C^1 regularity of solutions of the Monge-Ampère equation for optimal transport in dimension two

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Abstract

We prove C^1 regularity of c-convex weak Alexandrov solutions of the Monge-Ampère equation in dimension two assuming only a bound from above on the Monge-Ampère measure. The Monge-Ampère equations involved arise in the optimal transport problem. Our results hold true under a natural condition on the cost function, namely *non-negative cost-sectional curvature*, a condition introduced in [7], that was shown in [5] to be necessary for C^1 regularity. Such condition holds in particular for the case "cost = distance squared" which leads to the usual Monge-Ampère equation $\det D^2 u = f$. Our result is in some sense optimal, both for the assumptions on the density (thanks to the regularity counterexamples of Wang [11]) and for the assumptions on the cost-function (thanks to the results of Loeper [5]).

1 Introduction and preliminary results

Through a well established procedure, maps that solve optimal transport problem are shown to derive from a c-convex potential, itself solution to a Monge-Ampère type equation, which reads in its general form

$$\det(D^2\phi + A(x, \nabla\phi)) = f(x, \nabla\phi). \tag{1}$$

Here, $(x,p) \mapsto A(x,p)$ is a symmetric matrix valued function that depends on the cost function $c(x,y)$ through the formula

$$A(x,p) := D^2_{xx}c(x,y) \qquad \text{for } y \text{ such that } -\nabla_x c(x,y) = p.$$

That for any x there is indeed a unique y such that $-\nabla_x c(x,y) = p$ will be guaranteed by condition **A1** given hereafter. The optimal map will then be

$$x \mapsto [-\nabla_x c(x,\cdot)]^{-1}(\nabla\phi(x)).$$

Assuming that the data of the optimal transport problem are measures supported on sets satisfying necessary smoothness and convexity conditions, a necessary and sufficient condition on

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the cost function for smoothness of the optimal map (for arbitrary smooth positive data) has been given in ([7, 8, 5]) (see also [4] or [10, Chapter 12]). This is the so called condition **Aw** given below.

In this paper, under this condition (which is satisfied for instance by the cost $c(x, y) = -\langle x, y\rangle$, that leads to the usual Monge-Ampère equation), we prove C^1 regularity of the solution of the associated Monge-Ampère equation (and subsequently continuity of the optimal map) assuming only a bound from above on the right hand side of (1).

Let us recall the previous results available in this direction. First, as shown in [5], if **Aw** is not satisfied, no C^1 regularity can hold, even for C^∞ positive data. Under **Aw**, classical C^2 and higher regularity holds for smooth positive data [7, 8], while assuming the stronger condition **As**, $C^{1,\alpha}$ regularity holds for rough and possibly vanishing Monge-Ampère measures [5]. The only available weak regularity results under **Aw** are due to Caffarelli, for the particular case $c(x, y) = -x \cdot y$, and yield $C^{1,\alpha}$ regularity for Monge-Ampère measures bounded away from 0 and infinity. Hence our result can be seen as a step, in the two dimensional case, towards a general partial regularity result for weak solutions under **Aw**. We notice also that the case of two dimensional convex surfaces with curvature bounded only from above had already been addressed by Alexandrov [1], and this problem is very close to the optimal transport problem with quadratic cost.

Following [5], let us recall some definitions:

Definition 1.1 (c-transform and c-convex functions) *Given a lower semi continuous function $\phi : \Omega \subset \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, we define its* c-transform *by*

$$\phi^c(y) := \sup_{x\in\Omega} -c(x, y) - \phi(x).$$

Respectively, for $\psi : \Omega' \subset \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ lower semi continuous function, we define its c^*-transform *by*

$$\psi^{c^*}(x) := \sup_{y\in\Omega'} -c(x, y) - \psi(y).$$

A function is said to be c-convex *if it is the c^*-transform of some lower semi continuous function $\psi : \Omega' \subset \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$, that is $\phi = \psi^{c^*}$. Moreover, in this case, $(\phi^c)^{c^*} = \phi$ on Ω (see [9] or [10, Chapter 5]).*

Throughout this paper we will consider two bounded sets Ω, Ω' of $\mathbb{R}^2$. Our first assumption on the cost is:

- **A0** The cost function c belongs to $C^4(\Omega \times \Omega')$.

Definition 1.2 (Gradient mapping) *Let ϕ be a c-convex function. We define the* set-valued mapping G_ϕ *by*

$$G_\phi(x) = \{y \in \Omega' \mid \phi(x) + \phi^c(y) = -c(x, y)\}.$$

Noticing that for all $y \in G_\phi(x)$, $\phi(\cdot) + c(\cdot, y)$ has a global minimum at x, it is natural to introduce the following definition:

Definition 1.3 (Subdifferential and c-subdifferential) *For ϕ a locally semi convex function, the* subdifferential *of ϕ at x is the set*

$$\partial\phi(x) = \{p \in \mathbb{R}^n \mid \phi(y) \geq \phi(x) + \langle p, y - x\rangle + o(|x - y|)\}.$$

*If ϕ is c-convex, the c-*subdifferential *of ϕ at x is the set*

$$\partial^c\phi(x) = \{-\nabla_x c(x,y) \mid y \in G_\phi(x)\}.$$

The inclusion $\emptyset \neq \partial^c\phi(x) \subset \partial\phi(x)$ always holds.

We recall that a convex set is said strictly/uniformly convex if its boundary can be locally parameterized by the graph of a strictly/uniformly convex function.

Definition 1.4 ((strict/uniform) c-convexity) *Let $\Omega, \Omega' \subset \mathbb{R}^n$ be two open sets. We say that Ω' is* (strictly/uniformly) c-convex *with respect to Ω if, for all $x \in \Omega$, the set $-\nabla_x c(x, \Omega')$ is (strictly/uniformly) convex.*

Finally, before recalling the notion of *cost-sectional curvature* $\mathfrak{S}_\mathfrak{c}(x,y)$, we need to make some more assumptions on c:

- **A1** For any $x \in \Omega$, the mapping $\Omega' \ni y \mapsto -\nabla_x c(x,y) \in \mathbb{R}^n$ is injective.
- **A2** The cost function c satisfies $\det(D^2_{xy}c) \neq 0$ for all $(x,y) \in \Omega \times \Omega'$.

In particular, under conditions **A0**,**A1**, **A2**, one can define the c-exponential map (see [5]) by

$$p \to \text{c-exp}_x(p) = [-\nabla_x c(x,\cdot)]^{-1}(p), \tag{2}$$

it is C^3 smooth on its domain of definition.

Definition 1.5 *Under assumptions* **A0**-**A1**-**A2**, *one can define on $T_x\Omega \times T_x\Omega$ the real-valued map*

$$\mathfrak{S}_\mathfrak{c}(x_0,y_0)(\xi,\nu) = D^4_{p_\nu p_\nu x_\xi x_\xi}\Big[(x,p) \to -c(x, -[\nabla_x c(x_0,\cdot)]^{-1}(p))\Big]\Big|_{x_0, p_0 = -\nabla_x c(x_0,y_0)}. \tag{3}$$

When ξ, ν are unit orthogonal vectors, $\mathfrak{S}_\mathfrak{c}(x_0,y_0)(\xi,\nu)$ defines the cost-sectional curvature from x_0 to y_0 in directions (ξ,ν).

We also introduce the symmetric assumption to **A1**:

- **A1'** For any $y \in \Omega'$, the mapping $\Omega \ni x \mapsto -\nabla_y c(x,y) \in \mathbb{R}^n$ is injective.

Under assumption **A1'**, the operator $\mathfrak{S}_\mathfrak{c}$ is symmetric under the exchange of x and y, in the sense that $\mathfrak{S}_\mathfrak{c}(x,y)(\xi,\nu) = \mathfrak{S}_{\mathfrak{c}^*}(y,x)(\tilde{\nu},\tilde{\xi})$, where $c^*(x,y) = c(y,x)$, $\tilde{\nu} = [D_p(\text{c-exp}_x)] \cdot \nu$, $\tilde{\xi} = [D_p(\text{c}^*\text{-exp}_y)]^{-1} \cdot \xi$ (see [5]).

The last assumption that we make on the cost, which as we explained is necessary to prove a regularity result, is the following:

- **Aw** (**non-negative sectional curvature**) There exists $C_0 \geq 0$ such that for any $(x_0,y_0) \in \Omega \times \Omega'$, for all $\xi, \nu \in \mathbb{R}^n$ orthogonal vectors,

$$\mathfrak{S}_\mathfrak{c}(x_0,y_0)(\xi,\nu) \geq C_0|\xi|^2|\nu|^2.$$

If $C_0 > 0$, then c is said to satisfy **As** (**positive sectional curvature**).

We recall that, under assumptions **A0** and **A1**, the following existence and uniqueness result for optimal transport maps is well-known, (see [9] or [10, Chapter 10]) (actually, this result can be proved under much weaker assumptions on c and on the source measure μ_0):

Theorem 1.6 *Let c be a cost function satisfying **A0** and **A1**. Let μ_0 and μ_1 be two probability measures on Ω and Ω' respectively. Assume that*

$$\int_{\Omega\times\Omega'} c(x,y)\,d\mu_0(x)\,d\mu_1(y) < +\infty,$$

and that μ_0 is absolutely continuous with respect to the Lebesgue measure. Then there exists an optimal transport map $T : \Omega \to \Omega'$, that is a map such that $T_\#\mu_0 = \mu_1$ which minimizes the functional

$$\int_\Omega c(x,T(x))\,d\mu_0(x) = \min_{S_\sharp\mu_0=\mu_1} \left\{ \int_\Omega c(x,S(x))\,d\mu_0(x) \right\}.$$

This map T is unique μ_0-a.e. Moreover, there exists a c-convex function ϕ such that $T = G_\phi$. Finally, if ψ is c-convex and satisfies $(G_\psi)_\#\mu_0 = \mu_1$, then $\nabla\psi = \nabla\phi$ μ_0-a.e.

Now we observe that, if ϕ a c-convex function of class C^2 such that $(G_\phi)_\#\mu_0 = \mu_1$, with μ_0, μ_1 both absolutely continuous with respect to the Lebesgue measure (hence $\mu_0 = \rho_0\mathscr{L}^n$, $\mu_1 = \rho_1\mathscr{L}^n$ for some functions ρ_0, ρ_1) the conservation of mass is expressed in local coordinates by the following Monge-Ampère type equation:

$$\det(D^2\phi(x) + D^2_{xx}c(x,G_\phi(x)) = |\det(D^2_{xy}c)|(x,G_\phi(x))\frac{\rho_0(x)}{\rho_1(G_\phi(x))}. \tag{4}$$

Conditions **Aw** and **As** where first introduced in [7] and [8] as sufficient conditions to get C^2 (and subsequently C^∞) regularity, assuming the densities to be smooth together with c-convexity and smoothness of the domains (see [7, 8, 5] for more details). In [5] a geometric interpretation of these conditions is given, which allows to prove that **Aw** is indeed necessary for regularity.

Here we are interested in weak (or generalized) solutions of the equation (4). We recall two definitions of generalized solutions:

Definition 1.7 *Let $\phi : \mathbb{R}^n \to \mathbb{R}$ be c-convex.*

(i) ϕ is a solution of (4) in the Alexandrov sense *if*

$$\mu_0(B) = \mu_1(G_\phi(B)) \qquad \forall B \subset \Omega,$$

which will be denoted by $\mu_0 = (G_\phi)^\#\mu_1$;

(ii) ϕ is a solution of (4) in the Brenier sense *if*

$$\mu_0(G_\phi^{-1}(B)) = \mu_1(B) \qquad \forall B \subset \Omega,$$

that is $(G_\phi)_\#\mu_0 = \mu_1$.

The measure $(G_\phi)^\#\mathscr{L}^n$ is the c-Monge-Ampère *(in short Monge-Ampère) measure of ϕ.*

By Theorem 1.6, the optimal transportation problem yields an optimal transport map whenever μ_0 is absolutely continuous with respect to the Lebesgue measure. Moreover, the map G_ϕ given by the theorem will be a solution of (4) in the Brenier sense by construction. Using the c-convexity of ϕ it can be proven that, whenever μ_1 is also absolutely continuous with respect to the Lebesgue measure, $(G_\phi)^\# \mu_1$ is countably additive, and hence is a Radon measure (see [7, Lemmas 3.1-3.4]). However, in order to get equivalence between Brenier solutions and Alexandrov solutions, one has also to assume the c-convexity of the support of μ_1. More precisely, for μ_0 supported in Ω, if $(G_\phi)_\# \mu_0 = \mathscr{L}^n \llcorner \Omega'$ and $G_\phi(\mathbb{R}^n) = \Omega'$, then one can deduce $\mu_0 = (G_\phi)^\# \mathscr{L}^n$, provided Ω' is c-convex with respect to Ω (see [7]). In particular μ_0 is the Monge-Ampère measure of ϕ. Without this convexity assumption on the target, $\mu_0 = (G_\phi)^\# \mathscr{L}^n$ might not be implied by $(G_\phi)_\# \mu_0 = \mathscr{L}^n \llcorner \Omega'$, and counterexamples to regularity can be built (see [2]).

Let us from now focus on the case $\mu_0 = \rho_0 \mathscr{L}^n$ and $\mu_1 = \mathscr{L}^n \llcorner \Omega'$ with Ω' c-convex with respect to the support of μ_0 (we could also consider the case $\mu_1 = \rho_1 \mathscr{L}^n \llcorner \Omega'$ with ρ_1 bounded away from 0, but we assume $\rho_1 = 1$ only for simplicity of exposition). In the special case $c(x, y) = -\langle x, y\rangle$ (which is trivially equivalent to the quadratic cost $c(x, y) = |x - y|^2$), c-convexity reduces to the classical notion of convexity and (4) becomes the classical Monge-Ampère equation

$$\det(D^2\phi) = \rho_0,$$

with the constraint $\partial\phi(\mathbb{R}^2) = \Omega'$, with Ω' convex. In this case $C^{1,\alpha}$ regularity of ϕ can be deduced under the assumption $\frac{1}{M} \leq \rho_0 \leq M$ for a positive constant M (see [2]), while no C^1 regularity can be expected for arbitrary data when $n \geq 3$ if the lower bound on ρ_0 is removed (see [11]). In this paper, for $n = 2$, it is proven that one can get C^1 regularity assuming only an upper bound on the density for the class of costs which satisfies the **Aw** condition (see Theorem 3.3). We present a separate proof in the quadratic case, although the result in this particular case might be recovered from an old result of Alexandrov on convex two-dimensional surfaces [1].

In any case, the scheme of the proof is as follows: we will first prove a general lemma which, roughly speaking, says the following: let Ω' be a c-convex set, and let ϕ be a c-convex function in $\mathbb{R}^2$ such that the measure $(G_\phi)^\#(\mathscr{L}^2 \llcorner \Omega')$ has a bounded density. If c satisfies **Aw** and by contradiction ϕ is not C^1, then ϕ coincides with a "c-affine" function on a "c-line". In a second step, using again the condition **Aw** and assuming moreover Ω' to be strictly c-convex, we show that this implies $(G_\phi)^\#(\mathscr{L}^2 \llcorner \Omega') = 0$, which is absurd.

We also remark that, as an immediate consequence of our lemma, one can deduce C^1 regularity under assumption **As** (see Remark 3.4), although better regularity results under lower assumptions can be proven in this case (see [5]).

2 Regularity results for $c(x, y) = -x \cdot y$

We choose to present separately the proof for $c(x, y) = -x \cdot y$ as it is much shorter in this particular case, and might help the reader to follow the proof in the general case, since we will use the same strategy to prove the general case. We show here the following result:

Theorem 2.1 *Let $\Omega' \subset B(0, R)$ be a bounded and convex set in $\mathbb{R}^2$, and let ϕ be a convex solution of*

$$\begin{cases} \det(D^2\phi) = \mu & \text{in } \mathbb{R}^2, \\ \partial\phi(\mathbb{R}^2) = \Omega', \end{cases}$$

in the Alexandrov sense (or equivalently in the Brenier sense), with $\mu = \rho\mathscr{L}^2$ a positive measure. Assume μ supported in $B(0, K)$ for some $K > 0$ and $\rho \in L^\infty(\mathbb{R}^2)$. Then $\phi \in C^1(\mathbb{R}^2)$, and the modulus of continuity of $\nabla\phi$ depends only on $R, K, \|\rho\|_{L^\infty}$.

Proof of Theorem 2.1. We suppose that $\phi \notin C^1(\mathbb{R}^2)$. By an affine change of coordinates, and subtracting from ϕ an affine function, we can assume that

$$\{[-1, 1] \times 0\} \subset \partial\phi(0), \qquad \phi(0, 0) = 0,$$

i.e. that

$$\phi(x_1, x_2) \geq |x_1|, \qquad \phi(0, 0) = 0.$$

We remark that, since $\partial\phi(\mathbb{R}^2) = \Omega' \subset B(0, R)$, ϕ is R-Lipschitz. We will show that $\phi(0, x_2) \equiv 0$. For this, we assume by contradiction that there exist $h > 0, \delta > 0$ such that $\phi(0, \delta) = h$. We have first the following lemma:

Lemma 2.2 *Let $\phi : \mathbb{R}^2 \to \mathbb{R}$ be convex, R-Lipschitz, and such that $\phi(x_1, x_2) \geq |x_1|$, $\phi(0, 0) = 0$, $\phi(0, \delta) \geq h > 0$. Let*

$$S_{h,\delta} := \Big\{\{h\} \times [0, (1 + R)\delta]\Big\} \cup \Big\{[-h, h] \times \{(1 + R)\delta]\}\Big\} \cup \Big\{\{-h\} \times [0, (1 + R)\delta]\Big\}. \tag{5}$$

Then $\phi(x, y) \geq h$ on $S_{h,\delta}$.

Proof. Since $\phi(x_1, x_2) \geq |x_1|$, we clearly have

$$\phi \geq h \qquad \text{on } \{\pm h\} \times [0, (1 + R)\delta].$$

Moreover, as ϕ is R-Lipschitz, we get

$$\phi \leq h \qquad \text{on the segment } S = [-h/R, h/R] \times \{0\}. \tag{6}$$

We now recall the identity for convex functions:

$$f(y + t(y - x)) \geq f(y) + t(f(y) - f(x)) \qquad \forall t > 0, \forall x, y. \tag{7}$$

Since $\phi((0, \delta)) = h$, we have, using (6) and (7),

$$\phi(x) \geq h \qquad \text{for all } x = (0, \delta) + t((0, \delta) - y),\ t > 0,\ y \in S.$$

Applying the above inequality with $t = R$ we conclude that

$$\phi \geq h \qquad \text{on } [-h, h] \times \{(1 + R)\delta\}.$$

□

Lemma 2.3 *Let $R_{h,\delta}$ be the rectangle $[-h, h] \times [0, (1 + R)\delta]$. Consider $a, b \in \mathbb{R}$ such that*

$$a \in \Big(-\frac{1}{2}, \frac{1}{2}\Big), \qquad b \in \Big(0, \frac{h}{2\delta(1 + R)}\Big),$$

and let $L(x) := ax_1 + bx_2$. Then $\phi(x) - L(x)$ has a local minimum in $R_{h,\delta}$.

Proof. To get the result, we will first prove that $\phi \geq L$ on $\partial R_{h,\delta}$.

Since $\phi(x_1, x_2) \geq |x_1|$, we have $\phi \geq L$ on $[-h, h] \times \{0\}$. We check that $\phi \geq L$ on $S_{h,\delta}$ (with $S_{h,\delta}$ defined in (5)). By Lemma 2.2, $\phi \geq h$ on $S_{h,\delta}$, hence it is enough to show that $L \leq h$ on $S_{h,\delta}$. This follows from the fact that $|x_1| \leq h$ and $|x_2| \leq (1+R)\delta$ on $S_{h,\delta}$, and so

$$|ax_1 + bx_2| \leq \frac{1}{2}h + \frac{h}{2\delta(1+R)}(1+R)\delta = h.$$

We have therefore proved that $\phi \geq L$ on $\partial R_{h,\delta}$. There are now two possibilities: either $\phi < L$ in some interior point of $R_{h,\delta}$ or not.

In the first case, the thesis clearly follows.

In the second case we observe that $\phi(0) = L(0) = 0$, while

$$\phi(x) - L(x) \geq |x_1| - ax_1 - bx_2 \geq -bx_2 \geq 0 \qquad \text{for } x_2 \leq 0.$$

Therefore in this case, since by assumption $\phi \geq L$ on $R_{h,\delta}$, $\phi(x) - L(x)$ has a local minimum at 0 (which indeed is global by the convexity of ϕ). □

From Lemma 2.3 we have $E := [-1/2, 1/2] \times [0, h/(2\delta(1+R))] \subset \partial\phi(R_{h,\delta})$. On one hand we have

$$\mathscr{L}^2(E) = \frac{h}{2\delta(1+R)}, \tag{8}$$

and on the other hand

$$\mathscr{L}^2(R_{h,\delta}) = 2h\delta(1+R). \tag{9}$$

Therefore

$$\frac{h}{2\delta(1+R)} \leq \mathscr{L}^2\big(\partial\phi(R_{h,\delta})\big) = \mu(R_{h,\delta}) \leq \|\rho\|_{L^\infty} 2h\delta(1+R),$$

which implies

$$1 \leq \|\rho\|_{L^\infty} 4\delta^2(1+R)^2,$$

a contradiction if $\delta \leq \delta_0 := \sqrt{\|\rho\|_{L^\infty}}/(1+R)$.

We have thus proved that if $\phi(0,0) = 0$ and $\phi(x_1, x_2) \geq |x_1|$, then $\phi = 0$ on $\{0\} \times [0, \delta_0]$ for some $\delta_0 = \delta_0(R, \|\rho\|_{L^\infty}) > 0$. Iterating this argument this we see that $\phi \equiv 0$ on the set $\{0\} \times \mathbb{R}$.

We state now the following classical lemma:

Lemma 2.4 *Let f be convex on $\mathbb{R}^n$, and assume that $f = 0$ on the set $\{(0, \ldots, 0, t),\ t \in \mathbb{R}\}$. Then $\partial f(\mathbb{R}^n) \subset \{(y, 0) \mid y \in \mathbb{R}^{n-1}\}$.*

Proof. Fix $\bar{x} \in \mathbb{R}^n$, and take $p = (p_1, \ldots, p_n) \in \partial f(\bar{x})$. Then $f(x) \geq f(\bar{x}) + p \cdot (x - \bar{x})$ for all $x \in \mathbb{R}^n$. This implies that

$$0 \geq f(\bar{x}) + p_n t - p \cdot \bar{x} \qquad \forall t \in \mathbb{R},$$

and so $p_n = 0$. □

By the above lemma we have that $\partial\phi(\mathbb{R}^2) \subset \{(y, 0) \mid y \in \mathbb{R}\}$. This implies $\det D^2\phi \equiv 0$ in the Alexandrov sense, a contradiction that finishes the proof the C^1 regularity. The fact that the modulus of continuity of $\nabla\psi$ depends only on $R, K, \|\rho\|_{L^\infty}$ follows by a simple compactness argument.

3 Regularity results under Aw

Since many examples of costs that satisfies **Aw** are in general non-smooth on the whole $\mathbb{R}^2 \times \Omega'$, it is natural to study the regularity problem in bounded domains.

Thus we are going to consider a c-convex solution ϕ in $\Omega \subset \mathbb{R}^2$ of the equation $(G_\phi)^\#(\mathscr{L}^2 \llcorner \Omega') = \mu$, that is

$$\begin{cases} \mathscr{L}^2\big(G_\phi(B)\big) = \mu(B) \quad \forall B \subset \Omega \\ G_\phi(\Omega) = \Omega' \end{cases} \tag{10}$$

with $\mu \le C\mathscr{L}^2$, and $\Omega' \subset \mathbb{R}^2$ bounded and c-convex with respect to Ω. We will always assume in this section that the cost function satisfies **A0**, **A1**, **A2**, and **Aw**.

As we already said before, the strategy of the proof is to show that, if ϕ is not C^1, then it coincides with a "c-affine" function on a "c-line" (see Lemma 3.1 below for a precise statement). However, since now Ω can be bounded, we see that we cannot hope to obtain a contradiction using an analogous of Lemma 2.4. On the other hand, assuming that Ω is uniformly c-convex with respect to Ω', and Ω' is strictly c-convex with respect to Ω, we will use a recent result proved in [3] to get the desired contradiction.

We remark that ϕ is constructed in the following way: first one solves the optimal transport problem between μ and $\mathscr{L}^2 \llcorner \Omega'$, obtaining a c-convex function ψ on $\mathrm{supp}(\mu)$ (see Theorem 1.6). Then one can define

$$\phi(x) := \max_{y \in \Omega'} -c(x, y) - \psi^c(y) \qquad \forall x \in \Omega,$$

and, by the c-convexity of ψ and Ω', and the absolutely continuity of μ, one gets that ϕ solves (10) and coincides with ψ on $\mathrm{supp}(\mu)$. Moreover, again by the c-convexity of Ω', and thanks to assumption **Aw**, one has the equality $\partial^c \phi = \partial \phi$ (see [4, Theorem 3.1] or [10, Chapter 12]).

Suppose now that ϕ is not C^1. We can assume that $\phi(0) = 0$ and that

$$[-1, 1] \times \{0\} \subset \partial\phi(0) = \partial^c\phi(0), \tag{11}$$

or equivalently

$$\{y_\theta \mid \theta \in [-1, 1]\} \subset G_\phi(0),$$

where y_θ is the unique point such that

$$-\nabla_x c(0, y_\theta) = \theta e_1, \qquad \theta \in [-1, 1]$$

(here and in the sequel, e_1 and e_2 denote the vectors of the standard basis of $\mathbb{R}^2$). We consider the smooth curve

$$\Gamma = \{I \ni t \mapsto \gamma(t)\} \subset \overline{\Omega}$$

given by the maximal connected component of

$$\{x \in \overline{\Omega} | -c(x, y_{-1}) + c(0, y_{-1}) = -c(x, y_1) + c(0, y_1)\}$$

containing 0 (which is a C^1 graph with respect to $\{x_1 = 0\}$ in a neighborhood of 0). We can assume that $\gamma : I \to \overline{\Omega}$ is parameterized by arc length, that $\gamma(0) = 0$ and that $\dot\gamma(0) = e_2$.

Then the following result holds:

Lemma 3.1 *Let ϕ be a c-convex solution of (10) with $\mu \leq C\mathscr{L}^2$ and Ω' is bounded and c-convex with respect to Ω. Assume that $[-1,1]\times\{0\} \subset \partial^c\phi(0)$. Then $\phi = -c(x, y_0)+c(0, y_0)$ on Γ. Moreover Γ coincides with the maximal connected component of*

$$\cap_{\theta,\eta\in[-1,1]}\{x| - c(x, y_\theta) + c(0, y_\theta) = -c(x, y_\eta) + c(0, y_\eta)\}$$

containing 0, and $[-1, 1] \times \{0\} \in \partial^c\phi(x)$ for all $x \in \Gamma$. In particular

$$\phi(x) = -c(x, y_\theta) + c(0, y_\theta) \qquad \text{on } \Gamma$$

for all $\theta \in [-1, 1]$.

Since the proof of the above lemma is quite long and involved, we postpone it to the next paragraph. In [3], the following result is proved:

Lemma 3.2 *Let ϕ be a c-convex solution of (10) with $\mu \leq C\mathscr{L}^2$ and let Ω' be bounded and c-convex with respect to Ω. Assume moreover that Ω uniformly c-convex with respect to Ω', and that c satisfies* **A0**-**A1**-**A1'**-**A2**. *If $x \in \partial\Omega$ and $y \in G_\phi(x)$, then $y \in \partial\Omega'$.*

Combining these two lemmas, it is now difficult to prove the final result:

Theorem 3.3 *Let ϕ be a c-convex solution of (10), with $\mu \leq C\mathscr{L}^2$, Ω uniformly convex with respect to Ω', and Ω' is bounded and strictly c-convex with respect to Ω. If c satisfies* **A0**-**A1**-**A1'**-**A2**-**Aw**, *then ϕ is C^1. Furthermore, the modulus of continuity of $\nabla\phi$ depends only on the diameter of the support of μ, on the diameter of Ω', on the L^∞-bound on the density of μ, and on the regularity of the cost.*

Proof. With the same notations as above, we see that if ϕ is not C^1, then by Lemma 3.1

$$\{y_\theta \mid \theta \in [-1, 1]\} \subset G_\phi(x) \qquad \forall x \in \Gamma.$$

Claim: Γ necessarily intersects $\partial\Omega$.
Indeed, since by Lemma 3.1

$$-c(\gamma(t), y_\theta) + c(x_0, y_\theta) + c(\gamma(t), y_0) - c(x_0, y_0) \equiv 0 \qquad \forall t \in I,\ \theta \in (-1, 1),$$

differentiating with respect to θ at $\theta = 0$ gives

$$[-\nabla_y c(\gamma(t), y_0) + \nabla_y c(x_0, y_0)] \cdot \dot{y}_0 = 0 \qquad \forall t \in I,$$

where $\dot{y}_0 = [\nabla_{x,y}c(0, y_0)]^{-1}e_1 \neq 0$ (well defined by assumption **A2**). This implies that $-\nabla_y c(\Gamma, y_0)$ is a straight line, and the claim follows thanks to assumption **A1'**.

Let us consider $x \in \Gamma \cap \partial\Omega$. Thanks to Lemma 3.2 we know that $G_\phi(x) \subset \partial\Omega'$. This implies that $y_\theta \in \partial\Omega'$ for all $\theta \in [-1, 1]$, or equivalently

$$[-1, 1] \times \{0\} \subset -\nabla_x c(x, \partial\Omega'),$$

but this contradicts the c-strict convexity of Ω'.

Finally, thanks to a simple compactness argument, it is not difficult to see that the modulus of continuity of $\nabla\phi$ depends only on the diameter of the support of μ, on the diameter of Ω', on the L^∞-bound on the density of μ, and on the regularity of the cost. □

To conclude, we just recall some examples of smooth costs which satisfies **A0**, **A1**, **A1'**, **A2**, and either **Aw** or **As**:

(a) $c(x,y) = \frac{1}{2}|x-y|^2$ satisfies **Aw**;

(b) $c(x,y) = \frac{1}{2}|x-y|^2 + \frac{1}{2}|f(x)-g(y)|^2$ with $f,g : \mathbb{R}^n \to \mathbb{R}$ convex, smooth, with $\|\nabla f\|_\infty, \|\nabla g\|_\infty < 1$ satisfies **Aw** (**As** if f and g are strictly convex);

(c) $c(x,y) = \sqrt{1+|x-y|^2}$ satisfies **As**;

(d) $c(x,y) = \sqrt{1-|x-y|^2}$ satisfies **As**;

(e) $c(x,y) = (1+|x-y|^2)^{p/2}$ satisfies **As** for $1 \le p < 2$, $|x-y|^2 < \frac{1}{p-1}$;

(f) $c(x,y) = \pm\frac{1}{p}|x-y|^p, p \neq 0$ and satisfies **Aw** for $p = \pm 2$ and **As** for $-2 < p < 1$ ($-$ only);

(g) $c(x,y) = -\log|x-y|$ satisfies **As** on $\mathbb{R}^n \times \mathbb{R}^n \setminus \{(x,x) \mid x \in \mathbb{R}^n\}$;

(h) The reflector antenna problem corresponds to the case $c(x,y) = -\log|x-y|$ restricted to $\mathbb{S}^n$. As pointed out in [8], this cost satisfies **As** on $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \{x = y\}$;

(i) As shown in [6], the squared Riemannian distance on the sphere satisfies **As** on the set $\mathbb{S}^{n-1} \times \mathbb{S}^{n-1} \setminus \{x = -y\}$. Note that it is the restriction to $\mathbb{S}^{n-1}$ of the cost $c(x,y) = \theta^2(x,y)$, where θ is the angle formed by x and y. (For those two last cases, see [6].)

In particular, in the case (a), up to changing ϕ with $\frac{1}{2}|x|^2 + \phi$ one can equivalently consider the cost $c(x,y) = -\langle x, y\rangle$, and equation (10) reduces to the standard Monge-Ampère equation in the Alexandrov sense.

3.1 Proof of Lemma 3.1

We observe that (11) implies

$$\phi(x) \ge -c(x, y_\theta) + c(0, y_\theta) \qquad \text{for } \theta \in [-1,1]. \tag{12}$$

Let us consider the maximal interval $J \subset I$ such that $0 \in J$ and

$$\phi(x) = -c(x, y_0) + c(0, y_0) \qquad \text{on } \gamma(I).$$

Obviously J is closed in I. In order to conclude the proof of the lemma, we have to prove that J is open in I.

Fix $0 < \alpha \le \frac{1}{16}$. For $h > 0$ small, consider the family of functions

$$g_h^\alpha(x) := \max\{-c(x, y_\alpha) + c(0, y_\alpha), -c(x, y_{-\alpha}) + c(0, y_{-\alpha})\} + h.$$

By assumption **Aw** we remark that

$$g_h^\beta \le g_h^\alpha \qquad \text{if } \beta \le \alpha$$

(the inequality would be strict on Γ in a neighborhood of 0 under assumption **As**). Let $\gamma(\delta) \in \mathbb{R}^2$ be the first point (if it exists) of $\Gamma \cap \{x_2 \geq 0\}$ such that $\phi = g_h^\alpha$. Since $\partial\phi(x) \subset -\nabla_x c(x, \Omega')$ for any $x \in \mathbb{R}^2$, there exists $R > 0$ such that in a neighborhood of 0 we have $|p| \leq \frac{R}{2}$ for any $p \in \partial\phi(x)$, and $|\nabla_x c(x, y)| \leq \frac{R}{2}$ for all $y \in \Omega'$. So we get

$$\begin{aligned}\phi(\gamma(\delta) + te_1) &- c(\gamma(\delta) + te_1, y_{\pm\alpha}) + c(0, y_{\pm\alpha}) \\ &\geq \phi(\gamma(\delta)) - c(\gamma(\delta), y_{\pm\alpha}) + c(0, y_{\pm\alpha}) - Rt \\ &\geq h - Rt.\end{aligned}$$

This implies that, for all $\theta \in [-\alpha, \alpha]$,

$$\begin{aligned}\phi(x) &\geq \max\{-c(x, y_\alpha) + c(0, y_\alpha), -c(x, y_{-\alpha}) + c(0, y_{-\alpha})\} + \frac{h}{2} \\ &\geq -c(x, y_\theta) + c(0, y_\theta) + \frac{h}{2}\end{aligned}$$

for $x \in \{[-\frac{h}{2R}, \frac{h}{2R}] \times \{0\} + \gamma(\delta)\}$. Moreover, in a small neighborhood of 0, we have

$$\begin{cases} -\partial_{x_1} c(x, y_1) \geq \dfrac{7}{8} & (\text{since } -\partial_{x_1} c(0, y_1) = 1), \\ -\partial_{x_1} c(x, y_{-1}) \leq -\dfrac{7}{8}, & \\ |\partial_{x_1} c(x, y_\alpha)| \leq \dfrac{1}{8}, & (\text{since } \alpha \leq \dfrac{1}{16}). \end{cases} \tag{13}$$

Therefore, since

$$\phi(x) \geq \max\{-c(x, y_1) + c(0, y_1), -c(x, y_{-1)} + c(0, y_{-1})\} \geq -c(x, y_\theta) + c(0, y_\theta) \qquad \text{for } \theta \in [-1, 1],$$

by (13) we get

$$\phi(x) \geq -c(x, y_\alpha) + c(0, y_\alpha) + \frac{h}{4R} \qquad \text{on } \Big\{\pm\frac{h}{2R} e_1 + \gamma([0, \delta])\Big\}$$

for h small enough. So, if we define

$$S_{h,\delta} := \Big\{[-\frac{h}{2R}, \frac{h}{2R}] \times \{0\} + \gamma(\delta)\Big\} \cup \Big\{\pm\frac{h}{2R} e_1 + \gamma([0, \delta])\Big\},$$

we obtain

$$\phi(x) \geq -c(x, y_\theta) + c(0, y_\theta) + \frac{h}{4R} \qquad \text{on } S_{h,\delta} \tag{14}$$

for any $\theta \in [-\alpha, \alpha]$ (we can obviously assume $2R \geq 1$). Observe that the smallness of the neighborhood such that the above estimates hold depends uniquely on the regularity of the cost function. We can so assume that all the estimates hold uniformly for $\alpha \in (0, \frac{1}{16}]$ in a ball B_ε of radius $\varepsilon > 0$ centered at 0. Now we see that, by (14), for any $\theta \in [-\alpha, \alpha]$ we get

$$-c(x, y) + c(0, y) \leq -c(x, y_\theta) + c(0, y_\theta) + C|y - y_\theta||x| \leq \phi(x) \qquad \text{on } S_{h,\delta}$$

(where $C = \|D^2_{xy}c\|_{L^\infty(B_\varepsilon \times \Omega')}$) provided that

$$C|y - y_\theta||x| \leq \frac{h}{4R},$$

which is indeed true, for δ and h small, if

$$|y - y_\theta| \leq c\frac{h}{R(h+\delta)}$$

(since $|x| \leq C(h+\delta)$ on $S_{h,\delta}$). By (13) we also have

$$-c(x, y_\theta) + c(0, y_\theta) + \frac{1}{2}|x| \leq \phi(x) \qquad \text{on } \Big[-\frac{h}{2R}, \frac{h}{2R}\Big] \times \{0\},$$

and so

$$-c(x,y) + c(0,y) \leq -c(x, y_\theta) + c(0, y_\theta) + C|y - y_\theta||x| \leq \phi(x) \qquad \text{on } \Big[-\frac{h}{2R}, \frac{h}{2R}\Big] \times \{0\}$$

provided that

$$C|y - y_\theta||x| \leq \frac{1}{2}|x|,$$

i.e.

$$|y - y_\theta| \leq c'.$$

Therefore, calling $R_{h,\delta}$ the bounded set whose boundary is given by $S_{h,\delta} \cup [-\frac{h}{2R}, \frac{h}{2R}] \times \{0\}$ and using then $\partial^c\phi = \partial\phi$, by the argument used in Lemma 2.3 we get

$$G_\phi(R_{h,\delta}) \supset \bigg\{ y \mid \exists \theta \in [-\alpha, \alpha] \text{ such that } |y - y_\theta| \leq \min\Big\{c', c\frac{h}{R(h+\delta)}\Big\}\bigg\}.$$

Thus

$$\mathscr{L}^2\big(G_\phi(R_{h,\delta})\big) \geq \tilde{c}\min\Big\{c', c\frac{h}{R(h+\delta)}\Big\}\alpha,$$

where $\tilde{c}$ depends on the length of the curve $[-1,1] \ni \theta \mapsto y_\theta$ (which depends only on the cost function and is strictly positive thanks to assumption **A2**). Assuming without loss of generality $h + \delta \leq 1$, we have

$$\mathscr{L}^2\big(G_\phi(R_{h,\delta})\big) \geq \tilde{c}\min\Big\{c', c\frac{h}{R}\Big\}\alpha,$$

and combining this fact with the estimate

$$\mu(R_{h,\delta}) \leq C\mathscr{L}^2(R_{h,\delta}) \leq Ch\delta,$$

for h small enough we obtain

$$C\delta \geq \tilde{c}\frac{c\alpha}{R}.$$

This is absurd for $\delta \leq \alpha\delta_0$, with $\delta_0 = \frac{\tilde{c}c}{2CR}$. This argument can be used also in the half space $\{x_2 \leq 0\}$. Thus we have just proved that, if $G_\phi(\gamma(0)) \supset \{y_\theta \mid \theta \in [-1,1]\}$, then

$$\phi < g_h^\alpha \qquad \text{on } \gamma([-\alpha\delta_0, \alpha\delta_0]) \text{ for any } h \text{ sufficiently small.}$$

Letting $h \to 0$ and recalling (12), we get

$$\phi(x) = \max\{-c(x,y_\alpha)+c(0,y_\alpha), -c(x,y_{-\alpha)}+c(0,y_{-\alpha})\} \qquad \text{on } \gamma([-\alpha\delta_0,\alpha\delta_0]). \tag{15}$$

We now recall that

$$\begin{aligned}\phi(x) &\geq \max\{-c(x,y_\theta)+c(0,y_\theta), -c(x,y_{-\theta})+c(0,y_{-\theta})\}\\ &\geq \max\{-c(x,y_\alpha)+c(0,y_\alpha), -c(x,y_{-\alpha})+c(0,y_{-\alpha})\}\end{aligned}$$

for $\theta \in [-1,-\alpha]\cup[\alpha,1]$ (we remark that, under **As**, the second inequality above becomes strict on Γ in a neighborhood of 0, and so by (15) we would directly conclude an absurd, obtaining that ϕ is C^1, see Remark 3.4). Therefore, by the above inequality and (15), we get

$$-c(x,y_1)+c(0,y_1) = -c(x,y_{-1})+c(0,y_{-1}) = -c(x,y_\theta)+c(0,y_\theta) \qquad \text{on } \gamma([-\alpha\delta_0,\alpha\delta_0])$$

for $\theta \in [-1,-\alpha]\cup[\alpha,1]$.

This implies that

$$G_\phi(\gamma(t)) \supset \{y_\theta \mid \theta \in [-1,-\alpha]\cup[\alpha,1]\} \qquad \forall t \in [-\alpha\delta_0,\alpha\delta_0].$$

Moreover, since $\partial^c\phi(\gamma(t)) = [-\nabla_x c(\gamma(t),\cdot)]^{-1}\big(G_\phi(\gamma(t))\big)$ is convex (since it coincides with $\partial\phi(\gamma(t))$), $G_\phi(\gamma(t))$ must contain the so-called c-segment with respect to $\gamma(t)$ from $y_{-\alpha}$ to y_α which is given by the formula

$$y_\theta(t) := [-\nabla_x c(\gamma(t),\cdot)]^{-1}\Big(-\frac{1+\theta/\alpha}{2}\nabla_x c(\gamma(t),y_\alpha) - \frac{1-\theta/\alpha}{2}\nabla_x c(\gamma(t),y_{-\alpha})\Big).$$

Thus, defining also $y_\theta(t) = y_\theta$ for $\theta \in [-1,-\alpha]\cup[\alpha,1]$, we get

$$G_\phi(\gamma(t)) \supset \{y_\theta(t) \mid \theta \in [-1,1]\} \qquad \forall t \in [-\alpha\delta_0,\alpha\delta_0].$$

We can now argue in the same way as we did before starting from $\gamma(\alpha\delta_0)$ and considering as supporting functions

$$-c(x,y_\theta(\alpha\delta_0)) + c(\gamma(\alpha\delta_0),y_\theta(\alpha\delta_0)).$$

Indeed, since $y_\theta(\alpha\delta_0) = y_\theta$ for $\theta \in [-1,-\alpha]\cup[\alpha,1]$, we can use again (13) and all the subsequent estimates as soon as $\gamma(\alpha\delta_0) \subset B_\varepsilon$. Thus we can use exactly the same argument we used before to deduce that $\phi < g_h^\alpha$ on $\gamma([\alpha\delta_0,2\alpha\delta_0])$ for h small. Doing the same also in the half space $\{x_2 \leq 0\}$ starting from $\gamma(-\alpha\delta_0)$, by the arbitrariness of h we conclude

$$\phi(x) = \max\{-c(x,y_\alpha)+c(0,y_\alpha), -c(x,y_{-\alpha)}+c(0,y_{-\alpha})\} \qquad \text{on } \gamma([-2\alpha\delta_0,2\alpha\delta_0])$$

(here we use that, by definition, $y_{\pm\alpha}(\alpha\delta) = y_{\pm\alpha}$). Iterating the argument above a finite number of times (the number of iterations depending on α), we finally obtain

$$\phi(x) = \max\{-c(x,y_\alpha)+c(0,y_\alpha), -c(x,y_{-\alpha)}+c(0,y_{-\alpha})\} \qquad \text{on } \gamma([-\varepsilon,\varepsilon]) \subset B_\varepsilon,$$

where the inclusion $\gamma([-\varepsilon,\varepsilon]) \subset B_\varepsilon$ holds since γ was parameterized by arc length. Letting $\alpha \to 0$ we get

$$\phi(x) = -c(x,y_0)+c(0,y_0) \qquad \text{on } \gamma([-\varepsilon,\varepsilon]).$$

By this fact, the definition of Γ, and the inequality

$$\phi(x) \geq \max\{-c(x, y_{-\theta}) + c(0, y_{-\theta}), -c(x, y_\theta) + c(0, y_\theta)\} \geq -c(x, y_0) + c(0, y_0)$$

for $\theta \in [0, 1]$, we obtain

$$G_\phi(\gamma(t)) \supset \{y_\theta \mid \theta \in [-1, 1]\} \qquad \forall t \in [-\varepsilon, \varepsilon].$$

We can therefore conclude that J is open in I. Indeed it suffices to consider the maximal $\bar{t}$ in the interior of I such that $\bar{t} \in J$, and since $G_\phi(\gamma(\bar{t})) \supset \{y_\theta \mid \theta \in [-1, 1]\}$, the argument above shows that $\bar{t} \pm \varepsilon \in J$ for ε sufficiently small. So $J = I$. Moreover we observe that Γ coincides with the maximal connected component of

$$\cap_{\theta,\eta \in [-1,1]}\{x \mid -c(x, y_\theta) + c(0, y_\theta) = -c(x, y_\eta) + c(0, y_\eta)\}$$

containing 0, and $[-1, 1] \times \{0\} \in \partial^c\phi(x)$ for all $x \in \Gamma$.

Remark 3.4 As we remarked already during the proof of the lemma, under assumption **As**, equation (15) gives us the wanted contradiction, which implies that ϕ is C^1 (however, a better regularity result is true under **As**, see [5]). Thus we recover the following result:

Let ϕ be a c-convex solution of (10), with $\mu \leq C\mathscr{L}^2$ and $\Omega' \subset \mathbb{R}^2$ bounded and c-convex with respect to $\mathbb{R}^2$. If c satisfies **A0**-**A1**-**A2**-**As**, *then ϕ is C^1.*

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