors une preuve du Théorème 1.6.3 beaucoup plus simple que celle présentée dans [23].

Il est à noter que l'hyperbolicité des minimiseurs, qui résulte du Corollaire 1.6.4, semble suggérer une convergence exponentielle vers la mesure stationnaire pour l'équation de Burgers inviscide. Or, un passage à la limite lorsque ν tend vers 0 dans le Théorème 1.7.7 ne donne qu'une convergence en puissance négative de t.

1.8 Méthodes utilisées

1.8.1 Estimations pour les solutions et Burgulence

Dans cette sous-section, nous ne considérons, par souci de simplicité, que le cas de la nonlinéarité pour l'équation de Burgers classique uu_x. Nous nous restreignons également, dans la quasi-totalité de la sous-section, au cas où η est une force de type bruit blanc.

L'étude des normes de Sobolev pour les équations aux dérivées partielles d'évolution se divise en deux parties.

1) Les bornes supérieures sont surtout obtenues en utilisant le principe du maximum et/ou des relations de dissipation.

2) Pour avoir les bornes inférieures, il faut travailler davantage. Souvent, elles sont obtenues en intégrant en temps des relations de dissipation, ce qui donne des bornes sur les moyennes temporelles.

Ce schéma est encore valable ici malgré le fait que nous avons affaire à une équation stochastique. En effet, d'une part, les forces avec lesquelles nous travaillons sont typiquement lisses en espace et sont faciles à estimer, ce qui rend possible l'étape 1. D'autre part, les relations de dissipation découlent de la formule d'Itô pour le cas du bruit blanc et de considérations plus élémentaires, que l'on peut voir comme une formule d'Itô discrète, pour le cas d'une force de type « kick ». Pour une présentation de ces arguments dans un cadre plus général, voir [41, 42].

La borne supérieure cruciale est ici celle sur la partie positive de u_x. En effet, u_x vérifie l'équation

$$(u_x)_t + (u_x)^2 + u(u_x)_x = \nu(u_x)_{xx} + (w_x)_t.$$

Si l'on essaye d'appliquer le principe du maximum à la fonction $v = (u - w)_x$, on obtient des bornes supérieures par des arguments classiques, en considérant un point où v atteint son maximum. Ces bornes supérieures ont deux défauts notables :

- On a une forte dépendance par rapport à la condition initiale.

- Dans la mesure où ce sont des estimations sur $u - w$ et non sur u elle-même, elles deviennent moins utiles pour t grand.

Cependant, la présence du terme $(u_x)^2$ nous suggère qu'on a affaire à une quantité admettant une estimation en $1/t$. Ainsi, on applique le principe du maximum à tv. Cela nous permet d'avoir, pour $t \geq T_0 > 0$, des estimations sur les moments de $\max_x (u_x(t,x))^+$ qui sont valables uniformément en temps.

Les bornes supérieures pour les moments des normes de Sobolev H^m sont ensuite obtenues en utilisant des relations de dissipation suivantes :

$$\frac{d}{dt}\mathbb{E}\,\|u(t)\|_m^2 = I_m - 2\nu\mathbb{E}\,\|u(t)\|_{m+1}^2 - \mathbb{E}\;N_m(u(t)), \quad m \geq 1. \tag{1.18}$$

On estime la nonlinéarlité $N_m(u(t)) = \langle u^{(m)}, (uu_x)^{(m)} \rangle$ en utilisant l'inégalité de Gagliardo-Nirenberg (Lemme 1.1.1).

Ensuite, il reste à traiter les bornes inférieures. En utilisant encore une fois le Lemme 1.1.1, on prouve que comme on a déjà une borne supérieure pour les moments de $\max_x (u_x(t,x))^+$, et donc de $|u(t)|_{1,1}$, il suffit d'obtenir une borne inférieure pour $\|u\|_1$. Par la formule d'Itô on a

$$\begin{aligned}\langle |u(t+T_0)|^2 \rangle - \langle |u(t)|^2 \rangle &= -2\nu \int_t^{t+T_0} \langle \|u\|_1^2 \rangle + CT_0 \\ &= -2\nu T_0 \{\|u\|_1^2\} + CT_0.\end{aligned}$$

Or, on sait que pour $t \geq 1$:

$$\langle |u(t+T_0)|^2 \rangle \leq \langle (\max_x u_x(t+T_0,x))^2 \rangle \leq C.$$

Donc pour T_0 assez grand :

$$\{\|u\|_1^2\} \geq \frac{CT_0 - C}{2T_0}\nu^{-1} \geq C\nu^{-1}.$$

Les bornes supérieures sur les quantités à petite échelle découlent immédiatement des estimations sur les normes de Sobolev. Pour obtenir des bornes inférieures valables pour ν assez petit, on utilise les arguments de l'article [2], présentés dans la Sous-section 1.5. Pour montrer que ces arguments peuvent être appliqués rigoureusement dans notre cas, il suffit de prouver qu'avec une grande probabilité et sur un intervalle de temps assez grand, une solution est « typique ». Cela signifie que les quantités $\max u$, $\max u_x$ et $\min u_x$ sont, respectivement, du même ordre que 1, 1 et ν^{-1}, et $\max u_{xx}$ est au plus du même ordre que ν^{-2}.

Pour finir, disons quelques mots sur le cas $\eta = 0$, traité dans le Chapitre 2. Pour les bornes supérieures et la dérivation des quantités à petite échelle, on procède de même que pour l'équation de Burgers stochastique. Le seul point délicat est finalement l'obtention des bornes inférieures. Comme l'équation de Burgers non forcée est purement dissipative, on ne peut espérer une estimation uniforme par rapport à la condition initiale. Cela justifie l'introduction de la quantité D (voir (1.15)).

1.8.2 Mesure stationnaire et hyperbolicité

L'existence d'une mesure stationnaire dans L_1 est prouvée de façon standard, nos estimations étant largement suffisantes pour appliquer la méthode de Krylov-Bogolyubov (voir Section 1.3).

Pour prouver l'unicité de cette mesure et donner une estimation de la vitesse de convergence vers celle-ci, on utilise un argument qui est essentiellement une version simplifiée du Théorème 3.1.3. de [45]. Tout d'abord, on remarque que pour deux conditions initiales différentes, la distance entre les solutions correspondantes dans L_1 pour la même valeur de la force est une fonction décroissante du temps. Cette preuve utilise une version très légèrement modifiée du lemme de Crandall-Tartar [17]. Ensuite, il s'agit de prouver qu'avec une probabilité tendant vers 1 suffisamment vite, on peut rendre cette distance petite. Or, pour cela il suffit que la force η soit petite pendant assez longtemps.

Dans le Chapitre 5, la principale difficulté technique consiste à montrer le lemme suivant :

Lemme 1.8.1. *Il existe des constantes $c, T > 0$ telles que si $-\infty < s \leq t < +\infty$, alors on a p.s. l'inégalité suivante :*

$$\mathbb{P}\Big(d(\Omega_{s,t+T}) \leq d(\Omega_{s,t})/2 \mid \mathcal{F}_t\Big) \geq c,$$

où $\mathcal{F}_t$ est la σ-algèbre qui correspond au passé (comportement des v.a. avant le moment t).

En effet, comme $t \mapsto d(\Omega_{s,t})$ est une fonction décroissante, ce lemme implique la propriété de contraction exponentielle (Théorème 1.6.3).

Pour expliquer la démonstration du théorème, il faut donner quelques précisions sur la structure des minimiseurs, afin de le reformuler.

Pour $s < t$, définissons l'application S_s^t de S^1 dans lui-même, qui peut être vue comme une projection, au temps t, du flot Lagrangien generalisé correspondant à l'équation de Burgers. Ce flot dépend en particulier de la condition initiale ψ au temps $s-1$.

Si, au temps s, un point y est atteint par un ψ-minimiseur sur $[s-1,t]$ qui commence en x au temps t, alors $S_s^t(y)$ est le point x. Un tel point est unique, car les minimiseurs sur $[s-1,t]$ ne peuvent se couper en des temps autres que $s-1$ et t. Si un point y n'est pas atteint par un ψ-minimiseur, alors il appartient à un intervalle fermé correspondant à un choc au temps t. Alors, $S_s^t(y)$ est la position du choc correspondant. On peut vérifier que cette application est définie de façon unique pour tout y.

Il suffit donc de trouver des conditions sur le forçage impliquant que si $\Omega_{s,t}$ est de diamètre d, alors il existe un choc x tel que $(S_s^{t+T})^{-1}(x)$ couvre un sous-ensemble de $\Omega_{s,t}$ de diamètre au moins $d/2$. De telles conditions sont données dans la Section 5.3.

1.9 Perspectives de recherche

Un premier objectif serait d'obtenir des bornes inférieures sans la moyenne en temps pour les normes de Sobolev dans le cas général.

Il est aussi naturel d'étudier des solutions de l'équation de Burgers dans plusieurs dimensions dans le cas potentiel (1.10), où les résultats physiques [6] semblent indiquer le même comportement que dans le cas unidimensionnel. Il s'agit d'une question très intéressante, notamment à cause de ses applications en cosmologie. En effet, cette équation décrit le modèle de « poussière collante » (« sticky dust ») proposé par Shandarin et Zeldovich pour expliquer certaines étapes de l'évolution de l'univers [57].

On peut également modifier la nature du terme de forçage. Les forces de type bruit blanc en temps fournissent un cadre très répandu en physique mathématique. En effet, une telle force est physiquement acceptable car des arguments de type théorème de Donsker prouvent qu'il s'agit d'une limite pour de nombreux processus. Il s'agit également d'un objet commode d'un point de vue mathématique car on peut utiliser le calcul d'Itô. Cependant, il ne s'agit pas du modèle le plus réaliste d'un point de vue physique. En effet, les processus réels n'ont pas une corrélation nulle pour des temps différents mais très proches.

Ainsi, il serait souhaitable de considérer une autre classe de termes de forçage, avec des corrélations qui ne sont pas nulles pour des temps proches. Cela implique des difficultés techniques majeures, car les relations de type bilan d'énergie ne pourront plus s'écrire aussi facilement que pour une force de type bruit blanc en temps.

Une autre direction possible serait d'améliorer les résultats de convergence vers la mesure stationnaire (voir Sous-section 1.7.2). Il s'agirait de voir si la vitesse de convergence vers la mesure stationnaire est exponentielle, uniformément pour $\nu \to 0$.

Il s'agira également de généraliser nos résultats en sortant du cadre des conditions aux limites périodiques en x. On pourra par exemple s'intéresser au cas où la condition initiale est une variable aléatoire homogène en espace.

Finalement, il s'agira d'appliquer nos méthodes à des équations autres que l'équation de Burgers : un candidat naturel serait l'équation CGL (de Ginzburg-Landau complexe) : voir à ce sujet l'article [44].

Chapitre 2

Turbulence en déclin pour l'équation de Burgers généralisée

Ce chapitre correspond à la prépublication *Note on Decaying Turbulence in a Generalised Burgers Equation.*(arXiv :1208.5241).

Abstract. We consider a generalised Burgers equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0,\ t \geq 0,\ x \in S^1, \tag{2.1}$$

where f is strongly convex and ν is small and positive.

We obtain sharp upper and lower bounds for time-averaged Sobolev norms of solutions u of (2.1). These results yield estimates for dissipation length scale, small-scale increments and energy spectrum for those solutions, which characterise the turbulence in the Burgers equation.

This article extends the previous work by Biryuk, where very similar bounds have been obtained. The major difference is that here, we get sharp estimates for increments and energy spectrum, using some ideas of Aurell-Frisch-Lutsko-Vergassola.

2.1 Introduction

The Burgers equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \tag{2.2}$$

where $\nu > 0$ is a constant is, in some way, the most natural model for the Navier-Stokes equation. Indeed, both equations have similar nonlinearities and dissipative terms. However, the equation (2.2) can be integrated explicitly using the Hopf-Cole transformation. Nevertheless, for $\nu \ll 1$, these solutions display non-trivial small-scale behaviour, often referred to as Burgers turbulence or "Burgulence" [5, 6, 13].

In this paper, we consider a generalised one-dimensional space-periodic Burgers equation

$$\frac{\partial u}{\partial t} + \frac{df(u)}{dx} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad x \in S^1 = \mathbb{R}/\mathbb{Z} \tag{2.3}$$

with f C^∞-smooth and strongly convex, i.e. satisfying

$$f''(y) \geq \sigma > 0, \quad y \in \mathbb{R}. \tag{2.4}$$

The classical Burgers equation (2.2) corresponds to $f(u) = u^2/2$. For simplicity, we only consider solutions with zero space average:

$$\int_{S^1} u(t,x)dx = 0, \quad \forall t \geq 0. \tag{2.5}$$

In the paper [7], Biryuk studied norms in space of solutions u for small values of ν and obtained the following estimates for L_2 norms of the m-th derivatives:

$$\|u(t)\|_m^2 \leq C\nu^{-(2m-1)}, \ \frac{1}{T}\int_0^T \|u(t)\|_m^2 \geq c\nu^{-(2m-1)}, m \geq 1, \nu \leq \nu_0. \tag{2.6}$$

These estimates are sharp, in the sense that exponents for ν in lower and upper bounds are the same for each m. The constants ν_0, C, c, and T depend on the initial condition u_0 as well as on m and f. In [9, 10], we obtain similar results which are independent of the initial data, with the same exponents for ν as in (2.6). However, in both articles, we add in the right-hand side of

(2.3) a rough in time and smooth in space random forcing term (a "kicked" and a white force, respectively). Thus, we change the nature of the equation: the energy injection due to the random forcing now balances the dissipation due to the term $\nu\partial^2 u/\partial x^2$.

In this paper, instead of introducing a random forcing, we make some mild assumptions on Sobolev norms of the initial data. They suffice to obtain estimates very similar to the ones in [7, 9, 10], with the same quantities estimated by the same powers of ν.

Almost all proofs in this paper are either very similar to those in [7] (Section 2.4) or to those in [10] (Section 2.5). However, we include most of them, in order to make the article as self-contained as possible. In a way, this paper combines "light" versions of sharp u_0-dependent estimates in [7] and deterministic versions of estimates in [10]. Note that results of this type (but with different exponents for lower and upper bounds) have been pioneered using the same methods as here by Kuksin for more complex equations such as the non-linear Schrödinger and Complex Ginzburg-Landau equations [41, 42].

After introducing the notation and setup in Section 2.2, we formulate the main results in Section 2.3. In Section 2.4, we begin by estimating from above the quantity $\partial u/\partial x$. This crucial bound allows us to obtain upper bounds, as well as time-averaged lower bounds, of Sobolev norms $|u|_{m,p}$. Those bounds depend only on f and the constant

$$D = \max(|u_0|_1^{-1},\ |u_0|_{1,\infty}).$$

In Section 2.5, we give sharp upper and lower bounds for dissipation length scale, increments, flatness, and energy spectrum for the flow $u(t,x)$, which hold uniformly for $\nu \leq \nu_0$, and analyse the meaning of these results in terms of the theory of turbulence. Those bounds, as well as $\nu_0 > 0$ itself, still only depend on f and D.

These results rigorously justify classical predictions for small-scale statistical quantities for decaying Burgers turbulence [2, 14, 22, 32, 38]. In the proof of Lemma 2.5.8, we use an argument put forward by Aurell, Frisch, Lutsko, and Vergassola in [2]. When studying the typical behaviour of PDE solutions, one usually considers some averaging in the initial condition. The idea is to avoid pathological cases, since there is no random mechanism to get solutions out of "bad" regions. Here, no averaging in the initial condition is necessary. This is due to the particular structure of the deterministic

Burgers equation: an initial condition u_0 is "generic" if the Lebesgue norms of u_0 and of its space derivative both are of order 1.

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2.2 Notation and Setup

Agreement: In the whole paper, all functions that we consider are real-valued.

2.2.1 Sobolev Spaces

Consider a zero mean value integrable function v on S^1. For $p \in [1, \infty]$, we denote its L_p norm by $|v|_p$. The L_2 norm is denoted by $|v|$, and $\langle \cdot, \cdot \rangle$ stands for the L_2 scalar product. From now on L_p, $p \in [1, \infty]$ denotes the space of zero mean value functions in $L_p(S^1)$. Similarly, C^∞ is the space of C^∞-smooth zero mean value functions on S^1.

For a nonnegative integer m and $p \in [1, \infty]$, $W^{m,p}$ stands for the Sobolev space of zero mean value functions v on S^1 with finite norm

$$|v|_{m,p} = \left| \frac{d^m v}{dx^m} \right|_p .$$

In particular, $W^{0,p} = L_p$ for $p \in [1, \infty]$. For $p = 2$, we denote $W^{m,2}$ by H^m, and abbreviate the corresponding norm as $\|v\|_m$.

Note that since the length of S^1 is 1 and the mean value of v vanishes, we have

$$|v|_1 \leq |v|_\infty \leq |v|_{1,1} \leq |v|_{1,\infty} \leq \cdots \leq |v|_{m,1} \leq |v|_{m,\infty} \leq \ldots$$

We recall a version of the classical Gagliardo-Nirenberg inequality (see [21, Appendix]):

Lemma 2.2.1. *For a smooth zero mean value function v on S^1,*

$$|v|_{\beta,r} \le C\,|v|_{m,p}^{\theta}\,|v|_q^{1-\theta},$$

where $m > \beta$, and r is defined by

$$\frac{1}{r} = \beta - \theta\Big(m - \frac{1}{p}\Big) + (1-\theta)\frac{1}{q},$$

under the assumption $\theta = \beta/m$ if $p = 1$ or $p = \infty$, and $\beta/m \le \theta < 1$ otherwise. The constant C depends on m, p, q, β, θ.

For any $s \ge 0$, H^s stands for the Sobolev space of zero mean value functions v on S^1 with finite norm

$$\|v\|_s = (2\pi)^s\Big(\sum_{k\in\mathbb{Z}} |k|^{2s}|\hat{v}_k|^2\Big)^{1/2},$$

where $\hat{v}_k$ are the complex Fourier coefficients of $v(x)$. For integer values of $s = m$, this norm coincides with the previously defined H^m norm. For $s \in (0,1)$, $\|v\|_s$ is equivalent to the norm

$$\|v\|_s' = \Bigg(\int_{S^1}\Big(\int_0^1 \frac{|v(x+\ell) - v(x)|^2}{\ell^{2s+1}}d\ell\Big)dx\Bigg)^{1/2}$$

(see [1, 58]).

Subindices t and x, which can be repeated, denote partial differentiation with respect to the corresponding variables. We denote by $v^{(m)}$ the m-th derivative of v in the variable x. For brevity, the function $v(t,\cdot)$ is denoted by $v(t)$.

2.2.2 Preliminaries

In this paper, we study asymptotical properties of solutions to (2.3) for small values of ν, i.e. we suppose that

$$0 < \nu \le 1.$$

We assume that f is C^∞-smooth and satisfies (2.4). We recall that we restrict ourselves to the zero space average case: so the initial condition $u_0 := u(0,\cdot)$

satisfies (2.5). Consequently, $u(t)$ satisfies (2.5) for all t. Finally, we assume that $u_0 \in C^\infty$, and we denote by D the quantity

$$D = \max(|u_0|_1^{-1},\ |u_0|_{1,\infty}). \tag{2.7}$$

In particular, we assume that we are not in the case $u_0 \equiv 0$, corresponding to the trivial solution $u(t,x) \equiv 0$. Note that $D \geq 1$. Moreover, for $0 \leq m \leq 1$ and $1 \leq p \leq \infty$, we have

$$D^{-1} \leq |u_0|_{m,p} \leq D. \tag{2.8}$$

Existence, uniqueness, and smoothness of solutions to (2.3) is proved by standard arguments (see for instance [39]).

Agreements: From now on, all constants denoted by C with sub- or superindexes are strictly positive. Unless otherwise stated, they depend only on f and the constant D. By $C(a_1, \dots, a_k)$ we denote constants which also depend on parameters $a_1, \dots, a_k$. By $X \overset{a_1,\dots,a_k}{\lesssim} Y$ we mean that $X \leq C(a_1, \dots, a_k)Y$. The notation $X \overset{a_1,\dots,a_k}{\sim} Y$ stands for

$$Y \overset{a_1,\dots,a_k}{\lesssim} X \overset{a_1,\dots,a_k}{\lesssim} Y.$$

In particular, $X \lesssim Y$ and $X \sim Y$ mean that $X \leq CY$ and $C^{-1}Y \leq X \leq CY$, respectively.

All constants are independent of the viscosity ν. We denote by $u = u(t,x)$ a solution of (2.3) for an initial condition u_0. A relation where the admissible values of t (respectively, x) are not specified is assumed to hold for all $t \geq 0$ or $t > 0$, depending on the context (respectively, all $x \in S^1$).

The brackets $\{\cdot\}$ stand for averaging in time over an interval $[T_1, T_2]$, where T_1, T_2 only depend on f and D (see (2.20) for their definition.)

We use the notation $g^- = \max(-g, 0)$ and $g^+ = \max(g, 0)$.

2.3 Formulation of the Main Results

In Section 2.4, we prove sharp upper and lower bounds for a large class of Sobolev norms of u. The key estimate is obtained in Lemma 2.4.1: we show there that

$$u_x(t,x) \leq \min(D, \sigma^{-1}t^{-1}). \tag{2.9}$$

The main results are summed up in Theorem 2.4.8. Namely, for $0 \leq m \leq 1$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$ we have

$$\Big(\{|u(t)|^{\alpha}_{m,p}\}\Big)^{1/\alpha} \overset{m,p}{\sim} \nu^{-\gamma}, \quad \alpha > 0, \tag{2.10}$$

where $\gamma = \max(0, m - 1/p)$, and $\{\cdot\}$ denotes averaging in time over the interval $[T_1, T_2]$ defined by (2.20).

In Section 2.5 we obtain sharp estimates for analogues of the quantities characterising the hydrodynamical turbulence. In what follows, we assume that $\nu \in (0, \nu_0]$, where $\nu_0 \in (0,1]$ only depends on f and D.

To begin with, we define the non-empty and non-intersecting intervals

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1],$$

corresponding to the *dissipation range*, the *inertial range*, and the *energy range* from the Kolmogorov 1941 theory of turbulence, respectively [27].

Then we consider the averaged moments of increments in space for the flow $u(t,x)$:

$$S_p(\ell) = \left\{ \int_{S^1} |u(t,x+\ell) - u(t,x)|^p dx \right\},\ 0 < \ell \leq 1.$$

The quantity $S_p(\ell)$ is (up to averaging) the *structure function* of p-th order. As a consequence of (2.9-2.10), in Theorem 2.5.9 we prove that for $\ell \in J_1$:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p,\ 0 \leq p \leq 1. \\ \ell^p \nu^{-(p-1)},\ p \geq 1, \end{cases}$$

and for $\ell \in J_2$:

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p,\ 0 \leq p \leq 1. \\ \ell,\ p \geq 1. \end{cases}$$

Consequently, for $\ell \in J_2$ the flatness satisfies:

$$F(\ell) := S_4(\ell)/S_2^2(\ell) \sim \ell^{-1}.$$

Thus, u is highly intermittent in the inertial range (cf. [27]).

Finally, (2.9-2.10) yield estimates for the spectral asymptotics of Burgulence. On one hand, for $s \geq 1$ we have

$$\{|\hat{u}_k|^2\} \overset{s}{\lesssim} (k\nu)^{-2s}\nu.$$

In particular, $\{|\hat{u}_k|^2\}$ decreases at a faster-than-algebraic rate for $|k| \succeq \nu^{-1}$. On the other hand, for k such that $k^{-1} \in J_2$ the *energy spectrum* $E(k)$ satisfies

$$E(k) = \left\{ \frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}_n|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right\} \sim k^{-2},$$

where $M \geq 1$ is a constant depending only on f and D.

2.4 Estimates for Sobolev Norms

2.4.1 Upper Estimates

We begin by proving a key upper estimate for u_x.

LEMMA 2.4.1. *We have*

$$u_x(t,x) \leq \min(D, \sigma^{-1} t^{-1}).$$

Proof. Differentiating the equation (2.3) once in space we get

$$(u_x)_t + f''(u) u_x^2 + f'(u)(u_x)_x = \nu (u_x)_{xx}.$$

Now consider a strictly positive point of maximum (t_1, x_1) for u_x on the cylinder $S = [0,t] \times S^1$, such that $t_1 > 0$. At such a point, we would have $(u_x)_t \geq 0$, $(u_x)_x = 0$, and $(u_x)_{xx} \leq 0$. Consequently, by (2.4) we get

$$\sigma u_x^2 \leq f''(u) u_x^2 \leq 0,$$

which is impossible. Thus u_x can only reach a strictly positive maximum on S for $t_1 = 0$. In other words, we have

$$u_x(t,x) \leq \max_{x \in S^1} (u_0)_x(x) \leq D.$$

The inequality

$$u_x(t,x) \leq \sigma^{-1} t^{-1}$$

is proved in by a similar maximum principle argument applied to the function $t u_x$ (cf. [40]). □

Since the space averages of $u(t)$ and $u_x(t)$ vanish, we get

$$|u(t)|_p \leq \min(D, \sigma^{-1}t^{-1}), \quad p \in [1, \infty]. \tag{2.11}$$

$$|u(t)|_{1,1} = 2\int_{S^1} u_x^+(t) \leq 2\min(D, \sigma^{-1}t^{-1}). \tag{2.12}$$

Now we recall a standard estimate for the nonlinearity $\langle v^{(m)}, (f(v))^{(m+1)} \rangle$. For its proof, we refer to [10].

Lemma 2.4.2. *For $v \in C^\infty$ such that $|v|_\infty \leq A$, we have*

$$\left|\langle v^{(m)}, (f(v))^{(m+1)} \rangle\right| \leq \tilde{C} \|v\|_m \|v\|_{m+1}, \quad m \geq 1,$$

where $\tilde{C}$ depends only on m, A, and $|f|_{C^m([-A,A])}$.

Lemma 2.4.3. *We have*

$$\|u(t)\|_1^2 \lesssim \nu^{-1}.$$

On the other hand, for $m \geq 2$,

$$\|u(t)\|_m^2 \overset{m}{\lesssim} \max(\nu^{-(2m-1)}, t^{-(2m-1)}).$$

Proof. Fix $m \geq 1$. Denote

$$x(t) = \|u(t)\|_m^2.$$

We claim that the following implication holds:

$$x(t) \geq C'\nu^{-(2m-1)} \Longrightarrow \frac{d}{dt}x(t) \leq -(2m-1)x(t)^{2m/(2m-1)}, \tag{2.13}$$

where C' is a fixed strictly positive number, chosen later. Below, all constants denoted by C do not depend on C'.

Indeed, assume that $x(t) \geq C'\nu^{-(2m-1)}$. Integrating by parts in space and using (2.11) and Lemma 2.4.2, we get

$$\begin{aligned}\frac{d}{dt}x(t) &= -2\nu \|u(t)\|_{m+1}^2 - 2\langle u^{(m)}(t), (f(u(t)))^{(m+1)} \rangle \\ &\leq -2\nu \|u(t)\|_{m+1}^2 + C\|u(t)\|_m \|u(t)\|_{m+1}.\end{aligned} \tag{2.14}$$

Applying Lemma 2.2.1 to u_x and then using (2.12), we get

$$\begin{aligned}\|u(t)\|_m &\le C\,\|u(t)\|_{m+1}^{(2m-1)/(2m+1)}\,|u(t)|_{1,1}^{2/(2m+1)}\\ &\le C\,\|u(t)\|_{m+1}^{(2m-1)/(2m+1)}\,. \end{aligned} \tag{2.15}$$

Thus, we have the relation

$$\frac{d}{dt}x(t) \le (-2\nu\,\|u(t)\|_{m+1}^{2/(2m+1)} + C)\,\|u(t)\|_{m+1}^{4m/(2m+1)}\,. \tag{2.16}$$

The inequality (2.15) yields

$$\|u(t)\|_{m+1}^{2/(2m+1)} \ge Cx(t)^{1/(2m-1)}, \tag{2.17}$$

and then since $x(t) \ge C'\nu^{-(2m-1)}$ we get

$$\|u(t)\|_{m+1}^{2/(2m+1)} \ge CC'^{1/(2m-1)}\nu^{-1}. \tag{2.18}$$

Combining the inequalities (2.16-2.18), for C' large enough we get

$$\frac{d}{dt}x(t) \le (-CC'^{1/(2m-1)} + C)x(t)^{2m/(2m-1)}.$$

Thus we can choose C' in such a way that (2.13) holds.

For $m = 1$, (2.8) and (2.13) immediately yield that

$$x(t) \le \max(C'\nu^{-1}, D^2) \le \max(C', D^2)\nu^{-1},\ t \ge 0.$$

Now consider the case $m \ge 2$. We claim that

$$x(t) \le \max(C'\nu^{-(2m-1)}, t^{-(2m-1)}). \tag{2.19}$$

Indeed, if $x(s) \le C'\nu^{-(2m-1)}$ for some $s \in [0,t]$, then the assertion (2.13) ensures that $x(s)$ remains below this threshold up to time t.

Now, assume that $x(s) > C'\nu^{-(2m-1)}$ for all $s \in [0,t]$. Denote

$$\tilde{x}(s) = (x(s))^{-1/(2m-1)},\ s \in [0,t]\,.$$

By (2.13) we get $d\tilde{x}(s)/ds \ge 1$. Therefore $\tilde{x}(t) \ge t$, and $x(t) \le t^{-(2m-1)}$. Thus in this case, inequality (2.19) still holds. This proves the lemma's assertion. $\square$

Denote $\gamma = \max(0, m - 1/p)$.

LEMMA 2.4.4. *For $0 \le m \le 1$ and $p \in [1,\infty]$, or for $m \ge 2$ and $p \in (1,\infty]$,*

$$|u(t)|_{m,p}^2 \overset{m,p}{\lesssim} \max(\nu^{-2\gamma}, t^{-2\gamma}), \quad t \ge 0.$$

Proof. The case $m = 0$ follows from (2.11).

For $m \ge 1$ and $p \in [2,\infty]$, we interpolate $|u(t)|_{m,p}$ between $\|u(t)\|_m$ and $\|u(t)\|_{m+1}$. By Lemma 2.2.1 we have

$$|u(t)|_{m,p}^2 \overset{p}{\lesssim} (\|u(t)\|_m^2)^{1-\theta}(\|u(t)\|_{m+1}^2)^{\theta},\ \theta = \frac{1}{2} - \frac{1}{p}.$$

Then we use Lemma 2.4.3 and Hölder's inequality to complete the proof.

To prove the case $m = 1,\ p \in [1,2]$, we use the same method, combining (2.12) and the estimate for $\|u\|_1^2$ in Lemma 2.4.3. We also proceed similarly for $m \ge 2,\ p \in (1,2)$, combining (2.12) and the estimate for $|u|_{M,p}$ for a large value of M and some $p \ge 2$. □

Unfortunately, the proof of this theorem cannot be adapted to the case $m \ge 2$ and $p = 1$. Indeed, Lemma 2.2.1 only allows us to estimate a $W^{m,1}$ norm from above by other $W^{m,1}$ norms: we can only get that

$$|u|_{m,1} \overset{m,n,k}{\lesssim} |u|_{n,1}^{(m-k)/(n-k)} |u|_{k,1}^{(n-m)/(n-k)},\ 0 \le k < m < n,$$

and thus the upper estimates obtained above cannot be used. However, $|u|_{m,1} \le |u|_{m,1+\beta}$ for any $\beta > 0$. Consequently, the theorem's statement holds for $m \ge 2$ and $p = 1$, with γ replaced by $\gamma + \delta$, and $\overset{m,p,\alpha}{\lesssim}$ replaced by $\overset{m,p,\alpha,\delta}{\lesssim}$, for any $\delta > 0$.

2.4.2 Lower Estimates

Now we define

$$T_1 = \frac{1}{4} D^{-2} \tilde{C}^{-1}; \quad T_2 = \max\Big(\frac{3}{2} T_1, \quad 2D\sigma^{-1}\Big), \tag{2.20}$$

where $\tilde{C}$ is a constant such that for all t, $\|u(t)\|_1^2 \le \tilde{C}\nu^{-1}$ (see Lemma 2.4.3). From now on, $\{A(t)\}$ is by definition the time average

$$\{A(t)\} = \frac{1}{T_2 - T_1} \int_{T_1}^{T_2} A(t).$$

The first quantity that we estimate from below is $\{\|u(t)\|_1^2\}$.

Lemma 2.4.5. *We have*

$$\{\|u(t)\|_1^2\} \gtrsim \nu^{-1}.$$

Proof. Integrating by parts in space, we get the dissipation identity

$$\frac{d}{dt}|u(t)|^2 = \int_{S^1} (-2uf'(u)u_x + 2\nu u u_{xx}) = -2\nu \|u(t)\|_1^2. \tag{2.21}$$

Thus, integrating in time and using (2.7) and Lemma 2.4.3, we get

$$\begin{aligned} |u(T_1)|^2 &= |u_0|^2 - 2\nu \int_0^{T_1} \|u(t)\|_1^2 \\ &\geq D^{-2} - 2T_1\tilde{C} \geq \frac{1}{2}D^{-2}. \end{aligned}$$

Consequently, integrating (2.21) in time and using (2.11) for $p = 2$ we obtain that

$$\begin{aligned} \{\|u(t)\|_1^2\} &= \frac{1}{2\nu(T_2 - T_1)}(|u(T_1)|^2 - |u(T_2)|^2) \\ &\geq \frac{1}{2\nu(T_2 - T_1)}\Big(\frac{1}{2}D^{-2} - \sigma^{-2}T_2^{-2}\Big) \\ &\geq \frac{D^{-2}}{8(T_2 - T_1)}\nu^{-1}, \end{aligned}$$

which proves the lemma's assertion. □

This time-averaged lower bound yields similar bounds for other Sobolev norms.

Lemma 2.4.6. *For $m \geq 1$,*

$$\{\|u(t)\|_m^2\} \overset{m}{\gtrsim} \nu^{-(2m-1)}.$$

Proof. Since the case $m = 1$ has been treated in the previous lemma, we may assume that $m \geq 2$. By Lemma 2.2.1 and (2.12), we have

$$\{\|u(t)\|_m^2\} \overset{m}{\gtrsim} \{\|u(t)\|_m^2 \, |u(t)|_{1,1}^{(4m-4)}\} \overset{m}{\gtrsim} \{\|u(t)\|_1^{4m-2}\}.$$

Thus, using Hölder's inequality and Lemma 2.4.5, we get

$$\{\|u(t)\|_m^2\} \overset{m}{\gtrsim} \{\|u(t)\|_1^{4m-2}\} \overset{m}{\gtrsim} \{\|u(t)\|_1^2\}^{(2m-1)} \overset{m}{\gtrsim} \nu^{-(2m-1)}. \; \square$$

LEMMA 2.4.7. *For $m \geq 0$ and $p \in [1, \infty]$,*

$$\{|u(t)|_{m,p}^2\} \overset{m,p}{\gtrsim} \nu^{-2\gamma}.$$

Proof. First consider the case $m = 1,\ p \in [1, 2)$. By Hölder's inequality, Lemma 2.4.5, and Lemma 2.4.4 we have

$$\{|u(t)|_{1,p}^2\} \geq \{\|u(t)\|_1^2\}^{2/p} \{|u(t)|_{1,\infty}^2\}^{(p-2)/p} \overset{p}{\gtrsim} \nu^{-2\gamma}.$$

In the case $m = 1,\ p \geq 2$, it suffices to apply Hölder's inequality in place of Lemma 2.2.1 in the proof of an analogue for Lemma 2.4.6.

In the case $m \geq 2$, the proof is exactly the same as for Lemma 2.4.6. The only problematic case is $p = 1$, since then Lemma 2.2.1 does not allow us to estimate $|u(t)|_{m,p}^2$ from below using $|u(t)|_{1,1}^2$ and $\|u(t)\|_1^2$. However, it suffices to observe that $|u(t)|_{m,1} \geq |u(t)|_{m-1,\infty}$.

Finally, we study the case $m = 0$. We can take $p = 2$. Indeed, if we prove this case, then the result for $p > 2$ would follow immediately. On the other hand, the result for $p \in [1, 2)$ would follow as in the case $m = 1,\ p \in [1, 2)$ from Hölder's inequality, the lower estimate for the case $p = 2$, and the upper estimate in (2.11) (case $p = \infty$.)

For $t \in [T_1, 3T_1/2]$, we have

$$|u(t)|^2 \geq \frac{1}{3} D^{-2}.$$

This is proved in the same way as the estimate for $|u(T_1)|^2$ in the proof of Lemma 2.4.5. Thus, since $T_2 \geq 3T_1/2$, we have

$$\{|u(t)|^2\} \geq \frac{1}{T_2 - T_1} \int_{T_1}^{3T_1/2} |u(t)|^2 \geq \frac{T_1 D^{-2}}{6(T_2 - T_1)},$$

which proves the lemma's assertion. □

2.4.3 Main Result

The following theorem sums up the main results of this section, with the exception of Lemma 2.4.1.

THEOREM 2.4.8. *For $0 \leq m \leq 1$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$, we have*

$$\Big(\{|u(t)|^{\alpha}_{m,p}\}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \qquad \alpha > 0, \tag{2.22}$$

where $\gamma = \max(0, m-1/p)$, and $\{\cdot\}$ denotes time-averaging over $[T_1, T_2]$. The upper estimates in (2.22) hold without time-averaging, uniformly for t separated from 0. Namely, we have

$$|u(t)|_{m,p} \overset{m,p,}{\lesssim} \max(\nu^{-\gamma}, t^{-\gamma}).$$

On the other hand, the lower estimates hold for all m, p, α.

Proof. Upper estimates follow from Lemma 2.4.4. Lower estimates for $\alpha \geq 2$ follow from Lemma 2.4.7 by Hölder's inequality. Finally, we obtain lower estimates for $\alpha \in (0,2)$ and $p > 1$ using upper estimates for $\alpha = 3$, lower estimates for $\alpha = 2$, and Hölder's inequality; the case $p = 1$ follows from the case $p = \infty$ in the same way as previously. □

For $p = 2$, the relation (2.22) can be rewritten as:

$$\{\|u\|^2_s\} \overset{s}{\sim} \nu^{-(2s-1)}, \tag{2.23}$$

for integer values of s, $s \geq 1$. This relation also holds for non-integer values of s. Indeed, we obtain the upper bound by a standard interpolation argument, and the lower bound holds since for any integer $m > s$ we have

$$\{\|u\|^2_s\} \geq \{\|u\|^2_m\}^{m-s+1}\{\|u\|^2_{m+1}\}^{-(m-s)} \overset{s}{\gtrsim} \nu^{-(2s-1)}.$$

As a corollary of (2.23), for $s \geq 1$ we get

$$\{|\hat{u}_k|^2\} \lesssim k^{-2s}\{\|u\|^2_s\} \sim (k\nu)^{-2s}\nu. \tag{2.24}$$

We recall that we denote by $\hat{u}_k$ the k-th complex Fourier coefficient of u.

2.5 Estimates for Small-Scale Quantities

In this section, we study analogues of quantities which are important for the study of hydrodynamical turbulence. In Subsections 2.5.2 and 2.5.3, we consider, respectively, quantities in physical space (increments, flatness) and in Fourier space (energy spectrum).

2.5.1 Agreements and Notation

In this section we assume that $\nu \leq \nu_0$. The value of $\nu_0 > 0$ will be chosen in (2.33).

We define the intervals

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1].$$

The strictly positive constants C_1 and C_2 will be chosen in (2.32-2.33) in such a manner that $C_1\nu_0 < C_2 < 1$, which ensures that J_i are non-empty and non-intersecting.

By Theorem 2.4.8 we get $\{|u|^2\} \sim 1$ and (after integration by parts) $\{|\hat{u}_n|^2\} \leq \{|u|^2_{1,1}\}/(2\pi n)^2 \sim 1/n^2$. On the other hand, C_1 and C_2 can be made as small as desired (see (2.34)). Consequently, the proportion of the sum $\{\sum |\hat{u}_n|^2\}$ contained in Fourier modes corresponding to J_3 can be made as large as desired. For instance, we may assume that

$$\Big\{ \sum_{|n| \leq C_2^{-1}} |\hat{u}_n|^2 \Big\} \geq \frac{99}{100} \Big\{ \sum_{n \in \mathbb{Z}} |\hat{u}_n|^2 \Big\}.$$

For $p \geq 0$, we set:

$$S_p(\ell) = \Big\{ \int_{S^1} |u(t,x+\ell) - u(t,x)|^p dx \Big\}.$$

This quantity corresponds (up to averaging) to the structure function of p-th order. The flatness $F(\ell)$, which measures spatial intermittency, is given by

$$F(\ell) = S_4(\ell)/S_2^2(\ell) \tag{2.25}$$

(see [27]). Finally, for $k \geq 1$, we define the (layer-averaged) energy spectrum by

$$E(k) = \left\{ \frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}_n|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right\}, \tag{2.26}$$

where $M \geq 1$ is a constant which will be specified later (see Remark 2.5.13).

2.5.2 Results in Physical Space

We begin by estimating the functions $S_p(\ell)$ from above. In the proofs of the two following lemmas, constants denoted by C depend only on p.

Lemma 2.5.1. *For $\ell \in [0,1]$,*

$$S_p(\ell) \overset{p}{\lesssim} \begin{cases} \ell^p, \ 0 \le p \le 1. \\ \ell^p \nu^{-(p-1)}, \ p \ge 1. \end{cases}$$

Proof. We begin by considering the case $p \ge 1$. We have

$$\begin{aligned} S_p(\ell) &= \Big\{ \int_{S^1} |u(x+\ell) - u(x)|^p dx \Big\} \\ &\le \Big\{ \Big(\int_{S^1} |u(x+\ell) - u(x)| dx \Big) \Big(\max_x |u(x+\ell) - u(x)|^{p-1} \Big) \Big\}. \end{aligned}$$

Since the space average of $u(x+\ell) - u(x)$ vanishes, by Hölder's inequality we obtain that

$$\begin{aligned} S_p(\ell) \le & \Big\{ \Big(2 \int_{S^1} (u(x+\ell) - u(x))^+ dx \Big)^p \Big\}^{1/p} \\ & \times \Big\{ \max_x |u(x+\ell) - u(x)|^p \Big\}^{(p-1)/p} \\ \le & C\ell \Big\{ \max_x |u(x+\ell) - u(x)|^p \Big\}^{(p-1)/p}, \end{aligned} \tag{2.27}$$

where the second inequality follows from Lemma 2.4.1. Finally, by Theorem 2.4.8 we get

$$S_p(\ell) \le C\ell \Big\{ (\ell |u|_{1,\infty})^p \Big\}^{(p-1)/p} \le C\ell^p \nu^{-(p-1)}.$$

The case $p < 1$ follows immediately from the case $p = 1$ since now $S_p(\ell) \le (S_1(\ell))^p$, by Hölder's inequality. □

For $\ell \in J_2 \cup J_3$, we have a better upper bound if $p \ge 1$.

Lemma 2.5.2. *For $\ell \in J_2 \cup J_3$,*

$$S_p(\ell) \overset{p}{\lesssim} \begin{cases} \ell^p, \ 0 \le p \le 1. \\ \ell, \ p \ge 1. \end{cases}$$

Proof. The calculations are almost the same as in the previous lemma. The only difference is that we use another bound for the right-hand side of (2.27). Namely, by Theorem 2.4.8 we have

$$\begin{aligned} S_p(\ell) &\le C\ell \Big\{ \max_x |u(x+\ell) - u(x)|^p \Big\}^{(p-1)/p} \\ &\le C\ell \Big\{ (2|u|_\infty)^p \Big\}^{(p-1)/p} \le C\ell. \ \square \end{aligned}$$

REMARK 2.5.3. *It is easy to see that Lemmas 2.5.1 and 2.5.2 actually hold even if we drop the time-averaging, since we only use upper estimates which hold uniformly for $t \geq T_1$.*

To prove the lower estimates for S_p, we need a lemma. Loosely speaking, this lemma states that there exists a large enough set $L_K \subset [T_1, T_2]$ such that for $t \in L_K$, several Sobolev norms are of the same order as their time averages. Note that in the following definition, (2.28-2.29) contain lower and upper estimates, while (2.30) only contains an upper estimate. The inequality $|u(t)|_\infty \leq \max u_x(t)$ in (2.28) always holds, since $u(t)$ has zero mean value and the length of S^1 is 1.

DEFINITION 2.5.4. *For $K > 1$, we denote by L_K the set of all $t \in [T_1, T_2]$ such that the conditions*

$$K^{-1} \leq |u(t)|_\infty \leq \max u_x(t) \leq K \tag{2.28}$$

$$K^{-1}\nu^{-1} \leq |u(t)|_{1,\infty} \leq K\nu^{-1} \tag{2.29}$$

$$|u(t)|_{2,\infty} \leq K\nu^{-2} \tag{2.30}$$

hold.

LEMMA 2.5.5. *There exist constants $C, K_1 > 0$ such that for $K \geq K_1$, the Lebesgue measure of L_K verifies $\lambda(L_K) \geq C$.*

Proof. We begin by noting that if $K \leq K'$, $L_K \subset L_{K'}$. By Lemma 2.4.1 and Theorem 2.4.8, for K large enough the upper estimates in (2.28-2.30) hold for all t. Therefore, if we denote by B_K the set of t such that

"The lower estimates in (2.28-2.29) hold for a given value of K",

then it suffices to prove the lemma's statement with B_K in place of L_K. Now denote by D_K the set of t such that

"The lower estimate in (2.29) holds for a given value of K".

By Lemma 2.2.1 we have

$$|u|_\infty \geq C|u|_{2,\infty}^{-1}|u|_{1,\infty}^2.$$

Thus, if D_K holds, then $B_{K'}$ holds for K' large enough. Now it remains to show that there exists $C > 0$ such that for K large enough, $\lambda(D_K) \geq C$. We clearly have

$$\{|u|_{1,\infty}\mathbf{1}(|u|_{1,\infty} \leq K^{-1}\nu^{-1})\} \leq K^{-1}\nu^{-1}.$$

Here, $\mathbf{1}(A)$ denotes the indicator function of an event A. On the other hand, by the estimate for $\{|u|^2_{1,\infty}\}$ in Theorem 2.4.8 we get

$$\{|u|_{1,\infty}\mathbf{1}(|u|_{1,\infty} \geq K\nu^{-1})\} \leq CK^{-1}\nu^{-1}.$$

Now denote by f the function

$$f = |u|_{1,\infty}\mathbf{1}(K_0^{-1}\nu^{-1} \leq |u|_{1,\infty} \leq K_0\nu^{-1}).$$

The inequalities above and the estimate for $\{|u|_{1,\infty}\}$ in Theorem 2.4.8 imply that

$$\{f\} \geq (C - K_0^{-1} - CK_0^{-1})\nu^{-1} \geq C_0\nu^{-1},$$

for some suitable constants C_0 and K_0. Since $f \leq K_0\nu^{-1}$, then we get

$$\lambda(f \geq C_0\nu^{-1}/2) \geq C_0K_0^{-1}(T_2 - T_1)/2.$$

Thus, since $|u|_{1,\infty} \geq f$, we have the inequality

$$\lambda(|u|_{1,\infty} \geq C_0\nu^{-1}/2) \geq C_0K_0^{-1}(T_2 - T_1)/2,$$

which implies existence of $C, K_1 > 0$ such that $\lambda(D_K) \geq C$ for $K \geq K_1$. □

Let us denote by $O_K \subset [T_1, T_2]$ the set defined as L_K, but with relation (2.29) replaced by

$$K^{-1}\nu^{-1} \leq -\min u_x \leq K\nu^{-1}. \tag{2.31}$$

COROLLARY 2.5.6. *For $K \geq K_1$ and $\nu < K_1^{-2}$, we have $\lambda(O_K) \geq C$.*

Proof. For $K = K_1$ and $\nu < K_1^{-2}$, the estimates (2.28-2.29) tell us that

$$\max u_x(t) \leq K_1 < K_1^{-1}\nu^{-1} \leq |u_x(t)|_\infty, \quad t \in L_K.$$

Thus, in this case the assertion of Lemma 2.5.5 with (2.29) replaced by (2.31) holds for the set $O_K = L_K$. Since increasing K while keeping ν constant increases the measure of O_K, for $K \geq K_1$ and $\nu < K_1^{-2}$ we still have $\lambda(O_K) \geq C$. □

Now we fix

$$K = K_1, \tag{2.32}$$

and choose

$$\nu_0 = \frac{1}{6}K^{-2};\ C_1 = \frac{1}{4}K^{-2};\ C_2 = \frac{1}{20}K^{-4}. \tag{2.33}$$

In particular, we have $0 < C_1\nu_0 < C_2 < 1$: thus the intervals J_i are non-empty and non-intersecting for all $\nu \in (0, \nu_0]$. Everywhere below the constants depend on K.

Actually, we can choose any values of C_1, C_2, and ν_0, provided

$$C_1 \leq \frac{1}{4}K^{-2}; \quad 5K^2 \leq \frac{C_1}{C_2} < \frac{1}{\nu_0}. \tag{2.34}$$

LEMMA 2.5.7. *For $\ell \in J_1$,*

$$S_p(\ell) \overset{p}{\gtrsim} \begin{cases} \ell^p, \ 0 \leq p \leq 1. \\ \ell^p \nu^{-(p-1)}, \ p \geq 1. \end{cases}$$

Proof. By Corollary 2.5.6, it suffices to prove that the inequalities hold uniformly in t for $t \in O_K$, with $S_p(\ell)$ replaced by

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx.$$

Till the end of this proof, we assume that $t \in O_K$.

Denote by z the leftmost point on S^1 (considered as $[0,1)$) such that $u'(z) \leq -K^{-1}\nu^{-1}$. Since $|u|_{2,\infty} \leq K\nu^{-2}$, we have

$$u'(y) \leq -\frac{1}{2}K^{-1}\nu^{-1}, \quad y \in [z - \frac{1}{2}K^{-2}\nu, z + \frac{1}{2}K^{-2}\nu]. \tag{2.35}$$

Case $\mathbf{p} \geq \mathbf{1}$. Since $\ell \leq C_1\nu = \frac{1}{4}K^{-2}\nu$, then by Hölder's inequality we get

$$\begin{aligned} \int_{S^1} |u(x+\ell) - u(x)|^p dx &\geq \int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} |u(x+\ell) - u(x)|^p dx \\ &\geq (K^{-2}\nu/2)^{1-p} \Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} |u(x+\ell) - u(x)| dx \Big)^p \\ &= C(p)\nu^{1-p} \Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} \Big(\int_x^{x+\ell} -u'(y)dy \Big) dx \Big)^p \\ &\geq C(p)\nu^{1-p} \Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} \frac{1}{2}\ell K^{-1}\nu^{-1} \ dx \Big)^p = C(p)\nu^{1-p}\ell^p. \end{aligned}$$

Case p < 1. By Hölder's inequality we obtain that

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx \geq \int_{S^1} \Big((u(x+\ell) - u(x))^+ \Big)^p dx$$
$$\geq \Big(\int_{S^1} \Big((u(x+\ell) - u(x))^+ \Big)^2 dx \Big)^{p-1} \Big(\int_{S^1} (u(x+\ell) - u(x))^+ dx \Big)^{2-p}.$$

Using the upper estimate in (2.28) we get

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx$$
$$\geq \Big(\int_{S^1} \ell^2 K^2 dx \Big)^{p-1} \Big(\int_{S^1} (u(x+\ell) - u(x))^+ dx \Big)^{2-p}.$$

Since $\int_{S^1} (u(\cdot + \ell) - u(\cdot)) = 0$, we obtain that

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx$$
$$\geq C(p) \ell^{2(p-1)} \Big(\frac{1}{2} \int_{S^1} |u(x+\ell) - u(x)| dx \Big)^{2-p} \geq C(p) \ell^p.$$

The last inequality follows from the case $p = 1$. □

The proof of the following lemma uses an argument from [2], which becomes rigorous if we restrict ourselves to the set O_K.

Lemma 2.5.8. *For $m \geq 0$ and $\ell \in J_2$,*

$$S_p(\ell) \overset{p}{\gtrsim} \begin{cases} \ell^p, & 0 \leq p \leq 1. \\ \ell, & p \geq 1. \end{cases}$$

Proof. In the same way as above, it suffices to prove that the inequalities hold uniformly in t for $t \in O_K$, with $S_p(\ell)$ replaced by

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx,$$

and we can restrict ourselves to the case $p \geq 1$. Again, till the end of this proof, we assume that $t \in O_K$.

Defining z as in the proof of Lemma 2.5.7, we have

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx \geq$$
$$\int_{z-\frac{1}{2}\ell}^{z} \Big| \int_x^{x+\ell} u'^-(y)dy - \int_x^{x+\ell} u'^+(y)dy \Big|^p dx.$$

Since $\ell \geq C_1\nu = \frac{1}{4}K^{-2}\nu$, then by (2.35) for $x \in [z - \frac{1}{2}\ell, z]$ we get

$$\int_x^{x+\ell} u'^-(y)dy \geq \int_z^{z+\frac{1}{8}K^{-2}\nu} u'^-(y)dy \geq \frac{1}{16}K^{-3}.$$

.

On the other hand, since $\ell \leq C_2$, then by (2.28) and (2.33) we have

$$\int_x^{x+\ell} u'^+(y)dy \leq C_2 K = \frac{1}{20}K^{-3}.$$

Thus,

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx \geq \frac{1}{2}\ell\left(\left(\frac{1}{16} - \frac{1}{20}\right)K^{-3}\right)^p \geq C(p)\ell. \ \square$$

Summing up the results above we obtain the following theorem.

THEOREM 2.5.9. *For $\ell \in J_1$,*

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, & 0 \leq p \leq 1. \\ \ell^p \nu^{-(p-1)}, & p \geq 1. \end{cases}$$

On the other hand, for $\ell \in J_2$,

$$S_p(\ell) \overset{p}{\sim} \begin{cases} \ell^p, & 0 \leq p \leq 1. \\ \ell, & p \geq 1. \end{cases}$$

The following result follows immediately from the definition (2.25).

COROLLARY 2.5.10. *For $\ell \in J_2$, the flatness satisfies $F(\ell) \sim \ell^{-1}$.*

2.5.3 Results in Fourier Space

In this section, we only state the results. The proofs are exactly the same as in Subsection 4.6.3.

To begin with, we want to estimate the H^s norms of u for $s \in (0,1)$ (the case $s = 0$ is a particular case of Theorem 2.4.8).

LEMMA 2.5.11. *We have*

$$\{\|u\|_s^2\} \overset{s}{\sim} \begin{cases} 1, & 0 < s < 1/2. \\ |\log \nu|^{1/2}, & s = 1/2. \\ \nu^{-(2s-1)}, & 1/2 < s < 1. \end{cases}$$

The results above, as well as the relation (2.23), tell us that when $\nu \to 0^+$ the sums

$$\sum |k|^{2s} \{|\hat{u}_k|^2\}, \quad s \geq 0,$$

have exactly the same behaviour as the partial sums $\sum_{|k| \leq \nu^{-1}} |k|^{2s} |k|^{-2}$. Moreover, by (2.24), $\{|\hat{u}_k|^2\}$ decreases very fast for $|k| \gtrsim \nu^{-1}$.

Actually, as long as $|k|$ remains in a certain range, after layer-averaging, we have $\{|\hat{u}_k|^2\} \sim |k|^{-2}$.

THEOREM 2.5.12. *For k such that $k^{-1} \in J_2$, we have $E(k) \sim k^{-2}$.*

REMARK 2.5.13. *Theorem 2.5.12 holds for a certain choice of the constant M in the definition (2.26) of $E(k)$. From the proof in Subsection 4.6.3, it is clear that this constant depends only on f and D.*

Chapitre 3

Estimations pour des solutions de l'équation de Burgers généralisée avec forçage de type $\ll$ kick $\gg$

Ce chapitre correspond à l'article *Estimates for Solutions of a Low-Viscosity Kick-Forced Generalised Burgers Equation*, accepté pour une publication dans Proceedings of the Royal Society of Edinburgh, Section A.

Abstract. We consider a non-homogeneous generalised Burgers equation:

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = \eta^\omega, \quad t \in \mathbb{R},\ x \in S^1.$$

Here, ν is small and positive, f is strongly convex and satisfies a growth assumption, while η^ω is a space-smooth random "kicked" forcing term.

For any solution u of this equation, we consider the quasi-stationary regime, corresponding to $t \geq 2$. After taking the ensemble average, we obtain upper estimates as well as time-averaged lower estimates for a class of Sobolev norms of u. These estimates are of the form $C\nu^{-\beta}$ with the same values of β for bounds from above and from below. They depend on η and f, but do not depend on the time t or the initial condition.

3.1 Notation

Consider a zero mean value smooth function w on S^1. For $p \in [1,+\infty]$, we denote its L_p norm of by $|w|_p$. The L_2 norm will be denoted by $|w|$, and $\langle \cdot, \cdot \rangle$ stands for the L_2 scalar product. From now on, L_p, $p \in [1,+\infty]$ stands for the space of zero mean value functions in $L_p(S^1)$.

For a nonnegative integer n and $p \in [1,+\infty]$, $W^{n,p}$ stands for the Sobolev space of zero mean value functions w on S^1 with the norm

$$|w|_{n,p} = \left|w^{(n)}\right|_p,$$

where

$$w^{(n)} = \frac{d^n w}{dx^n}.$$

In particular, $W^{0,p} = L_p$ for $p \in [1,+\infty]$. For $p = 2$, we denote $W^{n,2}$ by H^n, and the corresponding norm is abbreviated as $\|w\|_n$.

We recall a version of the classical Gagliardo-Nirenberg inequality (see [51, p. 125]).

LEMMA 3.1.1. *For a smooth zero mean value function w on S^1,*

$$|w|_{\beta,r} \leq C\, |w|_{m,p}^{\theta}\, |w|_q^{1-\theta},$$

where $m > \beta$, and r is defined by

$$\frac{1}{r} = \beta + \theta\left(\frac{1}{p} - m\right) + (1-\theta)\frac{1}{q},$$

under the assumption that $\theta = \beta/m$ if $p = 1$ or $p = +\infty$, and $\beta/m \leq \theta < 1$ otherwise. Here $C = C(m,p,q,\beta,\theta) > 0$ is a constant.

For a smooth function $v(t,x)$ defined on $[0,+\infty) \times S^1$, v_t, v_x, and v_{xx} mean respectively $\frac{\partial v}{\partial t}$, $\frac{\partial v}{\partial x}$, and $\frac{\partial^2 v}{\partial x^2}$.

3.2 Introduction

The generalised one-dimensional space-periodic Burgers equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = 0, \quad \nu > 0 \tag{3.1}$$

(the classical Burgers equation corresponds to $f(u) = u^2$) appears in different domains of science, ranging from cosmology to traffic modelling (see [6]). It is sometimes called a viscous scalar conservation law. Historically, it has drawn most attention as a model for the Navier-Stokes equation (NSE). Indeed, it has a nonlinear term analogous to the nonlinearity $(u \cdot \nabla)u$ in the incompressible NSE. The dissipation term in (3.1) is also similar to the one in NSE. We note that the classical Burgers equation is explicitly solvable. This is done by the Hopf-Cole transformation (see [8]).

In [7], A.Biryuk considered equation (3.1) with f strongly convex, i.e. satisfying

$$f''(x) \geq \sigma > 0, \quad x \in \mathbb{R}. \tag{3.2}$$

He studied the behavior of the Sobolev norms of solutions u for small values of ν and obtained the following estimates:

$$\|u\|_m^2 \leq C\nu^{-(2m-1)/2}, \quad \frac{1}{T}\int_0^T \|u\|_m^2 \geq c\nu^{-(2m-1)/2}, \quad m \geq 1, \ \nu \leq \nu_0.$$

Note that exponents of ν in lower and upper estimates are the same. The quantities ν_0, C, c, and T depend on the deterministic initial condition u_0 as well as on m. To get results independent from the initial data, a natural idea is to introduce random forcing and to estimate ensemble-averaged norms of solutions.

In this article we consider (3.1) with a random kick force in the right-hand side. In Section 3.3 we recall classical existence and uniqueness results and introduce the probabilistic setting needed to define the kick force. Then, we estimate from above the moments of the $W^{1,1}$ norm of u. These estimates, valid after a certain damping time, are proved using ideas similar to those in [40]. Remarkably, this damping time and the estimate do not depend on the initial condition. This is the crucial result of this article.

Next, in Sections 3.4 and 3.5, this result allows us to obtain lower and upper estimates that are, up to taking the ensemble average, of the same type as in [7], for time $t \geq 2$. These estimates will only depend on the function f and the forcing. Let us emphasise that, for $t \geq 2$, we are in a quasi-stationary regime: all estimates hold independently of the initial condition. In Section 3.6, we give some additional estimates for the Sobolev norms.

In this paper, we use methods introduced by Kuksin in [41, 42], and developed by Biryuk in [7].

Equation (3.1) with $\nu \ll 1$ is a popular one-dimensional model for the theory of hydrodynamic turbulence. In Section 3.7, we present an interpretation of our results in terms of this theory.

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3.3 Preliminaries

In this section, we review properties of solutions of (3.1) used in our proof.

Physically, t corresponds to the time variable, whereas x corresponds to the one-dimensional space variable, and the constant $\nu > 0$ to a viscosity coefficient. The real-valued function $u(t,x)$ is defined on $[0,+\infty) \times \mathbb{R}$ and is Π-periodic in x. The function f is C^∞-smooth and strongly convex, i.e. it satisfies the condition (3.2) for some constant σ. Moreover, we assume that f, as well as its derivatives, has at most polynomial growth, i.e.

$$\forall m \geq 0,\ \exists n \geq 0,\ C_m > 0: \quad |f^{(m)}(x)| \leq C_m(1+|x|)^n, \quad x \in \mathbb{R}, \tag{3.3}$$

where $n = n(m)$. From now on, we fix $\Pi = 1$, which amounts to studying the problem on $[0,+\infty) \times S^1$. We note that L-periodic solutions of (3.1) with any L reduce, by means of scaling in x, to 1-periodic solutions with scaled f and ν.

Since we are mostly interested in the asymptotics of solutions of (3.1) as $\nu \to 0^+$, we assume that

$$\nu \in (0,1].$$

Moreover, it is enough to study the special case

$$\int_{S^1} u_0(y)dy = 0. \tag{3.4}$$

Indeed, if the mean value of u_0 on S^1 equals b, we may consider

$$v(t,x) = u(t, x+bt) - b.$$

Then v satisfies (3.4) and is a solution of (3.1) with $f(y)$ replaced with $g(y) = f(y+b) - by$.

Given a C^∞-smooth initial condition $u_0 = u(0,\cdot)$, equation (3.1) has a unique classical solution u, C^∞-smooth in both variables (see [39, Chapter 5]). Condition (3.4) implies that the mean value of a solution for (3.1) vanishes identically in t. Now provide each space $W^{n,p}(S^1)$ with the Borel σ-algebra. Consider a random variable ζ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ with values in $L^2(S^1)$, such that $\zeta^\omega \in C^\infty(S^1)$ for a.e. ω. We suppose that ζ satisfies the following three properties.

(i) (Non-triviality)

$$\mathbb{P}(\zeta \equiv 0) < 1.$$

(ii) (Finiteness of exponential moments for Sobolev norms) For every $m \geq 0$ there are constants $\alpha = \alpha(m) > 0,\ \beta = \beta(m)$ such that

$$\mathbb{E} \exp(\alpha \left\| \zeta \right\|_m^2) \leq \beta.$$

In particular

$$I_m = \mathbb{E} \left\| \zeta \right\|_m^2 < +\infty, \quad \forall m \geq 0.$$

(iii) (Vanishing of the expected value)

$$\mathbb{E}\zeta \equiv 0.$$

It is not difficult to construct explicitly ζ satisfying **(i)-(iii)**. For instance we could consider the real Fourier coefficients of ζ, defined for $k > 0$ by

$$a_k(\zeta) = \sqrt{2} \int_{S^1} \cos(2\pi kx) u(x),\ b_k(\zeta) = \sqrt{2} \int_{S^1} \sin(2\pi kx) u(x) \qquad (3.5)$$

as independent random variables with zero mean value and exponential moments tending to 1 fast enough as $k \to +\infty$.

Now let ζ_i, $i \in \mathbb{N}$ be independent identically distributed random variables having the same distribution as ζ. The sequence $(\zeta_i)_{i \geq 1}$ is a random variable, defined on a probability space which is a countable direct product of copies of Ω. From now on, this space will itself be called Ω. The meaning of $\mathcal{F}$ and $\mathbb{P}$ changes accordingly.

For $\omega \in \Omega$ and a time period $\theta > 0$, the kick force η^ω is a C^∞-smooth function in the variable x, with values in the space of distributions in the variable t, defined by

$$\eta^\omega(x) = \sum_{i=1}^{+\infty} \delta_{t=i\theta} \zeta_i^\omega(x),$$

where $\delta_{t=i\theta}$ denotes the Dirac measure at a time moment $i\theta$.

The kick-forced version of (3.1) corresponds to the case where, in the right-hand side, 0 is replaced with the kick force. This means that for integers $i \geq 1$, at the moments $i\theta$ the solution $u(x)$ instantly increases by the kick $\zeta_i^\omega(x)$, and that between these moments u solves (3.1). The equation is written as follows:

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu \frac{\partial^2 u}{\partial x^2} = \eta^\omega. \tag{3.6}$$

Derivatives are taken in the sense of distributions.

When studying solutions of (3.6), we will always assume that the initial condition $u_0 = u(0, \cdot)$ is C^∞-smooth. Moreover, we normalise those solutions to be right-continuous in time at the kick moments $i\theta$. Such a solution is uniquely defined for a given value of u_0, for a.e. ω.

For a given initial condition u_0, the function $u(t, x)$ always will denote such a solution of (3.1). The value of u before the i-th kick will be denoted by $u(i\theta^-, \cdot)$, or shortly u_i^-. We will also use the notation $u_i = u(i\theta, \cdot)$ and denote the function $u(t, \cdot)$ by $u(t)$. Finally, for a solution of (3.6), we consider time derivatives at the kick moments in the sense of right-sided time derivatives. Those derivatives are right-continuous in time.

Since space averages of the kicks vanish and $u_0(x)$ satisfies (3.4), the space average of $u(t),\ t \geq 0$ vanishes identically. For the sake of simplicity, we normalise the kick period: from now on $\theta = 1$.

We observe that, since the kicks are independent and between the kicks (3.6) is deterministic, the solutions of (3.6) make a random Markov process. For details, see [43], where a kick force is introduced in a similar setting.

Agreements. All constants denoted C with sub- or super-indexes are strictly positive. Unless otherwise stated, they depend only on f, on the distribution of the kicks, as well as on the parameters $a_1, \dots, a_k$ if they are denoted $C(a_1, \dots, a_k)$. By u we always denote a solution of (3.6) with any initial condition u_0. Averaging in ensemble corresponds to averaging in $\mathbb{P}$. All our estimates hold independently of the value of u_0.

We observe that for every integer i we have the following energy dissipation identity on the maximal kick-free intervals:

$$A_i = |u_i|^2 - \left|u_{i+1}^-\right|^2, \tag{3.7}$$

where

$$A_i = 2\nu \int_i^{(i+1)} \|u(t)\|_1^2 \; dt. \tag{3.8}$$

Indeed, for any $t \in (i, i+1)$ u satisfies

$$2\nu \|u(t)\|_1^2 = -2\nu \int_{S^1} u u_{xx} dx = -2 \int_{S^1} u f'(u) u_x dx - 2 \int_{S^1} u u_t dx.$$

The first term on the right-hand side vanishes since its integrand is a full derivative. The second term equals $-\frac{d}{dt}|u|^2$. Integrating in time we get (3.7). We note that energy dissipation between kicks A_i is always non-negative: energy can be added only at the kick points. We also note that an analogue of (3.7) holds on every kick-free time interval.

The following two lemmas are proved using the maximum principle in the same way as in [40].

Lemma 3.3.1. *We have the estimate*

$$u_x(t,x) \leq 2\sigma^{-1}, \quad t \in [k+1/2,\ k+1),\ k \in \mathbb{N},\ x \in S^1,$$

where σ is the constant in the assumption (3.2).

Proof. Consider the equation (3.6) on the kick-free time interval $[0, 1-\epsilon]$ for arbitrarily small ϵ and differentiate it once in space. We get

$$\frac{\partial u_x}{\partial t} + f''(u)u_x^2 + f'(u)\frac{\partial u_x}{\partial x} - \nu \frac{\partial^2 u_x}{\partial x^2} = 0. \tag{3.9}$$

Consider $v(t,x) = t u_x(t,x)$. For $t > 0$, v verifies

$$\frac{\partial v}{\partial t} + t^{-1}(-v + f''(u)v^2) + f'(u)\frac{\partial v}{\partial x} - \nu \frac{\partial^2 v}{\partial x^2} = 0. \tag{3.10}$$

Now observe that, if $v > 0$ somewhere on the domain $S_\epsilon = [0, 1-\epsilon] \times S^1$, then v attains its maximum M on S_ϵ at a point (t_1, x_1) such that $t_1 > 0$. At (t_1, x_1) we have $\frac{\partial v}{\partial t} \geq 0$, $\frac{\partial v}{\partial x} = 0$, and $\frac{\partial^2 v}{\partial x^2} \leq 0$. Therefore, (3.10) yields that

$$t_1^{-1}[-v(t_1,x_1) + f''(u(t_1,x_1))v^2(t_1,x_1)] \leq 0.$$

Since, by (3.2), $f'' \geq \sigma > 0$, then

$$-M + \sigma M^2 \leq 0,$$

and therefore

$$M \leq \sigma^{-1}.$$

Thus we have proved that $v \leq \sigma^{-1}$ everywhere on S_ϵ for every $\epsilon > 0$. In particular, by definition of v and S_ϵ, we get that

$$u_x(t,x) \leq 2\sigma^{-1}, \quad x \in S^1,\ t \in [1/2, 1).$$

Repeating the same argument on all the intervals $[k, k+1),\ k \in \mathbb{N}$ we get the lemma's assertion. □

Lemma 3.3.2. *There are constants C', C such that*

$$\mathbb{E} \exp(C' \sup_{t \in [k,k+1)} \max u_x(t,\cdot)) \leq C, \quad k \geq 1.$$

Proof. Fix $k \geq 1$. Since the $W^{1,\infty}$ norm is dominated by the H^2 norm, then for $C' > 0$ we get

$$\exp(C' u_x(k,x)) \leq \exp(C' u_x(k^-,x) + C' \|\zeta_k\|_2), \quad x \in S^1.$$

The same inequality holds when we maximise in x. Now denote by X_k the random variable

$$\max u_x(k,\cdot).$$

By Lemma 3.3.1 and Property **(ii)** of the kicks, for $C' = \alpha(2)$ we get

$$\mathbb{E} \exp(C' X_k) \leq \exp(2C' \sigma^{-1}) \mathbb{E} \exp(C' \|\zeta_k\|_2) \leq C, \tag{3.11}$$

for some constant C. Now consider the equation (3.9). An application of the maximum principle to the function u_x, which cannot be negative everywhere, yields

$$\max u_x(t,\cdot) \leq \max u_x(k,\cdot), \quad t \in [k, k+1)\,.$$

Therefore, in (3.11), we can replace X_k by $\sup_{t \in [k,k+1)} \max u_x(t,\cdot)$. This proves the lemma's assertion. □

Corollary 3.3.3. *For the same C', C as in Lemma 3.3.2 we have*

$$\mathbb{E}\exp\Big(\frac{C'}{2}\sup_{t\in[k,k+1)}|u(t)|_{1,1}\Big)\leq C,\quad k\geq 1.$$

Proof. Since the mean value of $u_x(t)$ is 0, then

$$\int_{S^1}|u_x(t)|=2\int_{S^1}\max(u_x(t),0).$$

□

Corollary 3.3.4. *For the same C', C as in Lemma 3.3.2 we have*

$$\mathbb{E}\exp(C'\sup_{t\in[k,k+1)}|u(t)|_p)\leq C,\quad k\geq 1,\ p\in[1,+\infty].$$

Note that C' and C do not depend on p.

3.4 Lower estimates of H^m norms

For a solution u of (3.6), the first quantity that we estimate from below is the expected value of

$$\frac{1}{N}\int_1^{N+1}\|u(t)\|_1^2=\frac{1}{N}(2\nu)^{-1}\sum_{i=1}^{N}A_i, \tag{3.12}$$

where N is a fixed natural number chosen later, and A_i is the same as in (3.8).

Lemma 3.4.1. *There exists a natural number $N\geq 1$, independent from u_0, such that*

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\,\|u(s)\|_1^2\geq C\nu^{-1}.$$

Proof. For $N \geq 1$ we have

$$\begin{aligned}
\mathbb{E}\left|u_{N+1}^-\right|^2 &\geq \mathbb{E}\Big(\left|u_{N+1}^-\right|^2 - \left|u_1^-\right|^2\Big) \\
&= \mathbb{E}\sum_{i=1}^{N}\Big(|u_{i+1}^-|^2 - |u_i|^2\Big) + \mathbb{E}\sum_{i=1}^{N}\Big(|u_i|^2 - |u_i^-|^2\Big) \\
&= -\mathbb{E}\sum_{i=1}^{N} A_i + \mathbb{E}\sum_{i=1}^{N}\Big(|u_i^- + \zeta_i|^2 - |u_i^-|^2\Big) \\
&= -\mathbb{E}\sum_{i=1}^{N} A_i + 2\mathbb{E}\sum_{i=1}^{N}\langle u_i^-, \zeta_i\rangle + \mathbb{E}\sum_{i=1}^{N}|\zeta_i|^2.
\end{aligned}$$

Since $\mathbb{E}\zeta_i \equiv 0$ (Property **(iii)** of the kicks), and u_i^- and ζ_i are independent, then $\mathbb{E}\langle u_i^-, \zeta_i\rangle = 0$. Therefore, by (3.8), we have

$$\mathbb{E}\left|u_{N+1}^-\right|^2 \geq -2\nu\mathbb{E}\int_1^{N+1}\|u(s)\|_1^2 + 0 + NI_0.$$

On the other hand, by Corollary 3.3.4 ($p = 2$) there is a constant C_1 such that

$$\mathbb{E}\left|u_{N+1}^-\right|^2 \leq C_1.$$

Consequently

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\,\|u(s)\|_1^2 \geq \frac{NI_0 - C_1}{2N}\nu^{-1}.$$

Choosing the smallest possible integer N verifying

$$N \geq \max\Big(1,\ \frac{C_1 + 1}{I_0}\Big),$$

we get the lemma's assertion. □

We have reached our first goal: estimating from below the expected value of (3.12). Thus, we have a time-averaged lower estimate for the H^1 norm, which enables us to obtain similar estimates of H^m norms for $m \geq 2$.

Lemma 3.4.2. *We have*

$$\frac{1}{N}\int_1^{N+1}\mathbb{E}\,\|u(s)\|_m^2 \geq C(m)\nu^{-(2m-1)}, \quad m \geq 1,$$

where N is the same as in Lemma 3.4.1.

Proof. This statement is already proved in the previous lemma for $m = 1$, so we may assume that $m \geq 2$. By Lemma 3.1.1 and Hölder's inequality we have

$$\left(\mathbb{E}\,\|u(s)\|_1^2\right)^{2m-1} \leq C'(m)\mathbb{E}\,\|u(s)\|_m^2 \left(\mathbb{E}\,|u(s)|_{1,1}^2\right)^{2m-2}. \tag{3.13}$$

Since by Corollary 3.3.3

$$\mathbb{E}\,|u(s)|_{1,1}^2 \leq K, \quad t \in [1, N+1],$$

where $K > 0$ is a constant, then, integrating (3.13) in time, we get

$$\frac{1}{N}\int_1^{N+1} \mathbb{E}\,\|u(s)\|_m^2 \geq \frac{\int_1^{N+1}[\mathbb{E}(\|u(s)\|_1^2)]^{(2m-1)}}{NC'(m)K^{2m-2}}.$$

By Hölder's inequality,

$$\int_1^{N+1}[\mathbb{E}(\|u(s)\|_1^2)]^{(2m-1)} \geq \left(\int_1^{N+1} \mathbb{E}\,\|u(s)\|_1^2\right)^{(2m-1)} N^{2-2m},$$

and then

$$\begin{aligned}\frac{1}{N}\int_1^{N+1} \mathbb{E}\,\|u(s)\|_m^2 \geq& \frac{\left(\int_1^{N+1}\mathbb{E}\,\|u(s)\|_1^2\right)^{(2m-1)} N^{2-2m}}{NC'(m)K^{2m-2}} \\ =& \frac{\left(\frac{1}{N}\int_1^{N+1}\mathbb{E}\,\|u(s)\|_1^2\right)^{(2m-1)}}{C'(m)K^{2m-2}}.\end{aligned}$$

Now the assertion follows from Lemma 3.4.1. □

Since we impose no conditions on u_0, we can consider a different positive integer "starting time". We may also consider a different averaging time interval of length $T \geq N$. Finally, we obtain a general result for a non-integer starting time $t \geq 1$ by considering the maximal interval $[m_1, m_2] \subset [t, t+T]$ such that m_1 and m_2 are positive integers.

Theorem 3.4.3. *We have*

$$\frac{1}{T}\int_t^{t+T} \mathbb{E}\,\|u(s)\|_m^2 \geq \frac{C(m)}{4}\nu^{-(2m-1)}, \quad t \geq 1,\ T \geq N+1,\ m \geq 1,$$

where N and $C(m)$ are the same as in Lemma 3.4.2.

3.5 Upper estimates of H^m norms

To estimate from above a Sobolev norm $\|u\|_m$, $m \geq 1$, of a solution u for (3.6), we differentiate between the kicks the quantity $\|u(t)\|_m^2$.

Denote by $B(u)$ the nonlinearity $2f'(u)u_x$, and by L the operator $-\partial_{xx}$. Integrating by parts, we get

$$\begin{aligned}\frac{d}{dt}\|u\|_m^2 &= 2\left\langle u^{(m)}, u_t^{(m)}\right\rangle \\ &= -2\nu\|u\|_{m+1}^2 - \langle L^m u, B(u)\rangle . \end{aligned} \tag{3.14}$$

We will need a standard estimate for the nonlinearity $\langle L^m u, B(u)\rangle$.

LEMMA 3.5.1. *For a zero mean value smooth function w such that $|w|_\infty \leq M$, we have*

$$|\langle L^m w, B(w)\rangle| \leq C\|w\|_m \|w\|_{m+1}, \quad m \geq 1,$$

with C satisfying

$$C \leq C_m(1+M)^n, \tag{3.15}$$

where C_m, as well as the natural number $n = n(m)$, depend only on m.

Proof. Let C' denote various positive constants satisfying an estimate for the type (3.15). Then we have

$$\begin{aligned}|\langle L^m w, B(w)\rangle| =& 2\left|\left\langle w^{(2m)}, (f(w))^{(1)}\right\rangle\right| \\ =& 2\left|\left\langle w^{(m+1)}, (f(w))^{(m)}\right\rangle\right| \\ \leq& C'\sum_{k=1}^{m}\sum_{\substack{1\leq a_1\leq\cdots\leq a_k\leq m\\ a_1+\cdots+a_k=m}}\int_{S^1}\left|w^{(m+1)}w^{(a_1)}\dots w^{(a_k)}f^{(k)}(w)\right| \\ \leq& C'|f|_{C^m[-M,M]}\sum_{k=1}^{m}\sum_{\substack{1\leq a_1\leq\cdots\leq a_k\leq m\\ a_1+\cdots+a_k=m}}\int_{S^1}|w^{(a_1)}\dots \\ &\dots w^{(a_k)}w^{(m+1)}|.\end{aligned}$$

By (3.3), $|f|_{C^m[-M,M]}$ satisfies an estimate of the type (3.15). By Hölder's inequality, we obtain that

$$\begin{aligned}|\langle L^m w, B(w)\rangle| \leq& C'\|w\|_{m+1}\sum_{\substack{1\leq a_1\leq\cdots\leq a_k\leq m\\ a_1+\cdots+a_k=m}}\Big(\left|w^{(a_1)}\right|_{2m/a_1}\dots \\ &\dots\left|w^{(a_k)}\right|_{2m/a_k}\Big).\end{aligned}$$

Finally, the Gagliardo-Nirenberg inequality yields

$$\begin{aligned}|\langle L^m w, B(w)\rangle| \le& C' \|w\|_{m+1} \sum_{k=1}^{m} \sum_{\substack{1\le a_1\le\cdots\le a_k\le m \\ a_1+\cdots+a_k=m}} \\ &\Big[(\|w\|_m^{a_1/m} |w|_\infty^{(m-a_1)/m}) \dots (\|w\|_m^{a_k/m} |w|_\infty^{(m-a_k)/m})\Big] \\ \le& C' |w|_\infty^{m-1} \|w\|_m \|w\|_{m+1} \\ \le& C' \|w\|_m \|w\|_{m+1}\,,\end{aligned}$$

which proves the lemma's assertion. □

THEOREM 3.5.2. *For any natural numbers m, n we have*

$$\mathbb{E}(\sup_{t\in[k,k+1)} \|u(t)\|_m^n) \le C(m,n)\nu^{-(2m-1)n/2}, \quad k \ge 2.$$

Proof. Fix $k \ge 2$ and $m \ge 1$. In this proof, Θ denotes various positive random constants which depend on m, such that all their moments are finite, and C denotes various positive deterministic constants, depending only on m.

We begin by noting that Corollary 3.3.3 and Property **(ii)** of the kicks imply the inequalities

$$|u(t)|_{1,1}, \|\zeta_k\|_m \le \Theta, \quad t \in [k-1, k+1). \tag{3.16}$$

We claim that when $\|u\|_m^2$ is too large, it decreases at least as fast as a solution of the differential equation

$$y' + (2m-1)y^{2m/(2m-1)} = 0,$$

i.e. as $t^{-(2m-1)}$. More precisely, we want to prove that for $t \in [k-1, k+1)$ we have

$$\begin{aligned}&\|u(t)\|_m^2 \ge \Theta_1 \nu^{-(2m-1)} \Longrightarrow \\ &\frac{d}{dt}\|u(t)\|_m^2 \le -(2m-1)\|u(t)\|_m^{4m/(2m-1)},\end{aligned} \tag{3.17}$$

where Θ_1 is a random positive constant, chosen later. Random constants Θ below do not depend on Θ_1.

Indeed, assume that

$$\|u(t)\|_m^2 \ge \Theta_1 \nu^{-(2m-1)}. \tag{3.18}$$

We begin by observing that by Lemma 3.1.1 we have

$$\|u\|_m \leq C\,\|u\|_{m+1}^{(2m-1)/(2m+1)}\,|u|_{1,1}^{2/(2m+1)},$$

and hence

$$\begin{aligned}\|u\|_{m+1} &\geq C\,|u|_{1,1}^{-2/(2m-1)}\,\|u\|_m^{(2m+1)/(2m-1)}\\ &\geq \Theta^{-1}\,\|u\|_m^{(2m+1)/(2m-1)}\end{aligned}\tag{3.19}$$

(we used (3.16)). Now, (3.14), (3.16), and Lemma 3.5.1 imply that

$$\begin{aligned}&\frac{d}{dt}\,\|u\|_m^2 \leq -2\nu\,\|u\|_{m+1}^2 + \Theta\,\|u\|_m\,\|u\|_{m+1}\\ &=(-2\nu\,\|u\|_{m+1}^{2/(2m+1)} + \Theta\,\|u\|_m\,\|u\|_{m+1}^{-(2m-1)/(2m+1)})\,\|u\|_{m+1}^{4m/(2m+1)}.\end{aligned}\tag{3.20}$$

Combining (3.20) and (3.19), we get

$$\frac{d}{dt}\,\|u\|_m^2 \leq(-2\nu\,\|u\|_{m+1}^{2/(2m+1)} + \Theta)\,\|u\|_{m+1}^{4m/(2m+1)}.$$

Therefore, by (3.19) and (3.18) we have

$$\begin{aligned}\frac{d}{dt}\,\|u\|_m^2 &\leq\Big(-\nu\Theta^{-1}\,\|u\|_m^{2/(2m-1)} + \Theta\Big)\,\|u\|_{m+1}^{4m/(2m+1)}\\ &\leq\Big(-\Theta^{-1}\Theta_1^{1/(2m-1)} + \Theta\Big)\,\|u\|_{m+1}^{4m/(2m+1)}.\end{aligned}$$

Now we choose Θ_1 in such a way that the quantity in the parentheses is negative. Under this assumption, we get from (3.19) that

$$\frac{d}{dt}\,\|u\|_m^2 \leq \Big(-\Theta^{-1}\Theta_1^{1/(2m-1)} + \Theta\Big)\,\Theta^{-1}\,\|u\|_m^{4m/(2m-1)}.$$

This relation implies (3.17) if we choose for Θ_1 a sufficiently big random constant with all moments finite.

Now we claim that

$$\left\|u_k^-\right\|_m^2 \leq \Theta_2\nu^{-(2m-1)},\tag{3.21}$$

where

$$\Theta_2 = \max(\Theta_1,\ 1)$$

has finite moments. Indeed, if $\|u(t)\|_m^2 \le \Theta_1 \nu^{-(2m-1)}$ for some $t \in [k-1, k)$, then (3.17) ensures that $\|u(t)\|_m^2$ remains under this threshold up to $t = k^-$. Otherwise, we consider the function

$$y(t) = \|u(t)\|_m^{-2/(2m-1)}, \quad t \in [k-1, k)\,.$$

By (3.17), since $\|u(t)\|_m^2 > \Theta_1 \nu^{-(2m-1)}$, $y(t)$ increases at least as fast as t. Indeed,

$$\begin{aligned} \frac{d}{dt} y(t) &= -\frac{1}{2m-1} \left(\|u(t)\|_m^2 \right)^{-2m/(2m-1)} \frac{d}{dt} \|u(t)\|_m^2 \\ &\ge \frac{1}{2m-1} \|u(t)\|_m^{-4m/(2m-1)} (2m-1) \|u(t)\|_m^{4m/(2m-1)} \\ &\ge 1. \end{aligned}$$

Therefore $\|y(k^-)\|_m^2 \ge 1$. Since $\nu \le 1$, then in this case we also have (3.21).

In exactly the same way, using (3.16), we obtain that for $t \in [k, k+1)$,

$$\begin{aligned} \|u(t)\|_m^2 &\le \max(\Theta_2 \nu^{-(2m-1)}, \|u(k)\|_m^2) \\ &\le \max \left[\Theta_2, \left(\Theta + \sqrt{\Theta_2} \right)^2 \right] \nu^{-(2m-1)} \\ &\le \left(\Theta + \sqrt{\Theta_2} \right)^2 \nu^{-(2m-1)}. \end{aligned}$$

Therefore $\|u(t)\|_m^2 \nu^{2m-1}$ is uniformly bounded by $\left(\Theta + \sqrt{\Theta_2}\right)^2$ for $t \in [k, k+1)$. Since all moments of this random variable are finite, the lemma's assertion is proved. $\square$

3.6 Estimates of other Sobolev norms.

The results in the three previous sections enable us to find upper and lower estimates for a large class of Sobolev norms. Unfortunately, while lower estimates extend to the whole Sobolev scale for $m \ge 0$ and $p \in [1, +\infty]$, there is a gap, corresponding to the case $m \ge 2$ and $p = 1$, for upper estimates.

LEMMA 3.6.1. *For $m \in \{0, 1\}$ and $p \in [1, +\infty]$, or for $m \ge 2$ and $p \in (1, +\infty]$, we have*

$$\left(\mathbb{E} \sup_{t \in [k, k+1)} |u(t)|_{m,p}^n \right)^{1/n} \le C(m, p, n) \nu^{-\gamma}, \quad n \ge 1, \; k \ge 2.$$

Here and later on,

$$\gamma = \gamma(m,p) = \max\Big(0,\ m - \frac{1}{p}\Big).$$

Proof. We begin by considering the case $m = 1$ and $p \in [2, +\infty]$. Since by Lemma 3.1.1 we have

$$|u(t)|_{m,p} \leq C(m,p) \, \|u(t)\|_m^{1-\theta} \, \|u(t)\|_{m+1}^{\theta},$$

where

$$\theta = \frac{1}{2} - \frac{1}{p},$$

then Theorem 3.5.2 and Hölder's inequality yield the wanted result.

The case $m = 1$ and $p \in [1, 2)$ is proved in exactly the same way, by combining Corollary 3.3.3 and Theorem 3.5.2 ($m = 1$). The same method is used to prove the case $m \geq 2$ and $p \in (1, 2)$, combining the case $p \in [2, +\infty]$ for a big enough value of m and Corollary 3.3.3. Unfortunately, it cannot be applied for $m \geq 2$ and $p = 1$, because Lemma 3.1.1 only allows us to estimate a $W^{n,1}$ norm from above by other $W^{n,1}$ norms.

Finally, the case $m = 0$ follows from Corollary 3.3.4. □

The first norm that we estimate from below is the L_2 norm.

Lemma 3.6.2. *We have*

$$\Big(\int_k^{k+1} \mathbb{E}|u(s)|^2\Big)^{1/2} \geq C, \quad k \geq 2.$$

Proof. Using Properties **(i)** and **(iii)** of the kicks (u_k^- and ζ_k being independent), we get

$$\begin{aligned}\mathbb{E}\left|u_k^+\right|^2 &= \mathbb{E}\left|u_k^-\right|^2 + 2\mathbb{E}\left\langle u_k^-,\ \zeta_k\right\rangle + \mathbb{E}\left|\zeta_k\right|^2 \\ &= \mathbb{E}\left|u_k^-\right|^2 + \mathbb{E}\left|\zeta_k\right|^2 \geq I_0.\end{aligned}$$

On the other hand, by Theorem 3.5.2 we have

$$\mathbb{E}\, \|u(t)\|_1^2 \leq C'\nu^{-1}, \quad t \in (k, k+1).$$

Since

$$\frac{d}{dt}\, |u(t)|^2 = -2\nu\, \|u(t)\|_1^2, \quad t \in (k, k+1),$$

then, integrating in time and setting

$$d = \min\Big(1, \frac{I_0}{4C'}\Big),$$

we obtain that, for $s \in [k, k+d]$,

$$\mathbb{E}|u(s)|^2 \geq \mathbb{E}\left|u_k^+\right|^2 - 2(s-k)C' \geq I_0 - 2C'd \geq \frac{I_0}{2}.$$

Therefore

$$\int_k^{k+1} \mathbb{E}|u(s)|^2 \geq \min\Big(\frac{I_0}{2},\ \frac{I_0^2}{8C'}\Big) > 0,$$

which proves the lemma's assertion. □

Now we can study the case $m = 0$ and $p \in [1, +\infty]$.

Corollary 3.6.3. *We have*

$$\Big(\int_k^{k+1} \mathbb{E}|u(s)|_p^2\Big)^{1/2} \geq C, \quad k \geq 2,\ p \in [1, +\infty],$$

where C does not depend on p.

Proof. It suffices to prove the inequality for $p = 1$. Using Hölder's inequality and integrating in time and in ensemble, and then using the Cauchy-Schwarz inequality, we get

$$\begin{aligned}\int_k^{k+1} \mathbb{E}|u|_1^2 &\geq \int_k^{k+1} \mathbb{E}|u|^4 |u|_\infty^{-2} \\ &\geq \Big(\int_k^{k+1} \mathbb{E}|u|^2\Big)^2 \Big(\int_k^{k+1} \mathbb{E}|u|_\infty^2\Big)^{-1}.\end{aligned}$$

Lemma 3.6.2 and Corollary 3.3.4 ($p = +\infty$) complete the proof. □

Since the $W^{1,1}$ norm dominates the L_∞ norm, we get

Corollary 3.6.4. *We have*

$$\Big(\int_k^{k+1} \mathbb{E}|u(s)|_{1,1}^2(t)\Big)^{1/2} \geq C, \quad k \geq 2.$$

The cases $m \geq 2$ and $m = 1$, $p \geq 2$ follow from Lemma 3.4.1 and Lemma 3.1.1 by interpolation in the same way as Lemma 3.4.2, for $p > 1$. The case $p = +\infty$ follows from the case $p = 1$, since $|u|_{m,1} \geq |u|_{m-1,\infty}$, and $\gamma(m,1) = \gamma(m-1,+\infty)$.

LEMMA 3.6.5. *If either $m \geq 2$ and $p \in [1,+\infty]$, or $m = 1$ and $p \in [2,+\infty]$, then*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^2_{m,p}\Big)^{1/2} \geq C(m,p)\nu^{-\gamma}, \quad t \geq 1,\ T \geq N+1,$$

where N is the same as in Lemma 3.4.1.

Now it remains to deal with the case $m = 1$ and $p \in (1,2)$.

LEMMA 3.6.6. *For $p \in (1,2)$ we have*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^2_{1,p}\Big)^{1/2} \geq C(p)\nu^{-\gamma}, \quad t \geq 2,\ T \geq N+1,$$

where N is the same as in Lemma 3.4.1. Note that here, $\gamma = 1 - 1/p$.

Proof. In the proof of this lemma, $C'(p)$ denotes various positive constants depending only on p. By Hölder's inequality in space we have

$$\|u(s)\|_1^2 \leq |u(s)|^p_{1,p}\,|u(s)|^{(2-p)}_{1,\infty}.$$

Therefore, using Hölder's inequality in time and in ensemble, as well as Lemma 3.6.1, we get

$$\begin{aligned}\frac{1}{T}\int_t^{t+T} \mathbb{E}\,\|u(s)\|_1^2 \leq &\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^2_{1,\infty}\Big)^{(2-p)/2}\cdot\\ &\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^2_{1,p}\Big)^{p/2}\\ &\leq C'(p)\nu^{(p-2)}\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^2_{1,p}\Big)^{p/2}.\end{aligned}$$

Furthermore, Lemma 3.4.1 implies that

$$\begin{aligned}\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^2_{1,p} &\geq C'(p)\Big(\nu^{(2-p)}\frac{1}{T}\int_t^{t+T} \mathbb{E}\,\|u(s)\|_1^2\Big)^{2/p}\\ &\geq C'(p)\Big(\nu^{(2-p)}\nu^{-1}\Big)^{2/p}\\ &\geq C'(p)\nu^{-(2p-2)/p}.\end{aligned}$$

□

Remark 3.6.7. *Upper estimates for*

$$\Big(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|^n_{m,p}\Big)^{1/n},\quad n\geq 2$$

follow from the lemmas above and Hölder's inequality.

3.7 Conclusion

Putting together the estimates that we have obtained, we formulate our main result.

Theorem 3.7.1. *For $m\in\{0,1\}$ and $p\in[1,+\infty]$, or for $m\geq 2$ and $p\in(1,+\infty]$, we have*

$$\Big(\mathbb{E}\sup_{t\in[k,k+1)}|u(t)|^n_{m,p}\Big)^{1/n}\leq C(m,p,n)\nu^{-\gamma},\quad n\geq 1,\ k\geq 2. \tag{3.22}$$

Moreover, there is an integer $N'\geq 1$ such that, for $m\geq 0$ and $p\in[1,+\infty]$, we have

$$\Big(\frac{1}{T}\int_t^{t+T}\mathbb{E}|u(s)|^n_{m,p}\Big)^{1/n}\geq C(m,p)\nu^{-\gamma},\quad n\geq 2,\ t\geq 2,\ T\geq N'. \tag{3.23}$$

In both inequalities

$$\gamma=\max\Big(0,\ m-\frac{1}{p}\ \Big).$$

For a solution u of (3.6), we have obtained asymptotic estimates for expectations of a large class of Sobolev norms. The power of ν is clearly optimal except for $m\geq 2$ and $p=1$, since it coincides for upper and lower estimates: we are in a *quasi-stationary regime*. Let us stress again that the upper bound $t=2$ for the time needed for a quasi-stationary regime to be established has no dependence on u_0. The condition $t\geq T_0$ for some time $T_0\geq 1$ is necessary: we need damping if u_0 is large and injection of energy at a kick point if u_0 is small.

Now put $\hat{u}_k=a_k(u)+ib_k(u)$ (see (3.5)). For $t\geq 2$ and T big enough (see Theorem 3.7.1), consider the averaged quantities

$$F_{s,\theta}=\frac{1}{T}\int_t^{t+T}\frac{\sum_{k\in I(s,\theta)}\mathbb{E}|\hat{u}_k|^2(\tau)}{\sum_{k\in I(s,\theta)}1},\qquad s,\theta>0,$$

where $I(s,\theta) = [\nu^{-s+\theta}, \nu^{-s-\theta})$. In the same way as in [7, formulas (1.6)-(1.8)], the inequalities (3.22-3.23) yield

$$F_{s,\theta} \leq C\nu^{2s} \tag{3.24}$$

$$F_{s,\theta} \leq C(m)\nu^{2+2m(s-1-\theta)}, \quad m > 0, \ s > 1+\theta \tag{3.25}$$

$$F_{1,\theta} > C\nu^{2+2\theta} \tag{3.26}$$

for $\nu \leq \nu(\theta)$ with some $\nu(\theta) > 0$. These results have some consequences for the energy spectrum of u.

Indeed, relation (3.25) implies that the energy of the k-th Fourier mode, $E_k = \frac{1}{2T}\int_t^{t+T} \mathbb{E}|\hat{u}_k|^2$, averaged around $k = l$, where $l \gg \nu^{-1}$, decays faster than any negative degree of l. On the other hand, by (3.24) and (3.26), the energy E_k, averaged around $k = \nu^{-1}$, behaves as k^{-2}. That is, the interval $k \in (\nu^{-1}, +\infty)$ is the *dissipation range*, where the energy E_k decays fast.

As the force η is smooth in x, then the energy is injected at frequencies $k \sim 1$. The estimate (3.24) readily implies that the energy $E = \sum E_k$ of a solution u is supported, when $\nu \to 0$, by any interval $(0, \nu^{-\gamma})$, $\gamma > 0$. That is, the *energy range* of the solution u is the interval $(0, \nu^0]$ (see [27]).

The complement to the energy and dissipation ranges is the *inertial range* (ν^0, ν^{-1}). At $k \sim \nu^{-1}$ we have $E_k \sim k^{-2}$. It is plausible that in this range E_k decays algebraically; possibly $E_k \sim k^{-2}$. The study of the energy spectrum of solutions u in the inertial range is one of the objectives of our future research.

We recall that the behavior of the energy spectrum E_k of turbulent fluid of the form "some negative degree of k in the inertial range, followed by fast decay in the dissipation range" is suggested by the Kolmogorov theory of turbulence (see [27]). Our results (following those of A.Biryuk in [7]) show that for the "Burgulence" (described by the Burgers equation, see [6]) the dissipation range is $(\nu^{-1}, +\infty)$ and suggest that the power-law in the inertial range is $E_k \sim k^{-2}$.

We also see that for $\nu \to 0^+$, solutions u display intermittency-type behavior (see [27, Chapter 8]). Indeed, in the quasi-stationary regime, up to averaging in time and in ensemble, $\max_{x \in S^1} u_x \sim 1$, whereas $\int_{S^1} u_x^2 \sim \nu^{-1}$. Thus, typically u has large negative gradients on a small subset of S^1, and small positive gradients on a large subset of S^1.

In a future paper, we will look at the same problem with the kick force replaced by a spatially smooth white noise in time (see [23] for a possible definition). This problem is, heuristically, the limit case of the kick-forced problem with more and more frequent appropriately scaled kicks.

Chapitre 4

Estimations précises pour la turbulence dans l'équation de Burgers généralisée avec forçage de type bruit blanc

Ce chapitre correspond à l'article *Sharp Estimates for Turbulence in White-Forced Generalised Burgers Equation*, soumis à Communications in Mathematical Physics.

Abstract. We consider a non-homogeneous generalised Burgers equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = \eta,\ t \geq 0,\ x \in S^1.$$

Here f is strongly convex, ν is small and positive, while η is a random forcing term, smooth in space and white in time.

For any solution u of this equation we consider the quasi-stationary regime, corresponding to $t \geq T_1$, where $T_1 > 0$ depends only on f and the distribution of η. We obtain sharp upper and lower bounds for Sobolev norms of u averaged in time and in ensemble. These results yield sharp upper and lower bounds for natural analogues of quantities characterising the hydrodynamical turbulence. All our bounds do not depend on the initial condition, and hold uniformly in ν.

Estimates similar to some of our results have been obtained by Aurell, Frisch, Lutsko, and Vergassola on a physical level of rigour; we use some arguments from their article.

Introduction

The generalised one-dimensional space-periodic Burgers equation

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = 0, \quad \nu > 0,\ x \in S^1 = \mathbb{R}/\mathbb{Z} \tag{4.1}$$

(the classical Burgers equation [13] corresponds to $f(u) = u^2/2$) is a popular model for the Navier-Stokes equation, since both of them have similar nonlinearities and dissipative terms. For $\nu \ll 1$ and f strongly convex, i.e. satisfying:

$$f''(x) \geq \sigma > 0, \quad x \in \mathbb{R}, \tag{4.2}$$

solutions of (4.1) display turbulent-like behaviour, called “Burgulence” [5, 6]. In this paper we are interested in qualitative and quantitative properties of the Burgulence.

To simplify presentation, we restrict ourselves to solutions with zero mean value in space:

$$\int_{S^1} u(t,x)dx = 0, \quad \forall t \geq 0. \tag{4.3}$$

Accordingly, we assume that the initial value $u(0,\cdot)$ satisfies (4.3).

In [7], Biryuk considered (4.1) with f satisfying (4.2). He studied norms in space of solutions u for small values of ν and obtained the following estimates for L_2 norms of the m-th derivatives:

$$\|u(t)\|_m^2 \leq C\nu^{-(2m-1)},\ \frac{1}{T}\int_0^T \|u(t)\|_m^2 \geq c\nu^{-(2m-1)}, \quad m \geq 1,\ \nu \leq \nu_0.$$

Note that the exponents for ν in lower and upper bounds are the same. The constants ν_0, C, c, and T depend on the deterministic initial condition u_0 as well as on m. To get results independent of the initial data, a natural idea is to introduce random forcing and to estimate ensemble-averaged characteristics of solutions. In the previous article [9], we have considered the case when 0 in the right-hand side of (4.1) is replaced by a random spatially smooth force, “kicked” in time. In this article we consider

$$\frac{\partial u}{\partial t} + f'(u)\frac{\partial u}{\partial x} - \nu\frac{\partial^2 u}{\partial x^2} = \eta^\omega, \tag{4.4}$$

where η^ω is a random force, white in time and smooth in space. Heuristically this force corresponds to a scaled limit of “kicked” forces with more and more frequent kicks. All forces that we consider satisfy (4.3).

Study of Sobolev norms of solutions for nonlinear PDEs with small viscosity (with or without random forcing) is motivated by the problem of turbulence. This research was initiated by Kuksin, who obtained lower and upper estimates of these norms by negative powers of the viscosity for a large class of equations (see [41, 42] and references in [42]), and continued by Biryuk [7]. We use some methods and ideas from those works. Note that for the Burgers equation considered in [7, 9] and in our work, estimates on Sobolev norms are asymptotically sharp in the sense that viscosity enters lower and upper bounds at the same negative power. Such estimates are not available for the more complicated equations considered in [41, 42].

In this work, after introducing the notation and setup in Section 4.1, we formulate the main results in Section 4.2. In Section 4.3, for $t \geq 1$, we estimate from above the moments of $\max \partial u/\partial x$ for solutions $u(t, x)$ of (4.4). Using these bounds, in Sections 4.3-4.5 we obtain estimates of the same type as in [7, 9], valid for time $t \geq T_1 = T_0 + 2$. Here, T_0 is a constant, independent of the initial condition and of ν. Actually, for $t \geq T_1$, we are in a quasi-stationary regime: all estimates hold uniformly in t, ν, and in the intial condition u_0.

In Section 4.6 we study implications of our results in terms of the theory of Burgulence. Namely, we give sharp upper and lower bounds for the dissipation length scale, increments, flatness, and spectral asymptotics for the flow $u(t, x)$. These bounds hold uniformly for $\nu \leq \nu_0$, where ν_0 is a constant which is independent of the initial condition.

The results of Section 4.6 rigorously justify the predictions for space increments of solutions $u(t, x)$ and for their spectral asymptotics made in [2, 22, 32, 38]; see also [14]. One proof in this section uses some constructions and arguments from [2]. Note that predictions for spectral asymptotics have been known since the 1950s: in [38], the author refers to some earlier results by Burgers and Tatsumi.

The rigorous proof of the asymptotics predicted by a physical argument, even for such a relatively simple model as the stochastic Burgers equation, is important since for the 3D or 2D incompressible Navier-Stokes equations there is no exact theory of this type, corresponding to the heuristic theories due to Kolmogorov and Kraichnan.

The stochastic Burgers equation admits a unique stationary measure μ_{stat}. Estimates in Sections 4.3-4.6 still hold if we replace averaging in time and probability by averaging with respect to μ_{stat}. Moreover, the rate of convergence to μ_{stat} in L_1 does not depend on the viscosity. We will give details of the proof in a future publication.

We are concerned with solutions for (4.13) with small but positive ν. For a detailed study of the limiting dynamics with $\nu = 0$, see [23]. Additional properties for both cases $\nu = 0$ and $\nu > 0$ have been established in [28, 31].

The results of Sections 4.6-4.7 also hold in the case of a "kicked" force, since for them we have estimates analogous to those in Sections 4.3-4.5 (see [9]). Finally, we would like to note that similar estimates hold in the case of the multidimensional potential randomly forced Burgers equation (see [6] for physical predictions). Those estimates will be the subject of a future publication.

4.1 Notation and Setup

Agreement: In the whole paper, all functions that we consider are real-valued.

4.1.1 Sobolev Spaces

Consider a zero mean value integrable function v on S^1. For $p \in [1, \infty]$, we denote its L_p norm by $|v|_p$. The L_2 norm is denoted by $|v|$, and $\langle \cdot, \cdot \rangle$ stands for the L_2 scalar product. From now on L_p, $p \in [1, \infty]$, denotes the space of zero mean value functions in $L_p(S^1)$. Similarly, C^∞ is the space of C^∞-smooth zero mean value functions on S^1.

For a nonnegative integer m and $p \in [1, \infty]$, $W^{m,p}$ stands for the Sobolev space of zero mean value functions v on S^1 with finite homogeneous norm

$$|v|_{m,p} = \left| \frac{d^m v}{dx^m} \right|_p .$$

In particular, $W^{0,p} = L_p$ for $p \in [1, \infty]$. For $p = 2$, we denote $W^{m,2}$ by H^m, and abbreviate the corresponding norm as $\|v\|_m$.

Note that since the length of S^1 is 1 and the mean value of v vanishes, we have

$$|v|_1 \leq |v|_\infty \leq |v|_{1,1} \leq |v|_{1,\infty} \leq \cdots \leq |v|_{m,1} \leq |v|_{m,\infty} \leq \dots$$

We recall a version of the classical Gagliardo-Nirenberg inequality (see [21, Appendix]):

LEMMA 4.1.1. *For a smooth zero mean value function v on S^1,*

$$|v|_{\beta,r} \leq C\,|v|_{m,p}^{\theta}\,|v|_q^{1-\theta}\,,$$

where $m > \beta \geq 0$, and r is defined by

$$\frac{1}{r} = \beta - \theta\Big(m - \frac{1}{p}\Big) + (1-\theta)\frac{1}{q},$$

under the assumption $\theta = \beta/m$ if $p = 1$ or $p = \infty$, and $\beta/m \leq \theta < 1$ otherwise. The constant C depends on m, p, q, β, θ.

For any $s \geq 0$, H^s stands for the Sobolev space of zero mean value functions v on S^1 with finite norm

$$\|v\|_s = (2\pi)^s \Big(\sum_{k\in\mathbb{Z}} |k|^{2s}|\hat{v}_k|^2\Big)^{1/2}, \tag{4.5}$$

where $\hat{v}_k$ are the complex Fourier coefficients of $v(x)$. For integer values of $s = m$, this norm coincides with the previously defined H^m norm. For $s \in (0,1)$, $\|v\|_s$ is equivalent to the norm

$$\|v\|_s' = \left(\int_{S^1}\Big(\int_0^1 \frac{|v(x+\ell)-v(x)|^2}{\ell^{2s+1}}d\ell\Big)dx\right)^{1/2} \tag{4.6}$$

(see [1, 58]).

Subindices t and x, which can be repeated, denote partial differentiation with respect to the corresponding variables. We denote $v^{(m)}$ the m-th derivative of v in the variable x. For brevity, the function $v(t,\cdot)$ is denoted by $v(t)$.

4.1.2 Random Setting

We provide each space $W^{m,p}$ with the Borel σ-algebra. Then we consider a random process $w(t) = w^{\omega}(t),\ \omega \in \Omega,\ t \geq 0$, defined on a complete probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and valued in L_2. We assume that $w(t)$ defines a Wiener process with respect to a filtration $\mathcal{F}_t,\ t \geq 0$, in each space $H^m,\ m \geq 0$. In particular, for $\zeta, \chi \in L_2$,

$$\mathbb{E}(\langle w(s),\zeta\rangle\,\langle w(t),\chi\rangle) = \min(s,t)\,\langle Q\zeta,\chi\rangle\,,$$

where Q is a symmetric operator which defines a continuous mapping $Q : L_2 \to H^m$ for each m. Thus, $w(t) \in C^\infty$ for every t, almost surely. We will denote $w(t)(x)$ by $w(t,x)$. For more details, see [19, Chapter 4]. For $m \geq 0$, we denote by I_m the quantity

$$I_m = Tr_{H^m}(Q) = \mathbb{E} \left\| w(1) \right\|_m^2 .$$

For example we could take

$$w(t) = \sqrt{2} \sum_{k \geq 1} b_k \beta_k(t) \sin(2\pi kx) + \sqrt{2} \sum_{k \leq -1} b_k \beta_k(t) \cos(2\pi kx),$$

where $\beta_k(t),\ k \neq 0$ are standard independent Wiener processes and $I_m = \sum_{k \neq 0} b_k^2 (2\pi k)^{2m} < \infty$ for each m. From now on, the term $dw(s)$ denotes the stochastic differential corresponding to the Wiener process $w(s)$ in the space L_2.

Now fix $m \geq 0$. By Fernique's Theorem [46, Theorem 3.3.1], there exist $\iota_m, C_m > 0$ such that

$$\mathbb{E} \exp\Big(\iota_m \left\| w(T) \right\|_m^2 / T \Big) \leq C_m. \tag{4.7}$$

Therefore by Doob's maximal inequality for infinite-dimensional submartingales [19, Theorem 3.8. (ii)] we have

$$\mathbb{E} \sup_{t \in [0,T]} \left\| w(t) \right\|_m^p \leq \Big(\frac{p}{p-1} \Big)^p \mathbb{E} \left\| w(T) \right\|_m^p < +\infty, \tag{4.8}$$

for any $T > 0$, and $p \in (1, \infty)$. Moreover, applying Doob's maximal inequality to $\exp(\alpha \left\| w(T) \right\|_m)$ and maximising in α, we prove the existence of $C'_m > 0$ such that

$$\mathbb{P}(\sup_{t \in [0,T]} \left\| w(t) \right\|_m \geq \iota) \leq \exp(-\iota^2 / 2C'_m T), \tag{4.9}$$

for any $T, \iota > 0$.

Now we consider the martingale

$$\xi(t) = \int_0^t L^m w(s) dw(s) = \frac{1}{2} \Big(\left\| w(t) \right\|_m^2 - t I_m \Big).$$

Denote by

$$\langle\langle \xi \rangle\rangle_{[0,t]}$$

its quadratic variation over the time interval $[0,t]$. Using an infinite-dimensional version of the supermartingale inequality (see [19, Lemma 10.15], [49, Section 2.9]), we get

$$\mathbb{P}\Big(\sup_{t\in[0,T]} \exp\Big(\iota\xi(t) - \frac{\iota^2}{2}\langle\langle\xi\rangle\rangle_{[0,t]}\Big) \geq \exp(\iota X)\Big) \leq \exp(-\iota X).$$

for any $T, \iota, X > 0$. Since I_m is the trace of the operator Q in H^m, we have the inequality:

$$\begin{aligned}
\iota\xi(s) - \frac{\iota^2}{2}\langle\langle\xi\rangle\rangle_{[0,t]} &= \frac{\iota}{2}\Big(\|w(t)\|_m^2 - tI_m\Big) - \frac{\iota^2}{2}\int_0^t \langle L^m w(s),\ Qw(s)\rangle ds \\
&= \frac{\iota}{2}\Big(\|w(t)\|_m^2 - tI_m\Big) - (-1)^m\frac{\iota^2}{2}\int_0^t \langle w^{(m)}(s),\ Qw^{(m)}(s)\rangle ds \\
&\geq \frac{\iota}{2}\Big(\|w(t)\|_m^2 - tI_m\Big) - \frac{I_m\iota^2}{2}\int_0^t \|w(s)\|_m^2 ds,
\end{aligned}$$

then we get

$$\begin{aligned}
&\mathbb{P}\Big(\sup_{t\in[0,T]} \exp\Big(\frac{\iota}{2}(\|w(t)\|_m^2 - tI_m) - \frac{I_m\iota^2}{2}\int_0^t \|w(s)\|_m^2 ds\Big) \geq \exp(\iota X)\Big) \\
&\leq \exp(-\iota X).
\end{aligned} \tag{4.10}$$

Since for $\beta > 0$ we have

$$\begin{aligned}
\mathbb{E}\sup_{t\in[0,T]} \exp\Big(\beta\int_0^t \|w(s)\|_m^2 ds\Big) &= \mathbb{E}\exp\Big(\beta\int_0^T \|w(s)\|_m^2 ds\Big) \\
&\leq \mathbb{E}\exp\Big(\beta T\|w(T)\|_m^2\Big),
\end{aligned}$$

then inequalities (4.7) and (4.10) imply existence, for every $T > 0$, of $\sigma_{m,T} > 0$ such that

$$\mathbb{E}\sup_{t\in[0,T]} \exp(\sigma\,\|w(t)\|_m^2) < +\infty. \tag{4.11}$$

4.1.3 Preliminaries

We begin by considering the free generalised Burgers-type parabolic equation (4.1). Here, $t \geq 0$, $x \in S^1 = \mathbb{R}/\mathbb{Z}$, and the viscosity coefficient satisfies

$\nu \in (0,1]$. The function f is C^∞-smooth and strongly convex, i.e. it verifies (4.2). We also assume that its derivatives satisfy:

$$\forall m \geq 0,\ \exists h \geq 0,\ C_m > 0 :\ |f^{(m)}(x)| \leq C_m(1+|x|)^h, \quad x \in \mathbb{R}, \tag{4.12}$$

where $h = h(m)$ is a function such that $h(1) = 2-\delta,\ \delta > 0$. Note that (4.2) yields that $\delta \in (0,1]$. The usual Burgers equation corresponds to $f(x) = x^2/2$.

The white-forced generalised Burgers equation is (4.1) with $\eta^\omega = \partial w^\omega/\partial t$, where $w^\omega(t),\ t \geq 0$ is the Wiener process with respect to $\mathcal{F}_t$ defined above.

Definition 4.1.2. *We say that an H^1-valued process $u(t,x) = u^\omega(t,x)$ is a solution of the equation*

$$\frac{\partial u^\omega}{\partial t} + f'(u^\omega)\frac{\partial u^\omega}{\partial x} - \nu\frac{\partial^2 u^\omega}{\partial x^2} = \eta^\omega \tag{4.13}$$

if

(i) For every t, $\omega \mapsto u^\omega(t,\cdot)$ is $\mathcal{F}_t$-measurable.

(ii) For a.e. (almost every) ω, $t \mapsto u^\omega(t,\cdot)$ is continuous in H^1 and satisfies

$$u^\omega(t) = u^\omega(0) - \int_0^t \Big(\nu L u^\omega(s) + \frac{1}{2}B(u^\omega)(s)\Big)ds + w^\omega(t), \tag{4.14}$$

where

$$B(u) = 2f'(u)u_x; \quad L = -\partial_{xx}.$$

When studying solutions of (4.13), we always assume that the initial condition $u_0 = u(0,\cdot)$ is C^∞-smooth, a.s. (almost surely). For a given random initial condition, we obtain that (4.13) has a unique solution, i.e. any two solutions coincide for a.e. ω. For brevity, this solution will be denoted by u. To prove this, we use the "mild solution" technique (cf. [20, Chapter 14]), a bootstrap argument, and finally uniform bounds of the same type as in Section 4.3.

Since the forcing and the initial condition are smooth in space, upper estimates of the same type as those proved in Section 4.3 allow us to show that $t \mapsto u(t)$ is time-continuous in H^m for every m, and $t \mapsto u(t) - w(t)$ has a time derivative in C^∞ for all t, a.s. In this paper, we always assume that u_0 satisfies (4.3); consequently, $u(t)$ satisfies (4.3) for all times.

Solutions of (4.13) make a time-continuous Markov process in H^1. For details, we refer to [45], where a white force is introduced in a similar setting.

Now consider, for a solution $u(t,x)$ of (4.13), the functional $G_m(u(t)) = \|u(t)\|_m^2$ and apply Itô's formula [19, Theorem 4.17] to (4.14):

$$\|u(t)\|_m^2 = \|u_0\|_m^2 + tI_m - \int_0^t \left(2\nu \|u(s)\|_{m+1}^2 + \langle L^m u(s),\ B(u)(s)\rangle\right) ds \\ + 2\int_0^t \langle L^m u(s),\ dw(s)\rangle \tag{4.15}$$

(we recall that $I_m = Tr(Q_m)$.) Consequently,

$$\frac{d}{dt}\mathbb{E}\|u(t)\|_m^2 = I_m - 2\nu\mathbb{E}\|u(t)\|_{m+1}^2 - \mathbb{E}\,\langle L^m u(t),\ B(u)(t)\rangle. \tag{4.16}$$

As $\langle u,\ B(u)\rangle = 0$, for $m = 0$ this relation becomes

$$\frac{d}{dt}\mathbb{E}\,|u(t)|^2 = I_0 - 2\nu\mathbb{E}\,\|u(t)\|_1^2. \tag{4.17}$$

4.1.4 Agreements

From now on, all constants denoted by C with sub- or superindexes are strictly positive and nonrandom. Unless otherwise stated, they depend only on f and the distribution of the Wiener process w. By $C(a_1,\dots,a_k)$ we denote constants which also depend on parameters $a_1,\dots,a_k$. By $X \overset{a_1,\dots,a_k}{\lesssim} Y$ we mean that $X \leq C(a_1,\dots,a_k)Y$. The notation $X \overset{a_1,\dots,a_k}{\sim} Y$ stands for

$$Y \overset{a_1,\dots,a_k}{\lesssim} X \overset{a_1,\dots,a_k}{\lesssim} Y.$$

In particular, $X \lesssim Y$ and $X \sim Y$ mean that $X \leq CY$ and $C^{-1}Y \leq X \leq CY$, respectively. All constants are independent of the viscosity ν and of the initial value u_0.

We denote by $u = u(t,x)$ a solution of (4.13) with an initial condition u_0. For simplicity, in Sections 4.3-4.6, we assume that u_0 is deterministic. However, using the Markov property we can easily generalise all results to the case of a random initial condition independent from $w(t), t \geq 0$. Indeed, for any measurable functional $\Phi(u(\cdot))$ we have

$$\mathbb{E}\Phi(u(\cdot)) = \int \mathbb{E}\Big(\Phi(u(\cdot))|u_0\Big) d\mu_{u_0},$$

where μ_{u_0} is the law of u_0.

All estimates which hold for time t or a time interval $[t, t+T]$ actually hold for time $t+\tau$ or a time interval $[t+\tau, t+\tau+T]$, uniformly in $\tau \geq 0$. Indeed, it is enough to consider the solution of (4.13) with initial condition $u(\tau)$. We will refer to this argument as the "starting time argument".

We use the notation $g^- = \max(-g, 0)$ and $g^+ = \max(g, 0)$. For the meaning of the brackets $\{\cdot\}$, see Subsection 4.6.1.

4.2 Formulation of the Main Results

In Sections 4.3-4.5, we prove sharp upper and lower estimates for a large class of Sobolev norms of u. A key result is proved in Theorem 4.3.1. Namely, there we obtain that for $k \geq 1$,

$$\mathbb{E}\,\Big(\max_{s\in[t,t+1]}\max_{x\in S^1} u_x(s,x)\Big)^k \overset{k}{\lesssim} 1, \quad t \geq 1. \tag{4.18}$$

The main estimates are those in the first part of Theorem 4.5.1. There we prove that for $m \in \{0, 1\}$ and $p \in [1, \infty]$ or for $m \geq 2$ and $p \in (1, \infty]$,

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{m,p}^\alpha\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq T_0 + 2,\ T \geq T_0, \tag{4.19}$$

where $\gamma = \max(0, m - 1/p)$ and T_0 is a constant, depending only on f and the distribution of the process w.

In Section 4.6 we obtain sharp estimates for analogues of quantities characterising hydrodynamical turbulence. In what follows, $\{\cdot\}$ denotes averaging in time and in ensemble (see Subsection 4.6.1). Although we only prove results for quantities averaged over a time period of length T_0, those results can be immediately extended to quantities averaged over time periods of length $T \geq T_0$.

To begin with, we assume that $\nu \in (0, \nu_0]$, where $\nu_0 \in (0, 1]$ only depends on f and the distribution of w. We define intervals

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1]$$

by analogy to the ranges in the Kolmogorov 1941 theory of turbulence [27]. The constants C_1, C_2, and ν_0 can take any value, as long as C_1/C_2 is large enough, and $C_1\nu_0 < C_2$. In particular, these assumptions ensure that the intervals J_1, J_2, and J_3 are non-empty and non-intersecting.

The interval J_1 corresponds to the *dissipation range*, i.e. for the Fourier modes k such that $|k|^{-1} \preceq C_1\nu$, $\{|\hat{u}_k|^2\}$ decreases super-algebraically in k. The interval J_2 corresponds to the *inertial range*, where the *energy spectrum*

$$E(k) = \left\{ \frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}_n|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right\}$$

behaves as a negative degree of k. Here $M \geq 1$ is a constant depending only on f and the distribution of w. The boundary $C_1\nu$ between these two ranges is the *dissipation length scale*. Finally, the interval J_3 corresponds to the *energy range*, i.e. the sum $\Sigma\{|\hat{u}_k|^2\}$ is mostly supported by the Fourier modes corresponding to $|k|^{-1} \in J_3$:

$$\sum_{|k| \leq C_2^{-1}} \{|\hat{u}_k|^2\} \geq \frac{99}{100} \sum_{k \in \mathbb{Z}} \{|\hat{u}_k|^2\}.$$

The proportion of energy contained in the modes from J_3 tends to 1 when C_2 tends to 0, uniformly in ν. Now consider the averaged moments of increments in the variable x for the flow $u(t,x)$:

$$S_{p,\alpha}(\ell) = \left\{ \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^{\alpha} \right\}, \ \alpha \geq 0, \ 0 < \ell \leq 1.$$

In particular, $S_{p,1}(\ell)$ is (up to averaging) the *structure function* of p-th order and is denoted by $S_p(\ell)$. As the first application of estimates (4.18-4.19), in Section 4.6 we obtain sharp estimates for the quantities $S_{p,\alpha}$. Namely, by Theorem 4.6.8, for $\ell \in J_1$:

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha p} \nu^{-\alpha(p-1)}, \ p \geq 1, \end{cases}$$

and on the other hand for $\ell \in J_2$:

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha}, \ p \geq 1. \end{cases}$$

Consequently, for $\ell \in J_2$ the flatness function $F(\ell) = S_4(\ell)/S_2^2(\ell)$ satisfies $F(\ell) \sim \ell^{-1}$. Thus, solutions u are highly intermittent in the inertial range (see [27]).

Using these results, we derive asymptotics for the energy spectrum of Burgulence. Namely, for all $m \geq 1$ and $k \in \mathbb{Z},\ k \neq 0$ we have

$$\{|\hat{u}_k|^2\} \overset{m}{\lesssim} (k\nu)^{-2m}\nu,$$

and for k such that $k^{-1} \in J_2$,

$$\left\{ \left(\frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}_n|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right)^{\alpha} \right\} \overset{\alpha}{\sim} k^{-2\alpha}, \quad \alpha > 0.$$

In particular, in the inertial range the energy spectrum satisfies $E(k) \sim k^{-2}$.

Finally, in Section 4.7, we observe that since (4.13) has a unique stationary measure μ_{stat}, then all estimates listed above still hold if we replace the brackets $\{\cdot\}$ with averaging with respect to μ_{stat}.

4.3 Upper Estimates for Sobolev Norms

The following theorem is proved using a stochastic version of Kruzhkov's maximum principle (cf. [40]). Note that in all results in Sections 4.3-4.6 the quantities estimated as functions of x for fixed t, ω, such as $\max_{x \in S^1} u_x$ or Sobolev norms, may be replaced by their suprema over all smooth initial conditions. For instance, the quantity

$$\mathbb{E} \max_{s \in [t,t+1]} |u(s)|_{m,p}$$

may be replaced by

$$\mathbb{E} \max_{s \in [t,t+1]} \sup_{u_0 \in C^\infty} |u(s)|_{m,p}.$$

Theorem 4.3.1. *Denote by X_t the random variable*

$$X_t = \max_{s \in [t,t+1]} \max_{x \in S^1} u_x(s,x).$$

For $k \geq 1$, we have

$$\mathbb{E}\ X_t^k \overset{k}{\lesssim} 1, \quad t \geq 1.$$

Proof. We take $t = 1$, denoting X_t by X, since by the "starting time argument" the general case $t \geq 1$ is proved in the same way.

Consider the equation (4.13) on the time interval $[0, 2]$. Putting $v = u - w$ and differentiating once in space, we get

$$\frac{\partial v_x}{\partial t} + f''(u)(v_x + w_x)^2 + f'(u)(v_x + w_x)_x = \nu(v_x + w_x)_{xx}. \tag{4.20}$$

Consider $\tilde{v}(t, x) = tv_x(t, x)$ and multiply (4.20) by t^2. For $t > 0$, $\tilde{v}$ verifies

$$\begin{aligned} & t\tilde{v}_t - \tilde{v} + f''(u)(\tilde{v} + tw_x)^2 + tf'(u)\tilde{v}_x + t^2 f'(u)w_{xx} \\ & = \nu t\tilde{v}_{xx} + \nu t^2 w_{xxx}. \end{aligned} \tag{4.21}$$

Now observe that if the zero mean function $\tilde{v}$ does not vanish identically on the domain $S = [0, 2] \times S^1$, then it attains its positive maximum N on S at a point (t_1, x_1) such that $t_1 > 0$. At (t_1, x_1) we have $\tilde{v}_t \geq 0$, $\tilde{v}_x = 0$, and $\tilde{v}_{xx} \leq 0$. By (4.21), at (t_1, x_1) we have the inequality

$$f''(u)(\tilde{v} + tw_x)^2 \leq \tilde{v} - t^2 f'(u)w_{xx} + \nu t^2 w_{xxx}. \tag{4.22}$$

Denote by A the random variable

$$A = \max_{t \in [0,2]} |w(t)|_{C^3}.$$

Since for every t, $tv(t)$ is the zero space average primitive of $\tilde{v}(t)$ on S^1, we get

$$\max_{t \in [0,2],\ x \in S^1} |tu| \leq \max_{t \in [0,2],\ x \in S^1} (|tv| + |tw|) \leq N + 2A.$$

Since $h(1) = 2 - \delta$ in (4.12), then we obtain that

$$\begin{aligned} \max_{t \in [0,2],\ x \in S^1} |t^2 f'(u)w_{xx}| & \leq At^\delta \max_{t \in [0,2],\ x \in S^1} t^{2-\delta}(|u| + 1)^{2-\delta} \\ & \leq CA((N + 2A)^{2-\delta} + 1). \end{aligned}$$

From now on, we assume that $N \geq 2A$. Since $\nu \in (0, 1]$ and $f'' \geq \sigma$, then the relation (4.22) yields

$$\sigma(N - 2A)^2 \leq N + CA(N + 2A)^{2-\delta} + CA.$$

Thus we have proved that if $N \geq 2A$, then $N \leq C(A + 1)^{1/\delta}$. Since by (4.8), all moments of A are finite, then all moments of N are as well finite. By definition of $\tilde{v}$ and S, the same is true for X. This proves the lemma's assertion. $\square$

REMARK 4.3.2. *Using an infinite-dimensional version of the supermartingale inequality (see [19, Lemma 10.15], [49, Section 2.9]), we can prove that there exist $\beta, \beta' > 0$ such that*

$$\mathbb{E}\exp(\beta X_t^{2\delta}) \leq \mathbb{E}\exp\left(\beta'(\max_{t\in[0,2]} |w(t)|_{C^3} + 1)^2\right) \lesssim 1, \quad t \geq 1.$$

COROLLARY 4.3.3. *For $k \geq 1$,*

$$\mathbb{E} \max_{s\in[t,t+1]} |u(s)|_{1,1}^k \overset{k}{\lesssim} 1, \quad t \geq 1.$$

Proof. The space average of $u_x(s)$ vanishes identically. Therefore

$$\int_{S^1} |u_x(s)| = 2\int_{S^1} (u_x(s))^+ \leq 2\max_{x\in S^1} u_x(s,x). \ \square$$

COROLLARY 4.3.4. *For $k \geq 1$,*

$$\mathbb{E} \max_{s\in[t,t+1]} |u(s)|_p^k \overset{k}{\lesssim} 1, \quad p \in [1,\infty], \ t \geq 1.$$

Now we recall a standard estimate for the nonlinearity $\langle L^m u, B(u)\rangle$ (see Subsection 4.1.3 for the definitions of L and B).

LEMMA 4.3.5. *For $w \in C^\infty$ such that $|w|_\infty \leq N$, we have*

$$N_m(w) = |\langle L^m w, B(w)\rangle| \leq C' \|w\|_m \|w\|_{m+1}, \quad m \geq 1,$$

with

$$C' = C_m(1+N)^{n'}, \tag{4.23}$$

where C_m, as well as the natural number $n' = n'(m)$, depend only on m.

Proof. Fix $m \geq 1$. Let C' denote various strictly positive constants of the form (4.23). We have

$$\begin{aligned} N_m(w) = 2\left|\left\langle w^{(2m)}, (f(w))^{(1)}\right\rangle\right| &= 2\left|\left\langle w^{(m+1)}, (f(w))^{(m)}\right\rangle\right| \\ \leq & C(m)\sum_{k=1}^m \sum_{\substack{1\leq a_1\leq\cdots\leq a_k\leq m\\ a_1+\cdots+a_k=m}} \int_{S^1} \left|w^{(m+1)}w^{(a_1)}\dots w^{(a_k)} f^{(k)}(w)\right| \\ \leq & C(m)\,|f|_{C^m([-N,N])} \sum_{k=1}^m \sum_{\substack{1\leq a_1\leq\cdots\leq a_k\leq m\\ a_1+\cdots+a_k=m}} \int_{S^1} |w^{(a_1)}\dots w^{(a_k)} w^{(m+1)}|. \end{aligned}$$

By (4.12), $|f|_{C^m([-N,N])}$ has an upper bound of the form (4.23). Using first Hölder's inequality and then Lemma 4.1.1, we get

$$\begin{aligned} N_m(w) \leq & C' \sum_{k=1}^{m} \sum_{\substack{1\leq a_1 \leq \dots \leq a_k \leq m \\ a_1+\dots+a_k=m}} \Big(\left|w^{(a_1)}\right|_{2m/a_1} \dots \left|w^{(a_k)}\right|_{2m/a_k} \|w\|_{m+1} \Big). \\ \leq & C' \|w\|_{m+1} \sum_{k=1}^{m} \sum_{\substack{1\leq a_1 \leq \dots \leq a_k \leq m \\ a_1+\dots+a_k=m}} \Big((\|w\|_m^{a_1/m} |w|_\infty^{(m-a_1)/m}) \times \dots \\ & \dots \times (\|w\|_m^{a_k/m} |w|_\infty^{(m-a_k)/m}) \Big) \\ \leq & C'(1+|w|_\infty)^{m-1} \|w\|_m \|w\|_{m+1} = C' \|w\|_m \|w\|_{m+1} . \ \square \end{aligned}$$

The following upper estimate for $\mathbb{E}\,\|u(t)\|_m^2$ holds uniformly for $t \geq 2$.

Lemma 4.3.6. *For $m \geq 1$,*

$$\mathbb{E}\,\|u(t)\|_m^2 \overset{m}{\lesssim} \nu^{-(2m-1)}, \quad t \geq 2.$$

Proof. Fix $m \geq 1$. We will use the notation

$$x(s) = \mathbb{E}\,\|u(s)\|_m^2; \quad y(s) = \mathbb{E}\,\|u(s)\|_{m+1}^2 .$$

We take $t = 2$; the general case follows by the "starting time argument". We claim that for $s \in [1, 2]$ we have the implication

$$\begin{aligned} & x(s) \geq C'\nu^{-(2m-1)} \Longrightarrow \\ & \frac{d}{ds}x(s) \leq -(2m-1)(x(s))^{2m/(2m-1)}, \end{aligned} \tag{4.24}$$

where $C' \geq 1$ is a fixed strictly positive number, chosen later. Below, all constants denoted by C are strictly positive and do not depend on C', and we denote by Z the quantity

$$Z = C'\nu^{-(2m-1)}.$$

Indeed, assume that $x(s) \geq Z$. By (4.16) and Lemma 4.3.5, we have

$$\begin{aligned} \frac{d}{ds}x(s) \leq & -2\nu\mathbb{E}\,\|u(s)\|_{m+1}^2 + C\mathbb{E}\Big((1+|u(s)|_\infty)^{n'} \|u(s)\|_m \\ & \times \|u(s)\|_{m+1}\Big) + I_m, \end{aligned}$$

with $n' = n'(m)$. Since by Lemma 4.1.1 applied to u_x, we get

$$\|u(s)\|_m \leq C \|u(s)\|_{m+1}^{(2m-1)/(2m+1)} |u(s)|_{1,1}^{2/(2m+1)}, \tag{4.25}$$

then

$$\begin{aligned}\frac{d}{ds}x(s) \leq & -2\nu\mathbb{E}\|u(s)\|_{m+1}^2 + C\mathbb{E}\Big((1+|u(s)|_{1,1})^{n'+2/(2m+1)} \\ & \times \|u(s)\|_{m+1}^{4m/(2m+1)}\Big) + I_m.\end{aligned}$$

Thus by Hölder's inequality and Corollary 4.3.3 we get

$$\frac{d}{ds}x(s) \leq \Big(-2\nu(y(s))^{1/(2m+1)} + C\Big)(y(s))^{2m/(2m+1)} + I_m.$$

On the other hand, (4.25), Hölder's inequality, and Corollary 4.3.3 yield

$$\begin{aligned}x(s) \leq & \; C(y(s))^{(2m-1)/(2m+1)}(\mathbb{E}|u(s)|_{1,1}^2)^{2/(2m+1)} \\ \leq & C(y(s))^{(2m-1)/(2m+1)},\end{aligned}$$

and thus

$$(y(s))^{1/(2m+1)} \geq C(x(s))^{1/(2m-1)}.$$

Consequently, since $x(s) \geq C'\nu^{-(2m-1)}$, then for C' large enough we have

$$\frac{d}{ds}x(s) \leq \left(-CC'^{1/(2m-1)} + C\right)(x(s))^{2m/(2m-1)} + I_m.$$

Thus we can choose C' in such a way that (4.24) holds.

Now we claim that

$$x(2) \leq Z. \tag{4.26}$$

Indeed, if $x(s) \leq Z$ for some $s \in [1,2]$, then the assertion (4.24) ensures that $x(s)$ remains below this threshold up to $s = 2$: thus we have proved (4.26).

Now, assume that $x(s) > Z$ for all $s \in [1,2]$. Denote

$$\tilde{x}(s) = (x(s))^{-1/(2m-1)}, \; s \in [1,2].$$

Using the implication (4.24) we get $d\tilde{x}(s)/ds \geq 1$. Therefore $\tilde{x}(2) \geq 1$. As $\nu \leq 1$ and $C' \geq 1$, we get $x(2) \leq Z$. Thus in both cases inequality (4.26) holds. This proves the lemma's assertion. □

Corollary 4.3.7. *For $m \geq 1$,*

$$\mathbb{E}\,\|u(t)\|_m^k \overset{m,k}{\lesssim} \nu^{-k(2m-1)/2}, \quad k \geq 1,\ t \geq 2.$$

Proof. The cases $k = 1, 2$ follow immediately from Lemma 4.3.6.

For $k \geq 3$, we consider only the case when k is odd, since the general case follows by Hölder's inequality. Setting $N = ((2m-1)k+1)/2$ and applying Lemma 4.1.1, we get

$$\|u(t)\|_m^k \overset{m,k}{\lesssim} \|u(t)\|_N\,|u(t)|_{1,1}^{k-1}.$$

Therefore, by Hölder's inequality, Lemma 4.3.6, and Corollary 4.3.3 we get

$$\begin{aligned}\mathbb{E}\,\|u(t)\|_m^k &\overset{m,k}{\lesssim} (\mathbb{E}\,\|u(t)\|_N^2)^{1/2}(\mathbb{E}\,|u(t)|_{1,1}^{2k-2})^{1/2} \\ &\overset{m,k}{\lesssim} \nu^{-(N-1/2)} = \nu^{-k(2m-1)/2}. \ \square\end{aligned}$$

Lemma 4.3.8. *For $m \geq 1$,*

$$\mathbb{E}\max_{s\in[t,t+1]}\|u(s)\|_m^2 \overset{m}{\lesssim} \nu^{-(2m-1)}, \quad t \geq 2.$$

Proof. We begin by fixing $m \geq 1$. As previously, we take $t = 2$. In this proof, the random constants $\Theta_i,\ i = 1, 2, 3$ are strictly positive and have finite moments.

By (4.15), for $s \geq 2$ we have

$$\begin{aligned}\|u(s)\|_m^2 &= \|u(2)\|_m^2 \\ &+ \int_2^s \Big(-2\nu\,\|u(s')\|_{m+1}^2 - \langle L^m u(s'),\ B(u)(s')\rangle + I_m\Big)\,ds' \\ &+ \int_2^s 2\langle L^m u(s'),\ dw(s')\rangle. \end{aligned} \tag{4.27}$$

Corollary 4.3.3 and Lemma 4.3.6 yield that

$$\max_{s'\in[2,3]}|u(s')|_{1,1} \leq \Theta_1; \qquad \|u(2)\|_m^2 \leq \Theta_2\nu^{-(2m-1)}, \tag{4.28}$$

respectively. By the same method as in the proof of Lemma 4.3.6 we show that:

$$\begin{aligned}&\|u(s')\|_m^2 \geq \Theta_3\nu^{-(2m-1)} \Longrightarrow \\ &-2\nu\,\|u(s')\|_{m+1}^2 - \langle L^m u(s'),\ B(u)(s')\rangle + I_m \leq 0.\end{aligned} \tag{4.29}$$

Here Θ_3 satisfies $\Theta_3 = C(1+\Theta_1)^{n'}$, where C, n' depend only on m. Now consider the random time moment τ defined by

$$\tau = \{\inf s \in [2,3] : \|u(s)\|_m^2 \geq \Theta_3 \nu^{-(2m-1)}\}.$$

By convention, $\tau = 3$ if the set in question is empty. Relations (4.27-4.29) yield that

$$\begin{aligned}\max_{s\in[2,3]} \|u(s)\|_m^2 &\leq \|u(\tau)\|_m^2 + \max_{s\in[\tau,3]} \int_\tau^s 2\langle L^m u(s'),\ dw(s')\rangle \\ &\leq \max(\Theta_2, \Theta_3)\nu^{-(2m-1)} + 4 \max_{s\in[2,3]} \Big| \int_2^s \langle L^m u(s'),\ dw(s')\rangle \Big|. \qquad (4.30)\end{aligned}$$

To complete the proof, it remains to estimate the expression

$$\max_{s\in[2,3]} \Big| \int_2^s \langle L^m u(s'),\ dw(s')\rangle \Big|$$

in the right-hand side of (4.30). Denote by $\xi(s)$ the stochastic integral

$$\xi(s) = \int_2^s \langle L^m u(s'),\ dw(s')\rangle.$$

By the Burkholder-Davis-Gundy inequality [19, Theorem 3.14.] and Lemma 4.3.6 we get

$$\begin{aligned}\mathbb{E} \max_{s\in[2,3]} |\xi(s)| &\leq \mathbb{E} \max_{s\in[2,3]} \xi(s) + \mathbb{E} \max_{s\in[2,3]} (-\xi(s)) \\ &\leq C\Big(\mathbb{E}\langle\langle \xi \rangle\rangle_{[2,3]}\Big)^{1/2} \\ &= C\Big(\mathbb{E} \int_2^3 \langle L^m u(s), Qu(s)\rangle\Big)^{1/2} \\ &\leq C(\max_{s\in[2,3]} I_m \mathbb{E}\, \|u(s)\|_m^2)^{1/2} \leq C\nu^{-(2m-1)/2}. \qquad (4.31)\end{aligned}$$

Here, $\langle\langle \xi \rangle\rangle_{[2,3]}$ denotes the quadratic variation of ξ over the time interval $[2,3]$. We recall that Q is the operator defined in Subsection 4.1.2, and that $I_m = Tr\ Q|_{H_m}$. Relations (4.30) and (4.31) imply that

$$\mathbb{E} \max_{s\in[2,3]} \|u(s)\|_m^2 \leq \mathbb{E} \max(\Theta_2, \Theta_3)\nu^{-(2m-1)} + C\nu^{-(2m-1)/2},$$

which proves the lemma's assertion. □

Repeating the proof of Corollary 4.3.7 we get that for $m \geq 1$,

$$\mathbb{E} \max_{s\in[t,t+1]} \|u(s)\|_m^k \overset{m,k}{\lesssim} \nu^{-k(2m-1)/2}, \quad k \geq 1,\ t \geq 2. \tag{4.32}$$

Denote $\gamma = \max(0, m - 1/p)$.

THEOREM 4.3.9. *For $m \in \{0,1\}$ and $p \in [1,+\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$,*

$$\Big(\mathbb{E} \max_{s\in[t,t+1]} |u(s)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\lesssim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq 2.$$

Proof. We consider only the case when α is an integer: the general case follows by Hölder's inequality.

For $m \geq 1$ and $p \in [2,\infty]$, we interpolate $|u(s)|_{m,p}$ between $\|u(s)\|_m$ and $\|u(s)\|_{m+1}$. By Lemma 4.1.1 we have

$$|u(s)|_{m,p}^{\alpha} \overset{p}{\lesssim} (\|u(s)\|_m^{\alpha})^{1-\theta} (\|u(s)\|_{m+1}^{\alpha})^{\theta},\ \theta = \frac{1}{2} - \frac{1}{p}.$$

Then we use (4.32) and Hölder's inequality to complete the proof.

We use the same method to prove the cases $m = 1,\ p \in [1,2]$, combining the inequality (4.32) and Corollary 4.3.3. We also proceed similarly for $m \geq 3, p \in (1,2)$, combining Corollary 4.3.3 and an estimate for $\|u\|_{M,p}^{\alpha}$ for a large value of M and some $p \geq 2$.

Finally, the case $m = 0$ follows from Corollary 4.3.4. □

Unfortunately, the proof of Theorem 4.3.9 cannot be adapted to the case $m \geq 2$ and $p = 1$. Indeed, Lemma 4.1.1 only allows us to estimate a $W^{m,1}$ norm from above by other $W^{m,1}$ norms: we can only get that

$$|w|_{m,1} \overset{m,n,k}{\lesssim} |w|_{n,1}^{(m-k)/(n-k)} |w|_{k,1}^{(n-m)/(n-k)},\ 0 \leq k < m < n,$$

and thus the upper estimates obtained above cannot be used. However, $|u|_{m,1} \leq |u|_{m,1+\beta}$ for any $\beta > 0$. Consequently, the theorem's statement holds for $m \geq 2$ and $p = 1$, with γ replaced by $\gamma + \iota$, and $\overset{m,p,\alpha}{\lesssim}$ replaced by $\overset{m,p,\alpha,\iota}{\lesssim}$, for any $\iota > 0$.

4.4 Lower Estimates for Sobolev Norms

For a solution $u(t)$ of (4.13), the first quantity that we estimate from below is the expected value of $\frac{1}{T}\int_1^{T+1}\|u(s)\|_1^2$, where $T > 0$ is sufficiently large.

LEMMA 4.4.1. *There exists a constant $T_0 > 0$ such that we have*

$$\frac{1}{T}\int_1^{T+1}\mathbb{E}\,\|u(s)\|_1^2 \gtrsim \nu^{-1}, \qquad T \geq T_0.$$

Proof. For $T > 0$, by (4.17) we get

$$\mathbb{E}\,|u(T+1)|^2 \geq \mathbb{E}(|u(T+1)|^2 - |u(1)|^2) = TI_0 - 2\nu\int_1^{T+1}\mathbb{E}\,\|u(s)\|_1^2.$$

On the other hand, by Corollary 4.3.4 there exists a constant $C' > 0$ such that $\mathbb{E}\,|u(T+1)|^2 \leq C'$. Consequently, for $T \geq T_0 := (C'+1)/I_0$,

$$\frac{1}{T}\int_1^{T+1}\mathbb{E}\,\|u(s)\|_1^2 \geq \frac{TI_0 - C'}{2T}\nu^{-1} \geq \frac{I_0}{2(C'+1)}\nu^{-1},$$

which proves the lemma's assertion. □

This time-averaged lower bound yields similar bounds for H^m norms, $m \geq 2$.

LEMMA 4.4.2. *For $m \geq 1$,*

$$\frac{1}{T}\int_t^{t+T}\mathbb{E}\,\|u(s)\|_m^2 \overset{m}{\gtrsim} \nu^{-(2m-1)}, \qquad t \geq 1,\ T \geq T_0.$$

Proof. By the "starting time argument", we can take $t = 1$. Since the case $m = 1$ has been treated in the previous lemma, we may assume that $m \geq 2$. By Lemma 4.1.1, Hölder's inequality, and Corollary 4.3.3 we have

$$\begin{aligned}(\mathbb{E}\,\|u(s)\|_1^2)^{(2m-1)} &\overset{m}{\lesssim} (\mathbb{E}\,\|u(s)\|_m^2)(\mathbb{E}\,|u(s)|_{1,1}^2)^{(2m-2)} \\ &\overset{m}{\lesssim} \mathbb{E}\,\|u(s)\|_m^2 .\end{aligned} \tag{4.33}$$

Integrating (4.33) in time, we get

$$\frac{1}{T}\int_1^{T+1} \mathbb{E}\,\|u(s)\|_m^2 \overset{m}{\gtrsim} \frac{1}{T}\int_1^{T+1} (\mathbb{E}\,\|u(s)\|_1^2)^{(2m-1)}$$
$$\overset{m}{\gtrsim} \Big(\frac{1}{T}\int_1^{T+1} \mathbb{E}\,\|u(s)\|_1^2\Big)^{(2m-1)}.$$

Now the theorem's assertion follows from Lemma 4.4.1. □

The following two results generalise Lemma 4.4.2. We recall that $\gamma = \max(0, m - 1/p)$.

LEMMA 4.4.3. *For $m = 0$ and $p = \infty$, or for $m \geq 1$ and $p \in [1, \infty]$,*

$$\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{m,p}^2 \overset{m,p}{\gtrsim} \nu^{-\gamma}, \quad t \geq 2,\ T \geq T_0.$$

Proof. In the case $m = 1,\ p \geq 2$, it suffices to apply Hölder's inequality in place of Lemma 4.1.1 in the proof of an analogue for Lemma 4.4.2.

In the case $m \geq 2$, the proof is exactly the same as for Lemma 4.4.2. The only problematic case is $p = 1$, since then Lemma 4.1.1 does not allow us to estimate $|u(s)|_{m,p}^2$ from below using $|u(s)|_{1,1}^2$ and $\|u(s)\|_1^2$. However, it suffices to observe that $|u(s)|_{m,1} \geq |u(s)|_{m-1,\infty}$.

Now consider the case $m = 1,\ p \in [1, 2)$. By Hölder's inequality we have

$$\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{1,p}^2 \geq \Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,\|u(s)\|_1^2\Big)^{2/p}$$
$$\times \Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{1,\infty}^2\Big)^{(p-2)/p}.$$

Using Lemma 4.4.1 and Theorem 4.3.9, we get the lemma's assertion.

We proceed similarly for the case $m = 0,\ p = \infty$. Indeed, by Lemma 4.1.1 we have $|u(s)|_{1,\infty} \leq C\,|u(s)|_\infty^{1/2}\,|u(s)|_{2,\infty}^{1/2}$, and the lemma's assertion follows from Hölder's inequality, the case $m = 1,\ p = \infty$, and Theorem 4.3.9. □

LEMMA 4.4.4. *For $m = 0$ and $p = \infty$, or for $m \geq 1$ and $p \in [1, \infty]$,*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{m,p}^\alpha\Big)^{1/\alpha} \overset{m,p,\alpha}{\gtrsim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq 2,\ T \geq T_0.$$

Proof. As previously, we may assume that $p > 1$. The case $\alpha \geq 2$ follows immediately from Lemma 4.4.3 and Hölder's inequality. The case $\alpha < 2$ follows from Hölder's inequality, the case $\alpha = 2$, and Theorem 4.3.9 (case $\alpha = 3$), since we have

$$\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^{\alpha}_{m,p} \geq \Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^{2}_{m,p}\Big)^{3-\alpha} \times \Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|^{3}_{m,p}\Big)^{\alpha-2}. \ \Box$$

Now we prove that for every $p \in [1,\infty)$, in a certain sense, $\mathbb{E}|u|_p$ is large if and only if $\mathbb{E}|u|_\infty$ is large.

LEMMA 4.4.5. *For $t \geq 1$, denote by A the quantity $\mathbb{E}|u(t)|^2_\infty$. Then there exists a constant $C' > 0$ such that for $p \in [1,\infty]$ we have*

$$\tilde{g}(A) := \min\Big(\frac{3A}{8}, \frac{3A^2}{16C'}\Big) \leq \mathbb{E}|u(t)|^2_p \leq A.$$

Proof. We may take $p = 1$. Denote by l the quantity

$$l = \min(\sqrt{A/2C'},\ 1),$$

where C' is the upper bound for $\mathbb{E}\ X_t^2$ in the statement of Theorem 4.3.1. Consider the random point $x = x_t$ where $|u(t,\cdot)|$ reaches its maximum. If this point is not unique, let x be the leftmost such point on S^1 considered as $[0,1)$. Let I be the interval $[x, x+l]$ if $u(t,x) < 0$, and the interval $[x-l, x]$ if $u(t,x) \geq 0$, respectively. We have

$$\begin{aligned}
\mathbb{E}|u(t)|^2_1 &\geq \mathbb{E}\left(\int_I |u(t,y)|dy\right)^2 \\
&\geq \mathbb{E}\left(l\left(|u(t)|_\infty - \frac{l\max_{x\in S^1} u_x(t)}{2}\right)\right)^2 \\
&\geq l^2\left(\frac{3}{4}\mathbb{E}|u(t)|^2_\infty - \frac{3l^2}{4}\mathbb{E}\Big(\big(\max_{x\in S^1} u_x(t)\big)^2\Big)\right).
\end{aligned}$$

By definition of A, C', and l, we get

$$\mathbb{E}|u(t)|^2_1 \geq l^2\left(\frac{3A}{4} - \frac{3l^2C'}{4}\right) \geq \frac{3l^2A}{8} = \tilde{g}(A).\Box$$

Finally we prove the following uniform lower estimate.

Lemma 4.4.6. *We have*

$$\mathbb{E}|u(t)|_p^2 \gtrsim 1, \quad t \geq T_0 + 2,\ p \in [1, \infty].$$

Proof. We can take $p = 2$. Indeed, the case $p \in (2, \infty]$ follows immediately from the case $p = 2$. On the other hand, the case $p \in [1, 2)$ follows from Hölder's inequality, the case $p = 2$, and the estimate for $\mathbb{E}|u|_\infty^2$ in Theorem 4.3.9, in the same way as in the proof of Lemma 4.4.4.

Let C' denote various strictly positive constants. From Lemma 4.4.4 (case $m = 0$ and $p = \infty$), it follows that for some $\tilde{t}$ in $[2, T_0 + 2]$ we have $\mathbb{E}|u(\tilde{t})|_\infty^2 \geq C'$. Then by Lemma 4.4.5 we get $\mathbb{E}|u(\tilde{t})|^2 \geq C'$. Thus it suffices to prove that

$$\mathbb{E}|u(t)|^2 \leq \kappa \Longrightarrow \frac{d}{dt}\mathbb{E}|u(t)|^2 \geq 0, \quad t \geq 2,$$

where κ is a fixed strictly positive number, chosen later.

If $\mathbb{E}|u(t)|^2 \leq \kappa$, then by Lemma 4.4.5, $\mathbb{E}|u(t)|_\infty^2 \leq \tilde{g}^{-1}(\kappa)$. On the other hand, by Hölder's inequality and Lemma 4.1.1, we have

$$\begin{aligned}\mathbb{E}\,\|u(t)\|_1^2 &\leq (\mathbb{E}|u(t)|_{1,\infty}^2)^{1/2}(\mathbb{E}|u(t)|_{1,1}^2)^{1/2} \\ &\leq C'(\mathbb{E}|u(t)|_\infty^2)^{1/4}(\mathbb{E}|u(t)|_{2,\infty}^2)^{1/4}(\mathbb{E}|u(t)|_{1,1}^2)^{1/2}.\end{aligned}$$

Therefore, by Theorem 4.3.9, $\mathbb{E}\,\|u(t)\|_1^2 \leq C'(\tilde{g}^{-1}(\kappa))^{1/4}\nu^{-1}$, and thus by (4.17), we get

$$\frac{d}{dt}\mathbb{E}|u(t)|^2 \geq I_0 - 2C'(\tilde{g}^{-1}(\kappa))^{1/4}.$$

Since $\tilde{g}^{-1}(\kappa) \underset{\kappa \to 0}{\longrightarrow} 0$, choosing κ small enough so that $2C'(\tilde{g}^{-1}(\kappa))^{1/4} \leq I_0$ proves the lemma's assertion. □

Since $|u(t)|_{1,1} \geq |u(t)|_\infty$, an analogue of Lemma 4.4.6 also holds for $|u(t)|_{1,1}$.

4.5 Sobolev Norms: Main Theorem

The following theorem sums up the main results of Sections 4.3-4.4, with the exception of Theorem 4.3.1. We recall that $\gamma = \max(0, m - 1/p)$.

For m and p such that $\gamma(m,p) = 0$, the lower estimates for $\alpha < 2$ are obtained from Hölder's inequality, the lower estimates for $\alpha = 2$, and the upper estimates for $\alpha = 3$ in the same way as in the proof of Lemma 4.4.4. The lower estimates for $\alpha > 2$ follow immediately from the lower estimates for $\alpha = 2$.

THEOREM 4.5.1. *For $m \in \{0,1\}$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$,*

$$\Big(\frac{1}{T}\int_t^{t+T} \mathbb{E}\,|u(s)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq T_0 + 2,\ T \geq T_0. \tag{4.34}$$

Moreover, the upper estimates hold with time-averaging replaced by maximising over $[t, t+1]$ for $t \geq 2$, i.e.

$$\Big(\mathbb{E}\max_{s\in[t,t+1]} |u(s)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\lesssim} \nu^{-\gamma}, \quad \alpha > 0,\ t \geq 2.$$

On the other hand, the lower estimates hold for all m, p. The asymptotics (4.34) hold without time-averaging if m and p are such that $\gamma(m,p) = 0$. Namely, in this case,

$$\Big(\mathbb{E}\,|u(t)|_{m,p}^{\alpha}\Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} 1, \quad \alpha > 0,\ t \geq T_0 + 2.$$

All these estimates remain true if we replace the Sobolev norms by their suprema over all smooth initial conditions.

For integers $m \geq 1$, this theorem yields the relation

$$\{\|u\|_m^2\} \overset{m}{\sim} \nu^{-(2m-1)}. \tag{4.35}$$

By a standard interpolation argument (see (4.5)) the upper bound in (4.35) also holds for non-integer numbers $s > 1$. Actually, the same is true for the lower bound, since for any integer $n > s$ we have

$$\{\|u\|_s^2\} \geq \{\|u\|_n^2\}^{n-s+1}\{\|u\|_{n+1}^2\}^{-(n-s)} \overset{s}{\gtrsim} \nu^{-(2s-1)}.$$

4.6 Estimates for Small-Scale Quantities

In this section, we estimate small-scale quantities which characterise Burgulence in physical space (increments, flatness) as well as in Fourier space (energy spectrum).

4.6.1 Agreements and Notation

In this section, we fix t satisfying $t \geq T_0 + 2$. Its precise value is not important, since all estimates in Sections 4.3-4.5 hold uniformly in t provided that $t \geq T_0 + 2$, and the same is true for all estimates in this section.

For a random function $A^\omega(t)$, we denote by $\{A\}$ the average of A in ensemble and in time over the interval $[t, t+T_0]$, T_0 being the same as in Theorem 4.5.1:

$$\{A\} = \mathbb{E}\frac{1}{T_0}\int_t^{t+T_0} A^\omega(s)ds.$$

We assume that $\nu \leq \nu_0$. Next, we define the intervals

$$J_1 = (0,\ C_1\nu];\ J_2 = (C_1\nu,\ C_2];\ J_3 = (C_2,\ 1]. \tag{4.36}$$

In other words, $J_1 = \{\ell :\ 0 < \ell \lesssim \nu\}$, $J_2 = \{\ell :\ \nu \lesssim \ell \lesssim 1\}$, $J_3 = \{\ell :\ \ell \sim 1\}$. In terms of the Kolmogorov 1941 theory of turbulence (cf. [27]), $C_1\nu$ corresponds to the dissipation length scale, while J_1, J_2, and J_3 correspond to the dissipation range, the inertial range, and the energy range, respectively.

The strictly positive constants C_1, C_2, and ν_0 will be chosen in (4.46). For them we can take any three positive numbers, satisfying relations:

$$C_1 \leq K^{-2}/4; \quad 5K^2 \leq \frac{C_1}{C_2} < \frac{1}{\nu_0}; \quad \nu_0 \leq 1. \tag{4.37}$$

Here K is a strictly positive constant, chosen in (4.45). Note that the intervals defined by (4.36-4.37) are non-empty and do not intersect each other for all values of $\nu \in (0, \nu_0]$.

The constants C_1 and C_2 can be made as small as desired. On the other hand, by Theorem 4.5.1 we have $\{|u|^2\} \sim 1$ and (after integration by parts) $\{|\hat{u}_n|^2\} \leq \{|u|^2_{1,1}\}/(2\pi n)^2 \sim 1/n^2$. We recall that we denote by $\hat{u}_n$ the complex Fourier coefficients of u. Thus, the proportion of the sum $\Sigma|\hat{u}_n|^2$ contained in Fourier modes corresponding to J_3 tends to 1 as C_2 tends to 0, uniformly in ν. For instance, we may assume that

$$\sum_{|n|\leq C_2^{-1}} \{|\hat{u}_n|^2\} \geq \frac{99}{100} \sum_{n\in\mathbb{Z}} \{|\hat{u}_n|^2\}.$$

For $p, \alpha \geq 0$, we consider the quantity

$$S_{p,\alpha}(\ell) = \Big\{ \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^{\alpha} \Big\}.$$

The quantity $S_{p,1}(\ell)$ is denoted by $S_p(\ell)$: it corresponds (up to averaging) to the structure function of p-th order. The flatness $F(\ell)$, given by

$$F(\ell) = S_4(\ell)/S_2^2(\ell), \tag{4.38}$$

measures spatial intermittency (see [27]). Finally, for $k \geq 1$, we define the (layer-averaged) energy spectrum by:

$$E(k) = \left\{ \frac{\sum_{|n|\in[M^{-1}k,Mk]} |\hat{u}_n|^2}{\sum_{|n|\in[M^{-1}k,Mk]} 1} \right\}. \tag{4.39}$$

The constant $M \geq 1$ will be chosen in the proof of Theorem 4.6.12.

4.6.2 Results in Physical Space

We begin by proving upper estimates for the functions $S_{p,\alpha}(\ell)$. In the proofs of the two following lemmas, constants denoted by C depend only on p, α.

LEMMA 4.6.1. *For $\alpha \geq 0$ and $\ell \in [0,1]$,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\lesssim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha p} \nu^{-\alpha(p-1)}, \ p \geq 1. \end{cases}$$

Proof. We begin by considering the case $p \geq 1$. We have

$$\begin{aligned} S_{p,\alpha}(\ell) &= \Big\{ \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^{\alpha} \Big\} \\ &\leq \Big\{ \Big(\max_x |u(x+\ell) - u(x)|^{p-1} \int_{S^1} |u(x+\ell) - u(x)| dx \Big)^{\alpha} \Big\}. \end{aligned}$$

By Hölder's inequality we get

$$\begin{aligned} S_{p,\alpha}(\ell) \leq & \Big\{ \Big(\int_{S^1} |u(x+\ell) - u(x)| dx \Big)^{\alpha p} \Big\}^{1/p} \\ & \times \Big\{ \max_x |u(x+\ell) - u(x)|^{\alpha p} \Big\}^{(p-1)/p}. \end{aligned}$$

Since the space average of $u(x+\ell) - u(x)$ vanishes, we obtain that

$$\begin{aligned} S_{p,\alpha}(\ell) \leq & \Big\{ \Big(2 \int_{S^1} (u(x+\ell) - u(x))^+ dx \Big)^{\alpha p} \Big\}^{1/p} \\ & \times \Big\{ \max_x |u(x+\ell) - u(x)|^{\alpha p} \Big\}^{(p-1)/p} \\ \leq & C\ell^{\alpha} \Big\{ \max_x |u(x+\ell) - u(x)|^{\alpha p} \Big\}^{(p-1)/p}, \end{aligned} \tag{4.40}$$

where the second inequality follows from Theorem 4.3.1. Finally, by Theorem 4.5.1 we get

$$S_{p,\alpha}(\ell) \leq C\ell^{\alpha} \Big\{ (\ell |u|_{1,\infty})^{\alpha p} \Big\}^{(p-1)/p} \leq C\ell^{\alpha p} \nu^{-\alpha(p-1)}.$$

The case $p < 1$ follows immediately from the case $p = 1$ since now $S_{p,\alpha}(\ell) \leq S_{1,\alpha p}(\ell)$, by Hölder's inequality. □

For $\ell \in J_2 \cup J_3$, we have a better upper bound if $p \geq 1$.

LEMMA 4.6.2. *For $\alpha \geq 0$ and $\ell \in J_2 \cup J_3$,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\lesssim} \begin{cases} \ell^{\alpha p}, & 0 \leq p \leq 1. \\ \ell^{\alpha}, & p \geq 1. \end{cases}$$

Proof. The calculations are almost the same as in the previous lemma. The only difference is that we use another upper bound for the right-hand side of (4.40). Namely, we have

$$\begin{aligned} S_{p,\alpha}(\ell) & \leq C\ell^{\alpha} \Big\{ \max_x |u(x+\ell) - u(x)|^{\alpha p} \Big\}^{(p-1)/p} \\ & \leq C\ell^{\alpha} \Big\{ (2|u|_{\infty})^{\alpha p} \Big\}^{(p-1)/p} \leq C\ell^{\alpha}, \end{aligned}$$

where the third inequality follows from Theorem 4.5.1. □

To prove lower estimates for $S_{p,\alpha}(\ell)$, we need a lemma. Loosely speaking, this lemma states that with a probability which is not too small, during a period of time which is not too small, several Sobolev norms are of the same order as their expected values. Note that in the following definition, (4.41 - 4.42) contain lower and upper estimates, while (4.43) only contains an upper estimate. The inequality $|u(s)|_\infty \leq \max u_x(s)$ in (4.41) always holds, since $u(s)$ has zero mean value and the length of S^1 is 1.

Definition 4.6.3. *For a given solution $u(s) = u^\omega(s)$ and $K > 1$, we denote by L_K the set of all $(s, \omega) \in [t, t+T_0] \times \Omega$ such that*

$$K^{-1} \leq |u(s)|_\infty \leq \max u_x(s) \leq K \tag{4.41}$$

$$K^{-1}\nu^{-1} \leq |u(s)|_{1,\infty} \leq K\nu^{-1} \tag{4.42}$$

$$|u(s)|_{2,\infty} \leq K\nu^{-2}. \tag{4.43}$$

Lemma 4.6.4. *There exist constants $C, K_1 > 0$ such that for $K \geq K_1$, $\rho(L_K) \geq C$. Here, ρ denotes the product measure of the Lebesgue measure and $\mathbb{P}$ on $[t, t+T_0] \times \Omega$.*

Proof. We denote by A_K, B_K, and D_K the set of (s, ω) satisfying

"The upper estimates in (4.41-4.43) hold for a given value of K",

"The lower estimates in (4.41-4.42) hold for a given value of K",

and

"The lower estimate in (4.42) holds for a given value of K",

respectively.

Now note that if $K \leq K'$, then $L_K \subset L_{K'}$, and similarly for the sets A_K, B_K, and D_K.

By Lemma 4.1.1 we get $|u|_\infty \geq C'|u|_{2,\infty}^{-1}|u|_{1,\infty}^2$ for some constant $C' > 0$. Thus, for $\tilde{K} \geq \max(C', 1)K^3$, we have $A_K \cap D_K \subset B_{\tilde{K}}$, and therefore:

$$A_K \cap D_K \subset A_{\tilde{K}} \cap B_{\tilde{K}} = L_{\tilde{K}}.$$

Consequently:

$$\rho(L_{\tilde{K}}) \geq \rho(A_K) + \rho(D_K) - 1.$$

By Theorem 4.3.1, Theorem 4.5.1, and Chebyshev's inequality, the measure of the set $A_{\tilde{K}}$ tends to T_0 as $\tilde{K}$ tends to $+\infty$. So to prove the lemma's assertion, it remains to show that there exists $C > 0$ such that for K large enough we have $\rho(D_K) \geq C$. Using the upper estimate for $\{|u|^2_{1,\infty}\}$ in Theorem 4.5.1, we get

$$\{|u|_{1,\infty}\mathbf{1}(|u|_{1,\infty} \geq K\nu^{-1})\} \leq CK^{-1}\nu^{-1}.$$

Here, $\mathbf{1}(A)$ denotes the indicator function of an event A. On the other hand, we clearly have

$$\{|u|_{1,\infty}\mathbf{1}(|u|_{1,\infty} \leq K^{-1}\nu^{-1})\} \leq K^{-1}\nu^{-1}.$$

Now consider the function

$$g = |u|_{1,\infty}\mathbf{1}(K_0^{-1}\nu^{-1} \leq |u|_{1,\infty} \leq K_0\nu^{-1}).$$

The lower estimate for $\{|u|_{1,\infty}\}$ in Theorem 4.5.1 and the relations above yield

$$\{g\} \geq (C - CK_0^{-1} - K_0^{-1})\nu^{-1} \geq C_0\nu^{-1},$$

for suitable constants K_0 and C_0. Since $g \leq K_0\nu^{-1}$, we get

$$\rho(g \geq C_0\nu^{-1}/2) \geq C_0K_0^{-1}T_0/2.$$

Since $g \leq |u|_{1,\infty}$, then

$$\rho(|u|_{1,\infty} \geq C_0\nu^{-1}/2) \geq C_0K_0^{-1}T_0/2,$$

which implies the existence of $C, K_1 > 0$ such that $\rho(D_K) \geq C$ for $K \geq K_1$. □

Let us denote by $O_K \subset [T_1, T_2]$ the set defined as L_K, but with relation (4.42) replaced by

$$K^{-1}\nu^{-1} \leq -\min u_x \leq K\nu^{-1}. \tag{4.44}$$

COROLLARY 4.6.5. *For $K \geq K_1$ and $\nu < K_1^{-2}$, we have $\lambda(O_K) \geq C$.*

Proof. For $K = K_1$ and $\nu < K_1^{-2}$, the estimates (4.41-4.42) tell us that for $(s,\omega) \in L_K$,

$$\max u_x(s) \leq K_1 < K_1^{-1}\nu^{-1} \leq |u_x(s)|_\infty.$$

Thus, in this case the assertion of Lemma 4.6.4 with (4.42) replaced by (4.44) holds for the set $O_K = L_K$. Finally, we observe that since increasing

K while keeping ν constant increases the measure of O_K, then the corollary's statement still holds for $K \geq K_1$ and $\nu < K_1^{-2}$. $\square$

Now we fix

$$K = K_1, \tag{4.45}$$

and choose

$$\nu_0 = \frac{1}{6}K^{-2};\ C_1 = \frac{1}{4}K^{-2};\ C_2 = \frac{1}{20}K^{-4}. \tag{4.46}$$

In particular, we have $0 < C_1\nu_0 < C_2 < 1$: thus the intervals J_i are non-empty and non-intersecting for all $\nu \in (0, \nu_0]$. Everywhere below, the constants depend on K.

Actually, we can choose any values of C_1, C_2, and ν_0, provided

$$C_1 \leq \frac{1}{4}K^{-2}; \quad 5K^2 \leq \frac{C_1}{C_2} < \frac{1}{\nu_0}.$$

LEMMA 4.6.6. *For $\alpha \geq 0$ and $\ell \in J_1$,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\gtrsim} \begin{cases} \ell^{\alpha p},\ 0 \leq p \leq 1. \\ \ell^{\alpha p}\nu^{-\alpha(p-1)},\ p \geq 1. \end{cases}$$

Proof. By Corollary 4.6.5, it suffices to prove that the inequalities hold uniformly in $(s,\omega) \in O_K$ with $S_{p,\alpha}(\ell)$ replaced by

$$\Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^{\alpha}.$$

The general case clearly follows from the case $\alpha = 1$. Till the end of the proof we assume that

$$(s,\omega) \in O_K.$$

Case $\mathbf{p \geq 1}$, $\alpha = \mathbf{1}$. Denote by z the leftmost point on S^1 (considered as $[0,1)$) such that $u'(z) \leq -K^{-1}\nu^{-1}$. Since $|u|_{2,\infty} \leq K\nu^{-2}$, we have

$$u'(y) \leq -\frac{1}{2}K^{-1}\nu^{-1}, \quad y \in [z - \frac{1}{2}K^{-2}\nu, z + \frac{1}{2}K^{-2}\nu]. \tag{4.47}$$

Since $\ell \le C_1\nu = \frac{1}{4}K^{-2}\nu$, then by Hölder's inequality we get

$$\begin{aligned}
\int_{S^1} |u(x+\ell)-u(x)|^p dx &\ge \int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} |u(x+\ell)-u(x)|^p dx \\
&\ge (K^{-2}\nu/2)^{1-p}\Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} |u(x+\ell)-u(x)|dx\Big)^p \\
&= C(p)\nu^{1-p}\Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} \Big(\int_x^{x+\ell} -u'(y)dy\Big)dx\Big)^p \\
&\ge C(p)\nu^{1-p}\Big(\int_{z-\frac{1}{4}K^{-2}\nu}^{z+\frac{1}{4}K^{-2}\nu} \frac{1}{2}\ell K^{-1}\nu^{-1}\, dx\Big)^p = C(p)\nu^{1-p}\ell^p.
\end{aligned}$$

Case $\mathbf{p<1}$, $\alpha = \mathbf{1}$. By Hölder's inequality we get

$$\begin{aligned}
&\int_{S^1} |u(x+\ell)-u(x)|^p dx \ge \int_{S^1} \Big((u(x+\ell)-u(x))^+\Big)^p dx \\
&\ge \Big(\int_{S^1} \Big((u(x+\ell)-u(x))^+\Big)^2 dx\Big)^{p-1}\Big(\int_{S^1} (u(x+\ell)-u(x))^+ dx\Big)^{2-p}.
\end{aligned}$$

Using the upper estimate in (4.41) we get

$$\begin{aligned}
&\int_{S^1} |u(x+\ell)-u(x)|^p dx \\
&\ge \Big(\int_{S^1} \ell^2K^2 dx\Big)^{p-1}\Big(\int_{S^1} (u(x+\ell)-u(x))^+ dx\Big)^{2-p}.
\end{aligned}$$

Finally, since $\int_{S^1} (u(\cdot+\ell)-u(\cdot)) = 0$, we obtain that

$$\begin{aligned}
\int_{S^1} |u(x+\ell)-u(x)|^p dx &\ge C(p)\ell^{2(p-1)}\Big(\frac{1}{2}\int_{S^1} |u(x+\ell)-u(x)|dx\Big)^{2-p} \\
&\ge C(p)\ell^p.
\end{aligned}$$

The last inequality follows from the case $p = 1,\ \alpha = 1$. □

The proof of the following lemma uses an argument from [2], which becomes rigorous if we restrict ourselves to the set O_K.

LEMMA 4.6.7. *For $\alpha \ge 0$ and $\ell \in J_2$,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\gtrsim} \begin{cases} \ell^{\alpha p},\ 0 \le p \le 1. \\ \ell^{\alpha},\ p \ge 1. \end{cases}$$

Proof. For the same reason as in the previous proof, it suffices to prove that as long as (s, ω) belongs to O_K, the inequalities hold uniformly for $p \geq 1$, $\alpha = 1$ and for $S_{p,\alpha}(\ell)$ replaced by

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx.$$

Once again, till the end of the proof we assume that $(s, \omega) \in O_K$.

Defining z in the same way as previously, we have

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx \geq \int_{z-\frac{1}{2}\ell}^{z} \left| \int_x^{x+\ell} u'^-(y)dy - \int_x^{x+\ell} u'^+(y)dy \right|^p dx.$$

We have $\ell \geq C_1\nu = \frac{1}{4}K^{-2}\nu$. Thus, by (4.47), for $x \in [z - \frac{1}{2}\ell, z]$ we get

$$\int_x^{x+\ell} u'^-(y)dy \geq \int_z^{z+\frac{1}{8}K^{-2}\nu} u'^-(y)dy \geq \frac{1}{16}K^{-3}.$$

.

On the other hand, since $\ell \leq C_2$, then using the lower estimate in (4.41) we get

$$\int_x^{x+\ell} u'^+(y)dy \leq C_2 K = \frac{1}{20}K^{-3}.$$

Thus,

$$\int_{S^1} |u(x+\ell) - u(x)|^p dx \geq \frac{1}{2}\ell\left(\left(\frac{1}{16} - \frac{1}{20}\right)K^{-3}\right)^p \geq C(p)\ell. \ \square$$

Summing up the results above we obtain the following theorem.

THEOREM 4.6.8. *For $\alpha \geq 0$ and $\ell \in J_1$,*

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha p}\nu^{-\alpha(p-1)}, \ p \geq 1. \end{cases}$$

On the other hand, for $\alpha \geq 0$ and $\ell \in J_2$,

$$S_{p,\alpha}(\ell) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha}, \ p \geq 1. \end{cases}$$

The following result follows immediately from the definition (4.38).

Corollary 4.6.9. *For $\ell \in J_2$, the flatness satisfies $F(\ell) \sim \ell^{-1}$.*

4.6.3 Results in Fourier Space

By (4.35), for $m \geq 1$ we have

$$\{|\hat{u}_k|^2\} \leq (2\pi k)^{-2m}\{\|u\|_m^2\} \overset{m}{\sim} (k\nu)^{-2m}\nu.$$

Thus, for $|k| \succeq \nu^{-1}$, $\{|\hat{u}_k|^2\}$ decreases super-algebraically.

Now we want to estimate the H^s norms of u for $s \in (0,1)$ (the case $s = 0$ is a particular case of Theorem 4.5.1).

Lemma 4.6.10. *We have*

$$\{\|u\|_{1/2}^2\} \sim |\log \nu|.$$

Proof. By (4.6) we have

$$\|u\|_{1/2} \sim \left(\int_{S^1} \Big(\int_0^1 \frac{|u(x+\ell) - u(x)|^2}{\ell^2} d\ell \Big) dx \right)^{1/2}.$$

Consequently, by Fubini's theorem,

$$\begin{aligned}\{\|u\|_{1/2}^2\} &\sim \int_0^1 \frac{1}{\ell^2} \Big\{ \int_{S^1} |u(x+\ell) - u(x)|^2 dx \Big\} d\ell \\ &= \int_0^1 \frac{S_2(\ell)}{\ell^2} d\ell = \int_{J_1} \frac{S_2(\ell)}{\ell^2} d\ell + \int_{J_2} \frac{S_2(\ell)}{\ell^2} d\ell + \int_{J_3} \frac{S_2(\ell)}{\ell^2} d\ell.\end{aligned}$$

By Theorem 4.6.8 we get

$$\int_{J_1} \frac{S_2(\ell)}{\ell^2} d\ell \sim \int_0^{C_1\nu} \frac{\ell^2 \nu^{-1}}{\ell^2} d\ell \sim 1$$

and

$$\int_{J_2} \frac{S_2(\ell)}{\ell^2} d\ell \sim \int_{C_1\nu}^{C_2} \frac{\ell}{\ell^2} d\ell \sim |\log \nu|,$$

respectively. Finally, by Lemma 4.6.2 we get

$$\int_{J_3} \frac{S_2(\ell)}{\ell^2} d\ell \leq C C_2^{-2} \leq C.$$

Thus,

$$\{\|u\|_{1/2}^2\} \sim |\log \nu|. \ \square$$

The proof of the following result follows the same lines.

LEMMA 4.6.11. *For $s \in (0, 1/2)$,*

$$\{\|u\|_s^2\} \overset{s}{\sim} 1.$$

On the other hand, for $s \in (1/2, 1)$,

$$\{\|u\|_s^2\} \overset{s}{\sim} \nu^{-(2s-1)}.$$

The results above and the relation (4.35) tell us that $\{|\hat{u}_k|^2\}$ decreases very fast for $|k| \gtrsim \nu^{-1}$, and that for $s \geq 0$ the sums $\sum |k|^{2s}\{|\hat{u}_k|^2\}$ have exactly the same behaviour as the partial sums $\sum_{|k| \leq \nu^{-1}} |k|^{2s}|k|^{-2}$ in the limit $\nu \to 0^+$. Therefore we can conjecture that for $|k| \lesssim \nu^{-1}$, we have $\{|\hat{u}_k|^2\} \sim |k|^{-2}$.

A result of this type actually holds (after layer-averaging), as long as $|k|$ is not too small. To prove it, we use a version of the Wiener-Khinchin theorem, stating that for any function $v \in L_2$ one has

$$|v(\cdot + y) - v(\cdot)|^2 = 4 \sum_{n \in \mathbb{Z}} \sin^2(\pi n y) |\hat{v}_n|^2. \tag{4.48}$$

THEOREM 4.6.12. *For k such that $k^{-1} \in J_2$, we have $E(k) \sim k^{-2}$.*

Proof. We recall that by definition (see (4.39)),

$$E(k) = \left\{ \frac{\sum_{|n| \in [M^{-1}k, Mk]} |\hat{u}_n|^2}{\sum_{|n| \in [M^{-1}k, Mk]} 1} \right\}.$$

Therefore proving the assertion of the theorem is the same as proving that

$$\sum_{|n| \in [M^{-1}k, Mk]} n^2 \{|\hat{u}_n|^2\} \sim k. \tag{4.49}$$

The upper estimate is an immediate corollary of the upper estimate for $|u|_{1,1}$ in Theorem 4.5.1 and holds without averaging over n such that $|n| \in [M^{-1}k, Mk]$. Indeed, integrating by parts we get

$$\{|\hat{u}_n|^2\} \leq (2\pi n)^{-2}\{|u_x|_1^2\} \leq Cn^{-2},$$

which proves the upper bound. Also, this inequality implies that

$$\sum_{|n|<M^{-1}k} n^2\{|\hat{u}_n|^2\} \leq CM^{-1}k \tag{4.50}$$

and

$$\sum_{|n|>Mk} \{|\hat{u}_n|^2\} \leq CM^{-1}k^{-1}. \tag{4.51}$$

To prove the lower bound we note that

$$\begin{aligned}\sum_{|n|\leq Mk} n^2\{|\hat{u}_n|^2\} &\geq \frac{k^2}{\pi^2} \sum_{|n|\leq Mk} \sin^2(\pi n k^{-1})\{|\hat{u}_n|^2\} \\ &\geq \frac{k^2}{\pi^2}\Big(\sum_{n\in\mathbb{Z}} \sin^2(\pi n k^{-1})\{|\hat{u}_n|^2\} - \sum_{|n|>Mk} \{|\hat{u}_n|^2\}\Big).\end{aligned}$$

Using (4.48) and (4.51) we get

$$\begin{aligned}\sum_{|n|\leq Mk} n^2\{|\hat{u}_n|^2\} &\geq \frac{k^2}{4\pi^2}\Big(\{|u(\cdot + k^{-1}) - u(\cdot)|^2\} - CM^{-1}k^{-1}\Big) \\ &\geq \frac{k^2}{4\pi^2}(S_2(k^{-1}) - CM^{-1}k^{-1}).\end{aligned}$$

Finally, using Theorem 4.6.8 we obtain that

$$\sum_{|n|\leq Mk} n^2\{|\hat{u}_n|^2\} \geq (C - CM^{-1})k.$$

Now we use (4.50) and we choose $M \geq 1$ large enough to obtain (4.49). ☐

REMARK 4.6.13. *We actually have*

$$\left\{\left(\frac{\sum_{|n|\in[M^{-1}k,Mk]} |\hat{u}_n|^2}{\sum_{|n|\in[M^{-1}k,Mk]} 1}\right)^\alpha\right\} \overset{\alpha}{\sim} k^{-2\alpha}, \quad \alpha > 0.$$

The upper bound is proved in the same way as previously, and then the lower bound follows from Hölder's inequality and the lower bound in Theorem 4.6.12.

4.7 Stationary Measure and Related Issues

Here we briefly discuss the stationary measure corresponding to (4.13). More details will be given in the next section, as well as in Appendix A.

Since (4.13) is a well-posed SPDE, its solutions form a Markov process in H^1, which induces a semigroup S_t in that space (see Appendix A). Now consider the corresponding semigroup S_t^* acting on probability measures on H^1. A *stationary measure* is a probability measure on H^1 invariant by S_t^* for every t. A *stationary solution* is a solution $u(t,x)$ of (4.13) such that the law of $u(t,\cdot)$ does not depend on t and thus is a stationary measure for (4.13).

Existence of a stationary measure for (4.13) is proved using the Bogolyubov-Krylov argument (see for instance [45]).

In Section 4.8, we prove that for two different initial conditions the distance between the corresponding solutions of (4.13) in L_1 is nonincreasing. Using some additional estimates on solutions and then applying [45, Theorem 3.1.3], we show that this contraction property implies uniqueness of the stationary measure μ_{stat}. Moreover, the distribution of $u(t,\cdot)$ converges to μ_{stat} as $t \to +\infty$, uniformly in u_0. Thus, estimates in Section 4.3 imply that μ_{stat} is supported in C^∞.

Estimates in the previous sections remain true for a stationary solution of (4.13). Indeed, it suffices to consider an initial condition u_0 distributed as μ_{stat}. It follows that those estimates still hold when averaging in time and in ensemble (denoted by $\{\cdot\}$) is replaced by averaging solely in ensemble. That is, by integrating with respect to μ_{stat}. Namely, Theorem 4.5.1, Theorem 4.6.8, and Theorem 4.6.12 imply, respectively, the following results.

THEOREM 4.7.1. *For $m \in \{0,1\}$ and $p \in [1,\infty]$, or for $m \geq 2$ and $p \in (1,\infty]$,*

$$\Big(\int |u|_{m,p}^{\alpha}\, d\mu_{stat}(u) \Big)^{1/\alpha} \overset{m,p,\alpha}{\sim} \nu^{-\gamma}, \quad \alpha > 0.$$

THEOREM 4.7.2. *For $\alpha \geq 0$ and $\ell \in J_1$,*

$$\int \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^{\alpha} d\mu_{stat}(u) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha p} \nu^{-\alpha(p-1)}, \ p \geq 1. \end{cases}$$

On the other hand, for $\alpha \geq 0$ and $\ell \in J_2$,

$$\int \Big(\int_{S^1} |u(x+\ell) - u(x)|^p dx \Big)^{\alpha} d\mu_{stat}(u) \overset{p,\alpha}{\sim} \begin{cases} \ell^{\alpha p}, \ 0 \leq p \leq 1. \\ \ell^{\alpha}, \ p \geq 1. \end{cases}$$

THEOREM 4.7.3. *For k such that $k^{-1} \in J_2$, we have*

$$\int \frac{\sum_{|n|\in[M^{-1}k,Mk]} |\hat{u}_n|^2}{\sum_{|n|\in[M^{-1}k,Mk]} 1} d\mu_{stat}(u) \sim k^{-2}.$$

4.8 Convergence to the stationary measure

In this section, we prove some statements in Section 4.7; see also Appendix A.

4.8.1 A Contraction Property

Agreements: In this subsection, we consider initial conditions which do not necessarily have zero mean value in space. Namely, here L_1 denotes the whole space $L_1(S^1)$ and not, as in the rest of the paper, the subspace of functions in $L_1(S^1)$ satisfying (4.3), and similarly for C^∞. This extension of the setting does not affect well-posedness for the equation (4.13). The functions u and $\tilde{u}$ always denote solutions corresponding to the same forcing, with initial conditions u_0 and $\tilde{u}_0$, respectively.

Contraction properties for solutions of scalar conservation laws have been known to hold since the works of Oleinik and Kruzhkov (cf. [18] and references therein). However, we will use a different method. For this, we need to prove an analog to one of the implications in the Crandall-Tartar Lemma [17, Proposition 1].

LEMMA 4.8.1. *Let O be any measurable space. Consider a transformation $\Phi : S \to S$, where S is a subset of $L_1(O)$, preserving the mean value, i.e. such that*

$$\int_O a = \int_O \Phi(a). \tag{4.52}$$

Then the assumption

$$(i)\ \forall a, \tilde{a} \in O,\ a \leq \tilde{a} \Rightarrow \Phi(a) \leq \Phi(\tilde{a}) \tag{4.53}$$

yields

$$(ii)\ \forall a, \tilde{a} \in O,\ \int_O |\Phi(a) - \Phi(\tilde{a})| \leq \int_O |a - \tilde{a}|, \tag{4.54}$$

provided

$$a, \tilde{a} \in S \Rightarrow \forall \delta > 0,\ \exists c_\delta \in S,\ \max(a, \tilde{a}) \le c_\delta \le \max(a, \tilde{a}) + \delta. \tag{4.55}$$

Proof. By (4.53), for every $\delta > 0$ we have

$$\Phi(a) \le \Phi(c_\delta),$$

and similarly for $\Phi(\tilde{a})$. By (4.52) and (4.55) we get

$$\begin{aligned}
\int_O |\Phi(a) - \Phi(\tilde{a})| &\le \int_O (\Phi(c_\delta) - \Phi(a)) + \int_O (\Phi(c_\delta) - \Phi(\tilde{a})) \\
&= \int_O (c_\delta - a) + \int_O (c_\delta - \tilde{a}) \\
&\le \int_O (\max(a, \tilde{a}) - a) + \int_O (\max(a, \tilde{a}) - \tilde{a}) + 2\delta \\
&= \int_O |a - \tilde{a}| + 2\delta. \ \square
\end{aligned}$$

REMARK 4.8.2. *In the original formulation of Crandall and Tartar, the assumption (4.55) is replaced by the stronger assumption*

$$a, \tilde{a} \in S \Rightarrow \max(a, \tilde{a}) \in S.$$

THEOREM 4.8.3. *Consider two solutions u, $\tilde{u}$ of (4.13), corresponding to the same random force but different initial conditions in C^∞. For all $t \ge 0$, we have*

$$|u(t) - \tilde{u}(t)|_1 \le |u(0) - \tilde{u}(0)|_1.$$

Proof. For $O = S^1$ and $S = C^\infty$, the transformation $\Phi : u(0) \mapsto u(t)$ verifies (4.52) for every $t \ge 0$. Now consider the function $v = u - \tilde{u}$. We have

$$v_t + \left(\frac{f(u) - f(\tilde{u})}{u - \tilde{u}} \right) v_x + \left(\frac{f(u) - f(\tilde{u})}{u - \tilde{u}} \right)_x v = \nu v_{xx}.$$

By the weak maximum principle for a *linear* parabolic equation [24] applied to the function $v \exp(-\kappa t)$ with large enough κ, if $v(0) \ge 0$, then $v(t) \ge 0$ for all t. Thus $u(0) \mapsto u(t)$ verifies the condition (4.53). An application of Lemma 4.8.1 ends the proof. $\square$

By the usual density argument, this theorem allows us to define solutions of (4.13) for any initial condition in L_1.

4.8.2 Setting and Definitions

The result above allows us to define for a.e. ω the semigroup S_s^{s+t} from L_1 to itself which maps an initial condition at time s to the corresponding solution of (4.13) at time $s+t$. Since these solutions verify a Markov property (see Appendix B), we can define the corresponding semigroup S_t^* acting on Borel measures on L_1. A *stationary measure* is a Borel probability measure on L_1 invariant by S_t^* for every t. A *stationary solution* is a random process v defined for $(t,\omega) \in [0,+\infty) \times \Omega$ and with values in L_1, such that its distribution does not depend on t. Such a distribution is automatically a stationary measure.

It remains to show existence and uniqueness of a stationary measure, which implies existence and uniqueness (up to the distribution) of a stationary solution. This fact has been proved in a slightly different setting: see [31] and references therein. Moreover, we obtain an additional result on the rate of convergence to the stationary measure in an appropriate distance. This rate does not depend on the viscosity or on the initial condition.

Definition 4.8.4. *For a function g on L_1, we denote by $|g|_{Lip}$ the semi-norm:*

$$|g|_{Lip} = \sup_{v_1,v_2 \in L_1} \frac{|g(v_1)-g(v_2)|}{|v_1-v_2|_1}.$$

The space of functions with finite Lipschitz norm

$$|g|_L := |g|_\infty + |g|_{Lip}$$

will be denoted by L.

Definition 4.8.5. *For two probability measures μ_1, μ_2 on L_1, we denote by $\|\mu_1-\mu_2\|_L^*$ the Lipschitz-dual distance:*

$$\|\mu_1-\mu_2\|_L^* := \sup_{g \in L,\ |g|_L \leq 1} \Big| \int_{S^1} g d\mu_1 - \int_{S^1} g d\mu_2 \Big|.$$

Existence of a stationary measure for the Burgers equation is proved using the Bogolyubov-Krylov argument (see [45]). Let us give a sketch of the proof.

For $s \geq 1$, $\mathbb{E}|u(s)|_{1,1}$ is uniformly bounded. Since by Helly's selection principle [37], $W^{1,1}$ is compactly embedded in L_1, then the family of measures $\{\mu_t,\ t \geq 1\}$ defined by:

$$\mu_t := \frac{1}{t} \int_1^{1+t} S_s^* u_0 \, ds$$

is tight for any initial condition u_0. Thus, we can extract a subsequence μ_{t_n} converging weakly to a stationary measure

Now we state the main result of this section, which is proved in Subsection 4.8.3, and immediately implies uniqueness of a stationary measure for equation (4.13).

THEOREM 4.8.6. *There exists a positive constant C' such that for $t \geq 0$, we have*

$$\|S_t^* \mu_1 - S_t^* \mu_2\|_L^* \leq \frac{C'}{t^{1/11}}, \qquad t \geq 1, \tag{4.56}$$

where $(\mu_1,\ \mu_2)$ are any pair of probability measures on L_1.

Finally, we observe that by Lemma 4.1.1 and Corollary 4.3.3, for any solutions $u, \tilde{u}$ of (4.13) and $p \in [1, \infty)$ we have for $t \geq 1$:

$$\mathbb{E}|u - \tilde{u}|_p \overset{p}{\lesssim} (\mathbb{E}|u - \tilde{u}|_1)^{1/p} (\mathbb{E}|u - \tilde{u}|_{1,1})^{(p-1)/p} \overset{p}{\lesssim} (\mathbb{E}|u - \tilde{u}|_1)^{1/p}.$$

Consequently, convergence to the stationary measure takes place at the rate $C(p) t^{-1/11p}$ in the Lipschitz-dual distance in L_p for any $p \in [1, \infty)$.

4.8.3 Proof of Theorem 4.8.6

To begin with, we need an auxiliary lemma. Its proof is very similar to the proof of Theorem 4.3.1.

LEMMA 4.8.7. *There exists a constant $C_1 \geq 2$ such that if $\tau \geq C_1$ and*

$$K = \max_{s \in [t, t+\tau]} |w(s) - w(t)|_{C^3}.$$

satisfies $K \leq \tau^{-2}$, then we have

$$\max_{x \in S^1} u_x(t + \tau, x) \leq \tau^{-1/2}. \tag{4.57}$$

Proof. Assume the converse of the lemma's statement.

In this proof, constants denoted by C' are strictly positive and independent of C_1. We abbreviate $w(s) - w(t)$ as $\tilde{w}(s)$ and use the notation from the proof of Theorem 4.3.1:

$$\tilde{v} = (s-t)(u_x(s,x) - \tilde{w}_x(s,x)); \quad M = \max_{s\in[t,t+\tau],\ x\in S^1} \tilde{v}(s,x). \tag{4.58}$$

In particular, since we assumed that (4.57) does not hold, we have

$$M > \tau^{1/2} - \tau K > \tau^{1/2}/2.$$

Now consider a point (t_1, x_1) at which the maximum M is attained. In the same way as in the proof of Theorem 4.3.1, we get

$$f''(u)(\tilde{v} + (t_1-t)\tilde{w}_x)^2 \leq \tilde{v} - (t_1-t)^2 f'(u)\tilde{w}_{xx} + \nu(t_1-t)^2\tilde{w}_{xxx}. \tag{4.59}$$

On the other hand, we have

$$\begin{aligned}(t_1-t)^2 f'(u(t_1,x_1)) &\leq C'(t_1-t)^{\delta}\Big((t_1-t) + (t_1-t)u(t_1,x_1)\Big)^{2-\delta}\\ &\leq C'\tau^{\delta}\Big(\tau^{2-\delta} + (M+\tau K)^{2-\delta}\Big),\end{aligned}$$

since $(t_1-t)u$ is the zero space average primitive of $\tilde{v} + (t_1-t)\tilde{w}$. Thus we get

$$\sigma(M-\tau K)^2 \leq M + C'K\tau^{\delta}(\tau^{2-\delta} + (M+\tau K)^{2-\delta}) + K\tau^2.$$

By assumption, we have $\tau \geq C_1$, $K \leq \tau^{-2}$, and $M > \tau^{1/2}/2$. Therefore we have, on one hand,

$$\sigma(M-\tau K)^2 \geq C'M^2,$$

and on the other hand,

$$M + C'K\tau^{\delta}(\tau^{2-\delta} + M^{2-\delta}) + K\tau^2 \leq C'M^{2-\delta}.$$

Thus, $M^{\delta} \leq C'$, and for C_1 large enough we have a contradiction with the fact that $M > \tau^{1/2}$. □

The following theorem yields the main result of this section: uniqueness of a stationary measure, and an estimation of the speed of convergence to it in the Lipschitz-dual distance. To prove this result, we use the "coupling method" [45, Chapter 3]. The situation is actually much simpler then for the stochastic 2D Navier Stokes equations, which are the main subject of [45]. Namely, in our setting the "damping time" needed to make the distance between two solutions not too large does not depend on the initial conditions.

Note that all estimates in the previous sections still hold for a stationary solution, since they hold uniformly for any initial condition in L_1 for large times, and a stationary solution has, by definition, the same statistical properties for any time.

Proof of Theorem 4.8.6. In this proof, C_1 is the constant in the statement of Lemma 4.8.7, and C' denotes various positive constants.

We can take $(\mu_1,\ \mu_2) = (\delta_{u_0}, \delta_{\tilde{u}_0})$: the general case follows by Fubini's theorem. By definition of the Lipschitz-dual distance, for $t \geq 1$ we have

$$\begin{aligned} \|S_t^* \delta_{u_0} - S_t^* \delta_{\tilde{u}_0}\|_L^* &\leq \mathbb{E} \sup_{g \in L,\ \|g\|_{Lip} \leq 1} \Big| g(u(t)) - g(\tilde{u}(t)) \Big| \\ &\leq \mathbb{E}\Big(\max(2, |u(t) - \tilde{u}(t)|_1) \Big). \end{aligned}$$

Since by Theorem 4.8.3 we know that $|u(t) - \tilde{u}(t)|_1$ is nonincreasing in t, it suffices to prove that for $n \geq C_1$, there exists $C_2 > 0$ such that we have

$$\mathbb{P}\Big(|u(n^{11}) - \tilde{u}(n^{11})|_1 \geq \frac{2}{n} \Big) \leq \frac{C_2}{n}, \tag{4.60}$$

since by definition of the Lipschitz-dual distance this inequality would yield that

$$\begin{aligned} &\mathbb{E}\Big(|u(n^{11}) - \tilde{u}(n^{11})|_1 \big) \\ &\leq \frac{2}{n} \mathbb{P}\Big(|u(n^{11}) - \tilde{u}(n^{11})|_1 < \frac{2}{n} \Big) + 2\mathbb{P}\Big(|u(n^{11}) - \tilde{u}(n^{11})|_1 \geq \frac{2}{n} \Big) \\ &\leq \frac{2 + 2C_2}{n}. \end{aligned}$$

Since increments of w on time intervals $[kn, (k+1)n]$ are independent, by

Lemma 4.8.7 for $n \geq \sqrt{C_1}$ we have

$$
\begin{aligned}
&\mathbb{P}\Big(|u(n^{11}) - \tilde{u}(n^{11})|_1 \geq \frac{2}{n}\Big) \\
&\leq \mathbb{P}\Big(\exists k \in [0, n^{10}-1]: \quad |u(kn^{10})| \geq \frac{1}{n} \quad or \quad |\tilde{u}(kn^{10})|_1 \geq \frac{1}{n}\Big) \\
&\leq \prod_{0 \leq k \leq n^{10}-1} \mathbb{P}\Big(\max_{t \in [kn,(k+1)n]} |w(t) - w(kn)|_{C^3} \geq \frac{1}{n^4}\Big) \\
&\leq \left(\exp\Big(-\frac{n^{-8}}{2C'n}\Big)\right)^{n^{10}} \leq e^{-C'n} \leq \frac{C'}{n},
\end{aligned}
$$

where the third inequality follows from (4.9). This proves (4.60). $\square$

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Chapitre 5

Hyperbolicité des minimiseurs pour les systèmes Lagrangiens aléatoires en une dimension

Ce chapitre correspond à l'article *On hyperbolicity of minimizers for 1D random Lagrangian systems*, écrit en collaboration avec K.Khanin et soumis à Nonlinearity.

Abstract. We prove hyperbolicity of global minimizers for random Lagrangian systems in dimension 1. The proof considerably simplifies a related result in [23]. The conditions for hyperbolicity are almost optimal: they are essentially the same as conditions for uniqueness of a global minimizer in [31].

5.1 Introduction

A large body of work on the random forced Burgers equation and Burgers turbulence in the last 10 years (see [6] and further references therein) has motivated closely related studies of random Lagrangian systems [23, 31]. The main object of analysis is a Lagrangian system which depends smoothly on position x and velocity v, but quite irregularly on time t:

$$L^{\omega}(x,v,t) = L_0(x,v) + F^{\omega}(x,t), \tag{5.1}$$

where $F^{\omega}(x,t)$ is a stationary random process in t. The Lagrangian is defined on the tangent bundle TM to a connected d-dimensional Riemannian manifold M. Most rigorous results available at the moment require that M be compact, which will also be the standing assumption in this paper. Since the potential $F^\omega(x,t)$ is smooth in x the most natural continuous time model is given by

$$F^\omega(x,t) = \sum_{k=1}^{K} \dot{W}_k^\omega(t) F^k(x), \tag{5.2}$$

where $F^k(x)$ are smooth non-random potentials on M, and $\dot{W}_k^\omega(t)$ are independent white noises. One can also consider "kicked" models:

$$F^\omega(x,t) = \sum_{j=-\infty}^{+\infty} F^\omega(j)(x)\delta(t-j), \tag{5.3}$$

where $\{F^\omega(j)(x),\ j \in \mathbb{Z}\}$ is a stationary sequence of random potentials. We shall assume that potentials $F^\omega(j)$ are picked independently for different $j \in \mathbb{Z}$ according to a given probability distribution μ on $C^n(M)$, where n is big enough. The Lagrangian dynamics corresponding to (5.3) can be described as follows. For non-integer times t the system evolves according to a non-random Lagrangian L_0, and at integer times $t = j \in \mathbb{Z}$ the velocity changes discontinuously:

$$v(j+0) = v(j-0) + \nabla F^\omega(j)(x).$$

Although the two models (5.2) and (5.3) look rather different, the theory and results for both cases are parallel.

Lagrangian systems (5.1) are related to random forced Hamilton-Jacobi equations. One has to first define the Hamiltonian

$$H^\omega(x,p,t) = \max_v \left[p \cdot v - L^\omega(x,v,t)\right] = H_0(x,p) - F^\omega(x,t),$$

and then to consider the corresponding Hamilton-Jacobi equation

$$\phi_t + H^\omega(x, \nabla\phi, t) = 0. \tag{5.4}$$

One of the most studied cases corresponds to $L_0 = v^2/2$. In this case $H_0 = p^2/2$ and the Hamilton-Jacobi equation (5.4) takes the form

$$\phi_t(x,t) + \tfrac{1}{2}|\nabla\phi|^2 - F^\omega(x,t) = 0.$$

Then for the velocity field $v(x,t) = \nabla\phi(x,t)$ one gets the inviscid Burgers equation:

$$v_t(x,t) + (v \cdot \nabla)v(x,t) - \nabla F^{\omega}(x,t) = 0.$$

Although all the results of this paper hold for any Lagrangian L_0 which is convex in v and grows super-linearly as $|v| \to \infty$, below we only consider the case $L_0 = v^2/2$.

It is well-known that minimizers for the Lagrangian L^{ω} generate the viscosity solution of the Hamilton-Jacobi equation (5.4). This connection is especially useful and important for the study of global solutions, that is solutions for $t \in (-\infty, \infty)$. In order to discuss a global solution one has to fix the value of the first integral

$$b = \int_M \nabla\phi(x,t)dx. \tag{5.5}$$

The theory developed in [31] states that under extremely mild conditions, with probability 1, for every value of the first integral $b \in \mathbb{R}^d$, there exists a unique (up to an additive constant) global solution to the Hamilton-Jacobi equation. This unique global solution can be viewed as a stationary solution. It plays the role of a global attractor for the dynamics corresponding to the Cauchy problem for the Hamilton-Jacobi equation. Under additional assumptions of non-degeneracy one can also prove that for every value of $b \in \mathbb{R}^d$, with probability 1, there exists a unique global minimizer for the Lagrangian L^{ω} (see [31]). A global minimizer can be defined as a smooth curve $\gamma : (-\infty, \infty) \to M$ such that for any compact perturbation $\tilde{\gamma}$ the difference between Lagrangian actions corresponding to $\tilde{\gamma}$ and to γ is non-negative. Namely, if $\tilde{\gamma} - \gamma$ is supported on $[T_1, T_2]$, then

$$A^{\omega,b}(\tilde{\gamma}) - A^{\omega,b}(\gamma) = \int_{T_1}^{T_2} L^{\omega}(\tilde{\gamma}, \dot{\tilde{\gamma}} - b, t)dt - \int_{T_1}^{T_2} L^{\omega}(\gamma, \dot{\gamma} - b, t)dt \geq 0.$$

It is expected that the global minimizer is a hyperbolic trajectory of the Lagrangian flow. Unfortunately such a result is not available at present in the multi-dimensional case $d > 1$. In our view hyperbolicity of the global minimizer is one of the most important open problems in the theory of random Lagrangian systems on compact manifolds. In the one-dimensional case hyperbolicity was established in [23]. However the proof in [23] is unnecessarily complicated and conditions are too restrictive. In this paper we present a new proof which is both elementary and conceptual. Here, conditions for hyperbolicity are almost the same as the conditions for uniqueness of a global minimizer (see [31]). This is another important advantage of the approach used in this article.

The following property is crucial for establishing hyperbolicity of the global minimizer. Define first backward minimizers as minimizers on semi-infinite time intervals $(-\infty, t]$ with one end point at t fixed. They can be defined in the same way as global minimizers. Now consider all backward minimizers which originate at time t, and denote by $\Omega_{s,t}$ the set of all points x which are reached by some backward minimizer at time $s \leq t$. We prove that the diameter of $\Omega_{s,t}$ tends to zero exponentially as $t \to \infty$. This property implies hyperbolicity by the standard argument, which also allows to construct corresponding stable and unstable manifolds. We shall not discuss these issues in the present paper and refer the readers to [23]. Instead, here we shall only deal with the key shrinking property formulated above.

We finish this section with several general remarks. First, we want to emphasize the importance of hyperbolicity of the global minimizer. It immediately implies many fundamental properties of the global solution to the Hamilton-Jacobi equation, such as piecewise smoothness, exponential rate of convergence to the global solution, and many others. It also allows to study the structure of singularities (shocks) (see [6]).

Our second remark is related to a general problem of hyperbolicity of minimizers for generic non-random Lagrangian systems. This is one of the central problems of the Aubry-Mather theory. Randomness is another way to introduce the notion of genericity. In this setting generic stands for properties which hold for almost all systems (with probability 1). Note however that in many respects, random and nonrandom (autonomous, or depending on time periodically) Lagrangian systems are very different. In particular, all number-theoretical aspects of the Aubry-Mather theory disappear in the random case.

Finally, we want to say a few words about the non-compact case. At present there are almost no rigorous results in that setting. It is believed that if the system exhibits any form of translation invariance, global minimizers do not exist. However, it is likely that backward minimizers do exist, and the study of their asymptotic scaling properties is an extremely interesting and important problem.

5.2 Hyperbolicity assumptions and main results

We begin by formulating the assumptions on potentials:

ASSUMPTION 5.2.1. *In the "kicked" case, we assume the following.*
(i) The kicks at integer times j are of the form

$$F^\omega(j) = \sum_{k=1}^{K} c_k^\omega(j) F^k,$$

where F^k are smooth potentials on $S^1 = \mathbb{R}/\mathbb{Z}$. The random vectors $(c_k^\omega(j))_{1\le k\le K}$ are independent identically distributed $\mathbb{R}^K$-valued random variables. Their distribution on $\mathbb{R}^K$, denoted by μ, is assumed to be absolutely continuous with respect to the Lebesgue measure.
(ii) We have $0 \in Supp\ \mu$.
(iii) The mapping from S^1 to $\mathbb{R}^K$ defined by

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding.

REMARK 5.2.2. *Let g be the function defined by*

$$g(c_1, ..., c_K) = \sum_{k=1}^{K} c_k F^k.$$

We denote by ξ the corresponding push-forward measure

$$\xi = g_*(\mu)$$

on a smooth Sobolev space. The assumption $0 \in Supp\ \mu$ can then be replaced by the slightly weaker assumption $0 \in Supp\ \xi$.

Assumption 5.2.3. *In the case of the white force potential, we assume the following.*
(i) The forcing has the form

$$F^\omega(x,t) = \sum_{k=1}^{K} \dot{W}_k^\omega(t) F^k(x),$$

where F^k are smooth potentials on S^1, and $\dot{W}_k^\omega$ are independent white noises, i.e. weak time derivatives of independent Wiener processes $W_k^\omega(t)$.
(ii) The mapping from S^1 to $\mathbb{R}^K$ defined by

$$x \mapsto (F^1(x), ..., F^K(x))$$

is an embedding.

We denote by G an antiderivative in time of the forcing:

$$G^\omega(x,t) = \sum_{k=1}^{K} W_k^\omega(t) F^k(x),$$

where $W_k^\omega(t)$ are independent standard Wiener processes with $W_k^\omega(0) = 0$. Since we will only consider time differences of G, the particular choice of antiderivative has no importance.

In both cases, F^ω will be abbreviated as F, and in the white force case $F(\cdot, t)$ will be abbreviated as $F(t)$, and similarly for G.

Remark 5.2.4. *The embedding conditions are consistent with the condition for uniqueness of the global minimizer (see [31]). In the "kicked" case, the condition for uniqueness in [31] is slightly weaker: the map*

$$x \mapsto (F^1(x), ..., F^K(x))$$

is only required to be one-to-one. However, we need to assume the embedding to prove hyperbolicity.

The following property, called the *separation property*, plays a crucial role in our construction.

Property 5.2.5. *There exist $\alpha_0 > 0$, three pairwise disjoint open intervals J_i, $i = 1, 2, 3$, and three potentials $\tilde{F}_i$, $i = 1, 2, 3$ with the following properties.*
1) In the "kicked" case, we have $\tilde{F}_i \in Supp\ \xi$ for every i. In the white force case, each $\tilde{F}_i$ is a linear combination of the F^k.
2) Each of the functions $-\tilde{F}_i$ reaches its minimum, denoted by m_i, at a single point x_i.
3) For every α, $0 < \alpha \leq \alpha_0$, there exist three open intervals $I_i(\alpha)$, $I_i \subset J_i$, $i = 1, 2, 3$ such that

$$\tilde{F}_i(S^1 - I_i) \subset (-\infty, -m_i - \alpha].$$

Note that for every i and α, the point x_i where $\min(-\tilde{F}_i)$ is reached belongs to I_i.

Lemma 5.2.6. *Assumptions 5.2.1 or 5.2.3 imply the separation property.*

Proof of Lemma 5.2.6:
"Kicked" case: We start by showing that, for Lebesgue-a.e. vector $(c_j)_{1 \leq j \leq K}$, the maximum of

$$\sum_{j=1}^{K} c_j F^j(x)$$

is reached at a single point $x \in S^1$. This follows from a rather standard argument (see [31, Corollary 5]). Indeed, the function

$$\Phi : (c_1, \ldots, c_K) \mapsto \max_{x \in S^1} \sum_{j=1}^{K} c_j F^j(x)$$

is Lipschitz and therefore differentiable a.e., with respect to the Lebesgue measure μ_{Leb}. On the other hand, at a point of differentiability of Φ,

$$\nabla\Phi(x_{max}) = (F^1(x_{max}), \ldots, F^K(x_{max}))$$

for every point of maximum x_{max}. Hence the embedding assumption 5.2.1 (iii) implies that the point of maximum is unique. Since μ is absolutely continuous with respect to μ_{Leb}, the maximum uniqueness set $O_1 \subset \mathbb{R}^K$ has full μ-measure.

Furthermore, by the Lebesgue points theorem [56, Theorem 7.7], $c = (c_j)_j$ is a Lebesgue point for the density

$$q = \frac{d\mu}{d\mu_{Leb}}$$

on a set O' of full μ_{Leb}-measure, and thus of full μ-measure.

Denote by $O_2 \subset O'$ the set of Lebesgue points c for q such that $q(c) > 0$. By definition, they belong to $Supp\ \mu$, and O_2 has full μ-measure.

Now consider $c^1 = (c^1_j)_j \in O_1 \cap O_2$. Denote by x_1 the point where the maximum of $\tilde{F}_1 = \sum_{j=1}^K c^1_j F^j(x)$ is reached: $x_1 = argmax\ \tilde{F}_1$.

Denote by V the set of vectors $(c_j)_j$ such that

$$\sum_{j=1}^{K} c_j \frac{dF^j}{dx}(x_1) \neq 0.$$

Denote by B_n the open ball with radius $1/n$ centered at c^1. We will also need $B_n' = B_n \cap (c^1 + V) \cap O_1 \cap O_2$. By the embedding assumption 5.2.1 (iii), $B_n \cap (c^1 + V)$ is just B_n itself with a removed hyperplane. Thus, since μ is continuous with respect to μ_{Leb}, we have $\mu(B_n') = \mu(B_n)$.

Using [56, Theorem 7.7] one more time, we obtain that there exists a constant N_0 such that for $n \geq N_0$,

$$\mu(B_n') = \mu(B_n) \geq \tfrac{q(c^1)}{2} \mu_{Leb}(B_n) > 0.$$

On the other hand, for small enough $\epsilon > 0$ there exists $N_1(\epsilon)$ such that for $n \geq N_1$, if $(c_j)_j \in B_n'$, then $\sum_{j=1}^K c_j F^j$ reaches its (unique) maximum in a point of the ϵ-neighbourhood of x different from x itself. Considering a smaller neighbourhood at each step, this argument can be repeated any finite number of times. It enables us to construct any number of potentials contained in $Supp\,\xi$ and attaining their respective maxima at different points: three suffice for our purposes. Denote them by $\tilde{F}_1, \tilde{F}_2, \tilde{F}_3$. Let J_1, J_2, J_3 be three non-intersecting open intervals around their respective points of maximum. Take as α_0 the minimum of $\max(\tilde{F}_i) - \max(\tilde{F}_i|_{S^1 - J_i})$. It is obvious that for any $\alpha \in (0, \alpha_0]$ we can construct the required intervals $I_i(\alpha)$.

White force case: The proof follows the same lines, but is much simpler since measure-theoretic arguments are trivialised. □

Definition 5.2.7. *Consider a closed subset Z of S^1. Let $m(Z)$ denote the maximal length of a connected component of $S^1 - Z$. We define the diameter of Z as*

$$d(Z) = 1 - m(Z).$$

The diameter of Z can be thought of as the minimal length of an interval on S^1 containing Z.

In what follows we use the function ψ^ω, either deterministic or random, as an initial condition at time s. Everywhere below, the value of the first integral b (see (5.5)) is fixed. For simplicity, we do not indicate dependence on b in our notation.

DEFINITION 5.2.8. *For a given value of $b \in \mathbb{R}$, a curve $\gamma_{s,t}^{y,x}(\tau)$ is a minimizer if it minimizes the action*

$$A(\gamma) = \frac{1}{2}\int_s^t (\dot{\gamma}(\tau) - b)^2 d\tau + \sum_{n \in [s,t)} \Big(- F(n)(\gamma(n))\Big)$$

in the "kicked" case and the action

$$A(\gamma) = \frac{1}{2}\int_s^t (\dot{\gamma}(\tau) - b)^2 d\tau + \int_s^t \left(\dot{\gamma}(\tau)\Big(\frac{\partial G}{\partial x}(\gamma(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma(\tau),t)\Big)\right) d\tau + \Big(G(\gamma(s),s) - G(\gamma(s),t)\Big)$$

in the white force case, respectively, over all absolutely continuous curves with endpoints x at time t and y at time s.

DEFINITION 5.2.9. *For any time interval $[s,t]$ and any continuous function $\psi : S^1 \to \mathbb{R}$, a curve $\gamma_{s,t,\psi}^x(\tau) : [s,t] \to S^1$ is a ψ-minimizer if it minimizes $A(\gamma) + \psi(\gamma(s))$ over all absolutely continuous curves with endpoint x at time t.*

DEFINITION 5.2.10. *For $-\infty < r < s \le t < +\infty$ and for a fixed function $\psi(\cdot, r) : S^1 \to \mathbb{R}$, let $\Omega_{r,s,t,\psi}$ be the set of points reached, at the time s, by ψ-minimizers on $[r,t]$:*

$$\Omega_{r,s,t,\psi} = \{\gamma_{r,t,\psi}^x(s),\ x \in S^1\}.$$

REMARK 5.2.11. *In what follows, the initial condition ψ will always be fixed, while t will increase to $+\infty$. It is important that we shall consider both deterministic and random initial conditions ψ. In the latter case, ψ should be measurable with respect to the past σ-algebra $\mathcal{F}_r = \mathcal{F}_{(-\infty,r]}$, which is defined in a standard way. It is important to take r smaller then s. Everywhere below, we set $r = s-1$. To simplify notation, $\Omega_{s-1,s,t,\psi}$ will be denoted by $\Omega_{s,t}$.*

It is well-known that $\Omega_{s,t}$ is a closed set. Obviously, $\Omega_{s,t_1} \supseteq \Omega_{s,t_2}$ for all $s \le t_1 \le t_2$. It follows that $t \mapsto d(\Omega_{s,t})$ is a non-increasing function.

We are now able to formulate the main results of this paper which are the following theorem and its corollary. Both results hold for a given value of $b \in \mathbb{R}$. However, all constants are uniformly bounded if b stays bounded. It is easy to see that in the "kicked" case, b is effectively defined modulo 1, since the action is invariant under the transformation $(b, \gamma) \mapsto (b+1, \gamma+t)$. Thus in this case all constants are uniformly bounded for all b.

THEOREM 5.2.12. *Assume that the separation property holds. Then there exist constants $\lambda, \tilde{C} > 0$ such that if $-\infty < s \le t < +\infty$, then*

$$\mathbb{E}(d(\Omega_{s,t})) \le \tilde{C} \exp(-\lambda(t-s)),$$

where $\mathbb{E}(\cdot)$ stands for the expectation with respect to the distribution of potentials.

COROLLARY 5.2.13. *Assume that the separation property holds. Fix $s \in \mathbb{R}$. Then, for a.e. ω, there exists a random constant $\tilde{C}(s, \omega) > 0$ such that*

$$d(\Omega_{s,t}) \le \tilde{C}(s,\omega) \exp(-\lambda(t-s)/2), \quad t \ge s.$$

Here, λ is the same as in Theorem 5.2.12.

As we have already pointed out in the introduction, Corollary 5.2.13 implies hyperbolicity (see [23] for details). The following lemma, called the *main lemma*, is proved in Section 5.3: the proof is quite involved, with additional technical difficulties in the white force case.

MAIN LEMMA. *Assume that the separation property holds. Fix $b \in \mathbb{R}$. Then there exist constants $c, T > 0$ such that if $-\infty < s \le t < +\infty$, then the following inequality holds a.s.:*

$$\mathbb{P}\left(d(\Omega_{s,t+T}) \le \tfrac{d(\Omega_{s,t})}{2} \mid \mathcal{F}_t \right) \ge c.$$

We finish this section by deriving Theorem 5.2.12 and Corollary 5.2.13 from the main lemma.

Proof of Theorem 5.2.12 : Consider the function

$$d(t) = \mathbb{E}(d(\Omega_{s,t}))\exp(\lambda(t-s)),$$

where λ is a fixed positive number, chosen later.

Since $t \mapsto d(\Omega_{s,t})$ is non-increasing, the main lemma implies that

$$\mathbb{E}(d(\Omega_{s,t+T})) \leq c\ \tfrac{\mathbb{E}(d(\Omega_{s,t}))}{2} + (1-c)\mathbb{E}(d(\Omega_{s,t})).$$

Thus

$$d(t+T) \leq \exp(\lambda T)\ \Big(1-\tfrac{c}{2}\Big)d(t).$$

Now put

$$\lambda = -\tfrac{1}{T}\ln\Big(1-\tfrac{c}{2}\Big).$$

It follows that $d(t+T) \leq d(t)$. But $d(s) = 1$. Therefore, for $t \in s + T\mathbb{N}$, we have $d(t) \leq 1$. Consequently, since $t \mapsto d(\Omega_{s,t})$ is non-increasing, we have

$$\mathbb{E}(d(\Omega_{s,t})) \leq \tilde{C}\exp(-\lambda(t-s)), \quad t \geq s,$$

with $\tilde{C} = \exp(\lambda T) = (1-\frac{c}{2})^{-1}$. This proves the theorem's assertion. □

Proof of Corollary 5.2.13 assuming Theorem 5.2.12: In the same way as in the previous proof, it is enough to prove the statement for $t \in s+T\mathbb{N}$. By Theorem 5.2.12 and Chebyshev's inequality, for every $X > 0$,

$$\mathbb{P}(d(\Omega_{s,s+nT}) \geq X\exp(-\lambda nT/2)) \leq \tfrac{\tilde{C}}{X}\exp(-n\lambda T/2), \quad n \geq 0.$$

An application of the Borel-Cantelli lemma ends the proof. □

5.3 Proof of the main lemma

For all $s < t$, let us define a map S_s^t from S^1 to S^1, which can be viewed as a coordinate projection at time t of the generalized Lagrangian flow corresponding to the Burgers equation. It certainly depends on the initial condition ψ at time $s-1$.

If, at time s, a point y belonging to S^1 is reached by a ψ-minimizer on $[s-1,t]$ starting in x at time t, then $S^t_s(y)$ is equal to the point x. Note that such an x is unique, since minimizers on the time interval $[s-1,t]$ cannot intersect outside of endpoints $s-1$ and t.

If a point y is not reached by such a ψ-minimizer, then it belongs to a closed interval corresponding to a shock at time t. In this case $S^t_s(y)$ is equal to the corresponding shock position. To define an interval at time s corresponding to a shock at time t, one has to consider rightmost and leftmost minimizers originating at (x,t). Intersections of those minimizers with $S^1 \times \{s\}$ generate a space interval of points absorbed by the shock (x,t). It is easy to see that every point (y,s) is reached by a minimizer or belongs to a shock interval generated by a uniquely defined shock.

Note that some points may correspond to both cases considered above. Namely, points corresponding to minimizers which originate from the shock positions. However, even in this case the map S^t_s is still uniquely defined.

5.3.1 Proof in the "kicked" case

Put

$$C = 3\left(\max_{i\in 1,2,3} \|\tilde{F}_i\|_{C^1} + 1\right). \tag{5.6}$$

Then put

$$\alpha = \min\left(\alpha_0, \frac{1}{10C}\right) \tag{5.7}$$

(see the separation property for the definition of α_0.) We keep in mind that $\alpha < 1/30$.

Consider integers

$$N' \in \left(2+\frac{1}{\alpha^3}, \ \frac{2}{\alpha^3}\right); \ N \in \left(\frac{2}{\alpha^{10}}+1, \ \frac{4}{\alpha^{10}}\right). \tag{5.8}$$

Denote by E_1 the event

$$\|F(t+k)\|_\infty \le \alpha^{20}, \quad 0 \le k \le N-1. \tag{5.9}$$

By Assumption 5.2.1 (ii) the zero potential belongs to $Supp\ \xi$. It follows that E_1 has positive probability.

Put $l = 1 - d(\Omega_{s,t})$. If $\Omega_{s,t} \neq S^1$, consider a connected component (y_1, y_2) of $S^1 - \Omega_{s,t}$ which has maximal length l. Let y_3 be the center of (y_1,y_2), and let y_4 be the point diametrically opposite to y_3. If $\Omega_{s,t} = S^1$, let y_3 and y_4 be any pair of diametrically opposite points in S^1. Then consider $z_1 = S_s^{(t+N)} y_3$ and $z_2 = S_s^{(t+N)} y_4$. Since the J_i (see the separation property for their definition) are pairwise disjoint, one of the J_i has an empty intersection with one of $[z_1, z_2]$ and $[z_2, z_1]$. Without loss of generality, we may suppose that $[z_1, z_2] \cap J_1 = \varnothing$.

Now consider the straight line defined by

$$\gamma(\tau) = x + b(\tau - t - N), \quad \tau \in [t + N, t + 2N - 1]$$

for some $x \in S^1$.

We claim that there exist (at least) N' different integers $0 = n_0 < \cdots < n_{N'-1} \leq N' - 1$ such that we have

$$\max_{j,j' \in [0,N-1]} |\gamma(t + N + n_j) - \gamma(t + N + n_{j'})| \leq \alpha^7. \tag{5.10}$$

Indeed, by the pigeonhole principle, since $N' \leq \alpha^7 N$, there exist integers $0 \leq \tilde{n}_0 < \cdots < \tilde{n}_{N'-1} \leq N - 1$ such that

$$\max_{j,j' \in [0,N'-1]} |\gamma(t + N + \tilde{n}_j) - \gamma(t + N + \tilde{n}_{j'})| \leq \alpha^7.$$

Then it suffices to take, for every j, $n_j = \tilde{n}_j - \tilde{n}_0$.

By definition of C and α, (5.10) yields that

$$\begin{aligned} \max_{j,j' \in [0,N'-1]} &|\tilde{F}_1(\gamma(t + N + n_j)) - \tilde{F}_1(\gamma(t + N + n_{j'}))| \\ &\leq \alpha^7 \|\tilde{F}_1\|_{C^1} \leq \alpha^6/10. \end{aligned} \tag{5.11}$$

Now consider the event E_2 defined by the system of inequalities:

$$\begin{cases} \|F(t + N + n_j) - \tilde{F}_1\|_\infty \leq \alpha^{20}, \quad 0 \leq j \leq N' - 1. \\ \|F(t + N + k)\|_\infty \leq \alpha^{20}, \\ k \in [0, N - 1] - \{n_0, \ldots, n_{N'-1}\}. \end{cases} \tag{5.12}$$

Since $\tilde{F}_1$ and 0 belong to *Supp* ξ, this event (independent from E_1) also has positive probability.

It remains to prove that for $\omega \in E_1 \cap E_2$ all minimizers on $[t, t + 2N]$ pass through $I_1(\alpha)$ at time $t + N$, which follows from Lemma 5.3.1 and Lemma 5.3.2. Indeed, if this statement holds, no such minimizers can pass through $[z_1, z_2]$ at $t + N$, since $I_1(\alpha) \subset J_1$. Consequently all the points that are in $[y_3, y_4]$ at time s will not be reached by minimizers originating at time $t+2N$. In particular, it follows that $[y_3, y_4]$ is contained in an interval generated by some shock at time $t+2N$. Therefore $(y_1, y_2) \cup [y_3, y_4] = (y_1, y_4]$ is contained in a connected component of $S^1 - \Omega_{s,t+2N}$. Thus

$$d(\Omega_{s,t+2N}) \leq 1 - \tfrac{1+l}{2} = \tfrac{1}{2} d(\Omega_{s,t})$$

with a positive conditional probability which equals at least $\mathbb{P}(E_1)\mathbb{P}(E_2)$. This proves the lemma's assertion. □

LEMMA 5.3.1. *Assume that* $\omega \in E_2$. *Then for every minimizer* γ_1 *on* $[t+N, t+2N]$ *there exists* j, $1 \leq j \leq N'-1$, *such that*

$$-\tilde{F}_1(\gamma_1(t+N+n_j)) \leq m_1 + \alpha^2.$$

Proof: We argue by contradiction. Suppose that there exists a minimizer γ_1 on $[t+N, t+2N]$ such that

$$-\tilde{F}_1(\gamma_1(t+N+n_j)) > m_1 + \alpha^2, \quad 1 \leq j \leq N'-1. \tag{5.13}$$

Consider a curve γ_2 with the same endpoints as γ_1, linear on intervals $[t+N+k, t+N+k+1]$. Moreover we suppose that $\gamma_2 = x_1 + b(\tau - t - N)$ on $[t+N+n_1, t+N+n_{N'-1}]$ (x_1 being the point where $\tilde{F}_1$ reaches its maximum), and that $|\dot{\gamma}_2 - b| \leq 1/2n_1$ and $|\dot{\gamma}_2 - b| \leq 1/2(N - n_{N'-1})$ on the extremal intervals $[t+N, t+N+n_1]$ and $[t+N+n_{N'-1}, t+2N]$, respectively.

From now on, for a curve γ we denote $\dot{\gamma} - b$ by $\dot{\gamma}^b$. We recall that the "kicked" case action A for $\gamma : [t_1, t_2] \to S^1$ equals

$$A(\gamma) = \tfrac{1}{2} \int_{t_1}^{t_2} (\dot{\gamma}^b(\tau))^2 d\tau - \textstyle\sum_{n \in [t_1, t_2)} F[\gamma(n)].$$

The first part of the right-hand side, corresponding to the kinetic energy, will be denoted by A^k. The remaining part, corresponding to the potential energy, will be denoted by A^p. We observe that

$$A^k(\gamma|_{[t_1,t_3]}) = A^k(\gamma|_{[t_1,t_2]}) + A^k(\gamma|_{[t_2,t_3]}), \tag{5.14}$$

and similarly for A^p. We have

$$A^k(\gamma_1) \geq 0; \; A^k(\gamma_2) \leq \tfrac{1}{4}.$$

On the other hand, using the inequalities (5.11-5.13), we get

$$A^p(\gamma_1) \geq (N'-1)(m_1+\alpha^2-\alpha^{20}) - (N-N')\alpha^{20} - F(\gamma(t+N)),$$
$$A^p(\gamma_2) \leq (N'-1)(m_1+\alpha^6/10+\alpha^{20}) + (N-N')\alpha^{20} - F(\gamma(t+N)).$$

Therefore, by (5.7-5.8), we get

$$\begin{aligned} A(\gamma_1)-A(\gamma_2) &= A^k(\gamma_1)-A^k(\gamma_2)+A^p(\gamma_1)-A^p(\gamma_2) \\ &\geq -\frac{1}{4}+(N'-1)(\alpha^2-\alpha^6/10)-2(N-1)\alpha^{20} \\ &\geq -\frac{1}{4}+\alpha^{-1}-\frac{\alpha^3}{10}-8\alpha^{10}>0. \end{aligned}$$

Thus we have a contradiction with the fact that γ_1 is a minimizer. This proves the lemma's assertion. □

LEMMA 5.3.2. *Assume that* $\omega \in E_1 \cap E_2$. *For some* j, $1 \leq j \leq N'-1$, *consider a minimizer* γ_1 *on* $[t, t+N+n_j]$ *such that* $y = \gamma_1(t+N+n_j)$ *satisfies:*

$$-\tilde{F}_1(y) \leq m_1+\alpha^2.$$

Then we have

$$\gamma_1(t+N) \in I_1(\alpha).$$

Proof: We argue by contradiction, supposing that $\gamma_1(t+N) \notin I_1(\alpha)$. We may also assume that

$$-\tilde{F}_1(\gamma_1(t+N+n_{j'})) > m_1+\alpha^2, \quad 1 \leq j' < j.$$

Indeed, otherwise we could consider a smaller value of j. In the same way as previously, we want to prove that γ_1 cannot be a minimizer, and we consider a curve γ_2 with the same endpoints as γ_1. Namely, we suppose that γ_2 satisfies $\dot{\gamma}_2^b = 0$ between $t+N$ and $t+N+n_j$, γ_2 is linear between t and $t+N$, and moreover $|\dot{\gamma}_2^b| \leq 1/2N$. We have the inequalities

$$A^k(\gamma_1) \geq 0; \;\; A^k(\gamma_2) \leq \frac{1}{8N}.$$

On the other hand, using the separation property, (5.9), (5.11), and (5.12), we get

$$\begin{aligned} A^p(\gamma_1) \geq & -N\alpha^{20} + (m_1 + \alpha - \alpha^{20}) + (j-1)(m_1 + \alpha^2 - \alpha^{20}) \\ & - (n_j - j)\alpha^{20}, \\ A^p(\gamma_2) \leq & N\alpha^{20} + j(m_1 + \alpha^2 + \alpha^6/10 + \alpha^{20}) + (n_j - j)\alpha^{20}. \end{aligned}$$

Therefore, by (5.7-5.8), we obtain that

$$A(\gamma_1) - A(\gamma_2) \geq -\frac{1}{8N} + \alpha - \alpha^2 - N'\alpha^6/10 - 4N\alpha^{20} > 0.$$

Again, we have a contradiction. This proves the lemma's assertion. □

5.3.2 Proof in the white force case

The scheme of the proof is very similar to the one in the "kicked" case. The major differences are auxiliary lemmas which are technically more involved and the conditions on the forcing, in some way much more restrictive.

The constants C, α, N', N are the same as in the proof of the "kicked" case, with the exception that now

$$\alpha = \min\left(\alpha_0, \frac{1}{10C}, \frac{1}{10(b+1)^2}\right), \tag{5.15}$$

and that the definitions of N' and N change accordingly. Denote by E_1 the event

$$\sup_{t_1, t_2 \in [t, t+N]} \|G(t_1) - G(t_2)\|_{C^1} \leq \alpha^{40}. \tag{5.16}$$

By classical properties of the Wiener process, E_1 has positive probability, uniformly in t.

Now we proceed exactly in the same way as in the "kicked" case, supposing with the same notation and without loss of generality that $[z_1, z_2] \cap J_1 = \varnothing$.

We assume that for every $j,\ j \in [0, N'-1]$ (we take $n_{N'} = N$), G satisfies:

$$\begin{cases} \|G(t+N+n_{j+1}) - G(t+N+n_j) - \tilde{F}_1\|_{C^1} \leq \alpha^{40}. \\ \|G(t+N+n_{j+1}) - G(t+N+n_j+\tau)\|_{C^1} \leq \alpha^{40}, \\ \tau \in [\alpha^{40}, n_{j+1} - n_j]. \\ \|G(t+N+n_j+\tau) - G(t+N+n_j+\tau')\|_{C^1} \\ \leq \frac{3}{2}\|\tilde{F}_1\|_{C^1} \leq \frac{C}{2},\ \tau, \tau' \in [0, n_{j+1} - n_j]. \end{cases} \tag{5.17}$$

This event, denoted by E_2, has positive probability and is independent from E_1.

Finally, in the same way as in the "kicked" case, the lemma's assertion follows from Lemma 5.3.3 and Lemma 5.3.4. □

LEMMA 5.3.3. *Consider a minimizer γ_1 on $[t+N, t+2N]$. Then, if $\omega \in E_2$, we have*

$$-\tilde{F}_1(\gamma_1(t+N+n_j)) \leq m_1 + \alpha^2 \tag{5.18}$$

for some j, $1 \leq j \leq N'-1$.

Proof: As previously, we argue by contradiction, considering a minimizer γ_1 on $[t+N, t+2N]$ such that (5.18) does not hold for any j, $1 \leq j \leq N'-1$. We recall that the action is given by:

$$A(\gamma) = \frac{1}{2}\int_{t_1}^{t_2} \dot{\gamma}^b(\tau)^2 d\tau + \int_{t_1}^{t_2} \Big(\dot{\gamma}(\tau)\Big(\frac{\partial G}{\partial x}(\gamma(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma(\tau),t_2)\Big)\Big)d\tau + \Big(G(\gamma(t_1),t_1) - G(\gamma(t_1),t_2)\Big).$$

The first term of the right-hand side, i.e. the kinetic energy, will be denoted by A^1. The second and the third terms, whose sum is the potential energy, will be denoted by A^2 and A^3, respectively. We observe that A as well as the quantities A^1 and $A^2 + A^3$ satisfy a relation of the same type as (5.14). To see it for $A^2 + A^3$, it suffices to write down this sum as a stochastic integral. $A(\gamma|_{[s,t]})$ is denoted by $A_{s,t}(\gamma)$, and similarly for A^i, $i = 1, 2, 3$.

Consider a curve γ_2 with the same endpoints as γ_1, defined exactly in the same way as in the proof of Lemma 5.3.1. Namely, $\gamma_2 = x_1 + b(\tau - t - N)$ on $[t+N+n_1, t+N+n_{N'-1}]$, and γ_2 is linear on $[t+N, t+N+n_1]$ and on $[t+N+n_{N'-1}, t+2N]$ with $|\dot{\gamma}_2^b| \leq 1/2n_1$ and $|\dot{\gamma}_2^b| \leq 1/2(N - n_{N'-1})$, respectively.

Now, for every $j \in [0, N'-1]$, consider a straight line γ_3 connecting $\gamma_1(t+N+n_j)$ and $\gamma_1(t+N+n_{j+1})$ with constant velocity $\dot{\gamma_3}$, $|\dot{\gamma_3}^b| \leq 1/2(n_{j+1} - n_j)$ (we take $n_{N'} = N$). Denote by R the quantity

$$R = A^2_{t+N+n_j, t+N+n_{j+1}}(\gamma_1).$$

Since

$$\int a(\tau)b(\tau)d\tau \geq -\frac{2\alpha^{20}}{C}\int\Big(\frac{a(\tau)}{2}\Big)^2 d\tau - \frac{C}{2\alpha^{20}}\int b(\tau)^2 d\tau,$$

then we have

$$
\begin{aligned}
R \geq & -\frac{2\alpha^{20}}{C} \int\limits_{t+N+n_j}^{t+N+n_{j+1}} \Big(\frac{\dot{\gamma}_1^b(\tau)+b}{2}\Big)^2 d\tau \\
& -\frac{C}{2\alpha^{20}} \int\limits_{t+N+n_j}^{t+N+n_{j+1}} \Big(\frac{\partial G}{\partial x}(\gamma_1(\tau),\tau)-\frac{\partial G}{\partial x}(\gamma_1(\tau),t+N+n_{j+1})\Big)^2 d\tau \\
\geq & -\frac{2\alpha^{20}}{C}\Big(A^1_{t+N+n_j,t+N+n_{j+1}}(\gamma_1)+\frac{b^2(n_{j+1}-n_j)}{2}\Big) \\
& -\frac{C}{2\alpha^{20}}\Bigg(\int\limits_{t+N+n_j}^{t+N+n_j+\alpha^{40}} \Big(\frac{\partial G}{\partial x}(\gamma_1(\tau),\tau)-\frac{\partial G}{\partial x}(\gamma_1(\tau),t+N+n_{j+1})\Big)^2 d\tau \\
& + \int\limits_{t+N+n_j+\alpha^{40}}^{t+N+n_{j+1}} \Big(\frac{\partial G}{\partial x}(\gamma_1(\tau),\tau)-\frac{\partial G}{\partial x}(\gamma_1(\tau),t+N+n_{j+1})\Big)^2 d\tau\Bigg).
\end{aligned}
$$

The first term of the right-hand side can be estimated by observing that the restriction of γ_1 to $[t+N+n_j,\ t+N+n_{j+1}]$ is still a minimizer, and that $A^3_{t+N+n_j,t+N+n_{j+1}}$, which only depends on the endpoint of the curve at $t+N+n_j$, is the same for γ_1 and γ_3.

On the other hand, the second and the third terms of the right-hand side can be estimated by using (5.17). Thus we obtain that

$$
\begin{aligned}
R \geq & -\frac{2\alpha^{20}}{C}\Big(-R+A^1_{t+N+n_j,t+N+n_{j+1}}(\gamma_3)+A^2_{t+N+n_j,t+N+n_{j+1}}(\gamma_3) \\
& +\frac{b^2N}{2}\Big)-\frac{C}{2\alpha^{20}}\Big(\frac{\alpha^{40}C^2}{4}+(n_{j+1}-n_j-\alpha^{40})\alpha^{80}\Big) \\
\geq & -\frac{2\alpha^{20}}{C}\Big(-R+\frac{1}{8(n_{j+1}-n_j)} \\
& +\int\limits_{t+N+n_j}^{t+N+n_{j+1}} \dot{\gamma}_3(\tau)\Big(\frac{\partial G}{\partial x}(\gamma_3(\tau),\tau)-\frac{\partial G}{\partial x}(\gamma_3(\tau),t+N+n_{j+1})\Big)d\tau \\
& +\frac{b^2N}{2}\Big)-\alpha^{20}C^3.
\end{aligned}
$$

Consequently,

$$\begin{aligned}R \geq &\frac{2\alpha^{20}}{C}R - \frac{\alpha^{20}}{4C}\\ &- \frac{2\alpha^{20}}{C}\Big(b+\frac{1}{2}\Big)\int\limits_{t+N+n_j}^{t+N+n_{j+1}} \Big|\frac{\partial G}{\partial x}(\gamma_3(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma_3(\tau), t+N+n_{j+1})\Big| d\tau\\ &- \frac{b^2 N\alpha^{20}}{C} - \alpha^{20}C^3.\end{aligned}$$

Using (5.6), (5.15), (5.8), and (5.17), we get

$$\begin{aligned}R &\geq \frac{2\alpha^{20}}{C}R - \frac{\alpha^{20}}{4C} - \Big(b+\frac{1}{2}\Big)N\alpha^{20} - \frac{b^2N\alpha^{20}}{C} - \alpha^{20}C^3\\ &\geq \frac{2\alpha^{20}}{C}R - N\alpha^{20}(b+1)^2 - \Big(C^3 + \frac{1}{4C}\Big)\alpha^{20}\\ &\geq \frac{2\alpha^{20}}{C}R - \frac{4\alpha^9}{10} - \frac{2\alpha^{17}}{10^3} \geq \frac{2\alpha^{20}}{C}R - \frac{\alpha^9}{2}.\end{aligned}$$

Consequently,

$$R \geq -\Big(1 - \frac{2\alpha^{20}}{C}\Big)^{-1}\frac{\alpha^9}{2} \geq -\alpha^9.$$

By (5.11), it follows that for $j \in [1, N'-2]$ we have

$$\begin{aligned}&A_{t+N+n_j,t+N+n_{j+1}}(\gamma_2) - A_{t+N+n_j,t+N+n_{j+1}}(\gamma_1)\\ =&(A^1_{t+N+n_j,t+N+n_{j+1}}(\gamma_2) - A^1_{t+N+n_j,t+N+n_{j+1}}(\gamma_1))\\ &+ (A^2_{t+N+n_j,t+N+n_{j+1}}(\gamma_2) - A^2_{t+N+n_j,t+N+n_{j+1}}(\gamma_1))\\ &+ (A^3_{t+N+n_j,t+N+n_{j+1}}(\gamma_2) - A^3_{t+N+n_j,t+N+n_{j+1}}(\gamma_1))\\ \leq&(0-0) + \Big(b(C\alpha^{40}/2 + N\alpha^{40}) - (-\alpha^9)\Big)\\ &+ \Big((m_1 + \alpha^6/10 + \alpha^{40}) - (m_1 + \alpha^2 - \alpha^{40})\Big) \leq -\frac{\alpha^2}{2}.\end{aligned} \tag{5.19}$$

Here, the estimate for $A^2_{t+N+n_j,t+N+n_{j+1}}(\gamma_2)$ follows from (5.17).

Similarly, since $A^3_{t+N,t+N+n_1}(\gamma_2) = A^3_{t+N,t+N+n_1}(\gamma_1)$, we have

$$\begin{aligned}&A_{t+N,t+N+n_1}(\gamma_2) - A_{t+N,t+N+n_1}(\gamma_1)\\ &\leq \frac{1}{8} + A^2_{t+N,t+N+n_1}(\gamma_2) + \alpha^9 \leq 1,\end{aligned} \tag{5.20}$$

and

$$A_{t+N+n_{N'-1},t+2N}(\gamma_2) - A_{t+N+n_{N'-1},t+2N}(\gamma_1)$$
$$\leq \frac{1}{8} + A^2_{t+N+n_{N'-1},t+2N}(\gamma_2) + \alpha^9 + C \leq 2C. \tag{5.21}$$

Here, we get

$$A^2_{t+N,t+N+n_1}(\gamma_2), A^2_{t+N+n_{N'-1},t+2N}(\gamma_2) \leq (b+1/2)(C\alpha^{40}/2 + N\alpha^{40})$$

in the same way as for the estimate for $A^2_{t+N+n_j,t+N+n_{j+1}}(\gamma_2)$ above.

It remains to add together the inequalities (5.19-5.21). Using (5.15) and (5.8) we get

$$A_{t+N,t+2N}(\gamma_2) - A_{t+N,t+2N}(\gamma_1)$$
$$\leq 2C + 1 - (N'-2)\frac{\alpha^2}{2} \leq 2C + 1 - \frac{1}{2\alpha} < 0.$$

This inequality is in contradiction with the fact that γ_1 is a minimizer. This proves the lemma's assertion. □

LEMMA 5.3.4. *For $\omega \in E_1 \cap E_2$, if for some minimizer γ_1 on $[t, t+N+n_j]$, $1 \leq j \leq N'-1$, $y = \gamma_1(t+N+n_j)$ satisfies:*

$$-\tilde{F}_1(y) \leq m_1 + \alpha^2,$$

then we have

$$\gamma_1(t+N) \in I_1(\alpha).$$

Proof: In the same way as in the proof of Lemma 5.3.2, we consider a "bad" minimizer γ_1. Without loss of generality, we assume that

$$-\tilde{F}_1[\gamma_1(t+N+n_{j'})] > m_1 + \alpha^2,\ 1 \leq j' < j. \tag{5.22}$$

We define γ_2 with the same endpoints as γ_1 in the same way as in the proof of Lemma 5.3.2, i.e. such that $\dot{\gamma_2}^b = 0$ between $t+N$ and $t+N+n_j$, linear between t and $t+N$, and satisfying $|\dot{\gamma_2}^b| \leq \frac{1}{2N}$. We get

$$A^3_{t,t+N}(\gamma_2) = A^3_{t,t+N}(\gamma_1).$$
$$A^1_{t,t+N}(\gamma_2) - A^1_{t,t+N}(\gamma_1) \leq \frac{N}{8N^2} - 0 \leq \frac{\alpha^{10}}{16}.$$
$$A^2_{t,t+N}(\gamma_2) \leq \left(b + \frac{1}{2N}\right) \int_t^{t+N} \left|\frac{\partial G}{\partial x}(\gamma_2(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma_2(\tau), t+N)\right| d\tau \leq \alpha^{29}.$$

The last inequality follows from (5.15), (5.8), and (5.16).

To estimate the quantity

$$R = A^2_{t,t+N}(\gamma_1),$$

we proceed in the same way as for $A^2_{t+N+n_j,t+N+n_{j+1}}(\gamma_1)$ in Lemma 5.3.3. Namely, we consider a straight line γ_3 with the same endpoints as $\gamma_1|_{[t,t+N]}$ satisfying $|\dot{\gamma_3}^b| \leq 1/2N$. We have

$$\begin{aligned}
R \geq & -2\alpha^{20} \int\limits_t^{t+N} \Big(\frac{\dot{\gamma}_1(\tau)}{2}\Big)^2 d\tau \\
& - \frac{1}{2\alpha^{20}} \int\limits_t^{t+N} \Big(\frac{\partial G}{\partial x}(\gamma_1(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma_1(\tau), t+N)\Big)^2 d\tau \\
\geq & -2\alpha^{20} \int\limits_t^{t+N} \frac{(\dot{\gamma}_1^b(\tau))^2 + b^2}{2} d\tau \\
& - \frac{1}{2\alpha^{20}} \int\limits_t^{t+N} \Big(\frac{\partial G}{\partial x}(\gamma_1(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma_1(\tau), t+N)\Big)^2 d\tau.
\end{aligned}$$

Since a restriction of γ_1 is still a minimizer, we get

$$\begin{aligned}
R \geq & -2\alpha^{20}\Big(A^1_{t,t+N}(\gamma_3) + A^2_{t,t+N}(\gamma_3) - R + \frac{b^2 N}{2}\Big) \\
& - \frac{1}{2\alpha^{20}} \int\limits_t^{t+N} \Big(\frac{\partial G}{\partial x}(\gamma_1(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma_1(\tau), t+N)\Big)^2 d\tau. \\
\geq & -2\alpha^{20}\Big(\frac{N}{8N^2} + \Big(b + \frac{1}{2N}\Big) \times \int\limits_t^{t+N} \Big|\frac{\partial G}{\partial x}(\gamma_3(\tau),\tau) - \frac{\partial G}{\partial x}(\gamma_3(\tau), t+N)\Big| d\tau \\
& - R + \frac{b^2 N}{2}\Big) - \frac{N}{2\alpha^{20}}\alpha^{80}.
\end{aligned}$$

Consequently,

$$\begin{aligned}
R & \geq 2\alpha^{20} R - 2\alpha^{20}\Big(\frac{\alpha^{10}}{16} + (b+1)N\alpha^{40} + \frac{2b^2}{\alpha^{10}}\Big) - \frac{N\alpha^{60}}{2} \\
& \geq 2\alpha^{20} R - (5b^2+1)\alpha^{10} \geq 2\alpha^{20} R - \frac{\alpha^9}{2}.
\end{aligned}$$

Therefore

$$A^2_{t,t+N}(\gamma_1) \geq -\alpha^9.$$

On the other hand, we have

$$A^1_{t+N,t+N+n_j}(\gamma_2) - A^1_{t+N,t+N+n_j}(\gamma_1) \leq 0.$$

By definition, the action difference

$$U = A_{t,t+N+n_j}(\gamma_2) - A_{t,t+N+n_j}(\gamma_1)$$

satisfies

$$\begin{aligned}U =&(A^1_{t,t+N}(\gamma_2) - A^1_{t,t+N}(\gamma_1)) + (A^1_{t+N,t+N+n_j}(\gamma_2) - A^1_{t+N,t+N+n_j}(\gamma_1))\\ &+ (A^2_{t,t+N}(\gamma_2) - A^2_{t,t+N}(\gamma_1)) + (A^3_{t,t+N}(\gamma_2) - A^3_{t,t+N}(\gamma_1))\\ &+ (A^2_{t+N,t+N+n_j}(\gamma_2) + A^3_{t+N,t+N+n_j}(\gamma_2)\\ &- A^2_{t+N,t+N+n_j}(\gamma_1) - A^3_{t+N,t+N+n_j}(\gamma_1)).\end{aligned}$$

Consequently,

$$\begin{aligned}U \leq& \frac{\alpha^{10}}{16} + 0 + \Big(\alpha^{29} + \alpha^9\Big) + 0 + (A^2_{t+N,t+N+n_j}(\gamma_2) + A^3_{t+N,t+N+n_j}(\gamma_2)\\ &- A^2_{t+N,t+N+n_j}(\gamma_1) - A^3_{t+N,t+N+n_j}(\gamma_1))\\ \leq& 2\alpha^9 + (A^2_{t+N,t+N+n_j}(\gamma_2) - A^2_{t+N,t+N+n_j}(\gamma_1)\\ &+ A^3_{t+N,t+N+n_j}(\gamma_2) - A^3_{t+N,t+N+n_j}(\gamma_1).\end{aligned} \tag{5.23}$$

In the same way as previously, we get

$$A^2_{t+N,t+N+n_j}(\gamma_2) \leq bj\Big(\frac{C\alpha^{40}}{2} + N\alpha^{40}\Big) \leq 5bN'\alpha^{30} \leq \alpha^{26}.$$

The estimates of $A^2_{t+N+n_{j'},t+N+n_{j'+1}}(\gamma_1),\ 0 \leq j' < j$ in Lemma 5.3.3 still hold in our case. Therefore

$$A^2_{t+N,t+N+n_j}(\gamma_1) \geq -N'\alpha^9 \geq -2\alpha^6.$$

By (5.11) and (5.22), for $1 \leq j' \leq j-1$ we get

$$\begin{aligned}&A^3_{t+N+n_{j'},t+N+n_{j'+1}}(\gamma_2) - A^3_{t+N+n_{j'},t+N+n_{j'+1}}(\gamma_1)\\ &\leq (m_1 + \alpha^2 + \frac{\alpha^6}{10} + \alpha^{40}) - (m_1 + \alpha^2 - \alpha^{40}) \leq \alpha^6.\end{aligned}$$

Finally, since we have supposed that $\gamma_1(t+N) \notin I_1(\alpha)$, we have

$$A^3_{t+N,t+N+n_1}(\gamma_2) - A^3_{t+N,t+N+n_1}(\gamma_1)$$
$$\leq (m_1 + \alpha^2 + \frac{\alpha^6}{10} + \alpha^{40}) - (m_1 + \alpha - \alpha^{40}) \leq -\alpha/2.$$

Combining all these inequalities with (5.23) we get

$$U \leq 2\alpha^9 + \alpha^{26} + 2\alpha^6 + (N'-1)\alpha^6 - \alpha/2 < 0.$$

We have a contradiction with the fact that γ_1 is a minimizer. This proves the lemma's assertion. □

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Appendice A. Le problème de Cauchy pour l'équation (1.14) avec force de type bruit blanc

In this appendix, we prove that the Cauchy problem for the equation (1.14) with white-noise forcing is well-posed in L_1, a.s. An analogous problem of well-posedness has been considered by Da Prato and Zabczyk in [20, Chapter 14]; however, their results are weaker than ours since they consider a white noise which is not smooth in space.

Here, the functions whose Sobolev norms we consider do not necessarily have zero mean value in space. The only thing that changes is that now in the expressions for the Sobolev norms $W^{m,p}$ (resp. H^s) we have to add the norm in L_p (resp. L_2) to the formulas in Subsection 1.1.2.

We begin by considering mild solutions in H^1, in the spirit of [19, 20]. Then, by a bootstrap argument, we prove that for an initial condition in C^∞, every mild solution is actually a weak solution in C^∞ (see Definition 4.1.2). The uniform estimates in Chapter 4 allow us to prove that such a weak solution is global. Finally, a contraction argument implies that (4.13) is well-posed in L_1.

We recall that there exists an event Ω_1 such that $\mathbb{P}(\Omega_1) = 1$ and for $\omega \in \Omega_1$, the Wiener process $w(t)$ belongs to $C([0,+\infty), H^s)$ for every $s \geq 0$. In this section, we use the same notation as in Chapter 4. In particular:

$$B(u) = 2f'(u)u_x; \quad L = -\partial_{xx}.$$

By the scaling argument, we can restrict ourselves to the equation (4.13) with $\nu = 1$. We will denote by $S_L(t)$ the heat semigroup e^{-tL}, and by D_x the operator $\partial/\partial x$. We recall that for $v \in L_2$ the function $S_L(t)v(x)$ is given by:

$$S_L(t)v(x) = \sum_{k \in \mathbb{Z}} e^{-4\pi^2 k^2 t} \hat{v}_k e^{2\pi i k x}. \tag{A.1}$$

Finally, we denote by w_L the stochastic convolution

$$w_L(t) = \int_0^t S_L(t-\tau) dw(\tau).$$

For $\omega \in \Omega_2,\ \mathbb{P}(\Omega_2) = 1$, this quantity belongs to $C([0, +\infty), C^\infty)$. From now on, we suppose that ω belongs to $\Omega_1 \cap \Omega_2$.

Following Da Prato and Zabczyk [20, Chapter 14], we consider a mild form of (4.13) for $Y(t) = u(t) - w_L(t)$:

$$Y(t) = S_L(t)u_0 + \int_0^t S_L(t-\tau) D_x(f(Y(\tau) + w_L(\tau)))d\tau. \tag{A.2}$$

The heat semigroup defines a contraction in each Sobolev space H^s, in particular in H^1. On the other hand, the mapping

$$Z \mapsto f(Z): \ H^1 \to H^1$$

is locally Lipschitz on bounded subsets of H^1. Indeed, consider Z_1 and Z_2 such that $\|Z_1\|_1, \|Z_2\|_1 \leq N$. Then we have

$$\begin{aligned} &\|f(Z_1) - f(Z_2)\|_1 = |f'(Z_1)Z_1' - f'(Z_2)Z_2'| + |f(Z_1) - f(Z_2)| \\ &\leq |(f'(Z_1) - f'(Z_2))Z_1'| + |f'(Z_2)(Z_1' - Z_2')| + |f(Z_1) - f(Z_2)|_\infty \\ &\leq |Z_1'||f'(Z_1) - f'(Z_2)|_\infty + |f'(Z_2)|_\infty |Z_1' - Z_2'| + C(N)|Z_1 - Z_2|_\infty \\ &\leq NC(N)|Z_1 - Z_2|_\infty + C(N)\|Z_1 - Z_2\|_1 \\ &\leq C(N)\|Z_1 - Z_2\|_1. \end{aligned} \tag{A.3}$$

Thus, the Cauchy-Lipschitz theorem implies that the equation (A.2) has a unique local solution in H^1.

Now consider such a solution Y. We want to prove that this solution belongs to C^∞ for all t, which would imply that $Y + w_L$ is a solution of (4.13). For this, it suffices to show that for $s \geq 1$, a solution $Y \in H^s$ lies in the space $H^{(s+1/2)}$. We will need the following lemmas:

Lemma A.0.5. *For $s > 1/2$, the mapping*

$$Z \mapsto f(Z): \ H^s \to H^s$$

is bounded on bounded subsets of H^s.

Proof: An analogous lemma is proved in a more general setting for Sobolev spaces on $\mathbb{R}^n$ in [11]. We use some arguments from this paper.

For integer values of s, we proceed in the same way as in the sequence of inequalities (A.3). For non-integer values of s, the proof is not much more difficult. We will give details only for the case $s \in (1/2, 1)$.

For a fixed value of s, consider Z such that $\|Z\|_s \leq N$. By (1.3), it follows that $|Z|_\infty \leq C(N)$. Thus by (1.2) we get

$$\|f(Z)\|_s^2 \lesssim \int_{S^1} \Big(\int_0^1 \frac{|f(Z(x+\ell)) - f(Z(x))|^2}{\ell^{2s+1}} d\ell \Big) dx + |f(Z)|^2 \tag{A.4}$$

$$\lesssim \max((f'(\tilde{Z}))^2, \ |\tilde{Z}|_\infty \leq C(N)) \int_{S^1} \Big(\int_0^1 \frac{|Z(x+\ell) - Z(x)|^2}{\ell^{2s+1}} d\ell \Big) dx + C(N)$$

$$\lesssim C(N)\|Z\|_s^2 + C(N). \ \square$$

Lemma A.0.6. *Consider a bounded function $Z : [0, T) \to H^s$ for some $s \geq 0$. The function*

$$t \mapsto \int_0^t S_L(t-\tau) Z(\tau) d\tau$$

belongs to $C([0, T), H^{(s+3/2)})$.

Proof: Fix $s \geq 0$. By (1.1) and (A.1), for $\tau \in [0, t)$ we have

$$\begin{aligned}\|S_L(t-\tau)Z(\tau)\|_{s+3/2}^2 &\sim \sum_{k \in \mathbb{Z}} |k|^{2s+3} e^{-4\pi^2 k^2 (t-\tau)} |(\hat{Z}(\tau))_k|^2 \\ &\lesssim \Big(\max_{k' \in \mathbb{Z}} |k'|^3 e^{-4\pi^2 k'^2 (t-\tau)} \Big) \sum_{k \in \mathbb{Z}} |k|^{2s} |(\hat{Z}(\tau))_k|^2 \\ &\lesssim \Big(\max_{k' \in \mathbb{Z}} |k'|^3 e^{-4\pi^2 k'^2 (t-\tau)} \Big) \|Z(\tau)\|_s^2. \\ &\lesssim C(t-\tau)^{-3/2} \|Z(\tau)\|_s^2.\end{aligned}$$

To prove the lemma's statement, it remains to observe that

$$\int_0^t (t-\tau)^{-3/4} d\tau < +\infty. \ \square$$

THEOREM A.0.7. *Consider a local solution Y of (A.2) in H^1 defined on an interval $[0,T)$. If for some $s \geq 1$, Y belongs to $C([0,T),H^s)$, then Y actually belongs to $C([0,T),H^{(s+1/2)})$.*

Proof: Since u_0 belongs to C^∞, $S_L(t)u_0$ belongs to $C([0,T),H^\delta)$ for every δ. On the other hand, by Lemma A.0.5 we have

$$D_x(f(Y(\tau)+w_L(\tau))) \in C([0,T),H^{s-1}),$$

and thus by Lemma A.0.6 we get

$$\int_0^t S_L(t-\tau)D_x(f(Y(\tau)+w_L(\tau)))d\tau \in C([0,T),H^{(s+1/2)}.$$

Since Y is a solution of (A.2),

$$Y(t) = S_L(t)u_0 + \int_0^t S_L(t-\tau)D_x(f(Y(\tau)+w_L(\tau)))d\tau$$

belongs to the space $C([0,T),H^{(s+1/2)})$. □

Thus, we have proved existence and uniqueness of a local solution to (4.13), which is C^∞-smooth in space for $t > 0$. To see that this solution is necessarily global, it suffices to observe that for any $\tau > 0$ it satisfies estimates which hold uniformly in time for $t \in [\tau, \tau+1]$. This fact is proved in the same way as Theorem 4.5.1.

Now we consider the mapping acting on H^1:

$$S(t) : u_0 \mapsto u(t),$$

which is only defined a priori for a.e. ω. On the "bad" set $\Omega - \Omega_1 \cap \Omega_2$ we can put $S(t) = Id$. The results in Subsection 4.8.1 tell us that $S(t)$ is a contraction in L_1. By density of C^∞, this allows us to consider solutions of (4.13) for any initial condition in L_1. Thus, we can define $S(t)$ as a mapping from L_1 to itself.

LEMMA A.0.8. *The function*

$$(t,u_0) \mapsto S(t)u_0 : \; \mathbb{R} \times L_1 \to L_1$$

is continuous in each variable.

Proof: By construction, this function is continuous (and 1-Lipschitz) in u_0. By density of C^∞ in L_1, to show that $t \mapsto S(t)u_0$ is continuous for any u_0, it suffices to do so for $u_0 \in C^\infty$. But the corresponding statement holds since in this case the equation (A.2) has a unique solution, which is continuous in H^1 and hence in L_1. □

Furthermore, in all estimates in Sections 4.3-4.6, we can replace the Sobolev norm (or the maximum of u_x) for a given value of t by its supremum over all smooth initial conditions. Consequently, using interpolation between L_1 and higher-order Sobolev spaces and the dominated convergence theorem, we can prove that solutions u with initial data u_0 in L_1 satisfy all estimates in Sections 4.3-4.6. In particular, for $t > 0$ solutions with initial data in L_1 are C^∞-smooth.

To give an example of a proof, consider a particular case of Theorem 4.5.1:

$$\frac{1}{T}\int_t^{t+T} \mathbb{E}\|u(s)\|_1^2 \sim \nu^{-1},\ t \geq T_0 + 2,\ T \geq T_0. \tag{A.5}$$

We choose a sequence $(u_0)_n \in C^\infty,\ n \geq 1$ such that

$$|u_0 - (u_0)_n|_1 \leq \frac{1}{n}, \quad n \geq 1.$$

Denote by u_n a solution of (4.13) with initial condition $(u_0)_n$. By Theorem 4.8.3 we have

$$|u(t) - u_n(t)|_1 \leq \frac{1}{n}, \quad n \geq 1.$$

By Theorem 4.5.1 there exists a constant $\tilde{C}$ such that

$$\sup_{n\geq 1} \mathbb{E}|u_n(t)|_{2,\infty} \leq \tilde{C}\nu^{-2}.$$

Thus, by Lemma 1.1.1, $u(t)$ is the limit of $u_n(t)$ in H^1. Finally, by Theorem 4.5.1 we know that uniformly on n, we have

$$\frac{1}{T}\int_t^{t+T} \mathbb{E}\|u_n(s)\|_1^2 \sim \nu^{-1},\ t \geq T_0 + 2,\ T \geq T_0.$$

To prove the estimate (A.5), it remains to apply the dominated convergence theorem.

Appendice B. Solutions de l'équation (1.14) vues comme processus de Markov

In this appendix, we give the terminology related to Markov processes, used to study solutions of the equation (1.14) with white-noise forcing. These solutions form such a process in L_1. This terminology can be adapted to the discrete setting, and hence used to study the equation with a "kicked" force: note that in that case randomness and evolution corresponding to the free equation are "decoupled" in time.

Moreover, the previously obtained upper estimates allow us to see that in both cases solutions form a Markov process in H^m for every $m \geq 0$.

First we recall some classical definitions and results about continuous Markov processes and RDS (random dynamical systems). A more complete presentation, with all the proofs, can be found in [45, Subsection 1.3]. Everywhere, $(X, |\cdot|_X)$ denotes a Polish (complete separable metric) space. We denote by $\mathcal{B}(X)$ the corresponding Borel σ-algebra.

Consider a measurable space $(\Omega, \mathcal{F})$. We say that a subset $A \subset \Omega$ is *universally measurable* if A belongs to the completion of $\mathcal{F}$ with respect to any probability measure on $(\Omega, \mathcal{F})$. A function $\Omega \to \mathbb{R}$ is said to be universally measurable if so are the preimages of all open subsets of $\mathbb{R}$.

Now consider a filtration $(\mathcal{F}_t, t \geq 0)$ for $(\Omega, \mathcal{F})$. We say that a random process $v : [0, +\infty) \to X$ is *adapted* to $\mathcal{F}_t$ if $v(t)$ is $\mathcal{F}_t$-measurable for every t.

Definition B.0.9. *A Markov process is the following family of objects:*

1) *A measurable space* $(\Omega, \mathcal{F})$ *with a filtration* $(\mathcal{F}_t, t \geq 0)$.

2) A Polish space X parametrising a family of measures $\{\mathbb{P}_v,\ v \in X\}$ such that $v \mapsto \mathbb{P}_v(A)$ is universally measurable for any $v \in X$ and $A \in \mathcal{F}$.

3) An X-valued random process $(v_t, t \geq 0)$ adapted to $(\mathcal{F}_t, t \geq 0)$ and satisfying the two following properties, for all $v \in X$, $\Gamma \in \mathcal{B}(X)$, $t, s \geq 0$:

$$\mathbb{P}_v(v_0 = v) = 1,$$

$$\mathbb{P}_v(v_{t+s} \in \Gamma | \mathcal{F}_s) = P_t(v_s, \Gamma), \quad \mathbb{P}_v - a.s.$$

Here, P_t denotes the transition function:

$$P_t(v, \Gamma) = \mathbb{P}_v(v_t \in \Gamma).$$

DEFINITION B.0.10. *Consider a Markov process, using the same notation as above. For a probability measure μ on $(\mathbb{P}, \Omega)$, $S_t^* \mu$ is the probability measure defined by:*

$$S_t^* \mu(\Gamma) = \int_X P_t(z, \Gamma) d\mu(z), \quad \Gamma \in \mathcal{B}(X).$$

LEMMA B.0.11. *The operators $(S_t^*,\ t \geq 0)$ form a semigroup (in other words, $S_0^* = Id$ and $S_t^* \circ S_s^* = S_{t+s}^*$).*

The following key theorem will be stated without proof. For a detailed proof of an analogous (but much more difficult) result in a very similar setting, see [45, Section 2.4.4]. There, the authors study the 2D Navier-Stokes equation: its solutions satisfy results of the same type as those proved in Appendix A, and in particular an analogue of Lemma A.0.8.

THEOREM B.0.12. *Solutions of the equation (1.14) in the sense of Appendix A form a Markov process in L_1.*

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