

# On the regularity of the polar factorization for time dependent maps

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## Abstract

We consider the polar factorization of vector valued mappings, introduced in [3], in the case of a family of mappings depending on a parameter. We investigate the regularity with respect to this parameter of the terms of the polar factorization by constructing some a priori bounds. To do so, we consider the linearization of the associated Monge-Ampère equation.

## 1 Introduction

### Polar factorization and Monge-Ampère equation

Brenier in [3] showed that given  $\Omega$  a bounded open set of  $\mathbb{R}^d$  such that  $|\partial\Omega| = 0$ , with  $|\cdot|$  the Lebesgue measure of  $\mathbb{R}^d$ , every Lebesgue measurable mapping  $\mathbf{X} \in L^2(\Omega, \mathbb{R}^d)$  satisfying the non-degeneracy condition

$$(1) \quad \forall B \subset \mathbb{R}^d \text{ measurable, } |B| = 0 \Rightarrow |\mathbf{X}^{-1}(B)| = 0$$

can be factorized in the following (unique) way:

$$(2) \quad \mathbf{X} = \nabla\Phi \circ \mathbf{g},$$

where  $\Phi$  is a convex function and  $\mathbf{g}$  belongs to  $G(\Omega)$  the set of Lebesgue-measure preserving mappings of  $\Omega$ , defined by

$$(3) \quad \mathbf{g} \in G(\Omega) \iff \forall f \in C_b(\Omega), \int_{\Omega} f(\mathbf{g}(x)) dx = \int_{\Omega} f(x) dx,$$

where  $C_b$  is the set of bounded continuous functions. If  $da$  denotes the Lebesgue measure of  $\Omega$ , the push-forward of  $da$  by  $\mathbf{X}$ , that we denote  $\mathbf{X}\#da$ , is the measure  $\rho$  defined by

$$(4) \quad \forall f \in C_b(\mathbb{R}^d), \int_{\mathbb{R}^d} f d\rho = \int_{\Omega} f(\mathbf{X}(a)) da.$$

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One sees first that the condition (1) is equivalent to the fact that  $\rho$  is absolutely continuous with respect to the Lebesgue measure, or has a density in  $L^1(\mathbb{R}^d, dx)$ . Then  $\Phi$  satisfies in  $\Omega$  the Monge-Ampère equation:

$$\rho(\nabla\Phi(x)) \det D^2\Phi(x) = 1$$

in the following weak sense:

$$(5) \quad \forall g \in C_b(\mathbb{R}^d), \int_{\Omega} g(\nabla\Phi(y)) dy = \int_{\mathbb{R}^d} g(x) d\rho(x).$$

$\Psi$ , the Legendre transform of  $\Phi$ , defined by

$$(6) \quad \Psi(y) = \sup_{x \in \Omega} \{x \cdot y - \Phi(x)\},$$

satisfies the Monge-Ampère equation

$$\det D^2\Psi(x) = \rho(x)$$

in the following weak sense:

$$(7) \quad \forall f \in C_b(\Omega), \int_{\mathbb{R}^d} f(\nabla\Psi(x)) d\rho(x) = \int_{\Omega} f(y) dy.$$

Note that the existence and uniqueness of the pair  $\nabla\Phi, \nabla\Psi$  and the validity of (5) is not subject to the condition (1) (see [22] Th 2.12 for this precise fact, and for a complete reference on polar factorization and optimal transportation). However (7) may not hold. Note also that this formulation of the second boundary value problem for the Monge-Ampère equation is strictly weaker than the Aleksandrov formulation (see [8] where the different formulations are compared and where it is shown that they may not coincide if some extra conditions are not satisfied).

**The periodic case** The polar factorization of maps on general Riemannian manifolds has been treated by [17], and also in the particular case of the flat torus by [10]. Given  $\mathbf{X}$  a mapping of  $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$  into itself, we look for a pair  $(\Phi, \mathbf{g})$  such that

1.  $\mathbf{g}$  is measure preserving from  $\mathbb{T}^d$  into itself,
2.  $\Phi$  is convex from  $\mathbb{R}^d$  to  $\mathbb{R}$  and  $\Phi - |x|^2/2$  is periodic,
3.  $\mathbf{X} = \nabla\Phi \circ \mathbf{g}$  (Note that the condition above ensures that  $\nabla\Phi - x$  is  $\mathbb{Z}^d$  periodic).

Then under the non-degeneracy condition (1), there exists a unique such pair  $(\mathbf{g}, \nabla\Phi)$ .

## Introducing the time-dependence

In this paper we are interested in the following problem: given a “time” dependent family of mappings  $t \rightarrow \mathbf{X}(t, \cdot)$ , where for all  $t$ ,  $\mathbf{X}(t)$  maps  $\Omega$  in  $\mathbb{R}^d$ , we investigate the regularity of the curve  $t \rightarrow (\mathbf{g}(t, \cdot), \Phi(t, \cdot), \Psi(t, \cdot))$ .

We state different results under different assumptions. The weakest assumption is that  $\rho = \mathbf{X}\#da$ ,  $\mathbf{X}$  and  $\partial_t \mathbf{X}$  belong to  $L^\infty$  in time and space. In this case  $\partial_t \nabla \Phi$  and  $\partial_t \mathbf{g}$  are bounded as measures (Th. 2.1).

Under the additional assumption that  $\rho$  is close to 1 (or actually to a continuous positive function) in  $L^\infty$  norm (but we do not ask for continuity), we obtain that  $\partial_t \Phi$  belongs to  $C^\alpha$  for some  $\alpha > 0$  (Th. 2.2). To this purpose we use a local maximum principle for solutions of degenerate elliptic equations (Theorem 3.5, Theorem 3.7) obtained by Murthy and Stampacchia ([18]) and Trudinger ([20]), and use a result by Caffarelli and Gutierrez ([9]) that establishes the Harnack inequality for solutions of the homogeneous linearized Monge-Ampère equation (Theorem 3.4).

The polar factorization has the following geometrical interpretation: if  $\mathbf{X} = \nabla \Phi \circ \mathbf{g}$ , as in (2), then  $\mathbf{g}$  is the projection, in the  $L^2(\Omega, \mathbb{R}^d)$  sense, of  $\mathbf{X}$  on  $G(\Omega)$ , the set of Lebesgue measure preserving mappings. Therefore our study amounts to examine the continuity and the differentiability of the projection operator on  $G(\Omega)$ . We also briefly discuss a variant of the Hodge decomposition of vector fields that appears naturally in this study.

Our results have an immediate application to the semi-geostrophic equations, a system arising in meteorology to model frontogenesis (see [12]). They allow in particular to define the velocity in the physical space, a fact that was not known for weak solutions. We discuss this application in a more extensive way in section 9.

## 1.1 Heuristics

We present here some formal computations, assuming that all the terms considered are smooth enough. Suppose that  $\Omega$  is bounded, and for any  $t$  we denote by  $d\rho(t, \cdot) = \mathbf{X}(t, \cdot)\#da$  (with  $da$  the Lebesgue measure on  $\Omega$ ) the measure defined by (4). Then for all  $t$ ,  $\Phi(t, \cdot)$ ,  $\Psi(t, \cdot)$  are as in (5,7).

### Parallel with the Hodge decomposition of vector fields

By differentiating (2) with respect to time one finds

$$\partial_t \mathbf{X}(t, a) = \partial_t \nabla \Phi(t, \mathbf{g}(t, a)) + D^2 \Phi(t, \mathbf{g}(t, a)) \partial_t \mathbf{g}(t, a).$$

If  $\mathbf{X}$  is invertible, one can write

$$(8) \quad \partial_t \mathbf{X}(t, a) = v(t, \mathbf{X}(t, a))$$

for some “Eulerian” vector field  $v(t, x)$  defined  $d\rho$  a.e. Note that  $\rho = \mathbf{X}\#da$  and  $v$  will be linked through the mass conservation constraint

$$(9) \quad \partial_t \rho + \nabla \cdot (\rho v) = 0.$$

$\mathbf{g}$  will then also be invertible and composing with  $\mathbf{g}^{-1}$  one gets:

$$(10) \quad v(t, \nabla \Phi(t, x)) = \partial_t \nabla \Phi(t, x) + D^2 \Phi(t, x) w(t, x)$$

with  $w = \partial_t \mathbf{g}(t, \mathbf{g}^{-1}(t, x))$ . Since for all  $t$ ,  $\mathbf{g}(t) \in G(\Omega)$ , it follows that  $w$  is divergence free. Composing with  $\nabla \Psi = \nabla \Phi^{-1}$  we obtain

$$v = \partial_t \nabla \Phi(\nabla \Psi) + D^2 \Phi w(\nabla \Psi).$$

It is easily checked that  $\bar{w} = D^2 \Phi w(\nabla \Psi)$  satisfies

$$\nabla \cdot (\rho \bar{w}) = 0,$$

therefore the second term in the decomposition (10) does not move mass. It plays the role of a divergence free vector field for a uniform density.

Note that a similar decomposition is performed in the study of the incompressible inhomogeneous Navier-Stokes equation in [15] where for a given velocity field  $v$ , and a density  $\rho > 0$ , one seeks to decompose  $v$  as

$$v = \frac{1}{\rho} \nabla p + w, \quad \nabla \cdot w = 0.$$

The next proposition shows that, in the non-degenerate case where  $\Phi$  is smooth and strictly convex, the decomposition (10) is defined in a unique way.

**Proposition 1.1** *Let  $v \in L^2(\mathbb{R}^d, d\rho; \mathbb{R}^d)$ , let  $\Phi : \bar{\Omega} \rightarrow \mathbb{R}^d$  be  $C^2$  and strictly convex on  $\bar{\Omega}$ , with  $\rho = \nabla \Phi \# da$ . Then there exists a unique decomposition of  $v$  such that*

$$(11) \quad v(\nabla \Phi) = \nabla p + D^2 \Phi w$$

with  $(\nabla p, w) \in L^2(\Omega; \mathbb{R}^d)$ ,  $\nabla \cdot w = 0$ ,  $w \cdot \partial \Omega = 0$ .

*Proof:* We only sketch the proof of this classical result.  $w$  can be found by looking for

$$(12) \quad \inf_{\substack{w \in L^2(\Omega; \mathbb{R}^d) \\ w \cdot \partial \Omega = 0 \\ \nabla \cdot w = 0}} \left\{ \int \frac{1}{2} w^t \cdot D^2 \Phi \cdot w - v(\nabla \Phi) \cdot w \right\}.$$

Using the strict convexity of  $\Phi$  we have  $D^2 \Phi \geq \lambda I$  on  $\bar{\Omega}$ , and we obtain that

$$\|w\|_{L^2(\Omega)} \leq \frac{2}{\lambda} \left[ \int \rho |v|^2 \right]^{1/2}.$$

The functional to minimize is strictly convex, and weakly lower semi continuous, therefore the problem admits a minimizer. For the uniqueness of the decomposition, notice that if

$$0 = \nabla p + D^2 \Phi w$$

for  $\nabla p, w \in L^2$ , multiplying by  $w$  and integrating over  $\Omega$ , we get that  $\nabla p, w = 0$ . Therefore, if  $v$  governs the evolution of  $\rho$  through the equation (9), the decomposition (11) will coincide with (10) and will yield  $\nabla p = \partial_t \nabla \Phi$ .

### The associated elliptic problems: The linearized Monge-Ampère equation

Multiplying (10) by  $D^2\Phi^{-1}$ , we find that  $\partial_t\Phi$  will be solution of the following elliptic problem:

$$\nabla \cdot (D^2\Phi^{-1}\nabla\partial_t\Phi) = \nabla \cdot (D^2\Phi^{-1}v(\nabla\Phi)).$$

On the other hand,  $\Psi = \Phi^*$  (see (6)) solves formally the equation

$$\det D^2\Psi = \rho.$$

Then for any  $(d \times d)$  matrices  $A, B$  we have

$$\det(A + tB) = \det A + t \operatorname{trace}({}^tA^*B) + o(t)$$

where  $A^*$  is the matrix of cofactors (or co-matrix) of  $A$  and thus, formally,  $\partial_t\Psi$  solves the elliptic equation

$$M_{ij}\partial_{ij}\partial_t\Psi = \partial_t\rho,$$

where  $(M_{ij})_{i,j \in [1..d]}$  is the co-matrix of  $D^2\Psi$ , given by

$$M = \det D^2\Psi [D^2\Psi]^{-1} = \rho D^2\Phi(\nabla\Psi).$$

Then if  $M$  is the co-matrix of a second derivative matrix, for all  $j \in [1..d]$

$$\sum_{i=1}^d \partial_i M_{ij}(x) \equiv 0,$$

and using this and the equation (9), we obtain a divergence formulation of the problem:

$$(13) \quad \nabla \cdot (M\nabla\partial_t\Psi) = \partial_t\rho = -\nabla \cdot (\rho v).$$

In the case where  $\rho$  is smooth and supported in a convex set, it will be shown using classical elliptic regularity and results on Monge-Ampère equation, that the decomposition holds (Proposition 4.1) and that the terms are smooth.

For a generic, non-necessarily smooth  $\rho$ , we see that the difficulty will be coming from the lack of regularity and ellipticity of this equation. Indeed we only know a-priori that  $D^2\Phi$  is a measure. If  $\rho$  is close to 1 in  $L^\infty$  norm, we get that  $D^2\Phi$  is in  $L^p_{loc}$  for some  $p < \infty$ , and thus non necessarily uniformly elliptic.

## 2 Results

### Notations

In the remainder of the paper  $\Omega$  will be kept fixed once for all and chosen bounded and convex. We will furthermore assume for simplicity (although one may possibly remove this assumption through approximation) that it is smooth and strictly convex.

The Lebesgue measure of  $\Omega$ ,  $\chi_\Omega \mathcal{L}^d$ , will be denoted in short  $da$ .

For compatibility  $\rho$  will be a probability measure on  $\mathbb{R}^d$  and  $\Omega$  of Lebesgue measure one.

$\mathcal{M}(\Omega)$  will design the set of (possibly vector valued) bounded measures on  $\Omega$ , with norm  $\|\cdot\|_{\mathcal{M}(\Omega)}$ .

For  $M$  a  $(d \times d)$  matrix, and  $u, v$  two vectors of  $\mathbb{R}^d$ ,  $uMv$  will denote  $\sum_{i,j} u_i M_{ij} v_j$ .

$I$  will be an non-empty open interval of  $\mathbb{R}$ .

We still use  $d\rho(t, \cdot) = \mathbf{X}(t, \cdot) \# da$ , the functions  $\Phi(t, \cdot), \Psi(t, \cdot)$  will be as in (5, 7) with  $(\rho(t, \cdot), \Omega)$ . Since they are defined only up to a constant, we will impose the condition:

$$(14) \quad \forall t \in I, \int_{\Omega} \Phi(t, x) dx = 0,$$

and this sets also  $\Psi$  through the relation  $\Psi = \Phi^*$ .

**Theorem 2.1** *Let  $\Omega, I$  be as above, let  $\mathbf{X} : I \times \Omega \rightarrow \mathbb{R}^d$ . Let, for any  $t \in I$ ,  $d\rho(t, \cdot) = \mathbf{X}(t, \cdot) \# da$  as in (4). Assume that  $(\mathbf{X}, \partial_t \mathbf{X}) \in L^\infty(I \times \Omega)$ , with  $R = \|\mathbf{X}\|_{L^\infty(I \times \Omega)}$ , and assume that  $\rho \in L^\infty(I \times \mathbb{R}^d)$ . Take*

$$\mathbf{X}(t) = \nabla \Phi(t) \circ \mathbf{g}(t), \quad \mathbf{g}(t) = \nabla \Psi(t) \circ \mathbf{X}(t)$$

to be the polar factorization of  $\mathbf{X}$  as in (2) where we impose (14). Then

1. for a.e.  $t \in I$ ,  $\partial_t \nabla \Phi(t, \cdot)$  is a bounded measure in  $\Omega$  with

$$\|\partial_t \nabla \Phi\|_{L^\infty(I; \mathcal{M}(\Omega))} \leq C(R, d, \Omega) \|\rho\|_{L^\infty(I \times B_R)}^{\frac{1}{2}} \|\partial_t \mathbf{X}\|_{L^\infty(I \times B_r)}$$

and  $\partial_t \Phi \in L^\infty(I, L^{1^*}(\Omega))$  with  $1^* = d/(d-1)$ .

2.  $\Phi$  (resp.  $\Psi$ ) belongs to  $C^\alpha(I; C^0(\bar{\Omega}))$  (resp. to  $C^\alpha(I; C^0(\bar{B}_R))$ ) for some  $\alpha \in ]0, 1[$ .
3. For a.e.  $t \in I$ ,  $\partial_t \mathbf{g}$  is a bounded measure on  $\Omega$  with

$$\|\partial_t \mathbf{g}\|_{L^\infty(I; \mathcal{M}(\Omega))} \leq C(R, d, \Omega) \|\rho\|_{L^\infty(I \times B_R)} \|\partial_t \mathbf{X}\|_{L^\infty(I \times \Omega)}.$$

4. If  $\rho$  is supported in  $\bar{\Omega}'$  for some open set  $\Omega'$ , and  $0 < \lambda \leq \rho(\cdot, \cdot) \leq \Lambda$  on  $\Omega'$ , for some  $(\lambda, \Lambda) \in \mathbb{R}_+^*$ , then there exists  $\beta \in ]0, 1[$  such that for any  $\omega' \subset\subset \Omega'$ ,

$$\nabla \Psi \in C^\beta(I; C^0(\omega')),$$

with  $\beta$  depending on  $\Lambda/\lambda$ .

5. If in addition  $\Omega'$  is convex, then there exists  $\beta' \in ]0, 1[$  such that for any  $\omega \subset\subset \Omega$ ,

$$\nabla \Phi \in C^{\beta'}(I; C^0(\omega)).$$

**Theorem 2.2** *Under the assumptions of Theorem 2.1, and assuming that  $\rho$  is supported in  $\bar{\Omega}'$ , for some open set  $\Omega'$ , we have:*

1. There exists  $\epsilon_0 > 0$  such that if  $|\rho - 1| \leq \epsilon < \epsilon_0$  in  $\Omega'$ , then there exists  $\alpha > 0$  (depending on  $\epsilon$ ) such that, for any  $w' \subset\subset \Omega'$ ,

$$\partial_t \Psi \in L^\infty(I; C^\alpha(\omega')).$$

If in addition  $\Omega'$  is convex, for any  $w \subset\subset \Omega$ ,

$$\partial_t \Phi \in L^\infty(I; C^\alpha(\omega)).$$

2. For any  $p < 2$ , there exists  $\epsilon(p) > 0$  such that, if  $|\rho - 1| \leq \epsilon(p)$  in  $\Omega'$ , for any  $w' \subset\subset \Omega'$ ,

$$\partial_t \nabla \Psi \in L^\infty(I; L^p(\omega')).$$

If in addition  $\Omega'$  is convex, for any  $w \subset\subset \Omega$ ,

$$\partial_t \nabla \Phi \in L^\infty(I; L^p(\omega)).$$

*Remark:* The Theorem remains true if one replaces the condition  $|\rho - 1| \leq \epsilon$  by  $|\rho - f| \leq \epsilon$  with  $f$  a positive continuous function and the bounds will then depend on the modulus of continuity of  $f$  (see [4]).

We also state the result in the periodic case: In this setting we have the following theorem, which is just an adaptation of the two previous:

**Theorem 2.3** *Under the assumptions that  $\rho \in L^\infty(I \times \mathbb{T}^d)$ ,  $\partial_t \mathbf{X} \in L^\infty(I \times \mathbb{T}^d)$ , we have:*

1. With the same bounds as in Theorem 2.1,

$$\begin{aligned} \partial_t \nabla \Phi &\in L^\infty(I; \mathcal{M}(\mathbb{T}^d)), \\ \partial_t \mathbf{g} &\in L^\infty(I; \mathcal{M}(\mathbb{T}^d)), \end{aligned}$$

and for some  $\alpha > 0$ , we have

$$\Phi, \Psi \in C^\alpha(I; C^0(\mathbb{T}^d)).$$

2. If for all  $(t, x) \in (I \times \mathbb{T}^d)$  we have  $0 < \lambda \leq \rho(t, x) \leq \Lambda$ , then for some  $\beta > 0$  depending on  $(\lambda, \Lambda) \in \mathbb{R}_+^*$ ,

$$\mathbf{g}, \nabla \Phi, \nabla \Psi \in C^\beta(I; L^\infty(\mathbb{T}^d)).$$

3. There exists  $\epsilon_0$  such that if  $|\rho - 1| \leq \epsilon \leq \epsilon_0$ , then for some  $\alpha > 0$  depending on  $\epsilon$ ,

$$\begin{aligned} \partial_t \Psi &\in L^\infty(I; C^\alpha(\mathbb{T}^d)), \\ \partial_t \Phi &\in L^\infty(I; C^\alpha(\mathbb{T}^d)). \end{aligned}$$

4. For any  $p < 2$  there exists  $\epsilon(p)$  such that if  $|\rho - 1| \leq \epsilon(p)$  then

$$\begin{aligned}\partial_t \nabla \Psi &\in L^\infty(I; L^p(\mathbb{T}^d)), \\ \partial_t \nabla \Phi &\in L^\infty(I; L^p(\mathbb{T}^d)), \\ \partial_t \mathbf{g} &\in L^\infty(I; L^p(\mathbb{T}^d)).\end{aligned}$$

*Remark:* in this case, the absence of boundary allows to have a bound over  $\mathbb{T}^d$  and not only interior estimates as in the previous results.

## 2.1 Related results

### The linearized Monge-Ampère equation

The linearized Monge-Ampère equation (*LMA*) is a well known equation, since it is used to carry out the continuity method, in order to obtain classical solutions of the Monge-Ampère equation (see [14], chapter 17). However for this purpose this is always made in the case where the densities and the domains considered are smooth, and thus the *LMA* equation is uniformly elliptic.

In the non-smooth case, [9] proved Harnack inequality for solutions of

$$M_{ij} \partial_{ij} u = 0$$

with  $M$  the co-matrix of  $D^2\Psi$ , for some  $\Psi$  convex, under the assumption that the measure  $\rho = \det D^2\Psi$  satisfies the following absolute continuity condition:

*C:* For any  $0 < \delta_1 < 1$  there exists  $0 < \delta_2 < 1$  such that for any section  $S$  and any measurable set  $E \subset S$ ,

$$(15) \quad \text{if } \frac{|E|}{|S|} \leq \delta_2 \text{ then } \frac{\rho(E)}{\rho(S)} \leq \delta_1,$$

(a section is a set of the form

$$S_t(x_0) = \{x | \Psi(x) - \Psi(x_0) \leq p \cdot (x - x_0) + t, p \in \partial\Psi(x_0)\}.$$

They showed that the solution of  $(\det D^2\Psi)(D^2\Psi)_{ij}^{-1} D_{ij} u = 0$  satisfies a Harnack inequality on the sections of  $\Psi$  and subsequently is  $C^\alpha$ . The precise result is stated below (Theorem 3.4). We will use this result to obtain the first part of Theorem 2.2. Note that the condition (15) implies  $C^{1,\alpha}$  regularity of the Aleksandrov solution of  $\det D^2\Psi = \rho$  ([6]). Note also that the condition (15) is satisfied when the density  $\rho$  is bounded between two positive constants. We will also obtain some results (Theorem 2.1) in the degenerate case when the condition (15) is not satisfied and show in some counterexamples (section 8) that when this condition is not fulfilled, the result of Theorem 2.2 does not hold.

### Maximum principles for degenerate elliptic equations

We will use a local maximum principle for degenerate elliptic equations to obtain Hölder continuity in Theorem 2.2. Consider the problem

$$\nabla \cdot (M(x) \nabla u(x)) = \nabla \cdot f(x)$$



where  $M(x) = M_{ij}(x)$ ,  $(i, j) \in [1..d]$  is a symmetric positive semi-definite, matrix,  $f(x) = (f_i(x))_{i \in [1..d]}$ . In the cases we will study, we will not have the usual uniform ellipticity condition

$$\lambda I \leq M \leq \Lambda I$$

with  $I$  the  $d \times d$  identity matrix, and for some positive numbers  $\lambda, \Lambda$ , but a condition of the form

$$(16) \quad \lambda(x)I \leq M \leq \Lambda(x)I$$

for some non negative measurable functions  $\lambda(x), \Lambda(x)$ . Under the assumption that  $(\lambda^{-1}(x), \Lambda(x)) \in L^p_{loc}(\Omega)$  for some  $p > d$  and that  $f \in L^\infty$ , we can obtain a bound on the solution  $u$  in  $L^\infty_{loc}$ . Properly localized, this bound with the Harnack inequality (Theorem 3.4) will yield Hölder continuity of the solution of the LMA equation (13). This type of maximum principles have been already obtained in [18], [20], (see also [19]), and we will use them under the forms of Theorems 3.5, 3.7, and Corollary 3.6. Note however that the condition (16) is not know by itself to guaranty Hölder continuity of the solution, but only a  $L^\infty$  bound.

It can be interesting to point out that we will thus use both the divergence and non-divergence structure of the LMA to obtain our results.

### 3 Some preliminary results

In this section we state the results that we are going to need for the proofs of the theorems. The reader may skip this section and come back to it whenever needed. Note that all these results can be extended to the periodic case.

#### 3.1 Regularity for solutions of Monge-Ampère equation

**Theorem 3.1** *Let  $\Omega, \Omega'$  be bounded,  $C^\infty$ , strictly convex, and  $|\Omega| = 1$ . Let  $\rho$  be a probability measure in  $\bar{\Omega}'$ , belong to  $C^\infty(\bar{\Omega}')$ , and satisfy  $0 < \lambda \leq \rho(t, x) \leq \Lambda$  for some pair  $(\lambda, \Lambda)$ . Then there exists a unique (up to a constant) solution of*

$$\begin{aligned} \det D^2\Psi &= \rho, \\ \nabla\Psi &\text{ maps } \Omega' \text{ to } \Omega, \end{aligned}$$

*in the sense of (7). The solution  $\Psi$  belongs to  $C^\infty(\bar{\Omega}')$ , and  $\Phi$ , defined as in (5), belongs to  $C^\infty(\bar{\Omega})$ .*

For this the reader can refer to [4]-[8], [13], [21].

The next Theorem can be found in [6], [8], [7].

**Theorem 3.2** *Let  $\rho$  be supported in  $\bar{\Omega}'$  with  $\Omega'$  open, satisfy  $0 < \lambda \leq \rho \leq \Lambda$ , and let  $\Psi$  be solution of*

$$\begin{aligned} \det D^2\Psi &= \rho, \\ \nabla\Psi &\text{ maps } \Omega' \text{ to } \Omega, \end{aligned}$$

in the sense of (7) with  $\Omega$  convex. Then for some  $\alpha \in ]0, 1[$  depending on  $\Lambda/\lambda$ ,  $\Psi \in C_{loc}^{1,\alpha}(\Omega')$ . If moreover  $\Omega'$  is also convex then  $\Psi$  (resp. its Legendre transform  $\Phi$ ) is in  $C^{1,\alpha}(\bar{\Omega}')$  (resp. in  $C^{1,\alpha}(\bar{\Omega})$ ).

The next Theorem can be found in [4].

**Theorem 3.3** *Let  $\Omega$  be normalized so that  $B_1 \subset \Omega \subset B_d$ . Let  $\Psi$  be a convex Aleksandrov solution of*

$$\begin{aligned} \det D^2\Psi &= \rho, \\ \Psi &= 0 \text{ on } \partial\Omega. \end{aligned}$$

Then for every  $p < \infty$  there exists  $\epsilon(p)$  such that if  $|\rho - 1| \leq \epsilon(p)$  then  $\Psi \in W_{loc}^{2,p}(\Omega)$  and

$$\|\Psi\|_{W^{2,p}(B_{1/2})} \leq C(\epsilon).$$

*Remark 1:* This implies also, maybe for a smaller value of  $\epsilon(p)$  that one can also have  $\|D^2\Psi^{-1}\|_{L^p(B_{1/2})} \leq C'(\epsilon)$ .

*Remark 2:* The theorem remains true if one replaces  $|\rho - 1| \leq \epsilon$  by  $|\rho - f| \leq \epsilon$ , for some continuous positive  $f$ , and the bounds depends on the modulus of continuity of  $f$ .

## 3.2 The linearized Monge-Ampère equation

We state here the result of [9] evoked in the previous section:

**Theorem 3.4** *Let  $\Omega$  be a domain in  $\mathbb{R}^d$ , let  $U$  be an Aleksandrov solution in  $\Omega$  of*

$$\det D^2U = \mu$$

where  $\mu$  the satisfies the condition (15). Let  $w$  be a solution in  $\Omega$  of the linearized homogeneous Monge-Ampère equation

$$A_{ij}\partial_{ij}w = 0$$

where  $A_{ij}$  is the co-matrix of  $D^2U$ , let  $R > 0$  and  $y \in \Omega$  be such that  $B_R(y) \subset \Omega$ , then for some  $\beta < 1$  depending only on the condition (15), for any  $r < R/4$ ,

$$\text{osc}(r/2) \leq \beta \text{osc}(r),$$

where

$$\begin{aligned} \text{osc}(r) &= M(r) - m(r), \\ M(r) &= \sup_{B_r(y)} w, \quad m(r) = \inf_{B_r(y)} w. \end{aligned}$$

### 3.3 Maximum principle for degenerate elliptic equations

We give here some results concerning degenerate elliptic equations of the form

$$(17) \quad \nabla \cdot (M(x)\nabla u(x)) = \nabla \cdot f(x)$$

where  $M$  is symmetric non-negative matrix,  $f = (f_i)$ ,  $i = 1..d$ . The equation can be written  $\partial_i(M_{ij}\partial_j u) = \partial_i f_i$  with summation over repeated indices. The usual strict ellipticity condition

$$\lambda|\xi|^2 \leq M_{ij}\xi_i\xi_j \leq \Lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d,$$

is replaced by the following

$$\sum_{i,j=1}^d |M_{ij}| + |M^{ij}| \in L^p_{loc}(\Omega) \text{ for some } p,$$

where  $M^{ij}$  denotes the inverse matrix of  $M$ . This is equivalent to the condition that there exists  $\lambda(x), \Lambda(x)$  such that  $\lambda^{-1}, \Lambda$  are in  $L^p_{loc}(\Omega)$  and such that  $\lambda(x)I \leq M(x) \leq \Lambda(x)I$ , in the sense of symmetric matrices.

The class of admissible test functions is

$$\mathcal{C}(\Omega) = \{v \in W_0^{1,1}(\Omega), M^{1/2}\nabla v \in L^2(\Omega)\}.$$

A subsolution (resp. supersolution)  $u$  of (17) is defined by the condition that for all non-negative  $v \in \mathcal{C}(\Omega)$ ,

$$\int_{\Omega} \nabla v M \nabla u - \nabla v \cdot f \leq (\geq) 0.$$

Then, following [18] and [20], we have the following results:

#### Bound for Dirichlet boundary data

We denote by  $S_d^+$  the set of  $d \times d$  non negative symmetric matrices.

**Theorem 3.5** *Let  $M : \Omega \rightarrow S_d^+$  be such that  $M^{-1}$  is in  $L^p(\Omega; S_d^+)$  for some  $p > d$ . Let  $f$  be in  $L^\infty(\Omega; \mathbb{R}^d)$ . Let  $u$  be a subsolution (supersolution) of*

$$\nabla \cdot (M(x)\nabla u(x)) = \nabla \cdot f(x)$$

*in  $\Omega$ , satisfying  $u \leq 0$  ( $u \geq 0$ ) on  $\partial\Omega$ . Then*

$$\sup_{\Omega} u(-u) \leq C(\|u^+(u^-)\|_{L^{a_0}(\Omega)} + \|f\|_{L^\infty(\Omega)})$$

*where  $C, C$  depends on  $|\Omega|, a_0 > 0, p > d, \|M^{-1}\|_{L^p(\Omega)}$ .*

This maximum principle can be precised in the following corollary, that will be crucial for the proof of Hölder continuity in Theorem 2.2.

**Corollary 3.6** *Under the previous assumptions, for  $y \in \Omega$ ,  $B_R(y) \subset \Omega$ , if  $u$  is a subsolution (supersolution) in  $B_R$  of (17) and  $u \leq 0$  ( $u \geq 0$ ) on  $\partial B_R$ , then*

$$\sup_{B_R} u(-u) \leq C\|M^{-1}\|_{L^p(B_R)}\|f\|_{L^\infty(B_R)}R^\delta,$$

*where  $\delta = 1 - \frac{n}{p}$ .*

### Bound without boundary data

Here we state a maximum principle that does not depend on the boundary data. Note that here we need to control the norm of both  $M$  and  $M^{-1}$  whereas we only needed to control  $M^{-1}$  above.

**Theorem 3.7** *Let  $M : \Omega \rightarrow S_d^+$  be such that  $M, M^{-1}$  are both in  $L_{loc}^p(\Omega)$ , with  $p > d$ . Let  $f$  be in  $L^\infty(\Omega)$ . Let  $u$  be a subsolution of*

$$\nabla \cdot (M(x)\nabla u(x)) = \nabla \cdot (f(x))$$

in  $\Omega$ . Then we have for any ball  $B_{2R} \subset\subset \Omega$  and  $a_0 > 0$

$$\sup_{B_R(y)} u \leq C_1 \|u^+\|_{L^{a_0}(B_{2R}(y))} + C_2 k$$

where  $k = \|f\|_{L^\infty(B_{2R})}$ ,  $C_1, C_2$  depend on  $R, a_0, p, \|M\|_{L^p(B_{2R})}, \|M^{-1}\|_{L^p(B_{2R})}$ .

### 3.4 Convex functions and Legendre transforms

We state first the following classical lemma on convex functions:

**Lemma 3.8** *Let  $\varphi$  be a convex function from  $\mathbb{R}^d$  to  $\mathbb{R}$ , globally Lipschitz with Lipschitz constant  $L$ . Then we have*

$$\|D^2\varphi\|_{\mathcal{M}(B_R)} \leq C(d)R^{d-1}L.$$

*Proof:* we have

$$\begin{aligned} \|D^2\varphi\|_{\mathcal{M}(B_R)} &\leq C \int_{B_R} \Delta\varphi \\ &= \int_{\partial B_R} \nabla\varphi \cdot n \\ &\leq C(d)R^{d-1}L. \end{aligned}$$

□

We recall here some useful properties of the Legendre transform. Let  $\Omega$  be a convex domain, let  $\phi : \Omega \mapsto \mathbb{R}$  be  $C^1$  convex. Let  $\phi^*$  be its Legendre transform defined by

$$\phi^*(y) = \sup_{x \in \Omega} x \cdot y - \phi(x).$$

Then, for all  $x \in \Omega$ ,

$$\nabla\phi^*(\nabla\phi(x)) = x.$$

If moreover  $\phi$  is  $C^2$  strictly convex, then, for all  $x \in \Omega$ ,

$$(18) \quad D^2\phi^*(\nabla\phi(x)) = D^2\phi^{-1}(x).$$

From this we deduce the following lemma:

**Lemma 3.9** *Let  $\Omega$  be convex, let  $(t, x) \mapsto \Phi(t, x) : I \times \Omega \mapsto \mathbb{R}$  and  $(t, y) \mapsto \Psi(t, y) : I \times \mathbb{R}^d \mapsto \mathbb{R}$  be such that*

1.  $\nabla \Phi$  (resp.  $\nabla \Psi$ ) belongs to  $C^1(I \times \Omega)$  (resp. belongs to  $C^1(I \times \mathbb{R}^d)$ ),
2. for all  $t \in I$ ,  $\Phi(t, \cdot)$  is convex and  $\Psi(t, \cdot)$  is the Legendre transform of  $\Phi(t, \cdot)$ .

then for every  $(t, x) \in I \times \Omega$ ,

$$(19) \quad \Phi(t, x) + \Psi(t, \nabla \Phi(t, x)) = x \cdot \nabla \Phi(t, x),$$

$$(20) \quad \partial_t \Phi + \partial_t \Psi(\nabla \Phi) = 0,$$

$$(21) \quad \partial_t \nabla \Phi + D^2 \Phi \partial_t \nabla \Psi(\nabla \Phi) = 0.$$

*Proof:* the first identity expresses just the fact that  $\Phi(t, \cdot)$ ,  $\Psi(t, \cdot)$  are Legendre transforms of each other (see (6)), then the two other come by differentiating with respect to time and then to space. □

## 4 Approximation by smooth functions

### 4.1 Construction of smooth solutions.

In this section we build an adequate smooth approximation of the problem. More precisely, given a mapping  $\mathbf{X}(t)$  and  $\rho(t) = \mathbf{X}(t) \# da$ , we construct an associated pair  $(\rho, v)$  satisfying

$$(22) \quad \partial_t \rho + \nabla \cdot (\rho v) = 0$$

and then find a “good” regularization of  $(\rho, v)$ . One of the problems is the following: it is known from a counterexample by Caffarelli (see [8]), that when transporting a (smooth) density  $\rho_1$  onto another (smooth) density  $\rho_2$  by the gradient of a convex function, one can not expect the convex function to be  $C^1$  unless  $\rho_2$  is supported and positive in a convex set. Therefore it is not enough to only regularize (by convolution for example) the density  $\rho = \mathbf{X} \# da$ , we must also approximate it by a density supported in a convex set.

The density  $\rho$  and  $\partial_t \rho$  are constructed from  $\mathbf{X}$ ,  $\partial_t \mathbf{X}$  respectively by the following procedure:

$$\begin{aligned} \forall f \in C_b^1(\mathbb{R}^d), \quad \int_{\mathbb{R}^d} \rho(t, x) f(x) dx &= \int_{\Omega} f(\mathbf{X}(t, a)) da \\ \int_{\mathbb{R}^d} \partial_t \rho(t, x) f(x) dx &= \int_{\Omega} \nabla f(\mathbf{X}(t, a)) \cdot \partial_t \mathbf{X}(t, a) da. \end{aligned}$$

To define  $v$  such that  $\partial_t \rho + \nabla \cdot (\rho v) = 0$ , we define the product  $\rho v$  as follows:

$$\forall \phi \in C_b^0(I \times \mathbb{R}^d; \mathbb{R}^d), \quad \int_{I \times \mathbb{R}^d} \rho v \cdot \phi dt dx = \int_{I \times \Omega} \phi(\mathbf{X}(t, a)) \cdot \partial_t \mathbf{X}(t, a) dt da.$$

Since  $\partial_t \mathbf{X} \in L^\infty$ ,  $v$  is well defined  $d\rho$  a.e. and we have

$$\|v(t, \cdot)\|_{L^\infty(\mathbb{R}^d, d\rho(t))} \leq \|\partial_t \mathbf{X}(t, \cdot)\|_{L^\infty(\Omega)}.$$

Now we construct  $(\rho_n, v_n)$  a smooth approximating sequence for  $(\rho, v)$  as follows: (remember that we have taken  $\rho(t, \cdot)$  to be supported in  $B_R$  at any time  $t \in I$ ). We take  $\eta \in C_c^\infty$  a standard convolution kernel, of integral 1, supported in  $B(0, 1)$  and positive. Take  $\eta_n = n^d \eta(nx)$ . We also note  $\chi_{R+1/n}$  the characteristic function of the ball  $B(0, R + 1/n)$ . Let

$$\begin{aligned}\rho_n &= \left( \frac{1}{n} \chi_{R+1/n} + \eta_n * \rho \right) c_n, \\ v_n &= c_n \frac{\eta_n * (\rho v)}{\rho_n},\end{aligned}$$

with  $c_n$  chosen such that  $\rho_n$  remains a probability measure. (Note that  $c_n$  is close to 1 for  $n$  large). The purpose of this construction is to have the following properties:

1.  $\|\rho_n, v_n\|_{L^\infty} \leq \|\rho, v\|_{L^\infty}$ ,
2.  $\rho_n, v_n$  satisfy the continuity equation (22),
3.  $\rho_n$  is supported and strictly positive in  $B(0, R + 1/n)$ , and belongs to  $C^\infty(\bar{B}(0, R + 1/n))$ .
4. If  $\Phi_n(t), \Psi_n(t)$  are associated to  $\rho_n(t)$  through (5,7), then, for every  $t \in I$ ,  $\Phi_n(t)$  converges uniformly on compact sets of  $\Omega$  to  $\Phi(t)$  and  $\Psi_n(t)$  converges uniformly on compact sets of  $\mathbb{R}^d$  to  $\Psi(t)$ . This last result can be found in [3]. Therefore,  $\partial_t \Phi_n, \partial_t \Psi_n$  will converge in the distribution sense to  $\partial_t \Phi, \partial_t \Psi$ .

Now we have the following regularity result, for smooth densities. Note that this result will only be used to legitimate the forthcoming computations, and not as an a-priori bound.

**Proposition 4.1** *let  $I, \Omega$  be as above, let  $\Omega'$  be  $C^\infty$  strictly convex. For any  $t \in I$ , let  $\rho(t, \cdot)$  be a probability density in  $\bar{\Omega}'$ , strictly positive in  $\bar{\Omega}'$  with  $\rho \in C^\infty(I \times \bar{\Omega}')$ . Let, for all  $t$ ,  $\Phi(t, \cdot), \Psi(t, \cdot)$  be as in (5,7) with  $(\rho(t), \Omega)$ . Then, for any  $0 < \alpha < 1$ ,*

$$\partial_t \Phi \in L^\infty(I, C^{2,\alpha}(\bar{\Omega})), \quad \partial_t \Psi \in L^\infty(I, C^{2,\alpha}(\bar{\Omega}')).$$

*Proof of Proposition 4.1:* Theorem 3.1 implies that for all  $t$ ,  $D^2 \Psi$  (resp.  $D^2 \Phi$ ) belongs to  $C^\infty(\bar{\Omega}')$  (resp. belongs to  $C^\infty(\bar{\Omega})$ ).

Now we wish to solve  $\det D^2 \Psi(t) = \rho(t)$  with  $t$  near  $t_0$ . We write a priori  $\Psi(t) = \Psi(t_0) + (t - t_0)u + o(|t - t_0|)$ , for some  $u$ , then we have

$$\det D^2 \Psi(t) = \det D^2 \Psi(t_0) + (t - t_0) \text{trace}(M D^2 u) + o(|t - t_0|)$$

where  $M$  is the comatrix of  $D^2 \Psi$  defined by

$$M(t, x) = \det D^2 \Psi(t, x) (D^2 \Psi(t, x))^{-1}.$$

Note that  $M$  belongs to  $C^\infty(\bar{\Omega}')$  and is uniformly elliptic. Let us now show that  $\partial_t \Psi$  can indeed be sought as the solution of

$$\text{trace}(MD^2u) = \partial_t \rho$$

with a suitable boundary condition. For this we introduce  $h$  a defining function for  $\Omega$ , (*i.e.*  $h \in C^\infty(\bar{\Omega})$  is strictly convex and vanishes on  $\partial\Omega$ , we can also impose  $|\nabla h|_{\partial\bar{\Omega}} \equiv 1$ ). The condition  $\nabla \Psi$  maps  $\Omega'$  on  $\Omega$  can be replaced by  $h(\nabla \Psi) = 0$  on  $\partial\Omega'$ . Now consider the operator

$$\mathcal{F} : \psi \mapsto (\det D^2\psi, h(\nabla\psi)|_{\partial\Omega'})$$

defined on  $\{\psi \in C^{2,\alpha}(\bar{\Omega}'), \psi \text{ convex}\}$  and ranging in  $C^\alpha(\bar{\Omega}') \times C^{1,\alpha}(\partial\Omega')$ . First note that a smooth solution of

$$(23) \quad \mathcal{F}(\psi) = (\rho(t), 0)$$

will satisfy (7) and thus coincide (up to a constant) with  $\Psi(t)$ . We now solve (23) around  $t_0$  by the implicit function Theorem. The derivative of  $\mathcal{F}$  at  $\Psi$  is defined by

$$d\mathcal{F}(\Psi)u = (I(u), B(u)) = (M_{ij}\partial_{ij}u, h_i(\nabla\Psi)\partial_iu).$$

The operator  $I = M_{ij}\partial_{ij}$  is uniformly elliptic with coefficients  $M_{ij}$  in  $C^\infty(\bar{\Omega}')$ . We need also to show that the boundary operator  $B$  is strictly oblique: First, note that  $\nabla h = \vec{n}_1$  on  $\partial\Omega$ , where  $\vec{n}_1$  is the outer unit normal to  $\partial\Omega$ . Moreover, if  $\vec{n}_2$  is the outer unit normal to  $\partial\Omega'$ , it has been established in [8], [13], [21], that there exists a constant  $C$  depending on  $\Omega, \|\rho\|_{C^2(\bar{\Omega}')}$ , and therefore uniform on  $I$ , such that

$$\vec{n}_2 \cdot \vec{n}_1(\nabla\Psi) \geq C > 0.$$

Thus the boundary condition is strictly oblique, uniformly with respect to  $t$ . It has been established in [13], p. 448, that the equation

$$d\mathcal{F}(\Psi)u = (\mu, 0)$$

with  $\mu \in C^\alpha(\bar{\Omega}')$  is solvable up to an additive constant if  $\int_{\Omega'} \mu = 0$ . This condition is met by  $\partial_t \rho$ , since  $\int \rho(t, x) dx \equiv 1$ .

We conclude that the operator  $d\mathcal{F}(\Psi)$  is invertible on the set

$$\left\{ \left\{ \mu \in C^\alpha(\bar{\Omega}'), \int \mu = 0 \right\} \times \left\{ \nu = 0 \right\} \right\}$$

*i.e.* for each  $\mu \in C^\alpha(\bar{\Omega}')$ , with  $\int_{\Omega'} \mu = 0$ , there exists a unique up to a constant solution  $u$  of  $d\mathcal{F}(\nabla\Psi)u = (\mu, 0)$ . Moreover, following [14], Theorem 6.30,  $u$  belongs to  $C^{2,\alpha}(\bar{\Omega}')$ . Therefore we can apply the implicit function Theorem and solve  $\mathcal{F}(\Psi(t)) = (\rho(t), 0)$  for  $t$  near  $t_0$ . By uniqueness of the solution of (7), this solution will coincide with the solution of Theorem 3.1. As we have built it,  $\partial_t \Psi(t, \cdot) = u$  is the unique (up to a constant) solution of

$$(24) \quad \text{trace}(MD^2u) = \partial_t \rho \text{ in } \Omega',$$

$$(25) \quad \nabla u \cdot \vec{n}_1(\nabla\Psi) = 0 \text{ in } \partial\Omega',$$

and since  $\partial_t \rho \in C^\infty(\bar{\Omega}')$ ,  $\partial_t \Psi$  belongs to  $C^{2,\alpha}(\bar{\Omega}')$  for any  $\alpha < 1$ . We also have, using the identity (20)

$$\partial_t \Phi + \partial_t \Psi(\nabla \Phi) = 0.$$

therefore  $\partial_t \Phi \in C^{2,\alpha}(\bar{\Omega})$  for any  $\alpha < 1$ . This achieves the proof of Proposition 4.1.  $\square$

## 5 Proof of Theorem 2.1

Theorem 2.1 will be deduced through approximation from the following proposition:

**Proposition 5.1** *Let  $\rho$  satisfy the assumptions of Proposition 4.1 above, with  $\Omega' = B_R$ , and  $\Phi, \Psi$  be as in (5, 7). Let  $v(t, x) \in \mathbb{R}^d$  be a smooth vector field on  $\bar{B}_R$  and satisfy on  $I \times \bar{B}_R$*

$$(26) \quad \partial_t \rho + \nabla \cdot (\rho v) = 0.$$

Take  $1 \leq p, r \leq \infty$ ,  $\frac{1}{r} + \frac{1}{r'} = 1$ ,  $q = \frac{2p}{1+p}$ . Then for any  $t \in I$ , for any  $\omega \subset \Omega$  we have:

$$(27) \quad \|\partial_t \nabla \Phi\|_{L^q(\omega)} \leq \left( \|\rho|v|^2\|_{L^{r'}} \|D^2 \Psi\|_{L^r(B_R)} \|D^2 \Phi\|_{L^p(\omega)} \right)^{1/2},$$

which implies in particular

$$(28) \quad \|\partial_t \nabla \Phi\|_{L^1(\Omega)} \leq C(R, d, \Omega) \left( \|\rho|v|^2\|_{L^\infty} \right)^{1/2},$$

and for any  $t \in I$ , for any  $\omega' \subset B_R$  we have:

$$(29) \quad \left[ \int_{\omega'} \rho |\partial_t \nabla \Psi|^q \right]^{1/q} \leq \left( \|\rho|v|^2\|_{L^{r'}} \|D^2 \Psi\|_{L^r(B_R)} \left[ \int_{\omega'} \rho |D^2 \Psi|^p \right]^{1/p} \right)^{1/2},$$

which implies in particular

$$(30) \quad \int_{\mathbb{R}^d} \rho |\partial_t \nabla \Psi| \leq C(R, d, \Omega) \|\rho\|_{L^\infty(\mathbb{R}^d)}^{1/2} \|\rho|v|^2\|_{L^\infty(\mathbb{R}^d)}^{1/2}.$$

*Proof of Proposition 5.1:*

Using Proposition 4.1, we can perform the following computations. We have from (5)

$$\int_{\mathbb{R}^d} \partial_t \Psi \rho = \int_{\Omega} \partial_t \Psi (\nabla \Phi)$$

Then we use the continuity equation:

$$\partial_t \rho + \nabla \cdot (\rho v) = 0$$



which implies for any smooth  $f$

$$\int_{\mathbb{R}^d} f \partial_t \rho = \int_{\mathbb{R}^d} \rho v \cdot \nabla f.$$

We obtain

$$\begin{aligned} \int_{\mathbb{R}^d} \partial_t \Psi \partial_t \rho &= \int_{\mathbb{R}^d} \partial_t \nabla \Psi \cdot \rho v \\ &= \int_{\Omega} \partial_t \nabla \Psi(\nabla \Phi) \cdot \partial_t \nabla \Phi \\ &= - \int_{\Omega} \partial_t \nabla^t \Psi(\nabla \Phi) \cdot D^2 \Phi \cdot \partial_t \nabla \Psi(\nabla \Phi) \end{aligned}$$

where we have used (21). Since we can write  $\sqrt{D^2 \Phi}$  because this is a positive symmetric matrix, we have

$$\begin{aligned} \|\sqrt{D^2 \Phi} \partial_t \nabla \Psi(\nabla \Phi)\|_{L^2(\Omega)}^2 &= - \int_{\mathbb{R}^d} \rho \partial_t \nabla \Psi \cdot v \\ &= - \int_{\Omega} \partial_t \nabla \Psi(\nabla \Phi) \cdot v(\nabla \Phi) \\ &= - \int_{\Omega} \sqrt{D^2 \Phi} \partial_t \nabla \Psi(\nabla \Phi) \cdot \sqrt{D^2 \Phi}^{-1} v(\nabla \Phi). \end{aligned}$$

This implies that

$$(31) \quad \|\sqrt{D^2 \Phi} \partial_t \nabla \Psi(\nabla \Phi)\|_{L^2(\Omega)} \leq \|\sqrt{D^2 \Phi}^{-1} v(\nabla \Phi)\|_{L^2(\Omega)}.$$

In order to estimate the right hand side, we write

$$\begin{aligned} \|\sqrt{D^2 \Phi}^{-1} v(\nabla \Phi)\|_{L^2(\Omega)} &= \left( \int_{\Omega} v^t(\nabla \Phi) \cdot (D^2 \Phi)^{-1} \cdot v(\nabla \Phi) \right)^{1/2} \\ &= \left( \int_{\mathbb{R}^d} \rho v^t \cdot (D^2 \Phi(\nabla \Psi))^{-1} \cdot v \right)^{1/2} \\ &= \left( \int_{\mathbb{R}^d} \rho v^t \cdot D^2 \Psi \cdot v \right)^{1/2} \\ (32) \quad &\leq \left( \|D^2 \Psi\|_{L^r(B_R)} \|\rho v^2\|_{L^{r'}(B_R)} \right)^{1/2}. \end{aligned}$$

In the second line we have used  $D^2 \Phi(\nabla \Psi) = (D^2 \Psi)^{-1}$ . From (21),

$$\begin{aligned} \|\sqrt{D^2 \Phi}^{-1} \partial_t \nabla \Phi\|_{L^2(\Omega)} &= \|\sqrt{D^2 \Phi} \partial_t \nabla \Psi(\nabla \Phi)\|_{L^2(\Omega)} \\ &\leq \|\sqrt{D^2 \Phi}^{-1} v(\nabla \Phi)\|_{L^2(\Omega)}. \end{aligned}$$

Writing

$$\partial_t \nabla \Phi = \sqrt{D^2 \Phi}^{-1} \sqrt{D^2 \Phi} \partial_t \nabla \Phi,$$

and, using Hölder's inequality, we obtain for  $\omega \subset \Omega$

$$\begin{aligned} \|\partial_t \nabla \Phi\|_{L^q(\omega)} &\leq \|\sqrt{D^2 \Phi}^{-1} \partial_t \nabla \Phi\|_{L^2(\omega)} \|\sqrt{D^2 \Phi}\|_{L^s(\omega)} \\ &\leq \left( \|\rho|v|^2\|_{L^r(B_R)} \|D^2 \Psi\|_{L^{r'}(B_R)} \|D^2 \Phi\|_{L^{s/2}(\omega)} \right)^{1/2} \end{aligned}$$

with  $q = \frac{2s}{2+s}$ . By taking  $p := s/2$  we have

$$\|\partial_t \nabla \Phi\|_{L^q(\omega)} \leq \left( \|\rho|v|^2\|_{L^r(B_R)} \|D^2 \Psi\|_{L^{r'}(B_R)} \|D^2 \Phi\|_{L^p(\omega)} \right)^{1/2}$$

and  $q = \frac{2p}{1+p}$ . This proves (27). To obtain a bound on  $\partial_t \Psi$  we write

$$\begin{aligned} \int_{\mathbb{R}^d} \rho \left| \sqrt{D^2 \Phi}(\nabla \Psi) \partial_t \nabla \Psi \right|^2 &= \int_{\mathbb{R}^d} \rho \partial_t \nabla^t \Psi \cdot D^2 \Phi(\nabla \Psi) \cdot \partial_t \nabla \Psi \\ &= \int_{\Omega} \partial_t \nabla^t \Psi(\nabla \Phi) \cdot D^2 \Phi \cdot \partial_t \nabla \Psi(\nabla \Phi) \\ &\leq \|D^2 \Psi\|_{L^r(B_R)} \|\rho|v|^2\|_{L^{r'}(B_R)} \end{aligned}$$

from (31) and (32). Then using Hölder's inequality, with  $q = \frac{2s}{2+s}$ , we obtain for  $\omega' \subset B_R$ ,

$$\begin{aligned} \left[ \int_{\omega'} \rho |\partial_t \nabla \Psi|^q \right]^{1/q} &\leq \\ &\left[ \int_{\omega'} \rho \left| \sqrt{D^2 \Phi}(\nabla \Psi) \partial_t \nabla \Psi \right|^2 \right]^{1/2} \left[ \int_{\omega'} \rho |[D^2 \Phi(\nabla \Psi)]^{-1}|^{s/2} \right]^{1/s}. \end{aligned}$$

The first factor of the right hand product has been estimated above, and the second is equal to  $\left( \int \rho |D^2 \Psi|^{s/2} \right)^{1/s}$ . We conclude that

$$\left[ \int_{\omega'} \rho |\partial_t \nabla \Psi|^q \right]^{1/q} \leq \left[ \|D^2 \Psi\|_{L^r(B_R)} \|\rho|v|^2\|_{L^{r'}(B_R)} \right]^{1/2} \left[ \int_{\omega'} \rho |D^2 \Psi|^{s/2} \right]^{1/s}.$$

Taking again  $p := s/2$ , we have proved (29).

The bounds (28, 30) are obtained as follows: we know from Lemma 3.8 that

$$\begin{aligned} \|D^2 \Psi\|_{L^1(B_R)} &\leq C(R, d, \Omega), \\ \|D^2 \Phi\|_{L^1(\Omega)} &\leq C(R, d, \Omega). \end{aligned}$$

Taking in (27, 29)  $r = +\infty, r' = 1, p = 1$  we obtain the desired bounds. This ends the proof of Proposition 5.1. □

## 5.1 Proof of Theorem 2.1

### Proof of the bound on $\partial_t \nabla \Phi$

Here we prove points 1,2,4,5 of Theorem 2.1. To obtain point 1, we just need to pass to the limit in the estimate (28). We need to have  $\liminf \|\rho_n |v_n|^2\|_{L^\infty} \leq \|\rho|v|^2\|_{L^\infty}$ : to prove

this, notice that  $F(\rho, v) = \rho|v|^2/2 = \frac{(\rho|v|)^2}{2\rho}$  is a convex functional in  $(\rho v, \rho)$  since it is expressed as:

$$\frac{(\rho|v|)^2}{2\rho} = \sup_{c+|m|^2/2 \leq 0} \{\rho c + \rho v \cdot m\}.$$

Then since  $\rho_n v_n = c_n \eta_n * (\rho v)$ ,  $\rho_n = c_n(\frac{1}{n} + \eta_n * \rho)$  we get that

$$F(\rho_n, \rho_n v_n) \leq c_n \eta_n * F(\rho, \rho v) \leq c_n \left\| \rho \frac{|v|^2}{2} \right\|_{L^\infty}$$

and letting  $n \rightarrow \infty$ :

$$\begin{aligned} \|\partial_t \nabla \Phi\|_{\mathcal{M}(\Omega)} &\leq (\|\rho|v|^2\|_{L^\infty})^{\frac{1}{2}} C(R, d, \Omega) \\ &\leq \|\rho\|_{L^\infty(B_R)}^{\frac{1}{2}} \|v\|_{L^\infty(B_{r,d\rho})} C(R, d, \Omega). \end{aligned}$$

Since we impose  $\int_{\Omega} \Phi(t, x) dx \equiv 0$ , and since  $\Omega$  is convex, (note that since  $\partial_t \Phi_n \notin W_0^{1,1}$ , a condition of this type is necessary, see [14], chap. 7) by Sobolev imbeddings we get also a bound on  $\|\partial_t \Phi_n\|_{L^{1^*}(\Omega)}$ . This proves the first point of Theorem 2.1.

Then we obtain points 2,4,5 by the following interpolation lemma:

**Lemma 5.2** *Let  $\Phi_1$  and  $\Phi_2$  be two  $R$ -Lipschitz convex functions on  $\Omega$  convex. Then 1- there exists  $C, \beta > 0$  depending on  $(\Omega, R, d, p)$  such that*

$$\|\Phi_1 - \Phi_2\|_{L^\infty(\Omega)} \leq C \|\Phi_1 - \Phi_2\|_{L^p(\Omega)}^\beta.$$

2- If moreover  $\Phi_1 \in C^{1,\alpha}$  for some  $0 < \alpha < 1$  then there exists  $C', \beta' > 0$  depending also on  $\alpha, \|\Phi_1\|_{C^{1,\alpha}}$ , such that, if  $\Omega_\delta = \{x \in \Omega, d(x, \partial\Omega) \geq \delta\}$ , with  $\delta$  going to 0 with  $\|\Phi_1 - \Phi_2\|_{L^p(\Omega)}$ , then

$$\|\nabla \Phi_1 - \nabla \Phi_2\|_{L^\infty(\Omega_\delta)} \leq C' \|\Phi_1 - \Phi_2\|_{L^p(\Omega)}^{\beta'}.$$

*Proof:* Suppose that  $\int_{\Omega} |\Phi_1 - \Phi_2|^p \leq \epsilon^p$ . Choose a point inside  $\Omega$  (say 0) such that  $|\Phi_1(0) - \Phi_2(0)| = M$ .  $\Phi_1$  and  $\Phi_2$  are globally Lipschitz with Lipschitz constant bounded by  $R$ . On  $B_{M/2R}(x) \cap \Omega$  we have  $|\Phi_1 - \Phi_2|(x) \geq M/2$  and thus

$$\int_{B_r} |\Phi_1 - \Phi_2|^p \geq \text{vol}(\Omega \cap B_{M/2R}(x)) (M/2)^p.$$

Next note that for  $\Omega$  convex,  $M$  small enough, for any  $x \in \Omega$ ,  $\text{vol}(\Omega \cap B_{M/2R}(x)) \geq C_\Omega \text{vol}(B_{M/2R}(x))$ . Finally we have

$$(33) \quad \epsilon^p \geq \int_{\Omega} |\Phi_1 - \Phi_2|^p \geq C(\Omega, R, d) M^{p+d},$$

and thus

$$M \leq C'(\Omega, R, d) \left[ \int_{B_r} |\Phi_1 - \Phi_2|^p \right]^{\frac{1}{p} \frac{p}{p+d}},$$

which gives the first part of the lemma, with  $\beta = \frac{p}{p+d}$ .

Now suppose that  $|\nabla \Phi_1(0) - \nabla \Phi_2(0)| = M$ . One can also set  $\Phi_1(0) = 0, \nabla \Phi_1(0) = 0$ . We know that  $\Phi_1$  is  $C^{1,\alpha}$  thus  $\Phi_1(x) \leq C|x|^{1+\alpha}$ . It follows that going in the direction of  $\nabla \Phi_2$  one will have

$$\Phi_2(x) - \Phi_1(x) \geq M|x| - C|x|^{1+\alpha} + \Phi_2(0).$$

Keeping in mind that  $|\Phi_1(x) - \Phi_2(x)| \leq C\epsilon^\beta$  yields  $M|x| - C|x|^{1+\alpha} \leq C\epsilon^\beta$ . The maximum of the left hand side is attained for  $|x| = \left(\frac{M}{(1+\alpha)C}\right)^{1/\alpha}$ , and is equal to  $\left(\frac{M}{(1+\alpha)C}\right)^{1/\alpha} \frac{\alpha}{1+\alpha} M$ . Therefore we have

$$M \leq C\epsilon^{\beta'}$$

in  $\Omega_\delta$  with  $\delta = \delta(\epsilon)$  going to 0 as  $\epsilon$  goes to 0 and with  $\beta' = \frac{\alpha\beta}{1+\alpha}$ .  $\square$

*Remark:* Suppose, as it is the case for  $\Psi$ , that we only know that  $\int \rho |\Psi_1 - \Psi_2|^p \leq \epsilon^p$ , then we have instead of (33),

$$\epsilon^p \geq \int_{B_r} \rho |\Psi_1 - \Psi_2|^p \geq \rho(B_{M/2R}(x)) M^{p+d}.$$

The first part of the lemma yields immediately that  $\Phi \in C^\alpha(I, C^0(\Omega))$  for some  $\alpha > 0$ . Moreover if  $\phi_1^*, \phi_2^*$  are the Legendre transform of  $\phi_1, \phi_2$ , then  $\|\phi_1^* - \phi_2^*\|_{L^\infty} \leq \|\phi_1 - \phi_2\|_{L^\infty}$ , thus  $\Psi \in C^\alpha(I, C^0(B_R))$ , and this gives the point 2.

The second point of the lemma will be used to prove point 4 and 5: Indeed, if  $\rho$  supported in  $\bar{\Omega}'$  for some open set  $\Omega'$ , and there exists  $0 < \lambda, \Lambda$  such that  $\lambda \leq \rho \leq \Lambda$  in  $\Omega'$ , from Theorem 3.2 we get that for any  $\omega' \subset \subset \Omega'$ ,  $\Psi(t, \cdot) \in C^{1,\alpha_1}(\omega')$  for some  $\alpha_1 > 0$ . Since  $\partial_t \Phi \in L^{1*}(\Omega)$ , using (21) we get that

$$\int \rho_n |\partial_t \Psi_n|^{1*} \leq C$$

uniformly in  $n$ , and thus that

$$\partial_t \Psi_n \in L^\infty(I, L^{1*}(\Omega')).$$

Therefore we can use Lemma 5.2 to obtain that for any  $\omega' \subset \subset \Omega'$ ,  $\nabla \Psi \in C^\beta(I, C^0(\omega'))$  (point 4 of Theorem 2.1).

Under the additional assumption that  $\Omega'$  is convex, Theorem 3.2 yields that  $\Phi(t, \cdot)$  in  $C^{1,\alpha_2}(\Omega)$  for some  $\alpha_2 > 0$ . The same procedure as above yields point 5.

Now we prove the point 3 of Theorem 2.1:

**Proof of the bound on  $\partial_t g$** 

Recall from Theorem 2.2:

$$\int_{\mathbb{R}^d} \rho_n |\partial_t \nabla \Psi_n| \leq C(d, R) \|\rho_n\|_{L^\infty(B_R)}^{\frac{1}{2}} \|\rho_n v_n^2\|_{L^\infty(B_R)}^{\frac{1}{2}}$$

We have  $g(t, a) = \nabla \Psi(t, \mathbf{X}(t, a))$  and thus formally

$$\partial_t g(t, a) = \partial_t \nabla \Psi(t, \mathbf{X}(t, a)) + D^2 \Psi(t, \mathbf{X}(t, a)) \partial_t \mathbf{X}(t, a).$$

Since  $\rho_n$  converges strongly (actually weakly would be enough) to  $\rho$ , we know that  $\nabla \Psi_n$  converges almost everywhere to  $\nabla \Psi$ . (See [3] for a proof of this fact, which relies on the convexity of  $\Psi_n$  and on the uniqueness of the polar factorization). Now consider

$$g_n(t, a) = \int_{\mathbb{R}^d} \nabla \Psi_n(t, y) \eta_n(y - \mathbf{X}(t, a)) dy = (\eta_n * \nabla \Psi_n)(t, \mathbf{X}(t, a))$$

with  $\eta_n$  a smoothing kernel as above. Then  $g_n$  converges almost everywhere to  $g$ . For  $f \in C^0(I \times \Omega, \mathbb{R}^d)$ , let us compute

$$\int_I \int_{\Omega} \partial_t g_n(t, a) \cdot f(t, a) dt da = T_1 + T_2,$$

with

$$\begin{aligned} T_1 &= \int_I \int_{\Omega} \int_{\mathbb{R}^d} \eta_n(y - \mathbf{X}(t, a)) \partial_t \nabla \Psi_n(t, y) \cdot f(t, a) dy dadt \\ T_2 &= - \int_I \int_{\Omega} \int_{\mathbb{R}^d} \nabla \Psi_n(t, y) \cdot f(t, a) \partial_t \mathbf{X}(T, a) \cdot \nabla \eta_n(y - \mathbf{X}(t, a)) dy dadt \end{aligned}$$

Let us evaluate  $T_1$  and  $T_2$ .

$$\begin{aligned} |T_1| &\leq \int_I \|f(t, \cdot)\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho(x) \eta_n(y - x) |\partial_t \nabla \Psi_n(t, y)| dx dy dt \\ &\leq \int_I \|f(t, \cdot)\|_{L^\infty(\Omega)} d_n \int_{\mathbb{R}^d \times \mathbb{R}^d} \rho_n(y) |\partial_t \nabla \Psi_n(t, y)| dx dy \\ &\leq \int_I \|f(t, \cdot)\|_{L^\infty(\Omega)} C(R, d) \|\rho_n\|_{L^\infty(I \times \mathbb{R}^d)} \|v_n\|_{L^\infty(I \times \mathbb{R}^d)} \end{aligned}$$

with  $d_n = 1/c_n$  and from Theorem 2.2. For  $T_2$  we have:

$$\begin{aligned} |T_2| &= \left| \int_I \int_{\Omega} \int_{\mathbb{R}^d} \nabla \Psi_n(t, y) \cdot f(t, a) \partial_t \mathbf{X}(T, a) \cdot \nabla \eta_n(y - \mathbf{X}(t, a)) dy dadt \right| \\ &= \left| \int_I \int_{\Omega} \int_{\mathbb{R}^d} \partial_t \mathbf{X}^t(T, a) \cdot (D^2 \Psi_n * \eta_n)(t, \mathbf{X}(t, a)) \cdot f(t, a) dy dadt \right| \\ &\leq \int_I \|f(t, \cdot) \partial_t \mathbf{X}(t, \cdot)\|_{L^\infty(\Omega)} \int_{\mathbb{R}^d} \rho(t, x) (|D^2 \Psi_n| * \eta_n)(x) dx dt \\ &\leq \int_I \|f(t, \cdot) \partial_t \mathbf{X}(t, \cdot)\|_{L^\infty(\Omega)} \|\rho(t, \cdot)\|_{L^\infty(\mathbb{R}^d)} C(R, d, \Omega) dt \end{aligned}$$

where we have used the bound on  $\|D^2 \Psi\|_{L^1_{loc}}$  (Lemma 3.8); we conclude that

$$\|\partial_t g\|_{L^\infty(I, \mathcal{M}(\Omega))} \leq C(R, d, \Omega) \|\rho\|_{L^\infty(I \times B_R)} \|\partial_t \mathbf{X}\|_{L^\infty(I \times \Omega)}.$$

This achieves the proof of Theorem 2.1.  $\square$

## 6 Proof of Theorem 2.2

### 6.1 Hölder regularity

It has been established ((13) and Theorem 3.1) that  $\partial_t \Psi_n$  satisfies

$$\nabla \cdot (M_n \partial_t \nabla \Psi_n) = \sum_{i,j} M_{n,ij} \partial_{ij} \partial_t \Psi_n = -\partial_t \rho_n = -\nabla \cdot (\rho_n v_n)$$

where  $M_n$  is the comatrix of  $D^2 \Psi_n$ . To establish the Hölder regularity of  $\partial_t \Psi_n$  we need to combine three preliminary results:

The first one (Theorem 3.4) asserts the Harnack inequality for solutions of the homogeneous linearized Monge-Ampère equation under a condition which is satisfied when the density  $\rho$  is between two positive constants.

The second one (Theorem 3.5, Theorem 3.7 and Corollary 3.6) is a local maximum principle that generalizes the local maximum principle for uniformly elliptic equations, to degenerate elliptic equations of the form  $\nabla \cdot (M \nabla u) = \nabla \cdot f$ . The uniform ellipticity is relaxed to the condition that the (positive symmetric matrix valued) functions  $M, M^{-1}$  belong to  $L^p$  for  $p$  large enough.  $p$  depends only on the dimension  $d$ .

The third one (Theorem 3.3) asserts that the comatrix of  $D^2 \Psi$ , and its inverse, are indeed in  $L^p_{loc}(\Omega')$  provided that the density  $\rho$  is close enough to a continuous positive function, the closeness being measured in  $L^\infty$  norm.

The result will be a consequence of the following propositions:

**Proposition 6.1** *Let  $\rho = \mathbf{X} \# da$  be supported in  $\bar{\Omega}'$ ,  $\lambda$  and  $\Lambda$  be two positive constants such that  $0 < \lambda \leq \rho(t, x) \leq \Lambda$  for all  $(t, x) \in I \times \Omega'$ . Let  $\rho_n, v_n$  be constructed from  $\mathbf{X}$  as above. Let  $(\Phi_n, \Psi_n)$  be associated to  $(\rho_n, \Omega)$  through (5, 7). Then there exists  $\beta < 1$ , and for any  $p > d$ , there exists  $C$  such that for any  $\omega' \subset\subset \Omega'$ , for any  $(y, r)$  with  $B_{4r}(y) \subset \omega'$ ,*

$$\text{osc}_{\partial_t \Psi_n}(r/2) \leq \beta \text{osc}_{\partial_t \Psi_n}(r) + Cr^\delta$$

for  $n$  large enough.  $\beta < 1$  depends on  $(\lambda, \Lambda)$  (see Theorem 3.4),  $C$  depend on  $(p, \lambda, \Lambda, \inf_{x \in \omega'} d(x, \partial\Omega'), \|D^2 \Psi\|_{L^p(B_r(y))})$ ,  $\delta = 1 - \frac{d}{p}$ , and

$$\text{osc}_u(r) = \max_{B_r} u - \min_{B_r} u.$$

*Remark:* The requirement  $n$  large enough is just to enforce that  $\lambda \leq \rho_n \leq \Lambda$ .

**Proposition 6.2** *Under the assumptions of Proposition 6.1, we have, for every  $\omega' \subset\subset \Omega'$ ,*

$$\|\partial_t \Psi_n\|_{L^\infty(\omega')} \leq C(K, p, \inf_{x \in \omega'} d(x, \partial\Omega'), \lambda, \Lambda, \|v_n\|_{L^\infty(d\rho(t))})$$

where  $K = \|D^2 \Psi_n + D^2 \Psi_n^{-1}\|_{L^p(\omega')}$ ,  $p > d$ .

**Proposition 6.3** *Under the assumptions of Proposition 6.1, for any  $p < \infty$ , there exists  $\epsilon > 0$  such that if  $|\rho - 1| \leq \epsilon$  in  $\Omega'$ , then for every  $K' \subset \Omega'$ ,  $K'$  compact, there exists  $C_{K'}$  such that*

$$\limsup_n \|D^2\Psi_n + D^2\Psi_n^{-1}\|_{L^p(K')} \leq C_{K'}.$$

Temporarily admitting these propositions we obtain the following:

### Proof of the first part of Theorem 2.2

From Propositions 6.1, 6.2, 6.3, we obtain that for any  $\omega' \subset \subset \Omega'$ , there exists  $C_{\omega'}$ ,  $\beta < 1$  independent of  $n$  such that, for  $n$  large enough, for any  $B_r = B_r(y) \subset \omega'$ , with  $B_{4r} \subset \Omega'$ , we have:

$$\text{osc}_{\partial_t\Psi_n}(r/2) \leq \beta \text{osc}_{\partial_t\Psi_n}(r) + C_{\omega'} r^\delta.$$

Moreover from Proposition 6.2,  $\partial_t\Psi_n$  is uniformly bounded for the sup norm inside  $\omega'$ . It is well known that this property implies Hölder continuity: using [14], Lemma 8.23, we obtain that for  $n$  large enough, for any  $\omega' \subset \subset \Omega'$ , there exists  $\alpha > 0$ ,  $C_{\omega'}$  that do not depend on  $n$ , such that for any  $(x, y) \in \omega'$ ,

$$|\partial_t\Psi_n(y) - \partial_t\Psi_n(x)| \leq C_{\omega'} |x - y|^\alpha.$$

Thus we have a uniform  $L^\infty(I; C^\alpha(\omega'))$  bound that will pass to the limit as  $n \rightarrow \infty$ . We thus obtain the  $C^\alpha$  estimate of Theorem 2.2.

To obtain Hölder continuity for  $\partial_t\Phi$ , in the case where  $\Omega'$  is convex, we just have to use the identity (19)

$$\partial_t\Phi = -\partial_t\Psi(\nabla\Phi)$$

and the Hölder regularity of  $\nabla\Phi$ , under the condition  $0 < \lambda \leq \rho \leq \Lambda$ ,  $\Omega'$  convex (Theorem 3.2), to conclude Hölder regularity for  $\partial_t\Phi$ . □

In the next proofs we drop the suffix  $n$  for simplicity.

*Proof of Proposition 6.2:* This proposition is a direct consequence of Theorem 3.7. It has been established that  $\partial_t\Psi$  satisfies

$$\nabla \cdot (M \partial_t \nabla \Psi) = \sum_{i,j} M_{ij} \partial_{ij} \partial_t \Psi = -\partial_t \rho = -\nabla \cdot (\rho v)$$

where  $M$  is the comatrix of  $D^2\Psi$ , given by  $M = \det D^2\Psi [D^2\Psi]^{-1}$  or  $M = \rho D^2\Phi(\nabla\Psi)$ . We remember that  $0 < \lambda \leq \rho \leq \Lambda$ . From Theorem 2.1, we have the a priori bound

$$\int_{\Omega} |\partial_t\Phi|^{1*} \leq C(\|\rho_n |v_n|^2\|_{L^\infty}, \Omega, R, d).$$

Using then that  $\partial_t \Psi = -\partial_t \Psi(\nabla \Phi)$  we have

$$\int \rho |\partial_t \Psi|^{1*} = \int_{\Omega} |\partial_t \Phi|^{1*}$$

and thus

$$\int_{\Omega'} |\partial_t \Psi|^{1*} \leq \frac{C}{\lambda}.$$

We can therefore apply Theorem 3.7 with  $a_0 = 1*$ .

□

*Proof of Proposition 6.1:*

We consider a ball  $B_{4r}(y)$  contained in  $\Omega$  and write  $\partial_t \Psi = u + w$  where  $u$  satisfies

$$\begin{aligned} \nabla \cdot (M \nabla u) &= -\nabla \cdot (\rho v), \\ u &= 0 \text{ on } \partial B_r(y), \end{aligned}$$

and  $w$  satisfies

$$\begin{aligned} \nabla \cdot (M \nabla w) &= 0 \\ w &= \partial_t \Psi \text{ on } \partial B_r(y). \end{aligned}$$

Note that  $w$  satisfies also  $M_{ij} \partial_{ij} w = 0$  which is the equation treated in [9].

We denote  $osc_f(r) = \sup_{B_r} f - \inf_{B_r} f$  and  $osc_f(\partial B_r) = \sup_{\partial B_r} f - \inf_{\partial B_r} f$ .

The assumptions of Theorem 3.4 are satisfied: indeed, in  $\omega' \subset \subset \Omega'$ , we have, for  $n$  large enough,  $\lambda \leq \rho_n \leq \Lambda$ . From Theorem 3.4, there exists  $\beta < 1$  such that

$$osc_w(r/2) \leq \beta osc_w(r).$$

From Corollary 3.6 we have

$$\sup_{B_r} |u| \leq C \|\rho v\|_{L^\infty} r^\alpha,$$

where  $\alpha = 1 - d/p$ ,  $C = C_0 \|M^{-1}\|_{L^p} = C_0 \|\rho^{-1} D^2 \Psi\|_{L^p(B_r)}$  (note that we have  $0 < \lambda \leq \rho \leq \Lambda$ ). Combining the two estimates, we have

$$\begin{aligned} osc_{\partial_t \Psi}(r/2) &\leq osc_w(r/2) + osc_u(r/2) \\ &\leq \beta osc_w(r) + Cr^\alpha \\ &\leq \beta osc_w(\partial B_r) + Cr^\alpha \\ &\leq \beta osc_{\partial_t \Psi}(\partial B_r) + Cr^\alpha \\ &\leq \beta osc_{\partial_t \Psi}(r) + Cr^\alpha \end{aligned}$$

where in the third line we have used the maximum principle to say that  $osc_w(r) = osc_w(\partial B_r)$  since  $w$  can not have interior extrema. Finally we conclude

$$osc_{\partial_t \Psi}(r/2) \leq \beta osc_{\partial_t \Psi}(r) + Cr^\alpha.$$



This achieves the proof of Proposition 6.1.  $\square$

*Proof of Proposition 6.3* We show here how to use the  $W^{2,p}$  regularity Theorem 3.3 to obtain estimates. First let us notice that if  $\nabla\Psi$  satisfies (7) for  $\rho$  supported in  $\bar{\Omega}'$ ,  $0 < \lambda \leq \rho \leq \Lambda$ , and since  $\Omega$  is convex, we know from [8] that  $\Psi$  is strictly convex in  $\Omega'$  and solution in the viscosity sense to

$$\det D^2\Psi = \rho$$

in  $\Omega'$ . Moreover  $\Psi$  is  $C_{loc}^{1,\alpha}$  in  $\Omega$  (Theorem 3.2). From the strict convexity, for any  $x \in \Omega'$ , there exists a section

$$S_{t_x,x} = \{y : \Psi(y) \leq \Psi(x) + \nabla\Psi(x) \cdot (y - x) + t_x\}$$

with non-empty interior and compactly contained in  $\Omega'$ . (Indeed the strict convexity means that diameter of the sections decreases to 0 as the height of the section  $t_x$  goes to 0). Then for every compact set  $K$  contained in  $\Omega'$  there exists a finite covering of  $K$  by sets  $\frac{1}{3d}S_i$ ,  $S_i = S_{t_{x_i},x_i}$ , and  $\frac{1}{3d}S_i$  means a contraction of  $S_i$  with respect to  $x_i$ . Then the functions  $u_i(y) = \Psi(y) - t_i - \nabla\Psi(x_i) \cdot (y - x_i)$  are solutions of

$$\begin{aligned} \det D^2u_i &= \rho \text{ in } S_i \\ u_i &= 0 \text{ on } \partial S_i. \end{aligned}$$

From John's lemma (see [5]), we can find an affine transformation  $T_i$ , with  $\det T_i = 1$  and a real number  $\mu_i$  such that  $B_1 \subset \mu_i^{-1}T_i^{-1}(S_i) = \tilde{S}_i \subset dB_1$ . Finally, considering  $\tilde{u}_i(y) = \frac{1}{\mu_i}u_i(\mu_i T_i y)$  we get that  $\tilde{u}_i$  is solution to

$$\begin{aligned} \det D^2\tilde{u}_i(y) &= \tilde{\rho}(y) = \rho(\mu_i T_i y) \text{ in } \tilde{S}_i \\ \tilde{u}_i &= 0 \text{ on } \partial\tilde{S}_i \\ B_1(x_i) &\subset \tilde{S}_i \subset dB_1(x_i). \end{aligned}$$

We can invoke Theorem 3.3 for  $\tilde{u}_i$ : For any  $0 < p < \infty$ , if  $|\tilde{\rho} - 1| \leq \epsilon(p)$  (this property is invariant under the renormalizations performed above), we have

$$\begin{aligned} \|D^2\tilde{u}_i + D^2\tilde{u}_i^{-1}\|_{L^p(B_{\frac{1}{2}})} &\leq C \\ (\text{meas}(S_i))^{-1/p} \|D^2u_i + D^2u_i^{-1}\|_{L^p(\frac{1}{2d}S_i)} &\leq C\|T_i\|^2. \end{aligned}$$

By our covering process, we have  $K \subset \bigcup_i T_i \mu_i B_{\frac{1}{3}}(x_i)$ . It follows that for every compact set  $K \subset \Omega'$ , there exists a constant  $C_K$  such that  $\|D^2\Psi\|_{L^p(K)} \leq C_K$  and  $\|D^2\Psi^{-1}\|_{L^p(K)} \leq C_K$ . The constant  $C_K$  depends on the supremum of the norm of the transformations  $T_i$  and can be taken (by compactness) uniformly bounded given  $\Omega, \Omega', K, \lambda, \Lambda$ .

Now we show that this covering process behaves uniformly well when we consider the regularization  $\rho_n$  of  $\rho$  and let  $n$  go to  $\infty$ . Indeed the corresponding  $\Psi_n$  will converge uniformly to  $\Psi$  and since the limit  $\Psi_n$  is  $C^1$  the sequence  $\nabla\Psi_n$  converges also uniformly in every compact set of  $\Omega'$ . Therefore the set  $S_i^n = \{y, \Psi_n(y) \leq \Psi_n(x_i) + \nabla\Psi_n(x_i) \cdot (y - x_i) + t_i\}$  converge uniformly to  $S_i$ . This means that for  $n$  large enough, the set  $K$  will be

covered by  $\bigcup_i \frac{1}{2d} S_i^n$ . Consider  $\mu_i^n, T_i^n$  the corresponding normalization. then we also have  $T_i^n, \mu_i^n$  converging to  $T_i, \mu_i$ , and  $K$  will be covered by  $\bigcup_i T_i^n \mu_i^n B_{\frac{1}{2}}(x_i)$ .

Moreover since we consider a compact set  $K$  contained in  $\Omega'$  and since  $|\rho - 1| \leq \epsilon$  in  $\Omega'$ , it follows from the construction of  $\rho_n$  that, for  $n$  large enough,  $|\rho_n - 1| \leq \epsilon$  in  $\Omega'$ . For  $n$  large enough, the functions  $\tilde{u}_i^n$  (obtained by the renormalization procedure) will thus all satisfy the assumptions of Theorem 3.3.

Therefore, for every  $K \subset\subset \Omega'$ , there exists  $C_K$  independent of  $n$  such that, for  $n$  large enough,

$$\|D^2\Psi_n + D^2\Psi_n^{-1}\|_{L^p(K)} \leq C_K.$$

This achieves the proof of Proposition 6.3. □

### Proof of the gradient bounds

This is point 2 of Theorem 2.2. The gradient bounds follow directly from Proposition 5.1 combined with Proposition 6.3. In estimates (27, 29) take  $r = \infty$ . Note that from Lemma 3.8 we have the bound  $\|D^2\Psi\|_{L^1(B_R)} \leq C(R, d, \Omega)$ . This ends the proof of Theorem 2.2. □

## 7 The periodic case: proof of Theorem 2.3

This result is only an adaptation of the two previous Theorems. All the regularity results used adapt to the periodic case as follows:

**Theorem 7.1** *Let  $\rho$  be a Lebesgue integrable probability measure on  $\mathbb{R}^d/\mathbb{Z}^d$ . There exists a unique  $\Psi$  convex on  $\mathbb{R}^d$ , with  $\Psi - |x|^2/2$  periodic, that satisfies*

$$\det D^2\Psi = \rho$$

*in the following sense:*

$$\forall f \in C^0(\mathbb{R}^d/\mathbb{Z}^d), \int_{\mathbb{T}^d} \rho f(\nabla\Psi) = \int_{\mathbb{T}^d} f.$$

*It has the following regularity properties:*

1. *If for some pair  $(\lambda, \Lambda) \in \mathbb{R}_+^*$ , we have  $\lambda \leq \rho \leq \Lambda$ , then for some  $\alpha > 0$  depending on  $\Lambda/\lambda$ ,  $\Psi - |x|^2/2$  is in  $C^{1,\alpha}(\mathbb{T}^d)$ .*
2. *For every  $p < \infty$ , there exists  $\epsilon(p)$  such that if  $|\rho - 1| \leq \epsilon(p)$ , then  $\Psi - |x|^2/2 \in W^{2,p}(\mathbb{T}^d)$ .*
3. *If  $\rho$  is positive and in  $C^\infty(\mathbb{T}^d)$ , then  $\Psi - |x|^2/2 \in C^\infty(\mathbb{T}^d)$ .*

We then modify the approximation procedure as follows: we take

$$\begin{aligned} \rho_n &= c_n(\eta_n * \rho + \frac{1}{n}) \\ \det D^2\Psi_n &= \rho_n \end{aligned}$$

with the constant  $c_n$  such that  $\int_{\mathbb{T}^d} \rho_n = 1$ . Then we use the same techniques as in the Theorems 2.1, 2.2.

We only mention the two new results that arise in this case:

In point 2, we obtain that  $\mathbf{g} \in C^\alpha(I, L^\infty(\mathbb{T}^d))$ . Indeed,  $\mathbf{g} = \nabla \Psi(t, \mathbf{X}(t))$ . We already know that, under the present assumptions,  $\nabla \Psi \in C^\alpha(I \times \mathbb{T}^d)$ , moreover  $\mathbf{X} \in W^{1,\infty}(I, L^\infty(\mathbb{T}^d))$  and the result follows.

In point 4, under the assumption that  $\|\rho - 1\|_{L^\infty(I \times \mathbb{T}^d)} \leq \epsilon$  for  $\epsilon$  small enough depending on  $q$ , we are able to obtain a bound in  $L^q(\mathbb{T}^d)$ ,  $q < 2$  for  $\partial_t \mathbf{g}$ . Indeed, writing

$$\mathbf{g}_n(t, a) = \nabla \Psi_n(t, X(t, a))$$

as in the proof of Theorem 2.1, and differentiating with respect to time, we obtain

$$\partial_t \mathbf{g}_n(t, a) = \partial_t \nabla \Psi_n(t, \mathbf{X}(t, a)) + D^2 \Psi_n(t, \mathbf{X}(t, a)) \partial_t \mathbf{X}(t, a).$$

with  $\Psi_n$  obtained from  $\rho_n$ , and thus in  $C^\infty(I \times \mathbb{T}^d)$ . If  $\rho$  is close enough to 1 so that  $D^2 \Psi_n$  is bounded in  $L^p(\mathbb{T}^d)$  (cf. Theorem 7.1 above), the first term is bounded in  $L^q(\mathbb{T}^d)$ , with  $q = \frac{2p}{1+p}$  (as in Proposition 4.1). The second term is bounded in  $L^p(\mathbb{T}^d)$ . Then we let  $\mathbf{g}_n$  converge to  $\mathbf{g}$ .

Note that this bound can not be obtained in the non periodic case since we have only interior regularity available for  $\Psi$ .

## 8 Counter-examples

Here we show through some examples that the bounds obtained in Theorem 2.1 are sharp under our present assumptions.

**Example 1:**  $\partial_t \nabla \Phi \notin L^1_{loc}$  and  $\partial_t \Phi \notin C^0$ .

Consider in  $\Omega = B(0, 1)$  in  $\mathbb{R}^2$ , and  $\mathbf{X}(t, \cdot) : B(0, 1) \rightarrow \mathbb{R}^2$  defined with complex notations  $\mathbf{X} = x + iy$  by

on  $y > 0$ ,

$$\mathbf{X}(t, (x, y)) = e^{it}(x + iy) + it,$$

on  $y < 0$ ,

$$\mathbf{X}(t, (x, y)) = e^{it}(x + iy) + t^2.$$

We check that  $\mathbf{X} \# da$  has a density bounded by 1, that  $\partial_t \mathbf{X} \in L^\infty(\Omega \times \mathbb{R}^+)$ . If  $\mathbf{X} = \nabla \Phi \circ g$  is the polar factorization of  $\mathbf{X}$  then up to a constant,  $\Phi$  is defined for  $t > 0$ ,  $(x, y) \in \Omega$  by:

$$\Phi(t, (x, y)) = \sup\left\{\frac{1}{2}(x^2 + y^2) + t^2 x, \frac{1}{2}(x^2 + y^2) + ty\right\}.$$

On  $\{y > tx\}$  we have

$$\Phi(t, (x, y)) = \frac{1}{2}(x^2 + y^2) + ty,$$

$$\nabla \Phi(t, (x, y)) = (x, y) + (0, t),$$

and on  $\{y < tx\}$

$$\begin{aligned}\Phi(t, (x, y)) &= \frac{1}{2}(x^2 + y^2) + t^2x, \\ \nabla\Phi(t, (x, y)) &= (x, y) + (t^2, 0).\end{aligned}$$

Thus

$$\begin{aligned}\partial_t\Phi(t, (x, y)) &= y\chi_{\{y>tx\}} + 2tx\chi_{\{y<tx\}} \notin C^0, \\ \partial_t\nabla\Phi(t, (x, y)) &= (0, 1)\chi_{\{y>tx\}} + (2t, 0)\chi_{\{y<tx\}} + (t^2, -t)\mathcal{H}^{d-1}\{y = tx\} \notin L^1_{loc}\end{aligned}$$

**Example 2:** Here we adapt a counterexample of Wang to build an example of a solution where  $\partial_t\Psi \notin C^0$ .

In  $\mathbb{R}^d$ , let  $x = (x_i)_{1 \leq i \leq d}$  and

$$\mathbf{X}(0, x) = \nabla\Phi_0(x)$$

$\Phi_0(x)$  convex Lipschitz on  $\Omega$ ,  $\Phi = +\infty$  outside, such that  $\rho = \nabla\Phi_0(x) \# dx$  has a density in  $L^\infty(\mathbb{R}^2)$ . Let

$$\mathbf{X}(t, x) = \nabla\Phi_0(x) + tv$$

for some fixed  $v \in \mathbb{R}^d$ .  $\mathbf{X}$  is Lipschitz with respect to time. Then

$$\begin{aligned}\Phi(t, x) &= \Phi_0(x) + tx \cdot v, \\ \nabla\Phi(t, x) &= \nabla\Phi_0(x) + tv.\end{aligned}$$

If  $\Psi_0$  is the Legendre transform of  $\Phi_0$ , the Legendre transform of  $\Phi(t, \cdot)$  is given by

$$\begin{aligned}\Psi(t, x) &= \Psi_0(x - tv), \\ \nabla\Psi(t, x) &= \nabla\Psi_0(x - tv),\end{aligned}$$

thus

$$\begin{aligned}\partial_t\Psi(t, x) &= v \cdot \nabla\Psi_0(x - tv), \\ \partial_t\nabla\Psi(t, x) &= D^2\Psi_0(x - tv) \cdot v.\end{aligned}$$

Wang has shown in [23] some counterexamples to the regularity of solutions of Monge-Ampère equations: namely, for  $d \geq 3$  he has exhibited a solution  $u$  of

$$\det D^2u = f$$

with  $f$  only bounded by above, such that  $u \notin C^1$ . By taking  $\Psi_0 = u$  one has an example of time dependent map such that

$$\partial_t\Psi(t, x) = v \cdot \nabla\Psi_0(x - tv) \notin C^0.$$

## 9 Application: the semi-geostrophic equations

The semi-geostrophic system is derived as an approximation to the primitive equations in meteorology, and is believed to model frontogenesis (see [12]). The formulation of the 3-d incompressible version is the following: we look for a time dependent probability measure  $\rho$  that satisfies the following *SG* system:

$$(34) \quad \partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$(35) \quad \mathbf{v}(t, x) = (\nabla \Psi(t, x) - x)^\perp$$

$$(36) \quad \det D^2 \Psi(t, x) = \rho(t, x).$$

Here  $\mathbf{v}^\perp$  means  $(-\mathbf{v}_2, \mathbf{v}_1, 0)$ . Equation (36) is understood in the sense of (7), where an open set  $\Omega$  of total mass 1 has been given before.

The system has also a periodic version in which  $\Omega = \mathbb{T}^3$  itself and equation (36) is solved with the condition that  $\Psi - |x|^2/2$  is  $\mathbb{Z}^3$  periodic.

The set  $\Omega$  is here called the physical space, whereas the space in which  $\rho$  lives is the dual space. Existence of global weak solutions for the *SG* system with initial data in  $L^1$  has been proved in [2], [11] and [16]. Note that uniqueness of weak solutions is still an open question.

### 9.1 The Lagrangian formulation of the (*SG*) system

Here we look for a mapping  $\mathbf{X} : \mathbb{R}^+ \times \Omega \rightarrow \mathbb{R}^3$  that satisfies

$$(37) \quad \partial_t \mathbf{X}(t, a) = (\nabla \Psi(t, \mathbf{X}(t, a)) - \mathbf{X}(t, a))^\perp$$

$$(38) \quad \nabla \Psi(t) \circ \mathbf{X}(t) = \mathbf{g}(t) \in G(\Omega), \quad \Psi \text{ convex.}$$

If we define  $\rho(t) = \mathbf{X}(t) \# da$ , the last equation means that for all  $t$ ,  $\Psi(t)$  solves  $\det D^2 \Psi(t) = \rho(t)$  in the sense of (7). Having  $\mathbf{X}$  solution of (37, 38) implies that  $\rho(t) = \mathbf{X}(t) \# da$  is solution of (34, 35, 36).  $\mathbf{X}$  defines the characteristics in the dual space whereas  $\mathbf{g}$  defines the characteristics in the physical space.

We expose briefly the arguments that allow to define the characteristics of the *SG* system:

1- First we check that  $\mathbf{X}(t)$  will satisfy for any time  $t$  the condition (1): indeed, the flow being incompressible, all the  $L^p$  norms of  $\rho$  are conserved. Therefore, given the potential  $\Psi(t)$ , if  $X_0$  satisfies the condition (1), or equivalently if  $\rho_0 \in L^1$ , then we know a priori that  $\mathbf{X}(t)$  satisfies the condition (1) for all time.

2- The velocity field is a priori bounded in *BV* because of the convexity of  $\Psi$  (see Lemma 3.8). Moreover it is incompressible. Therefore thanks to the result of [1], the characteristics of the corresponding ODE are uniquely defined for almost every initial data, which means that the curve  $t \mapsto \mathbf{X}(t, a)$  is uniquely defined for almost every  $a \in \Omega$ .

For  $\Omega$  bounded, it is easily checked (see [2]) that if  $\mathbf{X}_0 \in L^\infty(\Omega)$ , then  $(\mathbf{X}, \partial_t \mathbf{X}) \in L^\infty([0, T] \times \Omega)$  for all  $T > 0$ . The velocity field being incompressible, if  $\rho_0 \in L^\infty(\mathbb{R}^3)$ , then  $\rho \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^3)$ . Note that the Lagrangian system can also be defined in a periodic space, where  $\mathbf{X}$  is periodic in space for all time, and we require  $\Psi - |x|^2/2$  to be periodic. The bound of  $\mathbf{X}, \partial_t \mathbf{X}$  in  $L^\infty(\mathbb{R}^+ \times \mathbb{T}^3)$  is then independent of the initial data. Moreover, in this setting, if  $\rho_0$  is such that

$$(39) \quad 0 < \lambda \leq \rho_0 \leq \Lambda$$

for two constants  $\lambda, \Lambda$ , this property remains satisfied for all time, once again due to the incompressibility of the velocity field.

Thus we conclude the following:

**Lemma 9.1** *Let  $\mathbf{X}_0 \in L^\infty(\Omega; \mathbb{R}^3)$ ,  $\rho_0 = \mathbf{X}_0 \# da \in L^\infty(\mathbb{R}^3)$ . Then  $\rho, \mathbf{X}$  the corresponding solution of the SG system satisfies for all  $T > 0$ ,*

$$\begin{aligned} \mathbf{X}, \partial_t \mathbf{X} &\in L^\infty([0, T] \times \Omega) \\ \rho &\in L^\infty(\mathbb{R}^+ \times \mathbb{R}^3). \end{aligned}$$

*In the periodic case this remains true, and if moreover  $\rho_0$  satisfies (39), then for all time  $t$ ,  $\rho(t)$  satisfies (39).*

Under the assumptions of the above lemma, it is clear that  $\mathbf{X}$  satisfies the assumptions of Theorem 2.1. In the periodic case, if satisfied at time 0, all the assumptions of Theorem 2.3 are satisfied for all time. We can now state the following theorem of partial regularity. We restrict ourselves to the periodic case.

*Remark:* We also conjecture that the assumptions of Theorem 2.2 can be satisfied for some finite time, but the control the evolution of the support of  $\rho$  poses some some difficulties.

**Theorem 9.2** *Let  $\mathbf{X}, \rho, \mathbf{g}, \Psi, \Phi$  be as above, with  $\rho = \mathbf{X} \# da$  be a space-periodic solution of (34, 35, 36), and  $\mathbf{X}$  the corresponding space-periodic solution of (37, 38). Suppose that  $\rho_0 \in L^\infty(\mathbb{T}^3)$ , then*

$$\begin{aligned} \partial_t \mathbf{g} &\in L^\infty(\mathbb{R}^+, \mathcal{M}(\mathbb{T}^3)), \\ \partial_t \nabla \Phi &\in L^\infty(\mathbb{R}^+, \mathcal{M}(\mathbb{T}^3)). \end{aligned}$$

*If moreover there exists  $0 < \lambda, \Lambda$  such that  $\lambda \leq \rho_0 \leq \Lambda$ , then there exists  $\alpha > 0$  depending on  $(\lambda, \Lambda)$  such that*

$$\mathbf{g} \in C^\alpha(\mathbb{R}^+, L^\infty(\mathbb{T}^3)).$$

*For all  $p < 2$ , there exists  $\epsilon(p)$ , such that if  $|\rho_0 - 1| \leq \epsilon(p)$ , then*

$$\partial_t \mathbf{g} \in L^\infty([0, T], L^p(\mathbb{T}^3)).$$

*There exists  $\epsilon_0$ , such that if  $|\rho_0 - 1| \leq \epsilon < \epsilon_0$ , then*

$$\partial_t \Phi, \partial_t \Psi \in L^\infty(\mathbb{R}^+, C^\alpha(\mathbb{T}^3))$$

*where  $\alpha > 0$  depends on  $\epsilon$ .*

**Remark: The equations of motion in physical space** We derive here formally the equation giving the evolution of  $\mathbf{g}$ : writing (10) with  $\mathbf{v}$  as above, we have

$$\begin{aligned} (x - \nabla \Phi)^\perp &= \mathbf{v}(\nabla \Phi) = \partial_t \nabla \Phi + D^2 \Phi w, \\ \nabla \cdot w &= 0, \end{aligned}$$

where  $\partial_t \mathbf{g}(\mathbf{g}^{-1}) = w$ . This equation formally determines the evolution of the system, since the knowledge of  $\Phi(t)$  determines a unique pair  $\partial_t \nabla \Phi, w$  satisfying the above decomposition (see Proposition 1.1). One can see a parallel with the Euler incompressible equation where the evolution is given by solving the following decomposition problem:

$$\begin{aligned} -v \cdot \nabla v &= \partial_t v + \nabla p, \\ \nabla \cdot v &= 0. \end{aligned}$$

Thus the semi-geostrophic equations are associated to the decomposition of vector fields of Proposition 1.1 in a similar way as the Euler incompressible equations are associated to the Hodge “div-curl” decomposition.

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