ELECTRIC TURBULENCE IN A PLASMA SUBJECT TO A STRONG MAGNETIC FIELD

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Abstract

We consider in this paper a plasma subject to a strong deterministic magnetic field and we investigate the effect on this plasma of a stochastic electric field. We show that the limit behavior, which corresponds to the transfer of energy from the electric wave to the particles (Landau phenomena), is described by a Spherical Harmonics Expansion (SHE) model.

1   Introduction

This paper is concerned with the effect of a stochastic electric field on a plasma subject to a strong magnetic field. This is motivated by the study of the electric turbulence in a fusion machine as a Tokamak. Tokamaks are used to confine high energy plasmas in order to obtain the conditions needed for nuclear fusion reactions to take place. The plasma evolves in a toroidal reactor and is confined in the heart of the torus by the the mean of a strong magnetic field. A classical approximation is to suppose the ions to be at rest. Then only the electrons are moving. Another classical approximation argument in this type of study is the following: we are here interested only in interactions of particles over short distances of the order of the Larmor radius, moreover we suppose that at this scale the curvature of the magnetic field-lines can be neglected and that the plasma can be considered to be homogeneous along these field-lines. Thus we can restrict ourselves to a bidimensional problem. In this approach, the Vlasov equation describing the evolution of the repartition function f of the electrons is:

$$m \left(\frac{\partial f}{\partial t} + v \cdot \nabla_x f \right) + q \left(Bv^\perp + \nabla V^{\text{turb}}(t,x) \right) \cdot \nabla_v f = 0, \tag{1}$$

m stands for the electron's mass, q its electric charge, f the distribution function on $(t,x,v) \in \mathbb{R}^+ \times \mathbb{R}^2 \times \mathbb{R}^2$, with t the time variable, x the space variable and v the velocity variable. B is the (constant) norm of the transverse magnetic field, ∇V^{turb} is the turbulent electric field, $v^\perp$ is the velocity vector after a rotation of $\pi/2$. We denote

$$1/\epsilon = \frac{qB}{m}$$

the cyclotronic frequency, and we want to study the effect of ∇V^{turb} in the limit ϵ going to zero. In the deterministic case, the limits of related problems have been studied by several authors. In the case of the Vlasov-Poisson system (when the electric field is coupled with the density $\int f \, dv$), the limit has been studied, even in the 3D framework, by Frénod and Sonnendrucker [5]. In the 2D framework, using a slow time scale adapted to the problem,

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the convergence of the averaged motion, $\int_{\mathbb{R}^2} f(t, x, v) \, dv$ to the 2D Euler system of equations has been performed simultaneously by Brenier [2], Golse and Saint-Raymond [8] and Frénod and Sonnendrucker [6]. A general result in 3D taking into account the two effects has been performed by Saint-Raymond in [12].

In our case we neglect the Poisson non linear effect, concentrating on the stochastic behavior of the equation. Hamiltonian chaos method suggest that the modes of the turbulent electric field interacting with the electrons are those having a frequency of $\omega_n = 2\pi n \epsilon$ with n an integer. This is roughly speaking the Landau resonance. Then the quasi-linear theory, (see Garbet [7]) predicts a diffusive behavior with respect to the velocity variable. The diffusion coefficient obtained by this method being constant, it can not take into account the abnormal diffusion phenomena. In this paper we are interested in turbulent electric fields whose spectrum is spread around the Landau frequency and whose spatial fluctuations are of the scale of the Larmor radius (of order ϵ). We will show that the limit system is then governed by the following equation:

$$\partial_t \rho - \partial_e (a(e) \partial_e \rho) = 0 \tag{2}$$

where $e = |v|^2/2$, ρ is the average of f over a sphere $|v|^2 = 2e$ and the diffusion parameter $a(e)$ is an explicit function of the correlation of V^{turb}, the turbulent electric potential, and of the energy, thus allowing abnormal diffusion. This diffusion parameter is undimensionally defined by (5). This equation is similar to the so-called Spherical Harmonics Expansion (SHE) model in high field limit modeling microelectronics semiconductor devices (see P.Degond [3] or Ben Abdallah, Degond, Markowich and Schmeiser [1]). It describes the Landau phenomena of transfer of energy from the electric wave to the particles. This work uses the techniques introduced by Poupaud and Vasseur [11] to derive diffusive equation from transport in random media. This method works directly on the equation and, for this reason, is different from the method used in previous works (see Kesten and Papanicolaou [10], [9] and Fannjiang, Ryzhik and Papanicolaou [4]). The paper is organized as follows: the precise result is stated in Section 2. In Section 3 we show how we can compute explicitly the diffusion coefficients. Finally we give the proof of the theorem in Section 4.

2 Results

In the remainder of the paper we fix n and we denote

$$\nabla V^\epsilon(t, x) = \sqrt{\epsilon} \nabla V^{\mathrm{turb}}(2\pi n \epsilon t, \epsilon x) \tag{3}$$

for the stochastic potential. Equation (1) takes then the following undimensional form:

$$\frac{\partial f_\epsilon}{\partial t} + v \cdot \nabla_x f_\epsilon + \left(\frac{v^\perp}{\epsilon} + \frac{1}{\sqrt{\epsilon}} \nabla V^\epsilon(\frac{t}{2\pi n \epsilon}, \frac{x}{\epsilon}) \right) \cdot \nabla_v f_\epsilon = 0. \tag{4}$$

we denote $\mathbf{E}$ the expectation value of any variable and make the following assumptions on the electrostatic potential:

$(H1)$ $\quad V^\epsilon \in L^\infty(\mathbb{R}^+; W^{3,\infty}(\mathbb{R}^2))$ and $N(\epsilon) := \mathbf{E}\left(\left(\|V^\epsilon\|_{L^\infty(W^{3,\infty})}\right)^3\right) < \infty$,

$(H2)$ $\quad \mathbf{E}V^\epsilon(t,x) = 0$, for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}^2$,

$(H3)$ $\quad V^\epsilon(t,x)$, $V^\epsilon(s,y)$ are uncorrelated as soon as $|t-s| \geq 1$,

$(H4)$ $\quad \mathbf{E}(V^\epsilon(t,x)V^\epsilon(s,y)) = A(t-s, x-y) + g^\epsilon(t,s,x,y)$,

with:

$$\frac{\partial^{|\alpha|}}{\partial x^\alpha} A \in L^\infty(\mathbb{R} \times \mathbb{R}^2), \text{ for } \alpha \in \mathbb{N}^2, \ |\alpha| \leq 3,$$

$$\| \, |\nabla^2_{x,y} g^\epsilon| + |\nabla^3_{x,y,y} g^\epsilon| \, \|_{L^\infty((\mathbb{R}^+)^2 \times \mathbb{R}^4)} \xrightarrow{\epsilon \to 0} 0,$$

where $\nabla^2_{x,y} g^\epsilon$ is the matrix $\left(\partial_{x_i}\partial_{y_j} g^\epsilon\right)_{i,j}$ and $\nabla^3_{x,y,y} g^\epsilon$ is $\left(\partial_{x_i}\partial_{y_j}\partial_{y_k} g^\epsilon\right)_{i,j,k}$.

Those assumptions are the same than in [11]. Hypothesis (H1) is an assumption on the regularity of V^ϵ for ϵ fixed. Indeed the norm $N(\epsilon)$ can go to infinity when ϵ goes to 0. Hypothesis (H2) fixes the averaged potential at 0 which is not restrictive. In view of (3), Hypothesis (H3) determines $2\pi n \epsilon$ as the decorrelated lapse of time for the turbulent electric field. Namely, it is the bigger lapse of time $t-s$ such that the electric fields at time t and at time s can be dependent on each other. Finally Hypothesis (H4), which is very classical, can be seen as an homogeneity property which takes place at the local scale ϵ, since a quadratic quantity which depends on four variables (t,x,s,y), at the limit, depends only on two variables $(t-s, x-y)$.

We denote $R_s(v)$ the rotation of angle s with center 0 of v. We consider the angular average of A:

$$\tilde{A}(t,x) = \frac{1}{2\pi} \int_0^{2\pi} A(t, R_\theta x) \, d\theta.$$

We then have the following result:

Theorem 2.1 *Let V^ϵ be a stochastic potential satisfying assumptions (H) and independent of the initial data $f_\epsilon^0 \in L^2(\mathbb{R}^4)$. Let $a(e)$ be the function defined by:*

$$a(e) = \frac{1}{2\pi n^2} \int_0^{+\infty} (-\partial^2_{tt}\tilde{A})(-\frac{s}{2\pi n}, 2\sqrt{2e}|\sin \frac{s}{2}|) \, ds. \tag{5}$$

This function is non negative. Assume that there is a constant C_0 such that $\|f_\epsilon^0\|_{L^2(\mathbb{R}^4)} \leq C_0$ and:

$$\epsilon(1 + N(\epsilon)^2) \to 0. \tag{6}$$

Let ρ_ϵ be the gyro-average of f_ϵ defined by :

$$\rho_\epsilon(t,x,e) = \frac{1}{2\pi} \int_0^{2\pi} f_\epsilon(t, x, R_\theta v) \, d\theta, \tag{7}$$

for every v such that $|v|^2/2 = e$. Then up to extraction of a subsequence, $\mathbf{E}f_\epsilon^0$ converges weakly in $L^2(\mathbb{R}^4)$ to a function $f^0 \in L^2(\mathbb{R}^4)$, $\mathbf{E}f_\epsilon$ converges weakly in L^2 to a function $f \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^4))$, $\mathbf{E}\rho_\epsilon$ converges in $C^0([0,T], L^2(\mathbb{R}^2) - w)$ for all $T > 0$ toward a function $\rho \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2 \times \mathbb{R}^+)) \cap C^0(\mathbb{R}^+; L^2(\mathbb{R}^2 \times \mathbb{R}^+) - W)$ with $\rho(t = 0, x, e) = \int_{|v|^2 = 2e} f^0 \, dv$. This function ρ is solution to:

$$\partial_t \rho - \partial_e(a(e)\partial_e \rho) = 0 \qquad t > 0, x \in \mathbb{R}^2, e \in \mathbb{R}^+, \tag{8}$$

in the distribution sense. Finally $\mathbf{E}f_\epsilon$ converges weakly in $L^2(\mathbb{R}^+ \times \mathbb{R}^4)$ toward $\rho(t, x, |v|^2/2)$.

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Remark: depending on the regularity of the function $a(e)$ the solution of the Cauchy problem for equation (8) may be unique. In this case, the whole sequence ρ_ϵ converges to ρ the unique solution of (8).

3   Explicit computation of the diffusion parameter

In order to explicit the behavior of $a(e)$ we must need the correlation function A. We assume that it follows a "Richardson-like" law

$$A(t,x) = f(t)|x|^\alpha.$$

Then we have

$$
\begin{aligned}
a(e) &= -\frac{1}{2\pi n^2} \int_0^{+\infty} \partial_{tt}^2 \tilde{A}\left(\frac{-s}{2\pi n}, 2\sqrt{2e}|\sin\frac{s}{2}|\right) ds \\
&= -\frac{2^{\frac{3}{2}\alpha} e^{\alpha/2}}{2\pi n^2} \int_0^{+\infty} f''\left(\frac{-s}{2\pi n}\right)|\sin\frac{s}{2}|^\alpha ds \\
&= K e^{\alpha/2}.
\end{aligned}
$$

A necessary condition for a function of the form

$$\rho(t,e) = \gamma(t)\rho_0(e/t^\beta)$$

to be an auto-similar solution of (2) is that

$$\beta = \frac{2}{4-\alpha}.$$

we then have an abnormal diffusion in

$$e = t^{\frac{2}{4-\alpha}}.$$

For example if $\alpha = 4/3$ we find $a(e) = K e^{2/3}$ and an abnormal diffusion in $e = t^{3/4}$.

4   Proof of the result

We denote $\mathcal{S}(\mathbb{R}^4)$ the Schwartz space and $\mathcal{S}'(\mathbb{R}^4)$ its dual. We denote $\langle\cdot;\cdot\rangle$ the duality brackets between those two spaces. We recall that $L^2(\mathbb{R}^4) \subset \mathcal{S}'(\mathbb{R}^4)$ and by extension we will denote $\langle\cdot;\cdot\rangle$ as well for the scalar product on $L^2(\mathbb{R}^4)$. For every linear operator P on $\mathcal{S}(\mathbb{R}^4)$ we will denote in the same way P its extension on $\mathcal{S}'(\mathbb{R}^4)$ defined for every $\psi \in \mathcal{S}(\mathbb{R}^4)$ by:

$$\langle P\psi;\eta\rangle = \langle\psi;P^*\eta\rangle, \quad \eta \in \mathcal{S}(\mathbb{R}^4).$$

Finally we will say that $\psi_n \in \mathcal{S}'(\mathbb{R}^4)$ converges to $\psi \in \mathcal{S}'(\mathbb{R}^4)$ in $\mathcal{S}'(\mathbb{R}^4)$ if for every $\eta \in \mathcal{S}(\mathbb{R}^4)$, $\langle\psi_n;\eta\rangle$ converges to $\langle\psi;\eta\rangle$. (This is the weak convergence for $\mathcal{S}'(\mathbb{R}^4)$.)

Let us rewrite equation (4) in the following way:

$$
\begin{cases}
\partial_t f_\epsilon + C f_\epsilon + \dfrac{B f_\epsilon}{\epsilon} = -\theta_t^\epsilon f_\epsilon \\[2mm]
f_\epsilon|_{t=0} = f_\epsilon^0
\end{cases}
\tag{9}
$$

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where C, B, θ_t^ϵ are linear operators on $\mathcal{S}(\mathbb{R}^4)$ defined by:

$$C = v \cdot \nabla_x$$
$$B = v^\perp \cdot \nabla_v$$
$$\theta_t^\epsilon = \frac{1}{\sqrt{\epsilon}} \nabla V^\epsilon \left(\frac{t}{2\pi n \epsilon}, \frac{x}{\epsilon} \right) \cdot \nabla_v.$$

Notice that C and B are deterministic and non dependent on ϵ nor on t unlike θ_t^ϵ.

We introduce the projection operator J defined on $\mathcal{S}(\mathbb{R}^4)$ which averages the values of the function on the spheres $|v|^2/2 = e$. Namely, for $\eta \in \mathcal{S}(\mathbb{R}^4)$:

$$J\eta(x, v) = \frac{1}{2\pi} \int_0^{2\pi} \eta(x, R_\theta v) \, d\theta.$$

We call J the "gyroaverage operator". This operator is self adjoint for the L^2 scalar product. Applying the projection operator J on (9) and taking its expectation value leads to:

$$\partial_t \mathbf{E}(Jf_\epsilon) + \mathbf{E}(JCf_\epsilon) + \frac{\mathbf{E}(JBf_\epsilon)}{\epsilon} = -\mathbf{E}(J\theta_t^\epsilon f_\epsilon).$$

We first study some properties of the operators in order to pass to the limit in the left hand side of this equation. Then we investigate the limit of $\mathbf{E}(J\theta_t^\epsilon f_\epsilon)$ following the procedure of [11]. Finally we derive the SHE equation giving the explicit form of the diffusion coefficient $a(e)$.

4.1 Properties of the operators

We have the following properties on the operators C, B, θ_t^ϵ and J:

Lemma 4.1 *Operators C, B and θ_t^ϵ are skew adjoint for the L^2 scalar product. Operators C, B, and J commute with the expectation operator $\mathbf{E}$. The operator J is the restriction on $\mathcal{S}(\mathbb{R}^4)$ of the orthogonal projector of $L^2(\mathbb{R}^4)$ into $\mathrm{Ker}B$. In particular:*

$$\|J\|_{\mathcal{L}(L^2(\mathbb{R}^4))} = 1,$$
$$J^2 = J,$$
$$\mathrm{Ker}B = \mathrm{Im}J.$$

In addition:
$$JB = JCJ = 0.$$

Proof.
–Operators C, B and θ_t^ϵ can be rewritten as $b.D$, where D is a gradient operator and b a regular function verifying $D \cdot b = 0$. For every functions $\eta_1, \eta_2 \in \mathcal{S}(\mathbb{R}^4)$ we have:

$$\langle b \cdot D\eta_1; \eta_2 \rangle = -\langle \eta_1; D(b\eta_2) \rangle$$
$$= -\langle \eta_1; b \cdot D\eta_2 \rangle.$$

Hence they are skew adjoint operators for the L^2 scalar product.

–The operator J is clearly the L^2 projection on L^2 functions which depends only on $|v|^2/2$ with respect to v. In polar coordinates $v = (r \sin \theta, r \cos \theta)$, we have $B = \partial/\partial\theta$. So J is the projection on $\mathrm{Ker}B$.

–Operator J is the projector on $\text{Ker}B$, hence $BJ = 0$. Since B is skew adjoint and J is self adjoint, we have $(BJ)^* = -JB = 0$.

–Let us fix $\eta \in \mathcal{S}(\mathbb{R}^4)$. We denote: $J\eta(x,v) = \rho_\eta(x, \frac{|v|^2}{2})$. Hence

$$JCJ\eta(x,v) = \nabla_x \cdot \left(\rho_\eta(x, \frac{|v|^2}{2}) \frac{1}{2\pi} \int_0^{2\pi} R_\theta v \, d\theta \right) = 0.$$

Finally $JCJ = 0$.

–Since C, B and J are linear and deterministic, they commute with $\mathbf{E}$. $\qquad\square$.

From those properties we deduce the following proposition:

Proposition 4.2 *For every ϵ and every $t \in \mathbb{R}^+$ we have:*

$$\|f_\epsilon(t)\|_{L^2(\mathbb{R}^4)} = \|f_\epsilon^0\|_{L^2(\mathbb{R}^4)}.$$

There exists a function $f^0 \in L^2(\mathbb{R}^4)$ and a function $f \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^4))$ such that, up to a subsequence, $\mathbf{E}f_\epsilon^0$ converges weakly in $L^2(\mathbb{R}^4)$ to f^0, $\mathbf{E}f_\epsilon$ converges weakly in $L^2([0,T] \times \mathbb{R}^4)$ to f for every $T > 0$. For every $t > 0$, f verifies $Jf(t) = f(t)$. The function $J\mathbf{E}f_\epsilon$ is solution to:

$$\begin{cases} \partial_t J\mathbf{E}f_\epsilon + \mathbf{E}(J\theta_t^\epsilon f_\epsilon) = w_\epsilon \\ J\mathbf{E}f_\epsilon|_{t=0} = J\mathbf{E}f_\epsilon^0, \end{cases} \tag{10}$$

where w_ϵ converges to 0 in $\mathcal{S}'$.

Proof. Since C, B and θ_t^ϵ are skew adjoint operators, we have:

$$\partial_t \langle f_\epsilon(t); f_\epsilon(t) \rangle = 0,$$

which gives the first equality. By weak compactness there exists two functions $f^0 \in L^2(\mathbb{R}^4)$ and $f \in L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^4))$ such that, up to a subsequence, $\mathbf{E}f_\epsilon^0$ converges weakly in $L^2(\mathbb{R}^4)$ to f^0, $\mathbf{E}f_\epsilon$ converges weakly in $L^2([0,T] \times \mathbb{R}^4)$ to f for every $T > 0$. Notice that $\epsilon\theta_t^\epsilon$ converges to 0 in $\mathcal{S}'(\mathbb{R}^4)$. Multiplying Equation (9) by ϵ, taking its expectation value, and letting ϵ go to 0, we find:

$$Bf(t) = 0 \quad \text{on} \ \ \mathbb{R}^+,$$

since B and $\mathbf{E}$ commute. Thanks to Lemma 4.1 $f(t) \in \text{Im}J$, and since $J^2 = J$, we have $Jf(t) = f(t)$ for almost every $t > 0$. Since $JB = 0$, applying the operator J on equation (9) and taking its expectation value gives:

$$\partial_t J\mathbf{E}f_\epsilon + \mathbf{E}(J\theta_t^\epsilon f_\epsilon) = w_\epsilon,$$

with $w_\epsilon = -\mathbf{E}JCf_\epsilon$. This converges in $\mathcal{S}'$ to $-JCf = -JCJf = 0$, thanks to Lemma 4.1.
$\square$

Hence we are now concerned by the limit in $\mathcal{S}'$ of $\mathbf{E}(J\theta_t^\epsilon f_\epsilon)$.

4.2 Computation of $\mathbf{E}(J\theta_t^\epsilon f_\epsilon)$

Let us denote S_t^ϵ $t \in \mathbb{R}$ the group on $\mathcal{S}(\mathbb{R}^4)$ generated by the operator $C + B/\epsilon$. Namely, for every $h \in \mathcal{S}$, $S_t^\epsilon h$ is the unique solution on $\mathbb{R}$ to:

$$\begin{cases} \partial_t g + Cg + \dfrac{Bg}{\epsilon} = 0 \\ g|_{t=0} = h. \end{cases} \tag{11}$$

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The operator S_t^ϵ can be explicitly given by:

$$S_t^\epsilon h(x, v) = h(T_\epsilon(t)(x, v)),$$

where

$$T_\epsilon(t)(x, v) = (x + \epsilon v^\perp - \epsilon R_{-t/\epsilon} v^\perp, R_{-t/\epsilon} v).$$

The function $T_\epsilon(t)(x, v)$ gives the position at $-t$ of the particle being in x with speed v at time 0 and moving at constant speed $|v|$ on a circle of radius $\epsilon|v|$. In particular S_t^ϵ is $2\pi\epsilon$ periodic. Notice that the adjoint of S_t^ϵ is S_{-t}^ϵ.

Following the procedure of [11], we use a 2 times iterated Duhamel formula. The first iteration gives:

$$f_\epsilon(t) = S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon) - \int_0^{2\pi n\epsilon} (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon) S_\sigma^\epsilon f_\epsilon(t - \sigma) \, d\sigma$$

and then we write the Duhamel formula for the $f_\epsilon(t - \sigma)$ in the integral, and this yields:

$$
\begin{aligned}
f_\epsilon(t) &= S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon) - \int_0^{2\pi n\epsilon} (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon) S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon) \, d\sigma \\
&\quad + \int_0^{2\pi n\epsilon} \int_0^{4\pi n\epsilon - \sigma} (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon)(S_{s+\sigma}^\epsilon \theta_{t-\sigma-s}^\epsilon S_{-s-\sigma}^\epsilon) S_{s+\sigma}^\epsilon f_\epsilon(t - \sigma - s) \, ds \, d\sigma.
\end{aligned}
$$

We obtain:

$$
\begin{aligned}
\mathbf{E}(J\theta_t^\epsilon f_\epsilon(t)) &= J\mathbf{E}\left(\theta_t^\epsilon S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon)\right) \\
&\quad - \int_0^{2\pi n\epsilon} \mathbf{E}\left(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon) S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon)\right) \, d\sigma + r_t^\epsilon \qquad (12)
\end{aligned}
$$

with

$$r_t^\epsilon = \int_0^{2\pi n\epsilon} \int_0^{4\pi n\epsilon - \sigma} J\mathbf{E}\left(\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon)(S_{s+\sigma}^\epsilon \theta_{t-\sigma-s}^\epsilon S_{-s-\sigma}^\epsilon) S_{s+\sigma}^\epsilon f_\epsilon(t - \sigma - s)\right) \, ds \, d\sigma.$$

The function f_ϵ^0 is independent of the operators θ_t^ϵ, $t \in \mathbb{R}$. In particular, in view of the assumption (H3), θ_t^ϵ and $f_\epsilon(t - s)$ are independent as soon as $t \geq s + 2\pi n\epsilon$ and $s \geq 0$.

Combining this fact with (H2), Equation (12) becomes for $t \geq 4\pi n\epsilon$

$$
\begin{aligned}
\mathbf{E}(J\theta_t^\epsilon f_\epsilon) &= J\mathbf{E}(\theta_t^\epsilon)\mathbf{E}(S_{2\pi n\epsilon}^\epsilon f_\epsilon(t - 2\pi n\epsilon)) \\
&\quad - \int_0^{2\pi n\epsilon} \mathbf{E}(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))\mathbf{E}(S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon)) \, d\sigma + r_t^\epsilon, \\
\mathbf{E}(J\theta_t^\epsilon f_\epsilon) &= - \int_0^{2\pi n\epsilon} \mathbf{E}(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))\mathbf{E}f_\epsilon(t) \, d\sigma + r_t^\epsilon + e_t^\epsilon, \qquad (13)
\end{aligned}
$$

$$\text{with} \quad e_t^\epsilon = - \int_0^{2\pi n\epsilon} \mathbf{E}(J\theta_t^\epsilon (S_\sigma^\epsilon \theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))(\mathbf{E}S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon) - \mathbf{E}f_\epsilon(t)) \, d\sigma.$$

Since S_t^ϵ is $2\pi\epsilon$ periodic, $S_{4\pi n\epsilon}^\epsilon f_\epsilon(t - 4\pi n\epsilon) = f_\epsilon(t - 4\pi n\epsilon)$. We have:

$$S_s^\epsilon \theta_{t-s}^\epsilon S_{-s}^\epsilon = \frac{1}{\sqrt{\epsilon}}(S_s^\epsilon E^\epsilon(t - s)) \cdot D_s^\epsilon$$

where we denote
$$E^\epsilon(t, x) = \nabla V^\epsilon(\frac{t}{2\pi n\epsilon}, \frac{x}{\epsilon}),$$
and we define the differential operator D_s^ϵ by
$$D_s^\epsilon = R_{-s/\epsilon}\nabla_v + \epsilon R_{-s/\epsilon}\nabla_x^\perp - \epsilon\nabla_x^\perp.$$

Note that D_s^ϵ is skew adjoint. Let us introduce the operator L_t^ϵ on $\mathcal{S}(\mathbb{R}^4)$ (extended on $\mathcal{S}'(\mathbb{R}^4)$) defined for every $\eta \in \mathcal{S}(\mathbb{R}^4)$ by:
$$L_t^\epsilon\eta = -\int_0^{2\pi n\epsilon} \mathbf{E}(\theta_t^\epsilon(S_\sigma^\epsilon\theta_{t-\sigma}^\epsilon S_{-\sigma}^\epsilon))\eta \, d\sigma.$$

We can gather those results in the following way:

Lemma 4.3 *We have the following equality:*
$$\mathbf{E}(J\theta_t^\epsilon f_\epsilon) = JL_t^\epsilon\mathbf{E}f_\epsilon + r_t^\epsilon + e_t^\epsilon,$$
where the operator L_t^ϵ is defined for every $\eta \in \mathcal{S}(\mathbb{R}^4)$ by:
$$L_t^\epsilon\eta(x, v) = -\frac{1}{\epsilon}\int_0^{2\pi n\epsilon} \nabla_v \cdot (\mathbf{E}(S_\sigma^\epsilon E^\epsilon(t-\sigma) \otimes E^\epsilon(t)) \cdot D_\sigma^\epsilon\eta(x, v)) \, d\sigma, \quad (14)$$
and the remainders are defined by:
$$e_t^\epsilon = JL_t^\epsilon(\mathbf{E}f_\epsilon(t - 4\pi n\epsilon) - \mathbf{E}f_\epsilon(t)),$$
$$r_t^\epsilon = \frac{1}{\epsilon\sqrt{\epsilon}}\int_0^{2\pi n\epsilon}\int_0^{4\pi n\epsilon-\sigma} J\mathbf{E}\left(E^\epsilon(t) \cdot \nabla_v(S_\sigma^\epsilon E^\epsilon(t-\sigma) \cdot D_\sigma^\epsilon(S_{s+\sigma}^\epsilon E^\epsilon(t-s-\sigma)\right.$$
$$\left. \cdot D_{s+\sigma}^\epsilon(S_{s+\sigma}^\epsilon f_\epsilon(t-\sigma-s))))\right) \, ds \, d\sigma.$$

We can now show the following lemma:

Lemma 4.4 *For every $\eta \in \mathcal{S}(\mathbb{R}^4)$, the remainder r_t^ϵ verifies:*
$$|\langle r_t^\epsilon; \eta\rangle| \le C(\eta)\sqrt{\epsilon}N(\epsilon),$$
and $(L_t^\epsilon)^*\eta$ converges in $L^2(\mathbb{R}^4)$ to:
$$\int_0^{2\pi n} R_{-s}\nabla_v \cdot \left(\nabla_{xx}^2 A(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)\nabla_v\eta\right) ds.$$

Proof. We have:
$$\begin{aligned}(L_t^\epsilon)^*\eta &= -\frac{1}{\epsilon}\int_0^{2\pi n\epsilon} D_\sigma^\epsilon \cdot (\mathbf{E}(S_\sigma^\epsilon E^\epsilon(t-\sigma) \otimes E^\epsilon(t)) \cdot \nabla_v\eta) \, d\sigma\\ &= -\int_0^{2\pi n} D_{\epsilon\sigma}^\epsilon \cdot (\mathbf{E}(S_{\epsilon\sigma}^\epsilon E^\epsilon(t-\epsilon\sigma) \otimes E^\epsilon(t)) \cdot \nabla_v)\eta \, d\sigma.\end{aligned}$$

But thanks to the definition to $T_\epsilon(s)$, E^ϵ and Hypothesis (H3), the term
$$\mathbf{E}(S_{\epsilon\sigma}^\epsilon E^\epsilon(t-\epsilon\sigma) \otimes E^\epsilon(t)) = \mathbf{E}(\nabla V^\epsilon(\frac{t-\epsilon\sigma}{2\pi n\epsilon}, x/\epsilon + v^\perp - R_{-\sigma}v^\perp) \otimes \nabla V^\epsilon(\frac{t}{2\pi n\epsilon}, x/\epsilon))$$

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converges strongly to $(-\nabla^2_{xx}A)(-\sigma/(2\pi n), v^\perp - R_{-\sigma}v^\perp)$ in $L^\infty((\mathbb{R}^+)^2; W^{1,\infty}(\mathbb{R}^2))$. Hence thanks to the definition of D^ϵ_s, $(L^\epsilon_t)^*\eta$ converges strongly to:

$$\int_0^{2\pi n} R_{-s}\nabla_v \cdot \left(\nabla^2_{xx}A(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)\nabla_v\eta\right) ds$$

in $L^\infty(\mathbb{R}^+ \times \mathbb{R}^4)$. We recall that we have $D^\epsilon_{\epsilon s} = R_{-s}\nabla_v + \epsilon R_{-s}\nabla^\perp_x - \epsilon\nabla^\perp_x$ thus

$$\|D^\epsilon_{\epsilon s'}\Phi\|_{L^2(\mathbb{R}^+\times\mathbb{R}^4)} \leq C\|\Phi\|_{W^{1,2}(\mathbb{R}^+\times\mathbb{R}^4)},$$
$$\|D^\epsilon_{\epsilon s'}D_{\epsilon s}\Phi\|_{L^2(\mathbb{R}^+\times\mathbb{R}^4)} \leq C\|\Phi\|_{W^{2,2}(\mathbb{R}^+\times\mathbb{R}^4)},$$

hence

$$
\begin{aligned}
|\langle r^\epsilon_t; \eta\rangle| \leq{}& C(n)\sqrt{\epsilon}\|\Phi\|_{W^{3,2}}\|f_\epsilon\|_{L^2} \\
& \sup_{s,s'}\{\mathbf{E}(|E^\epsilon(T(\epsilon s'))|\,(|E^\epsilon(T(\epsilon s))| + |D^\epsilon_{\epsilon s'}[E^\epsilon(T(\epsilon s))]|) \\
& (|E^\epsilon(t,x)| + |D^\epsilon_{\epsilon s}E^\epsilon(t,x)| + |D^\epsilon_{\epsilon s'}E^\epsilon(t,x)| + |D^\epsilon_{\epsilon s'}D^\epsilon_{\epsilon s}E^\epsilon(t,x)|))\}.
\end{aligned}
$$

We then use the following bounds: (recall that $E^\epsilon(t,x) = \nabla V^\epsilon(\frac{t}{2\pi n\epsilon}, \frac{x}{\epsilon})$.)

$$
\begin{aligned}
|E^\epsilon| &\leq \|\nabla_x V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))} \\
|D^\epsilon_{\epsilon s}E^\epsilon(t,x)| &\leq \|\nabla^2_{xx}V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))} \\
|D^\epsilon_{\epsilon s'}D^\epsilon_{\epsilon s}E^\epsilon(t,x)| &\leq \|\nabla^3_{xxx}V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))} \\
|D^\epsilon_{\epsilon s'}[E^\epsilon(T(\epsilon s))]| &\leq C\epsilon|\nabla_x E^\epsilon|(T(\epsilon s)) \leq C\|\nabla^2_{xx}V^\epsilon\|_{L^\infty(\mathbb{R}^+\times\mathbb{R}^2))}.
\end{aligned}
$$

Then Hypothesis $(H1)$ ensures that

$$|\langle r^\epsilon_t; \eta\rangle| \leq C\sqrt{\epsilon}\|f^\epsilon_0\|_{L^2}N(\epsilon)\|\Phi\|_{W^{3,2}},$$

which ends the proof of the lemma. $\qquad\square$

We can now state the following proposition:

Proposition 4.5 *Assume that $\epsilon(N(\epsilon))^2$ converges to 0 when ϵ goes to 0. Then the convergence (up to a subsequence) of $J\mathbf{E}f_\epsilon$ to f holds in $C^0(\mathbb{R}^+; L^2(\mathbb{R}^4) - w)$, and f is solution to:*

$$\partial_t f + JL^0_t Jf = 0, \tag{15}$$

where the operator L^0_t is defined for every $\eta \in \mathcal{S}(\mathbb{R}^4)$ by:

$$(L^0_t)^*\eta = \int_0^{2\pi n} R_{-s}\nabla_v \cdot \left(\nabla^2_{xx}A(\frac{-s}{2\pi n}, v^\perp - R_{-s}v^\perp)\nabla_v\eta\right) ds.$$

Proof. Thanks to the previous lemma, for every test function $\eta \in \mathcal{S}(\mathbb{R}^4)$:

$$|\langle r^\epsilon_t; \eta\rangle| \xrightarrow{\epsilon\to 0} 0,$$

and $(L^\epsilon_t)^*\eta$ converges strongly in $L^2(\mathbb{R}^4)$ to $(L^0_t)^*\eta$. But, thanks to Proposition 4.2, f_ϵ converges weakly to f in $L^2 - w$. So

$$\langle JL^\epsilon_t\mathbf{E}f_\epsilon; \eta\rangle = \langle \mathbf{E}f_\epsilon; (L^\epsilon_t)^* J\eta\rangle$$

converges to:

$$\langle f; (L^0_t)^* J\eta\rangle = \langle JL^0_t f; \eta\rangle.$$

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The function $f_\epsilon(t) - S_{2\pi n\epsilon} f_\epsilon(t - 2\pi n\epsilon)$ converges to 0 in $L^2 - w$ as well. So e_t^ϵ converges to 0 in $\mathcal{S}'(\mathbb{R}^4)$. Passing to the limit in equation (10) gives equation (15). This shows that $\partial_t f_\epsilon$ is uniformly bounded in time in a negative Sobolev space. Hence f_ϵ converges to f in the space of continuous function in time with values in this Sobolev space. Finally since f_ϵ is bounded in $L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^4))$, the convergence holds in $C^0([0,T]; L^2(\mathbb{R}^4) - w)$ for every $T > 0$. $\qquad\square$

4.3 Convergence to the SHE model

Since $Jf = f$, we can introduce the gyroaverage function defined by:

$$\rho(t, x, e) = f(t, x, v),$$

for every v such that $2e = |v|^2$. This subsection is devoted to the proof of the following lemma:

Lemma 4.6 *The function ρ lies in $C^0(\mathbb{R}^+; L^2(\mathbb{R}^2 \times \mathbb{R}^+) - w) \cap L^\infty(\mathbb{R}^+; L^2(\mathbb{R}^2 \times \mathbb{R}^+))$. It is solution to:*

$$\begin{cases} \partial_t \rho - \partial_e(a(e)\partial_e \rho) = 0 \\[2mm] \rho|_{t=0} = \dfrac{1}{2\pi} \displaystyle\int_0^{2\pi} f^0(t, x, R_\theta v)\, d\theta \end{cases}$$

where the diffusion parameter is defined by:

$$a(e) = \int_0^{2\pi n} \int_0^{2\pi} R_\theta v \cdot (-\nabla_{xx}^2 A)(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{-s+\theta} v^\perp) \cdot R_{-s} R_\theta v\, ds\, d\theta,$$

for every v such that $e = |v|^2/2$.

Proof. Let us first compute the operator $JL_t^0 J$. Let η_1, η_1 be two test functions in $\mathcal{S}(\mathbb{R}^4)$. We have:

$$\begin{aligned} \langle \eta_1; JL_t^0 J\eta_2 \rangle &= \langle (L_t^0)^* J\eta_1; J\eta_2 \rangle \\ &= \int_{\mathbb{R}^4} \int_0^{2\pi n} \nabla_v J\eta_1 (-\nabla_{xx}^2 A)(\frac{-s}{2\pi n}, v^\perp - R_{-s} v^\perp) R_{-s} \nabla_v J\eta_2\, ds\, dx\, dv. \end{aligned}$$

Let us denote ρ_{η_i} for $i = 1, 2$ the functions defined by:

$$\rho_{\eta_i}\Big(x, \frac{|v|^2}{2}\Big) = J\eta_i(x, v).$$

Using polar coordinates and noticing that $dv = d\theta\, de$ we find:

$$\begin{aligned} \langle \eta_1; JL_t^0 J\eta_2 \rangle &= \int_{\mathbb{R}^2} \int_0^\infty \partial_e \rho_{\eta_1}(x, e) \partial_e \rho_{\eta_2}(x, e) \int_0^{2\pi n} \int_0^{2\pi} R_\theta \vec{e} \cdot \\ &\qquad (-\nabla_{xx}^2 A)(\frac{-s}{2\pi n}, -R_{-s+\theta+\frac{\pi}{2}} \vec{e} + R_{\theta+\frac{\pi}{2}} \vec{e}) \cdot R_{-s+\theta} \vec{e}\, ds\, d\theta\, de\, dx \\ &= \int_{\mathbb{R}^2} \int_0^\infty \partial_e \rho_{\eta_1}(x, e) \partial_e \rho_{\eta_2}(x, e) a(e)\, de\, dx, \end{aligned}$$

where $\vec{e} = (\sqrt{2e}, 0)$. Hence for every test function ρ_η, let us multiply it by Equation (15) and integrate with respect to x, v. Since $de\, d\theta = dv$ we find:

$$\partial_t \int_{\mathbb{R}^2} \int_0^\infty \rho(t, x, e)\rho_\eta(x, e)\, dx\, de = \int_{\mathbb{R}^2} \int_0^\infty \rho(t, x, e)\partial_e(a(e)\partial_e \rho_\eta(x, e))\, de\, dx.$$

This, with Proposition 4.5 gives the desired result. $\qquad\square$

Remark: we have a family of equations parametrized by $x \in \mathbb{R}^2$, and the solutions of two equations at two distinct x do not interact.

4.4 Explicit computation of the diffusion coefficient

We derive in the following a suitable form to the diffusion coefficient $a(e)$. We will show, in particular, that $a(e)$ is non negative. From $(H4)$ the correlation function $A(t, x)$ is even with respect to t and x. This with $(H3)$ gives:

Lemma 4.7 *The correlation function A satisfies:*

$$\text{Supp}A \subset [-2\pi n, 2\pi n] \times \mathbb{R}^2,$$
$$\nabla_x A(0,0) = 0,$$
$$\partial_s A(0,0) = 0.$$

This last subsection is devoted to the following proposition. Theorem 2.1 follows from this proposition, Proposition 4.2 and Proposition 4.5.

Proposition 4.8 *Let us denote*

$$\tilde{A}(t, x) = \frac{1}{2\pi} \int_0^{2\pi} A(R_\theta x, t)\, d\theta.$$

Then $a(e)$ is non negative and equal to:

$$\frac{1}{2\pi n^2} \int_0^\infty (-\partial_{tt}^2 \tilde{A})(\frac{-s}{2\pi n}, 2\sqrt{e}\sqrt{1 - \cos s})\, ds.$$

Proof. Thanks to Lemma 4.6 and lemma 4.7, we have

$$a(e) = \int_{s=0}^\infty \int_0^{2\pi} R_\theta v \cdot (-\nabla_{xx}^2 A)(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{-s} R_\theta v^\perp) \cdot R_{-s} R_\theta v\, ds\, d\theta.$$

Since

$$-\nabla_{xx}^2 A(-\frac{s}{2\pi n}, v^\perp - R_{-s} v^\perp) \cdot R_{-s} v$$
$$= \frac{1}{2\pi n} \nabla_x \partial_s A(-\frac{s}{2\pi n}, v^\perp - R_{-s} v^\perp) + \partial_s (\nabla_x A(-\frac{s}{2\pi n}, v^\perp - R_{-s} v^\perp)),$$

we find

$$
\begin{aligned}
a(e) &= \frac{1}{2\pi n} \int_{s=0}^\infty \int_0^{2\pi} R_\theta v \cdot \nabla_x \partial_s A(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{\theta - s} v^\perp)\, ds\, d\theta \\
&\quad - \int_0^{2\pi} R_\theta v \cdot \nabla_x A(0,0)\, d\theta \\
&= \frac{1}{2\pi n} \int_{s=0}^\infty \int_0^{2\pi} R_\theta \vec{e} \cdot \nabla_x \partial_s A(-\frac{s}{2\pi n}, R_{\theta + \pi/2} \vec{e} - R_{-s+\theta+\pi/2} \vec{e})\, ds\, d\theta
\end{aligned}
$$

where $\vec{e} = (\sqrt{2e}, 0)$. Let us do the change of variables $s' = \theta - s$ to get

$$a(e) = \frac{1}{2\pi n} \int_0^{2\pi} \int_{\mathbb{R}} \mathbf{1}_{\{s \le \theta\}} R_\theta \vec{e} \cdot \nabla_x \partial_s A(\frac{s - \theta}{2\pi n}, R_{\theta + \pi/2} \vec{e} - R_{s + \pi/2} \vec{e})\, d\theta\, ds.$$

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Next we have

$$R_\theta \vec{e} \cdot \nabla_x \partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e}) = -\frac{1}{2\pi n}\partial_{ss}^2 A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})$$

$$-\partial_\theta \left\{ \partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e}) \right\}.$$

Integrating by parts the second term of the RHS gives

$$\frac{1}{2\pi n}\int_0^{2\pi}\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}}\partial_\theta\left\{\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\right\} ds\, d\theta$$

$$= \frac{1}{2\pi n}\int_0^{2\pi}\partial_\theta\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}}\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\, ds\, d\theta$$

$$- \frac{1}{2\pi n}\int_0^{2\pi}\int_{\mathbb{R}} \delta_{s=\theta}\partial_s A(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\, ds\, d\theta$$

$$= \frac{1}{2\pi n}\int_{-\infty}^{2\pi}\partial_s A(\frac{s-2\pi}{2\pi n}, R_{\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\, ds$$

$$- \frac{1}{2\pi n}\int_{-\infty}^{0}\partial_s A(\frac{s}{2\pi n}, R_{\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\, ds$$

$$- \frac{1}{2\pi n}\int_0^{2\pi}\partial_s A(0,0)\, ds$$

The first two lines cancel by doing the change of variables $s' = s - 2\pi$ and the third line vanishes thanks to Lemma 4.7, thus

$$a(e) = \frac{1}{(2\pi n)^2}\int_0^{2\pi}\int_{\mathbb{R}} \mathbf{1}_{\{s\le\theta\}}(-\partial_{ss}^2 A)(\frac{s-\theta}{2\pi n}, R_{\theta+\pi/2}\vec{e} - R_{s+\pi/2}\vec{e})\, d\theta\, ds$$

Doing the change of variables $s' = \theta - s$ gives

$$a(e) = \frac{1}{(2\pi n)^2}\int_0^{2\pi}\int_0^{\infty} (-\partial_{ss}^2 A)(-\frac{s}{2\pi n}, R_{\theta+\pi/2}((I - R_{-s})\vec{e}))\, d\theta\, ds$$

$$= \frac{1}{2\pi n^2}\int_0^{\infty} (-\partial_{ss}^2 \tilde{A})(-\frac{s}{2\pi n}, |(I - R_{-s})\vec{e}|)\, ds.$$

Finally

$$|(I - R_s)\vec{e}| = \sqrt{(|1 - \cos s|^2 + \sin^2 s)}\sqrt{2e}$$

$$= \sqrt{2(1 - \cos s)}\sqrt{2e}$$

$$= 2\sqrt{e}\sqrt{1 - \cos s}$$

$$= 2\sqrt{2e}|\sin(s/2)|$$

which ends the proof of the second assertion. □

Computation of the sign of the diffusion coefficient.
Here we check the non-negativity of the diffusion coefficient by expressing it in another form. Thanks to lemma 4.6 and to hypothesis $(H3), (H4)$ we have

$$a(e) = \frac{1}{2N}\int_{s=0}^{+\infty}\int_{-2\pi N}^{2\pi N} R_\theta v \cdot (-\nabla_{xx}^2 A)(-\frac{s}{2\pi n}, R_\theta v^\perp - R_{\theta-s}v^\perp) \cdot R_{\theta-s}v\, ds\, d\theta$$

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for all N. Then doing the change of variable $s := \theta - s$ we find:

$$a(e) = \frac{1}{2N} \int_{s \in \mathbb{R}} \int_{-2\pi N}^{2\pi N} \mathbf{1}_{\{\theta \geq s\}} R_\theta v \cdot (-\nabla_{xx}^2 A)(\frac{s-\theta}{2\pi n}, R_\theta v^\perp - R_s v^\perp) \cdot R_s v ds d\theta$$

But we remind that thanks to hypothesis $(H4)$

$$-\nabla_{xx}^2 A(\frac{s-\theta}{2\pi n}, R_\theta v^\perp - R_s v^\perp) = \lim_{\epsilon \to 0} \mathbf{E}\left(\nabla_x V^\epsilon(-\frac{s}{2\pi n}, -R_s v^\perp) \otimes \nabla_x V^\epsilon(-\frac{\theta}{2\pi n}, -R_\theta v^\perp)\right)$$

Thus

$$a(e) = \lim_{\epsilon, N} \frac{1}{2N} \cdot$$

$$\int \int_{-2\pi N}^{2\pi N} \mathbf{1}_{\{\theta \geq s\}} \mathbf{E}\left([\nabla_x V^\epsilon(-\frac{s}{2\pi n}, -R_s v^\perp) \cdot R_s v][\nabla_x V^\epsilon(-\frac{\theta}{2\pi n}, -R_\theta v^\perp) \cdot R_\theta v]\right) ds d\theta$$

Interverting s and θ we see that we can replace $\mathbf{1}_{\{\theta-s \geq 0\}}$ by $\mathbf{1}_{\{s-\theta \geq 0\}}$ and finally by adding both we obtain:

$$a(e) = \lim_{\epsilon, N} \frac{1}{4N} \mathbf{E}\left(\left[\int_{-2\pi N}^{2\pi N} \nabla_x V^\epsilon(-\frac{s}{2\pi n}, -R_s v^\perp) \cdot R_s v ds\right]^2\right)$$

which is a positive quantity. $\qquad\square$

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