A fully non-linear version of the incompressible Euler equations: The semi-geostrophic system

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Abstract

The semi-geostrophic equations are used in meteorology. They appear as a variant of the two-dimensional Euler incompressible equations in vorticity form, where the Poisson equation that relates the stream function and the vorticity field is just replaced by the fully non linear elliptic Monge-Ampère equation. This work gathers new results concerning the semi-geostrophic equations: Existence and stability of measure valued solutions, existence and uniqueness of solutions under certain continuity conditions for the density, convergence to the incompressible Euler equations.

1 Introduction

The semi-geostrophic equations are an approximation to the Euler equations of fluid mechanics, used in meteorology to describe atmospheric flows. They are believed (see [12]) to be an efficient model to describe frontogenesis. Different versions (incompressible [1], shallow water [10] , compressible [11]) of this model have been studied, and we will focus here on the incompressible 2-d and 3-d versions. The 3-d model describes the behavior of an incompressible fluid in a domain $\Omega \subset \mathbb{R}^3$. To the evolution in Ω is associated a motion in a 'dual' space, described by the following non-linear transport equation:

$$\begin{aligned}
&\partial_t \rho + \nabla \cdot (\rho \mathbf{v}) = 0,\\
&\mathbf{v} = (\nabla \Psi(x) - x)^{\perp},\\
&\det D^2 \Psi = \rho,\\
&\rho(t=0) = \rho^0.
\end{aligned}$$

Here ρ^0 is a probability measure on $\mathbb{R}^3$, and for every $\mathbf{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$, $\mathbf{v}^{\perp}$ stands for $(-v_2, v_1, 0)$. The velocity field can be recovered at each time by solving a Monge-Ampère equation in the sense of the polar factorization of maps (see [3]), *i.e.* in the sense that Ψ is convex from $\mathbb{R}^3$ to $\mathbb{R}$ and satisfies $\nabla\Psi_{\#}\rho = \chi_\Omega \mathcal{L}^3$, where $\mathcal{L}^3$ is the Lebesgue measure of $\mathbb{R}^3$, and χ_Ω is the indicator function of Ω. It is imposed as a compatibility condition that Ω has Lebesgue-measure one. This model arises as an approximation to the primitive equations of meteorology, and we shall give a brief idea of the derivation of the model, although the reader interested in more details should refer to [12].

In this work we will deal with various questions related to the semi-geostrophic (hereafter SG) system: Existence and stability of measure-valued solutions, existence and uniqueness of smooth solutions, and finally convergence towards the incompressible Euler equations in 2-d. As stated in the title, we will all along the paper exploit the strong analogy with the 2-d incompressible Euler equations that we recall here:

$$\begin{aligned}
&\partial_t\omega + \nabla\cdot(\omega\mathbf{v}) = 0,\\
&\mathbf{v} = (\nabla\Phi)^\perp,\\
&\Delta\Phi = \omega,\\
&\omega(t=0) = \omega^0.
\end{aligned}$$

We recognize clearly that the vorticity ω plays here the role of the density ρ in the SG system. One obtains the SG system just by replacing the Poisson equation $\Delta\Phi = \omega$ by the Monge-Ampère equation $\det(I + D^2\Phi) = \omega$. (However, note that the density ρ does not have a clear physical interpretation since it is a density in a dual space). From this analogy, and inspired by the well developed mathematical theory on the 2-d Euler equations (see [21] for instance) the goal of this paper is twofold:

The first goal is to the study of the initial value problem for the SG system. We will first establish a global existence result for weak measure-valued solutions, hence giving a framework for weak solutions that strictly contains the results obtained in previous works. We will also obtain local smooth solutions, trying to lower as much as possible the requirement on the initial data, and prove uniqueness in a certain class of smooth solutions. This well posedness result for smooth initial data will be our main result.

The second goal is to give some rigorous mathematical justification of the derivation of the SG system from the 2-d Euler equations. As an attempt in this direction, we will show that in some asymptotic regime (namely 'small' solutions over a long time) the SG system and and the 2-d Euler system are asymptotically close.

We will use a combination of various techniques: The SG system is a transport equation, and we will study it as such, using either the Eulerian or the Lagrangian point of view. Since the coupling between density and velocity field involves a Monge-Ampère equation, we will also rely on the regularity theory developed for this fully non linear elliptic equation, which is much more recent and far less known than the results on solutions to the Poisson equation. More originally we will use optimal transportation and the technique developed in [18] to show uniqueness of certain solutions. Note (but this is a coïncidence) that optimal transportation will appear earlier in the paper in the derivation of the SG system. Finally the proof of the convergence toward the incompressible Euler equations will be done using modulated energy techniques, a general technique (also known as the 'relative entropy method', documented in [13]) used for the asymptotic study of hyperbolic systems.

The paper is organized as follows: In the next paragraph we give a short idea of the derivation of the Semi-Geostrophic system from the Euler incompressible equations. To formulate rigorously the system, we then review the results concerning optimal transportation and polar factorization of maps, that are key concepts used through the paper (section 1.2). We are then able to formulate the Semi-Geostrophic system, both in its Lagrangian, and Eulerian (or dual) form (sections 1.3 and 1.4).

Section 1.5 is then dedicated to a longer discussion on the results obtained, and gives a sketch of some of the crucial arguments. This section closes the introduction. Then each of the following sections is dedicated to the proof of one of the results.

Section 2 is devoted to the existence of weak measure-valued solution, in section 3 we show existence of Dini continuous solutions, in section 4 we show uniqueness of solutions with Hölder continuous density, and in section 5 we show the convergence of solutions of SG towards solutions of the 2-d Euler incompressible equations.

All those results will be reviewed and discussed in greater detail in section 1.5, after we have derived the Semi-Geostrophic equations.

1.1 Derivation of the semi-geostrophic equations

We now give for sake of completeness a brief and simplified idea of the derivation of the system, inspired from [1], and more complete arguments can be found in [12].

Lagrangian formulation

We start from the 3-d incompressible Euler equations with constant Coriolis parameter f in a domain Ω.

$$\begin{aligned}&\frac{D\mathbf{v}}{Dt}+f\mathbf{v}^{\perp}=\frac{1}{\rho}\nabla p-\nabla\varphi,\\&\nabla\cdot\mathbf{v}=0,\ \frac{D\rho}{Dt}=0,\\&\mathbf{v}\cdot\partial\Omega=0,\end{aligned}$$

where $\frac{D\cdot}{Dt}$ stands for $\partial_t+\mathbf{v}\cdot\nabla$, and we still use $\mathbf{v}^{\perp}=(-v_2,v_1,0)$. The term $\nabla\varphi$ denotes the gravitational effects (here we will take $\varphi=gx_3$ with constant g), and the term $f\mathbf{v}^{\perp}$ is the Coriolis force due to rotation of the Earth. For large scale atmospheric flows, the Coriolis force $f\mathbf{v}^{\perp}$ dominates the advection term $\frac{D\mathbf{v}}{Dt}$, and renders the flow mostly two-dimensional. We use the hydrostatic approximation: $\partial_{x_3}p=-\rho g$ and restrict ourselves to the case $\rho\equiv 1$.

Keeping only the leading order terms leads to the geostrophic balance

$$\mathbf{v}_g=-f^{-1}\nabla^{\perp}p,$$

that defines $\mathbf{v}_g$, the geostrophic wind. Decomposing $\mathbf{v}=\mathbf{v}_g+\mathbf{v}_{ag}$ where the second component is the ageostrophic wind, a supposed small departure from the geostrophic balance, the semi-geostrophic system reads:

$$\begin{aligned}&\frac{D\mathbf{v}_g}{Dt}+f\mathbf{v}^{\perp}=\nabla_H p,\\&\nabla\cdot\mathbf{v}=0,\end{aligned}$$

where $\nabla_H=(\partial_{x_1},\partial_{x_2},0)$. Note however that the advection operator $\partial_t+\mathbf{v}\cdot\nabla$ still uses the full velocity $\mathbf{v}$. Introducing the potential

$$\Phi=\frac{1}{2}|x_H|^2+f^{-2}p,$$

with $x_H=(x_1,x_2,0)$, we obtain the following

$$\frac{D}{Dt}\nabla\Phi(t,x)=f(x-\nabla\Phi(t,x))^{\perp}.$$

We introduce the Lagrangian map $\mathbf{g}:\Omega\times\mathbb{R}^+\mapsto\Omega$ giving the position at time t of the particle of fluid located at x_0 at time 0. The previous equation means that, if for fixed $x \in \Omega$ we consider the trajectory in the 'dual' space, defined by $X(t,x) = \nabla\Phi(t, \mathbf{g}(t,x))$, we have

$$\partial_t X(t,x) = f\left(\mathbf{g}(t,x) - X(t,x)\right)^{\perp}.$$

By rescaling the time, we can set $f = 1$. Under this form the system looks under-determined: Indeed Φ is unknown; however we have the condition $X(t,x) = \nabla\Phi(t, \mathbf{g}(t,x))$. Moreover, the motion of the fluid being incompressible and contained in Ω, the map $\mathbf{g}(t,\cdot)$ must be measure preserving in Ω for each t, *i.e.*

$$\mathcal{L}^3(\mathbf{g}(t)^{-1}(B)) = \mathcal{L}^3(B)$$

for each $B \subset \Omega$ measurable (where $\mathcal{L}^3$ denotes the Lebesgue measure of $\mathbb{R}^3$). We shall hereafter denote by $G(\Omega)$ the set of all such measure preserving maps. Then Cullen's stability criteria ([12]) asserts that the potential Φ should be convex for the system to be stable to small perturbations of particle's positions in the x space. Indeed the convexity of Φ asserts that $\nabla\Phi$ minimizes some potential energy (the reader interested in a more detailed explanation of this variational principle should refer to [12]).

Hence, for each t, Φ must be a convex function such that

$$X(t,\cdot) = \nabla\Phi(t, \mathbf{g}(t,\cdot)),$$

with $\mathbf{g}(t,\cdot) \in G(\Omega)$.

In the next paragraph we shall see that, under very mild assumptions on X, this decomposition, called polar factorization, can only happen for a unique choice of $\mathbf{g}$ and $\nabla\Phi$. Now if Φ^* is the Legendre transform of Φ,

$$\Phi^*(y) = \sup_{x\in\Omega} x \cdot y - \Phi(x),$$

then $\nabla\Phi$ and $\nabla\Phi^*$ are inverse maps of each other, and the semi-geostrophic system then reads

$$\begin{aligned} &\frac{DX}{Dt} = \left(\nabla\Phi^*(X(t)) - X(t)\right)^{\perp}, \\ &\nabla\Phi^*(t) \circ X(t) \in G(\Omega). \end{aligned}$$

In this context, $X(t)$ is thus the dual trajectory to the physical trajectory $\mathbf{g}(t)$, and $(\nabla\Phi^*(X(t)) - X(t))^{\perp}$ is up to a multiplicative constant, the geostrophic wind at point $\mathbf{g}(t) = \nabla\Phi^*(X(t))$.

In the next paragraph, we review the results concerning the existence and uniqueness of the gradients $\nabla\Phi, \nabla\Phi^*$.

1.2 Polar factorization of vector valued maps

The polar factorization of maps has been discovered by Brenier in [3]. It has later been extended to the case of general Riemannian manifolds by McCann in [23].

The Euclidean case

Let Ω be a fixed bounded domain of $\mathbb{R}^d$ of Lebesgue measure 1 and satisfying the condition $\mathcal{L}^d(\partial\Omega) = 0$. We consider a mapping $X \in L^2(\Omega; \mathbb{R}^d)$. We will also consider the push-forward of the Lebesgue measure of Ω by X, that we will denote by $X_\# \chi_\Omega \mathcal{L}^d = d\rho$ (or, in short, $X_\# dx$) and which is defined by

$$\forall f \in C_b^0(\mathbb{R}^d), \int_{\mathbb{R}^d} f(x)\; d\rho(x) = \int_\Omega f(X(x))\; dx.$$

Let $\mathcal{P}$ be the set of probability measures $\mathbb{R}^d$, and $\mathcal{P}_a^2$ the subset of $\mathcal{P}$ where the subscript a means absolutely continuous with respect to the Lebesgue measure (or equivalently that have a density in $L^1(\mathbb{R}^d)$), and the superscript 2 means with finite second moment. (*i.e.* such that

$$\int_{\mathbb{R}^d} |x|^2 d\rho(x) < +\infty.)$$

Note that for $X \in L^2(\Omega, \mathbb{R}^d)$, the measure $\rho = X_\# dx$ has necessarily finite second moment, and thus belongs to $\mathcal{P}^2$.

Theorem 1.1 (Brenier, [3]). *Let Ω be as above, $X \in L^2(\Omega; \mathbb{R}^d)$ and $\rho = X_\# dx$.*

1. *There exists a unique up to a constant convex function, that will be denoted $\Phi[\rho]$, such that:*

$$\forall f \in C_b^0(\mathbb{R}^d),\; \int_\Omega f(\nabla\Phi[\rho](x))\; dx = \int_{\mathbb{R}^d} f(x) d\rho(x).$$

2. *Let $\Psi[\rho]$ be the Legendre transform of $\Phi[\rho]$, if $\rho \in \mathcal{P}_a^2$, $\Psi[\rho]$ is the unique up to a constant convex function satisfying*

$$\forall f \in C_b^0(\Omega),\; \int_{\mathbb{R}^d} f(\nabla\Psi[\rho](x))\; d\rho(x) = \int_\Omega f(x) dx.$$

3. *If $\rho \in \mathcal{P}_a^2$, X admits the following unique polar factorization:*

$$X = \nabla\Phi[\rho] \circ g,$$

with $\Phi[\rho]$ convex, g measure preserving in Ω.

Remark: $\Psi[\rho], \Phi[\rho]$ depend only on ρ, and are solutions (in some weak sense) in $\mathbb{R}^d$ and Ω respectively, of the Monge-Ampère equations

$$\begin{aligned} &\det D^2\Psi = \rho, \\ &\rho(\nabla\Phi)\ \det D^2\Phi = 1. \end{aligned}$$

When Ψ and Φ are not in C^2_{loc} these equations can be understood in the viscosity (or Alexandrov) sense or in the sense of Theorem 1.1, which is strictly weaker. For the consistency of the different weak formulations and regularity issues the reader can refer to [8].

The periodic case

The polar factorization theorem has been extended to Riemannian manifolds in [23] (see also [9] for the case of the flat torus). In this case, we consider a mapping $X : \mathbb{R}^d \mapsto \mathbb{R}^d$ such that for all $\vec{p} \in \mathbb{Z}^d, X(\cdot + \vec{p}) = X + \vec{p}$. Then $\rho = X_\# dx$ is a probability measure on $\mathbb{T}^d$. We define $\Psi[\rho], \Phi[\rho]$ through the following:

Theorem 1.2. *Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be as above, with $\rho = X_\# dx$.*

1. *Up to an additive constant there exists a unique convex function $\Phi[\rho]$ such that $\Phi[\rho](x) - x^2/2$ is $\mathbb{Z}^d$-periodic (and thus $\nabla\Phi[\rho](x) - x$ is $\mathbb{Z}^d$ periodic), and*

$$\forall f \in C^0(\mathbb{T}^d),\ \int_{\mathbb{T}^d} f(\nabla\Phi[\rho](x))\ dx = \int_{\mathbb{T}^d} f(x)\ d\rho(x).$$

2. *Let $\Psi[\rho]$ be the Legendre transform of $\Phi[\rho]$. If ρ is Lebesgue integrable, $\Psi[\rho]$ is the unique up to a constant convex function satisfying $\Psi[\rho](x) - x^2/2$ is $\mathbb{Z}^d$-periodic (and thus $\nabla\Psi[\rho](x) - x$ is $\mathbb{Z}^d$ periodic), and*

$$\forall f \in C^0(\mathbb{T}^d),\ \int_{\mathbb{T}^d} f(\nabla\Psi[\rho](x))\ d\rho(x) = \int_{\mathbb{T}^d} f(x)\ dx.$$

3. *If ρ is Lebesgue integrable, X admits the following unique polar factorization:*

$$X = \nabla\Phi[\rho] \circ g$$

with g measure preserving from $\mathbb{T}^d$ into itself, and $\Phi[\rho]$ convex, $\Phi[\rho] - |x|^2/2$ periodic.

Remark 1: From the periodicity of $\nabla\Phi[\rho](x) - x, \nabla\Psi[\rho](x) - x$, for every f $\mathbb{Z}^d$-periodic, $f(\nabla\Psi[\rho]), f(\nabla\Phi[\rho])$ are well defined on $\mathbb{R}^d/\mathbb{Z}^d$.

Remark 2: Both in the periodic and non periodic case, the definitions of $\Psi[\rho]$ and $\Phi[\rho]$ make sense if ρ is absolutely continuous with respect to the Lebesgue measure. If not, the definition and uniqueness of $\Phi[\rho]$ is still valid, as well as the property $\nabla\Phi[\rho]_{\#}\rho = \chi_\Omega \mathcal{L}^d$. The definition of $\Psi[\rho]$ as the Legendre transform of $\Phi[\rho]$ is still valid also, but then the expression $\displaystyle\int f(\nabla\Psi[\rho](x))\, d\rho(x)$ does not necessarily make sense since $\nabla\Psi[\rho]$ is not necessarily continuous, and hence not defined $d\rho$ almost everywhere. Moreover the polar factorization does not hold any more.

Remark 3: We have (see [9]) the unconditional bound

$$\|\nabla\Psi[\rho](x) - x\|_{L^\infty(\mathbb{T}^d)} \leq \sqrt{d}/2$$

that will be useful later on.

1.3 Lagrangian formulation of the SG system

From Theorems 1.1, 1.2 the Lagrangian formulation of the semi-geostrophic equation then becomes

$$\frac{DX}{Dt} = [\nabla\Psi(X) - X]^\perp, \tag{1}$$

$$\Psi = \Psi[\rho], \quad \rho = X_{\#}dx. \tag{2}$$

1.4 Eulerian formulation in dual variables

In both cases (periodic and non periodic) we thus investigate the following system that will be referred to as SG in dual variables (but we will only say SG hereafter): We look for a time dependent probability measure $t \longrightarrow \rho(t,\cdot)$ satisfying

$$\partial_t\rho + \nabla\cdot(\rho\mathbf{v}) = 0, \tag{3}$$

$$\mathbf{v}(t,x) = (\nabla\Psi[\rho(t)](x) - x)^\perp, \tag{4}$$

$$\rho(t=0) = \rho^0. \tag{5}$$

Global existence of weak solutions (which are defined below) of this system with L^p initial data for $p \geq 1$ has been shown in [1], [10], [20].

1.5 Results

In this work we deal with various mathematical problems related to this system: We first extend the notion of weak solutions that had been shown to exist for $\rho \in L^\infty(\mathbb{R}_+, L^q(\mathbb{R}^3))$, $q > 1$ ([1], [10]), and then for $\rho \in L^\infty(\mathbb{R}_+, L^1(\mathbb{R}^3))$ ([20]), to the more general case of bounded measures. The question of existence of measure-valued solutions was raised and left unanswered in those papers, and we show here existence of global solutions to the Cauchy problem with initial data a bounded compactly supported measure, and show the weak stability/compactness of these *weak measure* solutions.

Then we show existence of continuous solutions, more precisely, we show local existence of solutions with Dini-continuous (see (12)) density. For this solutions, the velocity field is then C^1 and the Lagrangian system (1,2) is defined everywhere. This proof relies heavily on the available regularity results on solutions to the Monge-Ampère equation (Theorem 3.1). Note that the Dini condition is the lowest condition known on the right hand side of the Poisson equation that enforces C^2 regularity for the solution. Our result is not totally satisfactory since it does not provide existence of a global smooth solution, which is the case for the 2-d incompressible Euler equation. The reason for this more powerful result is that for the Poisson equation

$$\Delta\Phi = \omega,$$

ω bounded implies that $\nabla\Phi$ is Log-Lipschitz. This continuity is slightly weaker that the Lipschitz continuity, but allows to define a Holder continuous flow (see [21]). Moreover, the flow being incompressible, this implies (when $d = 2$) that the vorticity is just transported along the streamlines. The construction of global smooth solutions can then be achieved only using those two arguments.

For the SG system, solutions to

$$\det D^2\Psi = \rho$$

are only $C^{1,\alpha}$ when ρ is merely bounded. This is not enough to build a continuous flow, and prevents us from obtaining the same results as for Euler.

We also show uniqueness in the class of Hölder continuous solutions (a sub-class of Dini continuous solutions). This proof uses in an crucial way the optimal transportation of measures by convex gradients and its regularity properties, and can be adapted to give a new proof of uniqueness for solutions of the 2-d Euler equation with bounded vorticity, but also for a broad class of non-linearly coupled system. The typical application is a density evolving through a transport equation where the velocity field depends on the gradient of a potential. The potential is obtained by solving an elliptic equation, where the density appears in the right hand side. Well known examples of such cases are the Vlasov-Poisson and Euler-Poisson systems (see [18]). We point out that the results of existence and uniqueness obtained here are all obtained by working in a purely Lagrangian framework.

Finally, in the 2-d case, we study the convergence of the system to the Euler incompressible equations; this convergence is expected for ρ close to 1, since formally expanding $\Psi = x^2/2 + \epsilon\psi$, and linearizing the determinant around the identity matrix, we get

$$\rho = \det D^2\Psi = 1 + \epsilon\Delta\psi + O(\epsilon^2),$$

and the Monge-Ampère equation turns into the Poisson equation

$$\Delta\psi = \frac{\rho - 1}{\epsilon} =: \mu.$$

We then perform the change of time scale $t \longrightarrow t/\epsilon$, and consider now $\mu^\epsilon(t) := \mu(\frac{t}{\epsilon})$. Then μ^ϵ solves

$$\begin{aligned} &\partial_t\mu^\epsilon + \nabla\cdot(\mu^\epsilon\nabla^\perp\psi^\epsilon) = O(\epsilon), \\ &\Delta\psi^\epsilon = \mu^\epsilon, \end{aligned}$$

where, when we set $O(\epsilon) = 0$, we recognize as the vorticity formulation of the 2-d Euler incompressible equation.

Let us comment this scaling: We consider a small solution to SG, *i.e.* a solution where $\rho - 1$ is small. We then expand this solution by a factor ϵ^{-1}, and study it on a time scale of order ϵ^{-1}.

From a physical point of view, this asymptotic study may be seen as a justification of the consistency of the semi-geostrophic approximation when $d = 2$. Indeed, when $d = 2$, the Euler equations are not affected by the Coriolis force, i.e. the solutions to

$$\partial_t v + v\cdot\nabla v = -\nabla p, \tag{6}$$

$$\mathrm{div} v = 0, \tag{7}$$

and to

$$\partial_t v + v\cdot\nabla v + fv^\perp = -\nabla p, \tag{8}$$

$$\mathrm{div} v = 0, \tag{9}$$

are the same, since the term $v^\perp$ can be considered as a pressure term (remember that $v = \nabla^\perp\Phi$). The term f is just a time scale, and the geostrophic regime is the one where $\frac{v}{L} << f$, where v denotes here the typical size of v and L the typical space scale of the system. Then, note that if $v(t,x)$ is solution to (6), so is $v^\epsilon = \epsilon v(\epsilon t, x)$. But the ratio $\frac{v^\epsilon}{L}$ goes to 0 as ϵ goes to 0 (note that the space scales for v and v^ϵ are the same). Hence, in the limit $\epsilon \to 0$, i.e. for small solutions to Euler, the geostrophic approximation should be valid. This is precisely in this regime that we show that the SG system and the incompressible Euler system are asymptotically close to each other, since for SG, a small solution is a one where ρ is close to 1. Hence what we show is the following: Let ρ^0 be a 'small' initial data for SG. Consider μ^ϵ obtained from ρ as explained above, then μ^ϵ is close to some ω where ω solves the 2-d Euler incompressible equation in vorticity form

$$\begin{aligned}&\partial_t\omega + \nabla\cdot(\omega\nabla^\perp\phi) = 0,\\ &\Delta\phi = \omega.\end{aligned}$$

In other words, when ρ goes to 1, ρ is equivalent to a solution of Euler, on a time that goes to infinity.

The study of this 'quasi-neutral' limit is done by two different ways: One uses a modulated energy method similar as the one used in [4] and [5] and is valid for weak solutions. The other uses a more classical expansion of the solution, and regularity estimates, and is similar to the method used in [17]. The second method also yields almost global solutions: Indeed, it will be shown in this paper that smooth (say with Lipschitz density) solutions exists in short time. The asymptotic study of the convergence to Euler shows that the Lipschitz bound on the solution remains valid on a time that goes to infinity when the solution is chosen with an initial condition that converges toward the uniform density.

2 Measure valued solutions

2.1 A new definition of weak solutions

We have first the following classical weak formulation of equation (3):
$\rho \in C(\mathbb{R}+, L^1(\mathbb{R}^3) - w)$ is said to be a weak solution of SG if

$$\begin{aligned}&\forall T > 0,\ \forall\varphi \in C_c^\infty([0,T]\times\mathbb{R}^2),\\ &\int \partial_t\varphi\,\rho + \nabla\varphi\cdot(\nabla\Psi[\rho]-x)^\perp\,\rho\,dtdx\\ = \ &\int \varphi(T,x)\rho(T,x)dx - \int\varphi(0,x)\rho(0,x)dx,\end{aligned}$$

where for all t, $\Psi[\rho]$ is as in Theorem 1.1. The problematic part in the case of measure valued solutions is to give sense to the product $\rho\nabla\Psi[\rho]$ since at the point where ρ is singular $\nabla\Psi[\rho]$ is unlikely to be continuous. Therefore we use the Theorem 1.1 to write for any $\rho \in \mathcal{P}_a^2(\mathbb{R}^3)$

$$\forall\varphi \in C_c^\infty(\mathbb{R}^3), \int_{\mathbb{R}^3} \rho\nabla\Psi[\rho]^\perp \cdot \nabla\varphi = \int_\Omega x^\perp \cdot \nabla\varphi(\nabla\Phi[\rho])$$

(the integrals would be performed over $\mathbb{T}^3$ in the periodic case). The property $\nabla\Phi[\rho]_\# \chi_\Omega \mathcal{L}^3 = \rho$ is still valid when ρ is only a measure with finite second moment (see Remark 2 after Theorem 1.2). Therefore, the formulation on the right hand side extends unambiguously to the case where $\rho \notin L^1(\mathbb{R}^2)$.

Geometric interpretation

This weak formulation has a natural geometric interpretation: At a point where $\Psi[\rho]$ is not differentiable, and thus where $\partial\Psi[\rho]$ is not reduced to a single point, $\nabla\Psi[\rho]$ should be replaced by $\bar{\partial}\Psi[\rho]$ the center of mass of the (convex) set $\partial\Psi[\rho]$. The function $\bar{\partial}\Psi[\rho]$ coïncides Lebesgue almost everywhere with $\nabla\Psi$, and is defined as follows

Definition 2.1. *The map $\bar{\partial}\Psi[\rho]$ is defined at every point x by the center of mass with respect to the Lebesgue measure of the set $\partial\Psi[\rho](x)$. In other words, if $\partial\Psi[\rho](x)$ is a k-dimensional convex set, we have*

$$\bar{\partial}\Psi[\rho](x) = \int_{\partial\Psi[\rho](x)} y\, d\mathcal{L}^k(y).$$

This motivates the following definition of weak measure solutions

Definition 2.2. *Let, for all $t \in [0,T]$, $\rho(t)$ be a probability measure of $\mathbb{R}^3$. It is said to be a* **weak measure solution** *to SG if*

1- The time dependent probability measure ρ belongs to $C([0,T], \mathcal{P} - w*)$,

2- there exists $t \to R(t)$ non-decreasing such that for all $t \in [0,T]$, $\rho(t,\cdot)$ is supported in $B(0,R(t))$,

3- for all $T > 0$ and for all $\varphi \in C_c^\infty([0,T] \times \mathbb{R}^3)$ we have

$$\begin{aligned} & \int_{[0,T]\times\mathbb{R}^3} \partial_t\varphi(t,x)\ d\rho(dt,x) \qquad (10) \\ + & \int_{[0,T]\times\Omega} \nabla\varphi(t,\nabla\Phi[\rho(t)](x)) \cdot x^\perp\ dtdx - \int_{[0,T]\times\mathbb{R}^3} \nabla\varphi(t,x)\cdot x^\perp\ d\rho(dt,x) \\ = & \int \varphi(T,x)d\rho(T,x)\ dx - \int \varphi(0,x)d\rho(0,x)\ dx. \end{aligned}$$

This definition is consistent with the classical definition of weak solutions if for all t, $\rho(t,\cdot)$ is absolutely continuous with respect to the Lebesgue measure.

2.2 Result

Here we prove the following

Theorem 2.3. 1. *Let ρ^0 be a probability measure compactly supported. There exists a global weak measure solution to the system SG in the sense of Definition 2.2.*

2. *For any $T > 0$, if $(\rho_n)_{n\in\mathbb{N}}$ is a sequence of weak measure solutions on $[0,T]$ to SG with initial data $(\rho_n^0)_{n\in\mathbb{N}}$, supported in B_R for some $R > 0$ independent of n, the sequence $(\rho_n)_{n\in\mathbb{N}}$ is precompact in $C([0,T],\mathcal{P}-w*)$ and every converging subsequence converges to a weak measure solution of SG.*

Proof of Theorem 2.3

We first show the weak stability of the formulation of Definition 2.2, and the compactness of weak measure solutions. We then use this result to obtain global existence of solutions to the Cauchy problem with initial data a bounded measure.

Weak stability of solutions

We consider a sequence $(\rho_n)_{n\in\mathbb{N}}$ of solutions of SG in the sense of Definition 2.2. The sequence is uniformly compactly supported at time 0. We first show that there exists a non-decreasing function $R(t)$ such that $\rho_n(t)$ is supported in $B(R(t))$ for all t, n:

Lemma 2.4. *Let $\rho \in C([0,T],\mathcal{P}(\mathbb{R}^3)-w*)$ satisfy (10), let $\rho^0 = \rho(t=0)$ be supported in $B(0,R^0)$, then $\rho(t)$ is supported in $B(0,R^0+C_\Omega t)$, $C_\Omega = \sup_{y\in\Omega}\{|y|\}$.*

Proof. Consider any function $\xi_\epsilon(t,r) \in C^\infty([0,T]\times\mathbb{R})$ such that

$$\begin{aligned}
&\xi_\epsilon(0,r) \equiv 1 \text{ if } -\infty < r \le R^0,\\
&\xi_\epsilon(0,r) \equiv 0 \text{ if } r \ge R^0+\epsilon,\\
&\xi_\epsilon(t,r) = \xi_\epsilon(0,r - C_\Omega t),
\end{aligned}$$

with $\xi(0,\cdot)$ non increasing. Then applying (10) to the test function $\xi_\epsilon(t,|x|)$, we find

$$\begin{aligned}
&\frac{d}{dt}\int \xi_\epsilon(t,|x|)\ d\rho(t,x)\\
= &-\int \partial_r\xi_\epsilon(t,|x|)C_\Omega\ d\rho(t,x) + \int_\Omega \partial_r\xi_\epsilon(t,|\nabla\Phi[\rho(t)]|)\frac{\nabla\Phi[\rho(t)]}{|\nabla\Phi[\rho(t)]|}\cdot x^\perp\ dx\\
\geq &\int_\Omega \partial_r\xi_\epsilon(t,|\nabla\Phi[\rho(t)]|)(-C_\Omega+|x|)\ dx\\
\geq &\ 0
\end{aligned}$$

since, by definition of C_Ω, for $x\in\Omega$, $|x|\leq C_\Omega$ and ξ_ϵ is non increasing with respect to r. Note also that we have used $\int \nabla_x[\xi(t,|x|)]\cdot x^\perp d\rho(t,x)\ dx \equiv 0$. We know on the other hand that

$$\int_{\mathbb{R}^3}\xi_\epsilon(0,|x|)d\rho(0,x) = 1,$$
$$\int_{\mathbb{R}^3}\xi_\epsilon(t,|x|)d\rho(t,x) \leq 1,$$

therefore we conclude that $\int_{\mathbb{R}^3}\xi_\epsilon(t,|x|)d\rho(t,x) \equiv 1$, which concludes the lemma by letting ϵ go to 0.

□

From Lemma 2.4, we have:

$$\left|-\int_{[0,T]\times\mathbb{R}^3}\nabla\varphi(t,x)\cdot x^\perp\ d\rho_n(dt,x) + \int_{[0,T]\times\Omega}\nabla\varphi(t,\nabla\Phi[\rho_n(t)](x))\cdot x^\perp\ dtdx\right|$$
$$\leq C(T)\|\varphi\|_{L^1([0,T],C^1(B_{R(T)})}.$$

Thus from Definition 2.2 equation (10) we know that for any time $t\geq 0$, $\partial_t\rho_n(t,\cdot)$ is bounded in the dual of $L^1([0,T],C^1(\mathbb{R}^3))$ and thus in the dual of $L^1([0,T],W^{2,p}(\mathbb{R}^3))$ for $p>3$ by Sobolev embeddings. Thus for some $p'>1$ we have

$$\partial_t\rho_n \in L^\infty([0,T],W^{-2,p'}(\mathbb{R}^2)).$$

With the two above results, and using a classical compactness result (see [16, Chapter 1, lemma 5.1]), we can obtain the following lemma:

Lemma 2.5. *Let the sequence $(\rho_n)_{n\in\mathbb{N}}$ be as above, there exists $\rho\in C([0,T],\mathcal{P}-w*)$ and a subsequence $(\rho_{n_k})_{k\in\mathbb{N}}$, such that for all $t\in[0,T]$, $\rho_{n_k}(t)$ converges to $\rho(t)$ in the weak-$*$ topology of measures.*

With this lemma, we need to show that for all $\varphi \in C_c^\infty([0,T] \times \mathbb{R}^3)$ we have $\nabla\varphi(t, \nabla\Phi[\rho_n(t)])$ converging to $\nabla\varphi(t, \nabla\Phi[\rho(t)])$ whenever $\rho_n(t)$ converges weakly-$*$ to $\rho(t)$. This last step will be a consequence of the following stability theorem:

Theorem 2.6 (Brenier, [3]). *Let* Ω *be as above. Let* $(\rho_n)_{n\in\mathbb{N}}$ *be a sequence of probability measures on* $\mathbb{R}^d$*, such that* $\forall n$, $\int(1+|x|^2)d\rho_n \leq C$*, let* $\Phi_n = \Phi[\rho_n]$ *and* $\Psi_n = \Psi[\rho_n]$ *be as in Theorem 1.1. If for any* $f \in C^0(\mathbb{R}^d)$ *such that* $|f(x)| \leq C(1+|x|^2)$, $\int f\rho_n \to \int \rho f$*, then the sequence* Φ_n *can be chosen in such a way that* $\Phi_n \to \Phi[\rho]$ *uniformly on each compact set of* Ω *and strongly in* $W^{1,1}(\Omega;\mathbb{R}^d)$*, and* $\Psi_n \to \Psi[\rho]$ *uniformly on each compact set of* $\mathbb{R}^d$ *and strongly in* $W^{1,1}_{loc}(\mathbb{R}^d)$*.*

From this result, we obtain that the sequence $\nabla\Phi[\rho_n]$ converges strongly in $L^1(\Omega)$ and almost everywhere (because of the convexity of $\Phi[\rho]$) to $\nabla\Phi[\rho]$. Thus $\nabla\varphi(t, \nabla\Phi[\rho_n])$ converges to $\nabla\varphi(t, \nabla\Phi[\rho])$ in $L^1(\Omega)$ and one can pass to the limit in the formulation of Definition 2.2. This ends the proof of point 2 of Theorem 2.3.

Existence of solutions

We show briefly the existence of a solution to the Cauchy problem in the sense of Definition 2.2. Indeed given ρ^0 the initial data for the problem that we want to solve, by smoothing ρ^0, we can take a sequence ρ_n^0 of initial data belonging to $L^1(\mathbb{R}^2)$, uniformly compactly supported and converging weakly-$*$ to ρ^0. We know already from [1], [10], [20] that for every ρ_n^0, one can build a global weak solution of (3, 4, 5), that will be uniformly compactly supported on $[0,T]$ for all $T \geq 0$. This sequence will also be solution in the sense of Definition 2.2. We then use the stability Theorem 2.6, and conclude that, up to extraction of a subsequence, the sequence ρ_n converges in $C([0,T], \mathcal{P} - w*)$ to a weak measure solution of SG with initial data ρ^0. This achieves the proof of Theorem 2.3. $\square$

Remark: One can prove in fact the more general result, valid for non linear functionals:

Proposition 2.7. *Let* $F \in C^0(\Omega \times \mathbb{R}^d)$*, such that* $|F(x,y)| \leq C(1+|y|^2)$*, let* $(\rho_n)_{n\in\mathbb{N}}$ *be a bounded sequence of probability measures, Lebesgue integrable, with finite second moment. Let* ρ *be a probability measure with finite second moment, such that for all* $f \in C^0(\mathbb{R}^d)$ *such that* $|f(x)| \leq C(1+|x|^2)$*,*

$\int f d\rho_n \to \int f d\rho$. Then as n goes to ∞, we have

$$\begin{aligned} \int_{\mathbb{R}^d} F(\nabla\Psi[\rho_n](x), x)\ d\rho_n(x) &= \int_\Omega F(y, \nabla\Phi[\rho_n](y))\ dy \\ \to_n \quad \int_\Omega F(y, \nabla\Phi[\rho](y))\ dy &=: \int_{\mathbb{R}^d} F(\bar\partial\Psi[\rho](x), x)\ d\rho(x), \end{aligned}$$

where $\bar\partial\Psi[\rho]$ is given in Definition 2.1.

Remark: One checks easily that this definition of $\int_{\mathbb{R}^d} F(\bar\partial\Psi[\rho](x), x)\ d\rho(x)$ is consistent with the definition of $\int_{\mathbb{R}^d} F(\nabla\Psi[\rho](x), x)\ d\rho(x)$ whenever ρ is absolutely continuous with respect to the Lebesgue measure, or $\nabla\Psi[\rho]$ is continuous. Indeed, note that $\nabla\Psi$ and $\bar\partial\Psi$ always coïncide Lebesgue almost everywhere, since as a convex and hence Lipschitz function Ψ is differentiable Lebesgue almost everywhere (Rademacher's Theorem), hence $\partial\Psi$ is single valued Lebesgue almost everywhere.

3 Continuous solutions

What initial regularity is necessary in order to guarantee that the velocity fields remains Lipschitz, or that the flow remains continuous, at least for a short time ? The celebrated Youdovich's Theorem for the Euler incompressible equation shows that when $d = 2$, if the initial vorticity data is bounded in L^∞, the flow is Hölder continuous, with Hölder index decreasing to 0 as time goes to infinity. This proof relies on the following regularity property of the Poisson equation: If $\Delta\phi$ is bounded in L^∞, then $\nabla\phi$ is Log-Lipschitz. This continuity is enough to define a Hölder continuous flow for the vector field $\nabla\phi^\perp$. Such a result is not valid for the Monge-Ampère equation. As far as we know, the strongest regularity result for Monge-Ampère equations is the following:

3.1 Regularity of solutions to Monge-Ampère equation with Dini-continuous right hand side

Theorem 3.1 (Wang, [25]). *Let u be a strictly convex Alexandrov solution of*

$$\det D^2 u = \rho \tag{11}$$

with ρ strictly positive. If $w(r)$, the modulus of continuity of ρ, satisfies

$$\int_0^1 \frac{w(r)}{r} dr < \infty, \tag{12}$$

then u is in C^2_{loc}.

We will work here in the periodic case. In this case, u the solution of (11) will be $\Psi[\rho]$ of Theorem 1.2. The arguments of [7], [8], adapted to the periodic case, show that $\Psi[\rho]$ is indeed a strictly convex Alexandrov solution of solution of (11). Therefore we obtain the following corollary of Theorem 3.1:

Corollary 3.2. *Let $\rho \in \mathcal{P}(\mathbb{T}^d)$ be such that*

$$0 < m \leq \rho \leq M,$$
$$\int_0^1 \frac{w(r)}{r} dr = C < \infty.$$

where m, M, C are positive constants. Let $\Psi[\rho]$ be as in Theorem 1.2. We have, for some constant $\mathcal{H}$ depending only on m, M, C

$$\|\Psi[\rho]\|_{C^2(\mathbb{T}^d)} \leq \mathcal{H}.$$

3.2 Result

We will now prove the following:

Theorem 3.3. *Let ρ^0 be a probability on $\mathbb{T}^3$, such that ρ is strictly positive and satisfies the continuity condition (12). Then there exists $T > 0$ and C_1, C_2 depending on ρ^0, such that on $[0,T]$ there exists a solution $\rho(t,x)$ of SG that satisfies for all $t \in [0,T]$:*

$$\int_0^1 \frac{w(t,r)}{r} dr \leq C_1, \qquad \|\Psi(t,\cdot)\|_{C^2(\mathbb{T}^3)} \leq C_2,$$

where $w(t,r)$ is the modulus of continuity (in space) of $\rho(t,.)$.

Proof of Theorem 3.3

Let us first sketch the proof: If $\Psi \in C^2$, then the flow $t \to X(t,x)$ generated by the velocity field $[\nabla\Psi(x) - x]^\perp$ is Lipschitz in space. Since the flow is incompressible, we have $\rho(t,x) = \rho^0(X^{-1}(t,x))$.
Now we use the following property: If two functions f, g have modulus of continuity respectively w_f, w_g then $g \circ f$ has modulus $w_g \circ w_f$.
Thus if $X^{-1}(t)$ is Lipschitz, we have $w_{\rho^0 \circ X^{-1}(t)} \leq w_{\rho^0}(L\,\cdot)$ with L the Lipschitz constant of $X^{-1}(t)$ and condition (12) remains satisfied.

Remark 1: Note that Hölder continuous functions satisfy the condition (12).

Remark 2: Note also that we do not need any integrability on $\nabla\rho$ and the solution of the Eulerian system (3, 4, 5) still has to be understood in the distributional sense.

A fixed point argument

Let us introduce the semi-norm

$$\|\mu\|_{\mathcal{C}} = \int_0^1 \frac{w_\mu(r)}{r} dr \tag{13}$$

defined on $\mathcal{P}(\mathbb{T}^3)$, where we recall that w_μ is the modulus of continuity of μ. We denote $\mathcal{P}_C$ the set $\mathcal{P}$ equipped with this semi-norm, *i.e.*

$$\mathcal{P}_C = \{\mu \in \mathcal{P}(\mathbb{T}^3), \|\mu\|_{\mathcal{C}} < \infty\}.$$

From now, we fix ρ^0 a probability density in $\mathcal{P}_C$, satisfying $m \leq \rho^0 \leq M$, where m and M are strictly positive constants. Let μ be a time dependent probability density in $L^\infty([0,T];\mathcal{P}_C)$, such that $m \leq \mu(t) \leq M$ for all t, we consider the solution ρ of the initial value problem:

$$\partial_t\rho + (\nabla\Psi[\mu](x) - x)^\perp \cdot \nabla\rho = 0, \tag{14}$$

$$\rho(t=0) = \rho^0. \tag{15}$$

From Theorem 3.1 and its corollary, the vector field $\mathbf{v}[\mu] = (\nabla\Psi[\mu](x)-x)^\perp$ is C^1 uniformly in time, therefore there exists a unique solution to this equation, by Cauchy-Lipschitz Theorem. This solution can be built by the method of characteristics as follows: Consider the flow $X(t,x)$ of the vector field $\mathbf{v}[\mu]$, then $\rho(t)$ is ρ^0 pushed forward by $X(t)$, *i.e.* $\rho(t) = \rho^0 \circ X^{-1}(t)$. From the incompressibility of $\mathbf{v}[\mu]$ the condition $m \leq \rho^0 \leq M$ implies that for all $t \in [0,T]$, $m \leq \rho(t) \leq M$.

The initial data ρ^0 being fixed, the map $\mu \mapsto \rho$ will be denoted by $\mathcal{F}$.

The spatial derivative of X, D_xX satisfies

$$\partial_t D_xX(t) = D_x\mathbf{v}[\mu](X) D_xX(t), \tag{16}$$

therefore we have

$$|D_xX(t)| \leq \exp(t \sup_{s\in[0,t]} \|D_x\mathbf{v}[\mu](s)\|_{L^\infty}). \tag{17}$$

We have also from (16),

$$\partial_t \left[D_xX^{-1}(t,X(t,x))\right] = D_xX^{-1}(t,X(t,x)),$$

hence

$$|D_x X^{-1}(t)| \leq \exp(t \sup_{s\in[0,t]} \|D_x \mathbf{v}[\mu](s)\|_{L^\infty}). \tag{18}$$

Since $w_{f\circ g} \leq w_f \circ w_g$, and writing $C_t = \exp(t \sup_{s\in[0,t]} \|D_x \mathbf{v}[\mu]\|_{L^\infty})$, we obtain $w_{\rho(t)}(\cdot) \leq w_{\rho^0}(C_t \cdot)$, and

$$\begin{aligned}\int_0^1 \frac{w_{\rho(t)}(r)}{r} dr &\leq \int_0^{C_t} \frac{w_{\rho^0}(r)}{r} dr \\ &\leq \int_0^1 \frac{w_{\rho^0}(r)}{r} dr + (M-m)(C_t - 1),\end{aligned}$$

(using that $\forall r, w_\rho(r) \leq M - m$). Therefore,

$$\|\rho(t)\|_{\mathcal{C}} \leq \|\rho^0\|_{\mathcal{C}} + (M-m)(C_t - 1).$$

Now from Corollary 3.2, and m, M being fixed, there exists a non-decreasing function $\mathcal{H}$ such that

$$\|\mathbf{v}[\mu]\|_{C^1} \leq \mathcal{H}(\|\mu\|_{\mathcal{C}}),$$

and so $C_t \leq \exp(t\mathcal{H}(\|\mu\|_{L^\infty([0,t];\mathcal{P}_C)}))$. Hence we can chose $Q > 1$, and then T such that

$$\|\rho^0\|_{\mathcal{C}} + (M-m)\left(\exp(T\ \mathcal{H}(Q\|\rho^0\|_{\mathcal{C}})) - 1\right) = Q\|\rho^0\|_{\mathcal{C}}.$$

Note that for $Q > 1$, we necessarily have $T > 0$. Then the map $\mathcal{F} : \mu \mapsto \rho$ goes now from

$$\mathcal{A} = \left\{\mu,\ \|\mu\|_{L^\infty([0,T];\mathcal{P}_C)} \leq Q\|\rho^0\|_{\mathcal{C}},\ m \leq \mu \leq M\right\}$$

into

$$\mathcal{B} = \left\{\rho,\ \|\rho(t)\|_{\mathcal{C}} \leq \|\rho^0\|_{\mathcal{C}} + (M-m)\left(\exp(t\ \mathcal{H}(Q\|\rho^0\|_{\mathcal{C}})) - 1\ \right), \forall t \in [0,T]\right\},$$

and with our choice of $T = T(Q)$, we have $\mathcal{B} \subset \mathcal{A}$. Moreover from the unconditional bounds

$$\begin{aligned}&\rho \leq M, \\ &\|\mathbf{v}[\mu]\|_{L^\infty([0,T]\times\mathbb{T}^3)} \leq \sqrt{3}/2,\end{aligned}$$

(see the remark after Theorem 1.2 for the second bound) and using equation (14), we have also $\|\partial_t \rho\|_{L^\infty([0,T];W^{-1,\infty})} \leq K(M)$ whenever $\rho = \mathcal{F}(\mu)$.

Call $\tilde{\mathcal{A}}$ (resp. $\tilde{\mathcal{B}}$) the set $\mathcal{A} \cap \{\rho, \|\partial_t \rho\|_{L^\infty([0,T];W^{-1,\infty})} \leq K(M)\}$, (resp. $\mathcal{B} \cap \{\rho, \|\partial_t \rho\|_{L^\infty([0,T];W^{-1,\infty})} \leq K(M)\}$); we claim that

- $\mathcal{F}(\tilde{\mathcal{A}}) \subset \tilde{\mathcal{B}} \subset \tilde{\mathcal{A}}$,
- $\tilde{\mathcal{A}}$ is convex and compact for the $C^0([0,T] \times \mathbb{T}^3)$ topology,
- $\mathcal{F}$ is continuous for this topology,

so that we can apply the Schauder fixed point Theorem. We only check the last point, the second being a classical result of functional analysis. So let us consider a sequence $(\mu_n)_{n\in\mathbb{N}}$ converging to $\mu \in \mathcal{A}$, and the corresponding sequence $(\rho_n = \mathcal{F}(\mu_n))_{n\in\mathbb{N}}$. The sequence ρ_n is pre-compact in $C^0([0,T]\times\mathbb{T}^3)$, from the previous point, and we see (with the stability Theorem 2.6) that it converges to a solution ρ of

$$\partial_t \rho + \nabla \cdot (\rho \mathbf{v}[\mu]) = 0.$$

But, $\mathbf{v}[\mu]$ being Lipschitz, this solution is unique, and therefore $\mathcal{F}(\mu_n)$ converges to $\mathcal{F}(\mu)$, which proves the continuity of $\mathcal{F}$, and ends the proof of existence by the Schauder fixed point Theorem.

□

We state here some consequences of the previous result:

Corollary 3.4. *Let $\rho^0 \in \mathcal{P}(\mathbb{T}^3)$, such that $0 < m \leq \rho \leq M$.*

1. *If $\rho^0 \in C^\alpha, \alpha \in]0,1]$, for $T^* > 0$ depending on ρ^0, a solution $\rho(t,x)$ to (3,4,5) exists in $L^\infty([0,T^*[, C^\alpha(\mathbb{T}^3))$.*
2. *If $\rho^0 \in W^{1,p}, p > 3$, for $T^* > 0$ depending on ρ^0, a solution $\rho(t,x)$ to (3,4,5) exists in $L^\infty([0,T[, W^{1,p}(\mathbb{T}^3))$.*
3. *If $\rho^0 \in C^{k,\alpha}, \alpha \in]0,1], k \in \mathbb{N}$, for $T^* > 0$ depending on ρ^0, a solution $\rho(t,x)$ to (3,4,5) exists in $L^\infty([0,T^*[, C^{k,\alpha}(\mathbb{T}^3))$.*

Moreover, for these solutions, the velocity field is respectively in $C^{1,\alpha}(\mathbb{T}^3)$, $W^{2,p}(\mathbb{T}^3)$, and $C^{k+1,\alpha}(\mathbb{T}^3)$ on $[0,T^*[$.

Proof. We prove only the first point. We use the representation formula $\rho(t) = \rho^0(X^{-1}(t))$. Since Hölder continuous functions satisfy condition (12), we can construct a solution such that $X^{-1}(t)$ remains Lipschitz with respect to the x variable. Then composing a Hölder continuous function with a Lipschitz function, we obtain a Hölder continuous function, which yields the result.

4 Uniqueness of solutions to SG with Hölder continuous densities

4.1 Result

Here we prove the following theorem:

Theorem 4.1. *Suppose that $\rho^0 \in \mathcal{P}(\mathbb{T}^3)$ with $0 < m \leq \rho^0 \leq M$, and belongs to $C^\alpha(\mathbb{T}^3)$ for some $\alpha > 0$. From Theorem 3.3, for some $T > 0$ there exists a solution $\bar\rho$ to SG in $L^\infty([0,T], C^\alpha(\mathbb{T}^3))$. Then every solution of SG in $L^\infty([0,T'], C^\beta(\mathbb{T}^3))$ for $T' > 0, \beta > 0$ with same initial data coincides with $\bar\rho$ on $[0, \inf\{T, T'\}]$.*

Remark 1: The uniqueness of weak solutions is still an open question.

Remark 2: Our proof of uniqueness is thus valid in a smaller class of solutions than the one found in the previous section, the reason is the following: During the course of the proof, we will need to solve a Monge-Ampère equation, whose right-hand side is a function of the second derivatives of the solution of another Monge-Ampère equation. In Theorem 3.1, if u is solution to (11) with a right hand side satisfying (12), although $u \in C^2$, it is not clear that the second derivatives of u satisfy (12). Actually, it is even known to be wrong in the case of the Laplacian (for a precise discussion on the subject, the reader may refer to [15]). However, from Theorem 4.3 below, if $\rho \in C^\alpha$ then $u \in C^{2,\alpha}$.

What we actually need is a continuity condition on the right hand side of (11) such that the second derivative of the solution u satisfies (12). This may be a weaker condition than Hölder continuity, however the proof would not be affected, therefore it is enough to give it under the present form.

Proof of Theorem 4.1

Let ρ_1 and ρ_2 be two solutions of (3, 4, 5), in $L^\infty([0,T], C^\beta(\mathbb{T}^3))$ that coincide at time 0. Let X_1, X_2 be the two corresponding Lagrangian solutions, (*i.e.* solutions of (1,2)). The velocity field being C^1, for all $t \in [0,T]$, $X_1(t,\cdot)$ and $X_2(t,\cdot)$ are both C^1 diffeomorphisms of $\mathbb{T}^d$.

We call $\mathbf{v}_1$ (resp. $\mathbf{v}_2$) the velocity field associated to X_1 (resp. X_2), $\mathbf{v}_i(t,x) = [\nabla\Psi_i(t,x) - x]^\perp, i = 1,2$. We have

$$\begin{aligned}\partial_t(X_1 - X_2) &= \mathbf{v}_1(X_1) - \mathbf{v}_2(X_2)\\ &= (\mathbf{v}_1(X_1) - \mathbf{v}_1(X_2)) + (\mathbf{v}_1(X_2) - \mathbf{v}_2(X_2)).\end{aligned}$$

We want to obtain a Gronwall type inequality for $\|X_1 - X_2\|_{L^2}$. Since $\mathbf{v}_1$ is uniformly Lipschitz in space (from Theorem 3.3), the first bracket is estimated in L^2 norm by $C\|X_1 - X_2\|_{L^2}$.
We now need to estimate the second term. We first have that

$$\int |\mathbf{v}_1(X_2) - \mathbf{v}_2(X_2)|^2 = \int \rho_2 |\nabla\Psi_1 - \nabla\Psi_2|^2,$$

and since ρ_2 is bounded, we need to estimate $\|\nabla\Psi_1 - \nabla\Psi_2\|_{L^2}$. This will be done in the following Proposition:

Proposition 4.2. *Let X_1, X_2 be mappings from $\mathbb{T}^d$ into itself, such that the densities $\rho_i = X_{i\#}dx, i = 1,2$ are in $C^\alpha(\mathbb{T}^d)$ for some $\alpha > 0$, and satisfy $0 < m \le \rho_i \le M$. Let $\Psi_i, i = 1,2$ be convex such that*

$$\det D^2\Psi_i = \rho_i$$

in the sense of Theorem 1.1, i.e. $\Psi_i = \Psi[\rho_i]$. Then

$$\|\nabla\Psi_1 - \nabla\Psi_2\|_{L^2} \le C\|X_1 - X_2\|_{L^2},$$

where C depends only on α (the Hölder index of ρ_i), $\|\rho_i\|_{C^\alpha(\mathbb{T}^d)}$, m and M.

Before giving a proof of this result, we conclude the proof of the Theorem 4.1. The Proposition 4.2 implies immediately that

$$\|\partial_t(X_1 - X_2)\|_{L^2} \le C\|X_1 - X_2\|_{L^2},$$

and we conclude the proof of the Theorem by a standard Gronwall lemma.

□

4.2 Energy estimates along Wasserstein geodesics: Proof of Proposition 4.2.

In the proof of this result we will need the following result on optimal transportation of measures by gradient of convex functions:

Theorem 4.3 (Brenier, [3], McCann, [23], Cordero-Erausquin, [9], Caffarelli,[6]). *Let ρ_1, ρ_2 be two probability measures on $\mathbb{T}^d$, such that ρ_1 is absolutely continuous with respect to the Lebesgue measure.*

1. *There exists a convex function ϕ such that $\phi - |\cdot|^2/2$ is $\mathbb{Z}^d$ periodic, satisfying $\nabla\phi_{\#}\rho_1 = \rho_2$.*

2. *The map* $\nabla\phi$ *is the* $d\rho_1$ *a.e. unique solution of the minimization problem*

$$\inf_{T_\#\rho_1=\rho_2}\int_{\mathbb{T}^d}\rho_1(x)|T(x)-x|^2_{\mathbb{T}^d}\,dx, \tag{19}$$

and for all $x\in\mathbb{R}^d$, $|\nabla\phi(x)-x|_{\mathbb{T}^d}=|\nabla\phi(x)-x|_{\mathbb{R}^d}$.

3. *If* ρ_1, ρ_2 *are strictly positive and belong to* $C^\alpha(\mathbb{T}^d)$ *for some* $\alpha>0$ *then* $\phi\in C^{2,\alpha}(\mathbb{T}^d)$ *and satisfies pointwise*

$$\rho_2(\nabla\phi)\det D^2\phi=\rho_1.$$

For complete references on the optimal transportation problem (19) and its applications, the reader can refer to [24].

Remark 1: The expression $|\cdot|_{\mathbb{T}^d}$ denotes the Riemannian distance on the flat torus, whereas $|\cdot|_{\mathbb{R}^d}$ is the Euclidean distance on $\mathbb{R}^d$. The second assertion of point 2 means that, for all $x\in\mathbb{R}^d$, $|\nabla\phi(x)-x|\le\operatorname{diam}(\mathbb{T}^d)=\sqrt{d}/2$.

Remark 2: Here again, note that since $\phi-|\cdot|^2/2$ is periodic, the map $x\mapsto\nabla\phi(x)$ is compatible with the equivalence classes of $\mathbb{R}^d/\mathbb{Z}^d$, and therefore is defined without ambiguity on $\mathbb{T}^d$.

Wasserstein geodesics between probability measures

In this part we use results from [2], [22]. Using Theorem 4.3, we consider the unique (up to a constant) convex potential ϕ such that

$$\begin{aligned}&\nabla\phi_\#\rho_1=\rho_2,\\&\phi-|\cdot|^2/2 \text{ is } \mathbb{Z}^d-\text{periodic.}\end{aligned}$$

We consider, for $\theta\in[1,2]$, ϕ_θ defined by

$$\phi_\theta=(2-\theta)\frac{|x|^2}{2}+(\theta-1)\phi.$$

We also consider, for $\theta\in[1,2]$, ρ_θ defined by

$$\rho_\theta=\nabla\phi_{\theta\ \#}\rho_1.$$

Then ρ_θ interpolates between ρ_1 and ρ_2. This interpolation has been introduced in [2] and [22] as the time continuous formulation of the Monge-Kantorovitch mass transfer. In this construction, a velocity field v_θ is defined $d\rho_\theta$ a.e. as follows:

$$\begin{aligned}\forall f\in C^0(\mathbb{T}^d;\mathbb{R}^d),\ \int\rho_\theta v_\theta\cdot f\ &=\ \int\rho_1 f(\nabla\phi_\theta)\cdot\partial_\theta\nabla\phi_\theta\\&=\ \int\rho_1 f(\nabla\phi_\theta)\cdot(\nabla\phi(x)-x). \qquad (20)\end{aligned}$$

It is easily checked that the pair ρ_θ, v_θ satisfies

$$\partial_\theta \rho_\theta + \nabla \cdot (\rho_\theta v_\theta) = 0,$$

and for any $\theta \in [1,2]$, we have (see [2]):

$$\frac{1}{2}\int_{\mathbb{T}^d} \rho_\theta |v_\theta|^2 = \frac{1}{2}\int_{\mathbb{T}^d} \rho_1 |\nabla\phi(x) - x|^2 = W_2^2(\rho_1, \rho_2),$$

where $W_2(\rho_1, \rho_2)$ is the Wasserstein distance between ρ_1 and ρ_2, defined by

$$W_2^2(\rho_1, \rho_2) = \inf_{T_\# \rho_1 = \rho_2} \left\{ \int \rho_1(x) |T(x) - x|^2_{\mathbb{T}^d} \right\}.$$

The Wasserstein distance can also be formulated as follows:

$$W_2^2(\rho_1, \rho_2) = \inf_{Y_1, Y_2} \left\{ \int_{\mathbb{T}^d} |Y_1 - Y_2|^2_{\mathbb{T}^d} \right\}$$

where the infimum is performed over all maps $Y_1, Y_2 : \mathbb{T}^d \mapsto \mathbb{T}^d$ such that $Y_{i\#} dx = \rho_i, i = 1, 2$. From this definition we have easily

$$W_2^2(\rho_1, \rho_2) \leq \int |X_2(t,a) - X_1(t,a)|^2 \, da,$$

and it follows that, for every $\theta \in [1,2]$,

$$\int_{\mathbb{T}^d} \rho_\theta |v_\theta|^2 = W_2^2(\rho_1, \rho_2) \leq \|X_2 - X_1\|_{L^2}. \tag{21}$$

Regularity of the interpolant measure ρ_θ

From Theorem 4.3, for $\rho_1, \rho_2 \in C^\beta$ and pinched between the positive positive constants m and M , we know that $\phi \in C^{2,\beta}$ and satisfies

$$\det D^2 \phi = \frac{\rho_1}{\rho_2(\nabla\phi)}.$$

We now estimate $\rho_\theta = \rho_1 [\det D^2 \phi_\theta]^{-1}$. From the concavity of $\log(\det(\cdot))$ on symmetric positive matrices, we have

$$\begin{aligned} \det D^2 \phi_\theta &= \det((2-\theta)I + (\theta - 1) D^2\phi) \\ &\geq [\det D^2 \phi]^{\theta - 1} \\ &\geq \frac{m}{M}. \end{aligned}$$

Moreover, since $\phi \in C^2$, $\det D^2 \phi_\theta$ is bounded by above. Thus ρ_θ is uniformly bounded away from 0 and infinity, and uniformly Hölder continuous.

Final energy estimate

If we consider, for every $\theta \in [1,2]$, Ψ_θ solution of

$$\det D^2\Psi_\theta = \rho_\theta, \tag{22}$$

in the sense of Theorem 1.2, and we impose that

$$\int_{\mathbb{T}^d} \Phi_\theta = 0 \tag{23}$$

(see [19]). Then Ψ_θ interpolates between Ψ_1 and Ψ_2, and $\Psi_\theta \in C^{2,\beta}$ uniformly, from the regularity of ρ_θ. We will estimate $\partial_\theta \nabla \Psi_\theta$ by differentiating (22) with respect to θ. The fact that Ψ_θ, Φ_θ is differentiable with respect to θ is a consequence of the results of [19]. We will have, following the a priori estimate of [19, Propositon 5.1, Theorem 2.3]

$$\begin{aligned}
&\partial_\theta \nabla \Phi_\theta, \partial_\theta \nabla \Psi_\theta \in L^\infty([1,2], L^2(\mathbb{T}^d)),\\
&\partial_\theta \Phi_\theta, \partial_\theta \Psi_\theta \in L^\infty([1,2], C^\gamma(\mathbb{T}^d)),
\end{aligned}$$

for some $\gamma \in]0,1[$. (Note that we need the condition (23).)

Let us obtain a precise quantitative estimate in our present case. First we recall the following fact: For M, N two $d \times d$ matrices, $t \in \mathbb{R}$

$$\det(M + tN) = \det M + t\ (\text{trace } M_{co}^t N) + o(t),$$

where M_{co} is the co-matrix (or matrix of cofactors) of M. Moreover, for any $f \in C^2(\mathbb{R}^d; \mathbb{R})$, if M is the co-matrix of $D^2 f$, it is a common fact that

$$\forall j \in \{1..d\}, \sum_{i=1}^d \partial_i M_{ij} \equiv 0. \tag{24}$$

Hence, denoting M_θ the co-matrix of $D^2\Psi_\theta$, we obtain that $\partial_\theta \Psi_\theta$ satisfies

$$\begin{aligned}
\nabla \cdot (M_\theta \nabla \partial_\theta \Psi_\theta) &= \partial_\theta \rho_\theta(t)\\
&= -\nabla \cdot (\rho_\theta v_\theta),
\end{aligned} \tag{25}$$

where v_θ is given by (20). From the $C^{2,\beta}$ regularity of Ψ_θ, $D^2\Psi_\theta$ is a C^β smooth, positive definite matrix, and its co-matrix M_θ as well. Thus the problem (25) is uniformly elliptic. If we multiply by $\partial_\theta \Psi_\theta$, and integrate by parts we obtain

$$\int \nabla^t \partial_\theta \Psi_\theta\, M_\theta\, \nabla \partial_\theta \Psi_\theta = -\int \nabla \partial_\theta \Psi_\theta \cdot v_\theta \rho_\theta.$$

Using that $M_\theta \geq \lambda I$ for some $\lambda > 0$, and combining with the inequality (21) above, we obtain

$$\begin{aligned} \|\nabla \partial_\theta \Psi_\theta(t)\|_{L^2} &\leq \lambda^{-1} \|\rho_\theta v_\theta\|_{L^2} \\ &\leq \lambda^{-1} \|X_2 - X_1\|_{L^2} \left(\sup_\theta \|\rho_\theta\|_{L^\infty} \right)^{1/2} . \end{aligned}$$

The constant λ^{-1} depends only on $m, M, \beta, \{\|\rho_i\|_{C^\beta}, i = 1, 2\}$, and is thus bounded under our present assumptions. We have already seen that ρ_θ is uniformly bounded, and we finally obtain that

$$\|\nabla \Psi_1 - \nabla \Psi_2\|_{L^2} \leq C \|X_1 - X_2\|_{L^2}, \tag{26}$$

this ends the proof of Proposition 4.2.

□.

Remark 1. In [19], the author obtains also (weaker) estimates of the type of Proposition 4.2, for discontinuous densities ρ_1, ρ_2.

5 Uniqueness of solutions to the 2-d Euler equations with bounded vorticity: A new proof

As stated in the introduction, the method that we have presented here to show uniqueness of solutions to SG is in fact quite general. It has been shown in [18] to yield a uniqueness result for solutions to the Vlasov-Poisson system under the only condition that the density in the physical space is bounded. In that paper it was also shown that the method could give a new proof of Youdovich's theorem for solutions in the whole space $\mathbb{R}^2$.

We give here a simplified version of this proof in the periodic case.

We start now from the following system:

$$\partial_t \rho + \nabla \cdot (\rho \nabla \psi^\perp) = 0, \tag{27}$$

$$\rho = \Delta \psi, \tag{28}$$

$$\rho(t = 0) = \rho^0. \tag{29}$$

We restrict ourselves to the periodic case, *i.e.* $x \in \mathbb{T}^2$, ρ, ψ periodic, this implies that ρ has total mass equal to 0. We reprove the following classical result:

Theorem 5.1 (Youdovich, [26]). *Given an initial data $\rho^0 \in L^\infty(\mathbb{T}^2)$ satisfying $\int_{\mathbb{T}^2} \rho^0 = 0$, there exists a unique solution to (27, 28, 29) such that ρ belongs to $L^\infty_{loc}(\mathbb{R}^+ \times \mathbb{T}^2)$.*

Proof of Theorem 5.1

We consider two solutions ρ_1, ψ_1 and ρ_2, ψ_2, such that $\rho_i, i = 1, 2$ are bounded in $L^\infty([0,T] \times \mathbb{T}^d)$. In this case the velocity fields $\mathbf{v}_i = \nabla \psi_i^\perp$ both satisfy (see [21, Chapter 8])

$$\forall (x,y) \in \mathbb{T}^2,\ |x-y| \leq \frac{1}{2},\ |\mathbf{v}_i(x) - \mathbf{v}_i(y)| \leq C|x-y| \log \frac{1}{|x-y|}.$$

The flows $(t,x) \mapsto X_i(t,x)$ associated to the velocity fields $\mathbf{v}_i = \nabla \psi_i^\perp$ are then Hölder continuous, and one has, for all $t \in [0,T]$, $\rho_i(t) = X_i(t)_\# \rho^0$.

Applying the same technique as before, we need to estimate $\|\nabla \psi_1 - \nabla \psi_2\|_{L^2(\mathbb{T}^2)}$ in terms of $\|X_1 - X_2\|_{L^2(\mathbb{T}^2)}$. In the present case, the energy estimate of Proposition 4.2 will hold under the weaker assumptions that the two densities are bounded.

Proposition 5.2. *Let X_1, X_2 be continuous injective mappings from $\mathbb{T}^d$ into itself, let ρ^0 be a bounded measure, with $\int_{\mathbb{T}^d} \rho^0 = 0$. Let $\rho_i = X_{i\#}\rho^0, i = 1, 2$. Assume that ρ_1, ρ_2 have densities in L^∞ with respect to the Lebesgue measure. Let $\psi_i, i = 1, 2$ be periodic solutions of $\Delta \psi_i = \rho_i, i = 1, 2$, then we have*

$$\|\nabla \psi_1 - \nabla \psi_2\|_{L^2(\mathbb{T}^d)} \leq \left(2 \max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\} \|\rho_0\|_{L^\infty}\right)^{1/2} \|X_1 - X_2\|_{L^2(\mathbb{T}^d)}.$$

Remark 1: In other words, this proposition shows that for ρ_1, ρ_2 bounded, the H^{-1} norm of $\rho_1 - \rho_2$ is controlled by some 'generalized' (since here we have unsigned measures) Wasserstein distance between ρ_1 and ρ_2.

Remark 2: We see that we obtain a result as in Proposition 4.2 under the weaker condition that the densities are bounded in L^∞ (and not in C^α). This is because the Laplacian is uniformly elliptic, independently of the regularity of the solution, while the Monge-Ampère operator is uniformly elliptic only for C^2 solutions.

To conclude the proof of Theorem 5.1, note first that for all $C > 0$, we can take T small enough so that $\|X_2 - X_1\|_{L^\infty([0,T] \times \mathbb{T}^2)} \leq C$. Now we have for the difference $X_1 - X_2$, as long as $|X_1 - X_2| \leq 1/2$,

$$\begin{aligned} & \|\partial_t (X_1 - X_2)\|_{L^2} \\ \leq\ & \|\nabla \psi_1(X_1) - \nabla \psi_1(X_2)\|_{L^2} + \|\nabla \psi_1(X_2) - \nabla \psi_2(X_2)\|_{L^2} \\ \leq\ & C_1 \||X_1 - X_2| \log(|X_1 - X_2|)\|_{L^2} + C_2 \|X_1 - X_2\|_{L^2}, \end{aligned}$$

where we have used Proposition 5.2 to evaluate the second term. We just need to evaluate $\||X_1 - X_2| \log(|X_1 - X_2|)\|_{L^2}$. We take T small enough so that $\|X_2 - X_1\|_{L^\infty([0,T]\times\mathbb{T}^2)} \leq 1/e$ and notice that $x \mapsto x\log^2 x$ is concave for $0 \leq x \leq 1/e$, therefore by Jensen's inequality we have

$$\begin{aligned} & \int_{\mathbb{T}^2} |X_2 - X_2|^2 \log^2(|X_1 - X_2|) \\ = \; & \frac{1}{4}\int_{\mathbb{T}^2} |X_2 - X_2|^2 \log^2(|X_1 - X_2|^2) \\ \leq \; & \frac{1}{4}\int_{\mathbb{T}^2} |X_2 - X_1|^2 \, \log^2\left(\int_{\mathbb{T}^2} |X_2 - X_1|^2\right), \end{aligned}$$

and some elementary computations finally yield

$$\partial_t \|X_2 - X_1\|_{L^2} \leq C\|X_2 - X_1\|_{L^2} \log \frac{1}{\|X_2 - X_1\|_{L^2}}.$$

The conclusion $X_1 \equiv X_2$ follows then by standard arguments.

5.1 Energy estimates along Wasserstein geodesic: Proof of Proposition 5.2

The proof of this proposition is very close to the proof of Proposition 4.2, and we will only sketch it, insisting on the specific points. Here the densities ρ_i can not be of constant sign, since their mean value is zero, hence we introduce ρ_i^+ (resp. ρ_i^-) the positive (resp. negative) part of ρ_i, *i.e.* $\rho_i = \rho_i^+ - \rho_i^-$. The mappings X_i are supposed injective, therefore we have $X_{i\#}\rho_0^\pm = \rho_i^\pm$. Now, $\rho_i^\pm$ are positive measures of total mass equal to say M, with $M < \infty$.

Wasserstein geodesic

We interpolate between the positive parts ρ_i^+, and the negative part is handled in the same way. As before we introduce the density $\rho_\theta^+(t)$ that interpolates between $\rho_1^+(t)$ and $\rho_2^+(t)$. In this interpolation, we consider v_θ^+ such that

$$\partial_\theta \rho_\theta^+ + \nabla \cdot (\rho_\theta^+ v_\theta^+) = 0, \tag{30}$$

and we introduce as well $\rho_\theta^-, v_\theta^-$. Then $\rho_\theta = \rho_\theta^+ - \rho_\theta^-$ has mean value 0. Let the potential ψ_θ be solution to

$$\Delta\psi_\theta = \rho_\theta. \tag{31}$$

Note that ρ_θ has mean value zero therefore this equation is well posed on $\mathbb{T}^2$, moreover ψ_θ interpolates between ψ_1 and ψ_2.

Bound on the interpolant measure ρ_θ

Instead of interpolating between two smooth densities, we interpolate between bounded densities, and use the following result from [22]:

Proposition 5.3 (McCann, [22]). *Let ρ_θ^+ be the Wasserstein geodesic linking ρ_1^+ to ρ_2^+ defined above. Then, for all $\theta \in [1,2]$,*

$$\|\rho_\theta^+\|_{L^\infty} \leq \max\left\{\|\rho_1^+\|_{L^\infty}, \|\rho_2^+\|_{L^\infty}\right\}.$$

The same holds for ρ_i^-, ρ_θ^-.

Remark: This property is often referred to as 'displacement convexity'.

Energy estimates

Now we impose that $\int_{\mathbb{T}^d} \phi_\theta = 0$. Since $\rho_\theta^\pm, v_\theta^\pm$ are uniformly bounded in L^∞, we have using (30) that $\partial_\theta \rho_\theta \in L^\infty([1,2]; W^{-1,\infty}(\mathbb{T}^d))$. We can thus differentiate (31) with respect to θ, to obtain

$$\Delta \partial_\theta \psi_\theta = \partial_\theta \rho_\theta = -\nabla \cdot (\rho_\theta^+ v_\theta^+ - \rho_\theta^- v_\theta^-), \tag{32}$$

with $v_\theta^\pm$ the interpolating velocity defined as in (20), and satisfying for all $\theta \in [1,2]$,

$$\int \rho_\theta^\pm(t) |v_\theta^\pm|^2(t) = W_2^2(\rho_1^\pm(t), \rho_2^\pm(t)).$$

Multiplying (32) by $\partial_\theta \psi_\theta$, and integrating over $\theta \in [1,2]$, we obtain

$$\begin{aligned}
\|\nabla\psi_1 - \nabla\psi_2\|_{L^2(\mathbb{T}^d)} &\leq \int_{\theta=1}^2 \|\rho_\theta^+ v_\theta^+\|_{L^2} + \|\rho_\theta^- v_\theta^-\|_{L^2} \\
&\leq W_2(\rho_1^+, \rho_2^+) \left(\sup_\theta \|\rho_\theta^+\|_{L^\infty}\right)^{1/2} \\
&+ W_2(\rho_1^-, \rho_2^-) \left(\sup_\theta \|\rho_\theta^-\|_{L^\infty}\right)^{1/2}.
\end{aligned}$$

Note that the energy estimate is easier here than in the Monge-Ampère case, since the problem is immediately uniformly elliptic.

The mappings X_i are injective and satisfy $X_{i\#}\rho_0 = \rho_i$, therefore we have $X_{i\#}(\rho_0^\pm) = \rho_i^\pm$. Hence,

$$\begin{aligned}
W_2^2(\rho_1^\pm, \rho_2^\pm) &\leq \int \rho_0^\pm |X_1 - X_2|^2 \\
&\leq \|\rho_0\|_{L^\infty} \|X_1 - X_2\|_{L^2}.
\end{aligned}$$

Using Proposition 5.3, we conclude:

$$\begin{aligned} & \|\nabla\psi_1 - \nabla\psi_2\|_{L^2(\mathbb{T}^d)} \\ \leq\ & 2\|\rho_0\|_{L^\infty}^{1/2}\|X_2 - X_1\|_{L^2}\left(\max\left\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\right\}\right)^{1/2}. \end{aligned}$$

This ends the proof of Proposition 5.2. Note that in our specific case, X_i are Lebesgue measure preserving invertible mappings, therefore $\|\rho_i^\pm\|_{L^\infty} = \|\rho_0^\pm\|_{L^\infty}$, and the estimate can be simplified in

$$\|\nabla\psi_1 - \nabla\psi_2\|_{L^2(\mathbb{T}^d)} \leq 2\|\rho_0\|_{L^\infty}\|X_2 - X_1\|_{L^2(\mathbb{T}^d)}.$$

□

6 Convergence to the Euler equation

6.1 Scaling of the system

Here we present a rescaled version of the 2-d *SG* system and some formal arguments to motivate the next convergence results. Here $x \in \mathbb{T}^2, t \in \mathbb{R}^+$ and for $\mathbf{v} = (v_1, v_2) \in \mathbb{R}^2$, $\mathbf{v}^\perp$ now means $(-v_2, v_1)$. Introducing $\psi[\rho] = \Psi[\rho] - |x|^2/2$, where $\Psi[\rho]$ is given by Theorem 1.2, the periodic 2-d *SG* system now reads

$$\begin{aligned} &\partial_t\rho + \nabla\cdot(\rho\nabla\psi^\perp) = 0, \\ &\det(I + D^2\psi) = \rho. \end{aligned}$$

If ρ is close to one then ψ should be small, and therefore one may consider the linearization $\det(I+D^2\psi) = 1+\Delta\psi+O(|D^2\psi|^2)$, that yields $\Delta\psi \simeq \rho-1$. Thus for small initial data, *i.e.* ρ^0-1 small, one expects $\psi, \mu = \rho-1$ to stay close to a solution of the Euler incompressible equation EI

$$\partial_t\bar\rho + \nabla\cdot(\bar\rho\nabla\bar\phi^\perp) = 0, \tag{33}$$

$$\Delta\bar\phi = \bar\rho. \tag{34}$$

We shall rescale the equation, in order to consider quantities of order one. We introduce the new unknown

$$\begin{aligned} \rho^\epsilon(t,x) &= \frac{1}{\epsilon}(\rho(\frac{t}{\epsilon},x) - 1), \\ \psi^\epsilon(t,x) &= \frac{1}{\epsilon}\psi(\frac{t}{\epsilon},x). \end{aligned}$$

Then we have

$$\rho(t) = 1 + \epsilon\rho^\epsilon(\epsilon t),$$
$$\Psi[\rho](t) = |x|^2/2 + \epsilon\psi^\epsilon(\epsilon t),$$

and we define ϕ^ϵ by

$$\epsilon\phi^\epsilon = |x|^2/2 - \Phi[\rho],$$

so that

$$\nabla\phi^\epsilon = \nabla\psi^\epsilon(\nabla\Phi[\rho]). \tag{35}$$

Hence, at a point $x \in \mathbb{T}^2$, $\nabla\phi^{\epsilon\perp}$ is the velocity of the associated dual point $\nabla\Phi[\rho](x)$. The evolution of this quantities is then governed by the system SG_ϵ

$$\partial_t\rho^\epsilon + \nabla\cdot(\rho^\epsilon\nabla\psi^{\epsilon\perp}) = 0, \tag{36}$$
$$\det(I + \epsilon D^2\psi^\epsilon) = 1 + \epsilon\rho^\epsilon. \tag{37}$$

Remark: Note that this system admits global weak solutions with initial data any bounded measure $\rho^{\epsilon\,0}$, as long as

$$\int_{\mathbb{T}^2}\rho^{\epsilon\,0} = 0, \tag{38}$$
$$\rho^{\epsilon\,0} \geq -\frac{1}{\epsilon}. \tag{39}$$

Note also that if the pair $(\bar{\rho}, \bar{\phi})$ is solution to the EI system (33, 34), so is the pair $\left(\frac{1}{\epsilon}\bar{\rho}(\frac{t}{\epsilon}, x),\ \frac{1}{\epsilon}\bar{\phi}(\frac{t}{\epsilon}, x)\right)$.

We now present the convergence results. We show that solutions of SG_ϵ converge to solutions of EI in the following sense: If $\rho^{\epsilon\,0}$, the initial data of SG_ϵ, is close (in some sense depending on the type of convergence we wish to show) to a smooth initial data $\bar{\rho}^0$ for EI, then ρ^ϵ and $\bar{\rho}$ remain close for some time. This time goes to ∞ when ϵ goes to 0.

We present two different versions of this result: The first one is for weak solutions of SG_ϵ, and the second one is for Lipschitz solutions.

6.2 Convergence of weak solutions

Theorem 6.1. *Let $(\rho^\epsilon, \psi^\epsilon)$ be a weak solution of the SG_ϵ system (36, 37). Let $(\bar{\rho}, \bar{\phi})$ be a smooth $C^3([0,T]\times\mathbb{T}^2)$ solution of the EI system (33, 34). Let ϕ^ϵ be obtained from ψ^ϵ as in (35), let $H_\epsilon(t)$ be defined by*

$$H_\epsilon(t) = \frac{1}{2}\int_{\mathbb{T}^2}\left|\nabla\phi^\epsilon - \nabla\bar{\phi}\right|^2,$$

then

$$H_\epsilon(t) \leq (H_\epsilon(0) + C\epsilon^{2/3}(1+t)) \exp Ct$$

where C *depends only on* $\sup_{0\leq s\leq t}\{\|D^3\bar\phi(s), D^2\partial_t\bar\phi(s)\ \|_{L^\infty(\mathbb{T}^2)}\}$.

In particular, if $H_\epsilon(0) \leq C_0\epsilon^{2/3}$*, we have for all* $T > 0, t \in [0,T]$,

$$H_\epsilon(t) \leq C_T\epsilon^{2/3}.$$

where C_T *depends on* T, C, C_0 *above.*

Remark 1: Note that $\nabla\phi^{\epsilon\perp}(t,x)$ is the velocity at point $\nabla\Phi[\rho] = x - \epsilon\nabla\phi^\epsilon$. Thus we compare the SG_ϵ velocity at point $x-\epsilon\nabla\phi^\epsilon$ (the dual point of x) with the EI velocity at point x. Our result allows also to compare the velocities at the same point, by noticing that

$$\begin{aligned} G_\epsilon(t) &= \frac{1}{2}\int_{\mathbb{T}^2} \rho\left|\nabla\psi^\epsilon - \nabla\bar\phi\right|^2 \\ &= \frac{1}{2}\int_{\mathbb{T}^2} \left|\nabla\phi^\epsilon - \nabla\bar\phi(x-\epsilon\nabla\phi^\epsilon)\right|^2 \\ &\leq C(H_\epsilon(t) + \epsilon^2) \end{aligned}$$

using the smoothness of $\bar\phi$, and if $\mathbf{v}_{sg_\epsilon}, \mathbf{v}_{ei}$ are the respective velocities of the SG_ϵ and EI systems, $G_\epsilon = \displaystyle\int_{\mathbb{T}^2} \rho^\epsilon|\mathbf{v}_{sg_\epsilon} - \mathbf{v}_{ei}|^2$.

Remark 2: The expansion $\det(I + D^2\psi) = 1 + \Delta\psi + O(|D^2\psi|^2)$, used in the heuristic argument above to justify the convergence relies *a priori* on the control of $D^2\psi$ in the sup norm. But in the Theorem 6.1, the initial data must satisfy $\nabla\psi^\epsilon$ close in L^2 norm to $\nabla\bar\phi$; this condition means that $D^2\psi^\epsilon$ is close in H^{-1} norm to $D^2\bar\phi$, which is smooth. This control does not allow to justify the expansion $\det(I + D^2\psi) = 1 + \Delta\psi + O(|D^2\psi|^2)$, but we see that the result remains valid.

Proof of Theorem 6.1

In all the proof, we use C to denote any quantity that depends only on $\bar\phi$. We use the conservation of the energy of the SG_ϵ system, given by

$$E(t) = \int_{\mathbb{T}^2} |\nabla\phi^\epsilon|^2. \tag{40}$$

This fact, although formally easily justified, is actually not so straightforward for weak solutions, and has been proved by F. Otto in an unpublished work.

The argument is explained in [5]. Therefore $E(t) = E_0$. The energy of the smooth solution of EI is given by

$$\bar{E}(t) = \int_{\mathbb{T}^2} |\nabla \bar{\phi}|^2 \tag{41}$$

and also conserved. For all smooth θ, we will use the notation:

$$< D^2\theta > (t,x) = \int_{s=0}^{1} (1-s) D^2\theta(t, x - s\epsilon\nabla\phi^\epsilon(t,x)).$$

Thus we have the identity

$$\begin{aligned} \int_{\mathbb{T}^2} \rho^\epsilon \theta &= \int_{\mathbb{T}^2} \theta(x - \epsilon\nabla\phi^\epsilon) && (42) \\ &= \int_{\mathbb{T}^2} \theta - \epsilon \int_{\mathbb{T}^2} \nabla\theta \cdot \nabla\phi^\epsilon + \epsilon^2 \int_{\mathbb{T}^2} < D^2\theta > \nabla\phi^\epsilon \nabla\phi^\epsilon. && (43) \end{aligned}$$

Using the energy bound, the last term is bounded by $\epsilon^2 \|D^2\theta\|_{L^\infty(\mathbb{T}^2)} E_0$. Then, using the conservation of the energies E and $\bar{E}$ defined respectively in (40, 41), we have

$$\frac{d}{dt} H_\epsilon(t) = -\frac{d}{dt} \int_{\mathbb{T}^2} \nabla\bar{\phi} \cdot \nabla\phi^\epsilon.$$

Using the identity (43), we have for all smooth θ,

$$\epsilon \int_{\mathbb{T}^2} \nabla\theta \cdot \nabla\phi^\epsilon = -\int_{\mathbb{T}^2} \rho^\epsilon\theta + \int_{\mathbb{T}^2} \theta + \epsilon^2 \int_{\mathbb{T}^2} < D^2\theta > \nabla\phi^\epsilon\nabla\phi^\epsilon,$$

hence, replacing θ by $\bar{\phi}$ in this identity, we get

$$\frac{d}{dt} H_\epsilon(t) = \frac{1}{\epsilon}\frac{d}{dt} \int_{\mathbb{T}^2} [\rho^\epsilon\bar{\phi} - \bar{\phi} - \epsilon^2 < D^2\bar{\phi} > \nabla\phi^\epsilon\nabla\phi^\epsilon].$$

We can suppose without loss of generality that $\int_{\mathbb{T}^2} \bar{\phi}(t,x)\ dx \equiv 0$. Then if we define

$$Q_\epsilon(t) = \int_{\mathbb{T}^2} \epsilon < D^2\bar{\phi} > \nabla\phi^\epsilon\nabla\phi^\epsilon,$$

(note that $|Q_\epsilon(t)| \leq C\epsilon$), we have

$$\frac{d}{dt}(H_\epsilon + Q_\epsilon) = \frac{1}{\epsilon}\frac{d}{dt} \int_{\mathbb{T}^2} \rho^\epsilon\bar{\phi}.$$

Hence we are left to compute

$$\begin{aligned}
\frac{1}{\epsilon}\frac{d}{dt}\int_{\mathbb{T}^2}\rho^\epsilon\bar{\phi} &= \frac{1}{\epsilon}\int_{\mathbb{T}^2}\partial_t\rho^\epsilon\bar{\phi}+\rho^\epsilon\partial_t\bar{\phi} \\
&= \frac{1}{\epsilon}\int_{\mathbb{T}^2}\rho^\epsilon\nabla\psi^{\epsilon\perp}\cdot\nabla\bar{\phi}-\epsilon\nabla\phi^\epsilon\cdot\nabla\partial_t\bar{\phi}+\epsilon^2 < D^2\partial_t\bar{\phi} > \nabla\bar{\phi}\nabla\bar{\phi} \\
&= \frac{1}{\epsilon}\int_{\mathbb{T}^2}\rho^\epsilon\nabla\psi^{\epsilon\perp}\cdot\nabla\bar{\phi}-\int_{\mathbb{T}^2}\nabla\phi^\epsilon\cdot\nabla\partial_t\bar{\phi}+O(\epsilon) \\
&= T_1+T_2+O(\epsilon),
\end{aligned}$$

where at the second line we have used (36) for the first term and (43) with $\theta = \partial_t\bar{\phi}$ for the second and third term. (Remember also that we assume $\int\partial_t\bar{\phi}\equiv 0$.)

We will now use the other formulation of the Euler equation: $\mathbf{v} = \nabla\bar{\phi}^\perp$ satisfies

$$\partial_t\mathbf{v}+\mathbf{v}\cdot\nabla\mathbf{v} = -\nabla p.$$

After a rotation of $\pi/2$, this equation becomes:

$$\partial_t\nabla\bar{\phi}+D^2\bar{\phi}\nabla\bar{\phi}^\perp = \nabla p^\perp,$$

thus for T_2 we have

$$\begin{aligned}
T_2 &= -\int_{\mathbb{T}^2}\nabla\phi^\epsilon\cdot\nabla\partial_t\bar{\phi} \\
&= \int_{\mathbb{T}^2}\nabla\phi^\epsilon D^2\bar{\phi}\nabla\bar{\phi}^\perp.
\end{aligned}$$

For T_1, using (35) and (43), we have

$$\begin{aligned}
\epsilon T_1 &= \int_{\mathbb{T}^2}\rho^\epsilon\nabla\psi^{\epsilon\perp}\cdot\nabla\bar{\phi} \\
&= \int_{\mathbb{T}^2}\nabla\psi^{\epsilon\perp}(x-\epsilon\nabla\phi^\epsilon)\cdot\nabla\bar{\phi}(x-\epsilon\nabla\phi^\epsilon) \\
&= \int_{\mathbb{T}^2}\nabla\phi^{\epsilon\perp}\cdot\nabla\bar{\phi}-\epsilon\nabla\phi^{\epsilon\perp}D^2\bar{\phi}\nabla\phi^\epsilon+\epsilon\Xi
\end{aligned}$$

where Ξ is defined by

$$\Xi = \int_{\mathbb{T}^2}\nabla\phi^{\epsilon\perp}\left(D^2\bar{\phi}-\int_{s=0}^1 D^2\bar{\phi}(x-s\epsilon\nabla\phi^\epsilon)\,ds\right)\nabla\phi^\epsilon. \tag{44}$$

The term $\int_{\mathbb{T}^2}\nabla\phi^{\epsilon\perp}\cdot\nabla\bar{\phi}$ vanishes identically. Concerning Ξ, we claim the following estimate:

Lemma 6.2. *Let Ξ be defined by (44), then*

$$|\Xi| \leq C(\epsilon^{\frac{2}{3}} + H_\epsilon),$$

where C depends on $\|D^3\bar{\phi}\|_{L^\infty}$.

We postpone the proof of this lemma after the proof of Theorem 6.1. We now obtain

$$\frac{d}{dt}(H_\epsilon(t) + Q_\epsilon(t)) \ \leq \ \int_{\mathbb{T}^2} (\nabla\bar{\phi}^\perp - \nabla\phi^{\epsilon\perp}) D^2\bar{\phi}\nabla\phi^\epsilon + CH_\epsilon + C\epsilon^{2/3}.$$

Noticing that for every $\theta : \mathbb{T}^2 \mapsto \mathbb{R}$ we have

$$\int_{\mathbb{T}^2} \nabla\theta^\perp D^2\bar{\phi}\nabla\bar{\phi} = \int_{\mathbb{T}^2} \nabla\theta^\perp \cdot \nabla(\frac{1}{2}|\nabla\bar{\phi}|^2) = 0,$$

we find that

$$\int_{\mathbb{T}^2} (\nabla\bar{\phi}^\perp - \nabla\phi^{\epsilon\perp}) D^2\bar{\phi}\nabla\phi^\epsilon = \int_{\mathbb{T}^2} (\nabla\phi^\perp - \nabla\bar{\phi}^{\epsilon\perp}) D^2\bar{\phi}(\nabla\phi^\epsilon - \nabla\bar{\phi}),$$

hence

$$\begin{aligned}\frac{d}{dt}(H_\epsilon(t) + Q_\epsilon(t)) &\leq -\int_{\mathbb{T}^2} (\nabla\phi^{\epsilon\perp} - \nabla\bar{\phi}^\perp) D^2\bar{\phi}(\nabla\phi^\epsilon - \nabla\bar{\phi}) + CH_\epsilon + C\epsilon^{2/3} \\ &\leq C(H_\epsilon(t) + Q_\epsilon(t) + \epsilon^{2/3})\end{aligned}$$

using that $Q_\epsilon(t) \leq C\epsilon$. Therefore

$$H_\epsilon(t) + Q_\epsilon(t) \leq (H_\epsilon(0) + Q_\epsilon(0) + C\epsilon^{2/3}t)\exp(Ct)$$

and finally

$$H_\epsilon(t) \leq (H_\epsilon(0) + C\epsilon^{2/3}(1+t))\exp(Ct)$$

and the result follows. Check that the constant C depends only on $\sup_{0\leq s\leq t}\{\|D^3\bar{\phi}, D^2\partial_t\bar{\phi}\|_{L^\infty(\mathbb{T}^2)}\}$. This ends the proof of Theorem 6.1

□

Proof of Lemma 6.2

First we show that if $\Theta(R) = \displaystyle\int_{\{|\nabla\phi^\epsilon| \geq R\}} |\nabla\phi^\epsilon|^2$, then, for some $C > 0$

$$\Theta(R) \leq C\int |\nabla\phi^\epsilon - \nabla\bar{\phi}|^2 + \frac{C}{R^2}. \tag{45}$$

Indeed, $\int |\nabla\phi^\epsilon|^2 \leq C$, implies that $\mathrm{meas}\{|\nabla\phi^\epsilon| \geq R\} \leq C\frac{1}{R^2}$. Since $|\nabla\bar{\phi}(t,x)| \leq C$ for $(t,x) \in [0,T'] \times \mathbb{T}^d$, we have

$$\begin{aligned}\Theta(R) &\leq 2\int_{\{|\nabla\phi^\epsilon| \geq R\}} |\nabla\bar{\phi}|^2 + 2\int_{\{|\nabla\phi^\epsilon| \geq R\}} |\nabla\phi^\epsilon - \nabla\bar{\phi}|^2 \\ &\leq \frac{2C}{R^2} + 2\int |\nabla\phi^\epsilon - \nabla\bar{\phi}|^2.\end{aligned}$$

Hence (45) is proved, for C replaced by $\max\{2, 2C\}$.

Then, letting

$$K(x) = D^2\bar{\phi} - \int_{s=0}^{1} D^2\bar{\phi}(x - s\epsilon\nabla\phi^\epsilon)\ ds,$$

we have

$$\Xi \leq C\Theta(R) + \int_{|\nabla\phi^\epsilon| \leq R} |K(x)||\nabla\phi^\epsilon|^2$$

with $|K(x)| \leq C\epsilon|\nabla\phi^\epsilon|$ thus

$$\begin{aligned}&\Xi \leq C\epsilon\int_{|\nabla\phi^\epsilon| \leq R} |\nabla\phi^\epsilon|^3 + C\Theta(R) \\ &\leq C\left(\epsilon R\int |\nabla\phi^\epsilon|^2 + \frac{1}{R^2} + \int |\nabla\phi^\epsilon - \nabla\bar{\phi}|^2\right) \\ &\leq C\left(\epsilon R + \frac{1}{R^2} + \int |\nabla\phi^\epsilon - \nabla\bar{\phi}|^2\right)\end{aligned}$$

for all R, so for $R = \epsilon^{-1/3}$ we obtain:

$$\Xi \leq C\epsilon^{2/3} + C\int |\nabla\phi^\epsilon - \nabla\bar{\phi}|^2.$$

This proves Lemma 6.2

□

6.3 Convergence of strong solutions

We present here another proof of convergence, that holds for stronger norms. Let us consider as above the solution $(\bar{\rho}, \bar{\phi})$ to Euler:

$$\begin{aligned}&\partial_t\bar{\rho} + \nabla\cdot(\bar{\rho}\nabla\bar{\phi}^\perp) = 0, \\ &\Delta\bar{\phi} = \bar{\rho},\end{aligned}$$

and we recall the SG_ϵ system

$$\begin{aligned}&\partial_t \rho^\epsilon + \nabla \cdot (\rho^\epsilon \nabla \psi^{\epsilon\perp}) = 0,\\ &\det(I + \epsilon D^2 \psi^\epsilon) = 1 + \epsilon \rho^\epsilon.\end{aligned}$$

We have then

Theorem 6.3. *Let $(\bar\rho, \bar\phi)$ be a solution of EI, such that that $\bar\rho \in C^2_{loc}(\mathbb{R}^+ \times \mathbb{T}^2)$. Let $\rho^{\epsilon\, 0}$ be a sequence of initial data for SG_ϵ satisfying (38, 39), and such that $\dfrac{\rho^{\epsilon\, 0} - \bar\rho^0}{\epsilon}$ is bounded in $W^{1,\infty}(\mathbb{T}^2)$. Then there exists a sequence $(\rho^\epsilon, \psi^\epsilon)$ of solutions to SG_ϵ that satisfies: For all $T > 0$, there exists $\epsilon_T > 0$, such that the sequence*

$$\frac{\rho^\epsilon - \bar\rho}{\epsilon}, \frac{\nabla\psi^\epsilon - \nabla\bar\phi}{\epsilon}$$

for $0 < \epsilon < \epsilon_T$ is uniformly bounded in $L^\infty([0, T], W^{1,\infty}(\mathbb{T}^2))$.

Remark: In the previous theorem, we obtained estimates in L^2 norm, here we obtain estimates in Lipschitz norm. Estimates of higher derivatives follow in the same way.

Proof of Theorem 6.3

We expand the solution of SG_ϵ as the solution of EI plus a small perturbation of order ϵ and show that this perturbation remains bounded in large norms (at least Lipschitz). We first remark the the assumption on $\bar\rho$ implies that $\forall T > 0$, $\bar\phi \in L^\infty([0, T]; C^3(\mathbb{T}^2))$. Let us write

$$\begin{aligned}\rho^\epsilon &= \bar\rho + \epsilon\rho_1\\ \psi^\epsilon &= \bar\phi + \epsilon\psi_1.\end{aligned}$$

Rewritten in terms of ρ_1, ψ_1, the SG_ϵ system reads:

$$\begin{aligned}&\partial_t \rho_1 + (\nabla\bar\phi + \epsilon\nabla\psi_1)^\perp \cdot \nabla\rho_1 = -\nabla\psi_1^\perp \cdot \nabla\bar\rho,\\ &\Delta\psi_1 + \epsilon \text{ trace } [D^2\psi_1 D^2\bar\phi] + \epsilon^2 \det D^2\psi_1 = \rho_1 - \det D^2\bar\phi.\end{aligned}$$

Differentiating the first equation with respect to space, we find the evolution equation for $\nabla\rho_1$:

$$\begin{aligned}&\partial_t \nabla\rho_1 + ((\nabla\bar\phi + \epsilon\nabla\psi_1)^\perp \cdot \nabla)\nabla\rho_1\\ = \; &-(D^2\bar\phi + \epsilon D^2\psi_1)\nabla\rho_1^\perp - D^2\psi_1\nabla\bar\rho^\perp - D^2\bar\rho\nabla\psi_1^\perp. \qquad (46)\end{aligned}$$

We claim that in order to conclude the proof it is enough to have an estimate of the form

$$\|\psi_1(t,\cdot)\|_{C^{1,1}(\mathbb{T}^2)} \leq C(1+\|\rho_1(t,\cdot)\|_{C^{0,1}(\mathbb{T}^2)}), \tag{47}$$

where C depends on $\bar{\phi}$. Let us admit this bound temporarily, and finish the proof of the theorem: Using (47) and (46), we obtain

$$\frac{d}{dt}\|\nabla\rho_1\|_{L^\infty} \leq C(t)(1+\|\nabla\rho_1\|_{L^\infty}+\epsilon\|\nabla\rho_1\|^2_{L^\infty}),$$

where the constant $C(t)$ depends on the $C^2(\mathbb{T}^2)$ norm of $(\bar{\rho}(t,\cdot),\bar{\phi}(t,\cdot))$. This quantity is bounded on every interval $[0,T]$.

Thus we conclude using Gronwall's lemma that $\|\nabla\rho_1(t,\cdot)\|_{L^\infty(\mathbb{T}^2)}$ remains bounded on $[0,T_\epsilon]$ with T_ϵ going to T as ϵ goes to 0. We then choose T as large as we want, since when $d=2$ the smooth solution to EI is global in time. From estimate (47) the $W^{1,\infty}$ bound on ρ_1 implies a $W^{2,\infty}$ bound on ψ_1. Then, we remember that

$$\rho_1 = \frac{\rho^\epsilon-\bar{\rho}}{\epsilon}, \quad \nabla\psi_1 = \frac{\nabla\psi^\epsilon-\nabla\bar{\phi}}{\epsilon}$$

to conclude the proof of Theorem 6.3.

□

Proof of the estimate (47)

We write the equation followed by ψ_1 as follows:

$$\Delta\psi_1 = -\text{ trace }[\epsilon D^2\psi_1 D^2\bar{\phi}] - \epsilon^2 \det D^2\psi_1 + \rho_1 - \det D^2\bar{\phi}.$$

We recall that

$$\|fg\|_{C^{2,\alpha}} \leq \|f\|_{C^{2,\alpha}}\|g\|_{C^{2,\alpha}},$$

hence, using Schauder $C^{2,\alpha}$ estimates for solutions to Laplace equation (see [14]), we have

$$\|\psi_1\|_{C^{2,\alpha}} \leq C_1(1+\epsilon\|\psi_1\|_{C^{2,\alpha}}+\epsilon^2\|\psi_1\|^2_{C^{2,\alpha}}), \tag{48}$$

where C_1 depends on $\|\bar{\phi}\|_{C^{2,\alpha}}, \|\rho_1\|_{C^\alpha}$. The inequality (48) will be satisfied in two cases: Either for $\|\psi_1\|_{C^{2,\alpha}} \leq C_2$ or for $\|\psi_1\|_{C^{2,\alpha}} \geq C_3\epsilon^{-2}$ where C_2, C_3 are positive constants that depend on C_1.

Now we show that ψ^ϵ, solution of (37), is bounded in $C^{2,\alpha}$ for ρ^ϵ bounded in C^α norm. We consider for $t \in [0,1]$ ψ^ϵ_t the unique up to a constant periodic solution of

$$\det(I + \epsilon D^2\psi^\epsilon_t) = 1 + t\epsilon\rho^\epsilon.$$

Differentiating this equation with respect to t, we find

$$M_{ij}D_{ij}\partial_t\psi^\epsilon_t = \rho^\epsilon,$$

where M is the co-matrix of $I+\epsilon D^2\psi^\epsilon_t$. From the regularity result of Theorem 4.3, M is C^α and strictly elliptic. From Schauder estimates, we have then $\|\partial_t\psi^\epsilon_t\|_{C^{2,\alpha}} \leq C\|\rho^\epsilon\|_{C^{2,\alpha}}$, and integrated over $t \in [0,1]$, we get

$$\|\psi^\epsilon\|_{C^{2,\alpha}} \leq C\|\rho^\epsilon\|_{C^{2,\alpha}}.$$

Hence, since $\psi^\epsilon = \bar{\phi} + \epsilon\psi_1$, we have ψ_1 bounded by C/ϵ in $C^{2,\alpha}$. Hence it can not be bigger than C_3/ϵ^2, and to satisfy (48), we must have

$$\|\psi_1\|_{C^{2,\alpha}} \leq C_2,$$

where C_2 as above depends on $\|\bar{\phi}\|_{C^{2,\alpha}}$, $\|\rho_1\|_{C^\alpha}$. This proves estimate (47).

□

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