Uniqueness of the solution to the Vlasov-Poisson system with bounded density

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Abstract

In this note, we show uniqueness of weak solutions to the Vlasov-Poisson system on the only condition that the macroscopic density ρ defined by $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi)d\xi$ is bounded in L^∞. Our proof is based on optimal transportation.

Résumé Français

Dans ce papier, on montre l'unicité des solutions faibles du système Vlasov-Poisson sous la seule condition que la densité macroscopique ρ définie par $\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi)d\xi$ reste bornée dans L^∞. Notre preuve est basée sur le transport optimal. Elle fournit également une preuve alternative du Théorème de Youdovich pour l'unicité des solutions du système Euler 2-d incompressible avec vorticité dans L^∞.

key words: Vlasov-Poisson system, Optimal transportation, Transport equations.

1 Introduction

The Vlasov-Poisson system (hereafter (VP)) describes the evolution of a cloud of electrons or gravitational matter through the equations

$$\partial_t f + \xi \cdot \nabla_x f - \nabla\Psi \cdot \nabla_\xi f = 0, \tag{1}$$

$$-\Delta\Psi = \epsilon\rho, \tag{2}$$

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where $\rho(t,x) = \int f(t,x,\xi)d\xi$, and $\epsilon > 0$ in the electrostatic (repulsive) case, $\epsilon < 0$ in the gravitational (attractive) case. Here $f(t,x,\xi) \geq 0$ denotes the density of electrons (or matter) at time $t \in \mathbb{R}^+$, position $x \in \mathbb{R}^3$, velocity $\xi \in \mathbb{R}^3$. Equation (2) is understood in the following sense:

$$\Psi(t,x) = \epsilon \int_{\mathbb{R}^3} \rho(t,y) \frac{1}{4\pi|x-y|} dy. \tag{3}$$

We denote by $\mathcal{M}(\mathbb{R}^6)$ (resp. $\mathcal{M}^+(\mathbb{R}^6)$) the set of bounded (resp. bounded and positive) measures on $\mathbb{R}^6$. Given an initial datum $f^0 \in \mathcal{M}^+(\mathbb{R}^6)$, we look for solutions to (1, 2) such that

$$f|_{t=0} = f^0. \tag{4}$$

Definition 1.1 (Weak solutions to (VP)). *For $T > 0$, we will call f a solution to (1, 2, 4) in $\mathcal{D}'([0,T) \times \mathbb{R}^6)$, if*

- *$f \in C([0,T), \mathcal{M}^+(\mathbb{R}^6) - w*)$,*
- *$\forall \varphi \in C_c^\infty([0,T) \times \mathbb{R}^6)$,*

$$\int_{[0,T)\times\mathbb{R}^6} f(\partial_t \varphi + \xi \cdot \nabla_x \varphi - \nabla_x \Psi \cdot \nabla_\xi \varphi) dt dx d\xi = -\int_{\mathbb{R}^6} f^0 \varphi|_{t=0} dx d\xi, \tag{5}$$

- *for all $t \in [0,T[$, $\Psi(t)$ is given by (3).*

We will not discuss the conditions needed on Ψ, f to give sense to the product $f\nabla_x\Psi$ or to the singular integral (3), since we will only consider the case where $\rho \in L^1 \cap L^\infty$. In this case, $\nabla_x \Psi$ will be continuous, and the product $f\nabla_x\Psi$ will be well defined for f a bounded measure.

Our main result is the following:

Theorem 1.2. *Given f^0 in $\mathcal{M}^+(\mathbb{R}^6)$, given $T > 0$, there exists at most one weak solution to (1, 2, 4) in $\mathcal{D}'([0,T) \times \mathbb{R}^6)$ such that*

$$\|\rho\|_{L^\infty([0,T)\times\mathbb{R}^d)} < +\infty. \tag{6}$$

Remark 1. Note that we do not ask for any bound on the moments of f, and also that we do not ask the energy, given by

$$\mathcal{E}(t) = \int_{\mathbb{R}^6} f(t,x,\xi) \frac{|\xi|^2}{2} + \epsilon \frac{|\nabla \Psi(t,x)|^2}{2} \, dx d\xi \tag{7}$$

to be finite.

Remark 2. To establish the existence of a solution to (VP) satisfying the bound (6) might require much more assumptions on the initial datum than what we need here ! This question is treated in [5]. From their results, one can build solutions with bounded density ρ, and as corollary of our result, such solution will be unique.

Theorem 1.3 (Lions & Perthame [5]). *Let $f^0 \in L^\infty(\mathbb{R}^6)$ satisfy*

$$\int_{\mathbb{R}^6} f^0(t,x,\xi)|\xi|^{m_0} dx d\xi < +\infty \text{ for some } m_0 > 6.$$

Assume that

$$\begin{aligned} \forall R > 0, \forall T > 0, \qquad & \text{ess sup}\ \{f^0(y+t\xi, w), |y-x| \le Rt^2, |\xi - w| \le Rt\} \\ & \in L^\infty([0,T) \times \mathbb{R}^3_x; L^1(\mathbb{R}^3_\xi)), \end{aligned} \tag{8}$$

then there exist a weak solution to (1, 2, 4) such that $\rho \in L^\infty_{loc}(\mathbb{R}^+; L^\infty(\mathbb{R}^3))$.

Remark. Note that condition (8) is satisfied for $f^0(x,\xi) \le C(1+|\xi|^p)^{-1}$, $p > 3$.

A sufficient condition for uniqueness had been given by Lions and Perthame in [5], relying on Lipschitz bounds on the initial data f^0, but they expected a uniqueness result without this assumption. The Lipschitz condition had indeed later been relaxed by Robert in [9] down to $f \in L^\infty$ compactly supported in x and ξ for $t \in [0,T]$. Here we relax the bound on the support of f, and we do not ask either f to be bounded in L^∞. We only need a $L^\infty([0,T) \times \mathbb{R}^3)$ bound on $\rho(t,x)$. Hence our result applies also to monokinetic solutions of (1, 2). In that case, we have $f(t,x,\xi) = \rho(t,x)\delta(\xi - v(t,x))$ for some vector field v, and this gives formally a solution to the Euler-Poisson system

$$\partial_t \rho + \nabla \cdot (\rho v) = 0, \tag{9}$$

$$\partial_t(\rho v) + \nabla \cdot (\rho v \otimes v) = -\rho \nabla \Psi, \tag{10}$$

$$-\Delta \Psi = \epsilon \rho. \tag{11}$$

Our proof will rely on optimal transportation, and the next section is devoted to recall some facts concerning this subject. The reader can find a complete reference on this topic in [10]. The technique we will use adapts to many similar problems, where a transport equation and an elliptic equation are coupled. The velocity field is the gradient of a potential satisfying an elliptic equation whose right hand side depends smoothly on the density. A typical example of such a system is the 2-d incompressible Euler equations, for which we present an alternate proof of Youdovich's theorem in section 4. This technique will be used in a forthcoming paper [6] on the semi-geostrophic equations.

Our result is based on a new functional inequality that relates the Wasserstein distance between two measures, the H^{-1} norm of their difference, and the L^∞ norm of their densities, see Theorem 2.7.

It is interesting to notice that our technique yields a new proof of Youdovich's Theorem [11] (section 4), while the technique used by Robert in [9] was and adaptation of Youdovich's original proof.

2 Preliminary results on optimal transportation and Wasserstein distances

2.1 Definitions

Definition 2.1. *Let ρ_1, ρ_2 be two Borel probability measures on $\mathbb{R}^d$. We define the Wasserstein distance of order 2 between ρ_1 and ρ_2, that we denote $W_2(\rho_1, \rho_2)$, by*

$$W_2(\rho_1, \rho_2) = \left(\inf_\gamma \int_{\mathbb{R}^d \times \mathbb{R}^d} d\gamma(x,y)|x-y|^2 \right)^{\frac{1}{2}},$$

where the infimum runs over probability measures γ on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals $\mathbb{P}_x\gamma$ and $\mathbb{P}_y\gamma$ equal respectively to ρ_1 and ρ_2.

(The Wasserstein distance of order $p \geq 1$ would have been defined in the same way, replacing $|x-y|^2$ by $|x-y|^p$.)

We now show why this distance is related to optimal transportation. Let us first recall the definition of the push-forward of a measure by a mapping:

Definition 2.2. *Let ρ_1 be a Borel measure on $\mathbb{R}^d$ and $T : \mathbb{R}^d \to \mathbb{R}^d$ be a measurable mapping. The push-forward of ρ_1 by T is the measure ρ_2 defined by*

$$\forall B \subset \mathbb{R}^d \text{ Borel}, \ \rho_2(B) = \rho_1(T^{-1}(B)).$$

We will use the notation $\rho_2 = T_\# \rho_1$.

Remark. Let (Ω, μ) be a probability space, and consider X_1, X_2 mappings from (Ω, μ) to $\mathbb{R}^d$. Assume that $X_{1\#}d\mu = \rho_1$, $X_{2\#}d\mu = \rho_2$, then $\gamma = (X_1, X_2)_\# d\mu$ has marginals ρ_1 and ρ_2, and one deduces that

$$\int_\Omega |X_1 - X_2|^2 d\mu = \int_{\mathbb{R}^d \times \mathbb{R}^d} d\gamma(x,y)|x-y|^2 \leq W_2^2(\rho_1, \rho_2).$$

This remark will be useful later on.

Then we have the fundamental theorem of existence/characterization of the minimizer in Definition (2.1). This result is due to Brenier in [2]. We state it in a version due to McCann and Gangbo, [4, Theorem 1.2], that does not require that ρ_1 and ρ_2 have finite moments of order 2. In this case of course, their Wasserstein distance might be infinite.

Theorem 2.3. *Assume that in Definition 2.1, ρ_1 is absolutely continuous with respect to the Lebesgue measure. Then*

$$W_2(\rho_1,\rho_2)=\left(\inf_{T_\#\rho_1=\rho_2}\int_{\mathbb{R}^d}|T(x)-x|^2\rho_1(x)dx\right)^{\frac{1}{2}},$$

where the infimum runs over all measurable mappings $T:\mathbb{R}^d\to\mathbb{R}^d$ that push forward ρ_1 onto ρ_2. Moreover, the infimum is reached by a $d\rho_1$ a.e. unique mapping T, and there exists a convex function ϕ such that $T=\nabla\phi$.

Remark. One sees immediately that if $\rho_2=T_\#\rho_1$, then the joint measure $\gamma(x,y)=\rho_1(x)\delta(y=T(x))$ has marginals ρ_1 and ρ_2. Hence it is not difficult to see that the infimum of Definition 2.1 is lower than the infimum in the above theorem. To prove the converse is more difficult.

2.2 Wasserstein distance and H^{-1} norm

In this paragraph we establish an inequality between the Wasserstein distance and the H^{-1} norm. The fact that those two quantities are somehow comparable had also been noticed in the asymptotic case where we consider perturbation of a given measure. In fact, it can be shown that (see [10, Theorem 7.26]) given $\mu\in\mathcal{P}_2(\mathbb{R}^d)$ (i.e. with finite second moment), for all $\nu\in L^\infty(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d}hd\mu=0$,

$$\|\nu\|_{H^{-1}(d\mu)}\le\liminf_{\epsilon\to0}\frac{W_2(\mu,\mu(1+\epsilon\nu))}{\epsilon},$$

where

$$\|\nu\|_{H^{-1}(d\mu)}=\sup\Big\{\int\nu f d\mu;f\in C_c^\infty(\mathbb{R}^d),\int|\nabla f|^2d\mu\le1\Big\}.$$

This result can be compared with the one that we are going to show below (Theorem 2.7).

Optimal transportation induces a natural interpolation between two measures ρ_1 and ρ_2: indeed consider for $\theta\in[1,2]$,

$$\rho_\theta=((\theta-1)T+(2-\theta)x)_\#\rho_1, \tag{12}$$

where T is the optimal transport map between ρ_1 and ρ_2. This path ρ_θ has some interesting properties, the following one being now refered to as *displacement convexity.*

Theorem 2.4 (McCann, [8]). *Let ρ_1, ρ_2 be two probability measures on $\mathbb{R}^d$, with ρ_1 absolutely continuous with respect to the Lebesgue measure. Let $\rho_{\theta,\theta\in[1,2]}$ be the interpolant betwen ρ_1 and ρ_2 defined above. Then $\theta \to \log(\|\rho_\theta\|_{L^p})$ is convex on $[1,2]$ for all $p \geq 1$.*

Using the well known fact that for any L^1 function f, $\limsup_{p\to\infty} \|f\|_{L^p} = \|f\|_{L^\infty}$, we deduce immediatly the following corollary:

Corollary 2.5. *Under the previous notations, we have*

$$\forall\theta \in [1,2], \|\rho_\theta\|_{L^\infty} \leq \max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\}. \tag{13}$$

We now establish an estimate concerning the θ derivative of the path ρ_θ. We will obtain this estimate in the following H^{-1} norm:

$$\|f\|_{H^{-1}(\mathbb{R}^d)} = \sup\Big\{\int_{\mathbb{R}^d} fg,\ g \in C_c^\infty(\mathbb{R}^d), \int |\nabla g|^2(x)dx \leq 1\Big\}. \tag{14}$$

We suppose hereafter that $W_2(\rho_1,\rho_2) < +\infty$ otherwise there is nothing to prove.

Proposition 2.6. *We take ρ_1, ρ_2 in $\mathcal{P}(\mathbb{R}^d) \cap L^\infty(\mathbb{R}^d)$, and $\rho_{\theta,\theta\in[1,2]}$ the interpolant between ρ_1 and ρ_2 defined in (12). Then*

$$\partial_\theta\rho_\theta \in L^\infty([1,2]; H^{-1}(\mathbb{R}^d))$$

and

$$\|\partial_\theta\rho_\theta\|_{L^\infty([1,2];H^{-1}(\mathbb{R}^d))} \leq \max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\}^{1/2} W_2(\rho_1,\rho_2).$$

Consequently we have

$$\|\rho_1 - \rho_2\|_{H^{-1}(\mathbb{R}^d)} \leq \max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\}^{1/2} W_2(\rho_1,\rho_2).$$

Proof. By definition of ρ_θ we have, for all $f \in C_c^\infty(\mathbb{R}^d)$,

$$\int \rho_\theta(x) f(x)\, dx = \int \rho_1(x) f((\theta-1)\nabla\phi(x) + (2-\theta)x)\, dx,$$

where $\nabla\phi_\#\rho_1 = \rho_2$, ϕ is the convex potential such that $T = \nabla\phi$ is the optimal map between ρ_1 and ρ_2. We can differentiate this expression with respect to θ and obtain

$$\frac{d}{d\theta}\int \rho_\theta(x) f(x)\, dx = \int \rho_1(x) \nabla f((\theta-1)\nabla\phi(x) + (2-\theta)x)\cdot(\nabla\phi(x) - x)\, dx.$$

Using Cauchy-Schwartz inequality, we then have

$$\frac{d}{d\theta}\int \rho_\theta(x) f(x)\, dx \le \left(\int \rho_1(x)|\nabla\phi(x) - x|^2\, dx\right)^{1/2} \left(\int \rho_\theta(x)|\nabla f(x)|^2\, dx\right)^{1/2}.$$

In the first term of the right hand side, we recognize the Wasserstein distance between ρ_1 and ρ_2, and we then use the bound (13) for the second term. $\square$

We now deduce the following estimate at the core of our result:

Theorem 2.7. *Let ρ_1, ρ_2 be two probability measures on $\mathbb{R}^d$ with L^∞ densities with respect to the Lebesgue measure. Let $\Psi_i, i = 1, 2$ solve*

$$-\Delta\Psi_i = \rho_i \text{ in } \mathbb{R}^d,$$
$$\Psi_i(x) \to 0 \text{ as } |x| \to \infty,$$

i.e. in the sense of (3). Then

$$\|\nabla\Psi_1 - \nabla\Psi_2\|_{L^2(\mathbb{R}^d)} \le \left[\max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\}\right]^{\frac{1}{2}} W_2(\rho_1, \rho_2), \qquad (15)$$

where $W_2(\rho_1, \rho_2)$ is the Wasserstein distance between ρ_1 and ρ_2 given in Definition 2.1.

Proof of Theorem 2.7. The proof is an immediate consequence of Proposition 2.6, and of the following lemma:

Lemma 2.8. *Let f belong to $L^1 \cap L^\infty(\mathbb{R}^d)$, and F satisfy $-\Delta F = f$ in the sense of (3). We have (with possibly infinite values) $\|f\|_{H^{-1}(\mathbb{R}^d)} = \|\nabla F\|_{L^2(\mathbb{R}^d)}$.*

Remark. In particular, $\|f\|_{H^{-1}}$ is infinite when $d = 2$ and $\int_{\mathbb{R}^2} f \neq 0$.

Applying this lemma to $f = \rho_1 - \rho_2$ leads to the conclusion of the Theorem. $\square$

Proof of Lemma 2.8.

We have, for all $g \in C_c^\infty(\mathbb{R}^d)$,

$$\begin{aligned}\int f(x)g(x)dx &= -\int \Delta F(x) g(x) dx \\ &= \int \nabla F(x)\cdot\nabla g(x) dx.\end{aligned}$$

Taking the supremum of the last line on the set $\{g \in C_c^\infty(\mathbb{R}^d), \|\nabla g\|_{L^2} \le 1\}$, we reach the desired conclusion. $\square$

3 Proof of Theorem 1.2

From now on, we assume for simplicity that $\int_{\mathbb{R}^6} f^0(x,\xi) = 1$, $\epsilon = 1$, and the reader can check that this choice does not play any role in the proof. In particular, the result of the previous section adapt with minor changes to the case of two positive measures of equal total mass.

3.1 Lagrangian formulation of the Vlasov-Poisson system

Given a solution of (VP) with bounded density ρ on $[0,T)$, we consider for $t \in [0,T)$ the characteristics of equation (1), that solve the ODE

$$\dot{X} = \Xi, \tag{16}$$

$$\dot{\Xi} = -\nabla\Psi(t,X). \tag{17}$$

Since we assume an $L^1 \cap L^\infty([0,T) \times \mathbb{R}^3)$ bound on the density ρ, the field $\nabla\Psi$ classically satisfies the following (see [7, Chapter 8]):

Lemma 3.1. *Let Ψ be obtained from ρ through (3). Then there exists C_Δ that depends on $\|\rho\|_{L^\infty} + \|\rho\|_{L^1}$ such that*

$$\|\nabla\Psi\|_{L^\infty} \leq C_\Delta, \tag{18}$$

$$\forall t \in [0,T),\ \forall (x,y) \in \mathbb{R}^3 \times \mathbb{R}^3, |x-y| \leq \frac{1}{2},$$

$$|\nabla\Psi(t,x) - \nabla\Psi(t,y)| \leq C_\Delta |x-y| \log\frac{1}{|x-y|}. \tag{19}$$

This condition is enough to define a Hölder continuous flow (see [7, Chapter 8])

$$Y(t,x,\xi) = (X,\Xi)(t,x,\xi)$$

for the ODE (16,17), where (X,Ξ) is the pair (velocity, position) at time t of the trajectory having (velocity, position) equal to (x,ξ) at time 0. Note that Y_t will be Hölder continuous with respect to (x,ξ), with Hölder index decaying exponentially to 0 as $t \to +\infty$.

Then we use the following Theorem, proved in [1]:

Theorem 3.2. *Let $u(t,x)$ be a vector field on $\mathbb{R}^d$. Consider the ODE*

$$\dot{\gamma}(t) = u(t,\gamma(t)),$$

and the PDE

$$\partial_t \mu(t,x) + \nabla \cdot (\mu(t,x)u(t,x)) = 0.$$

Let $B \subset \mathbb{R}^d$ be a Borel set. The following are equivalent:

(a) For all x in B, there exists a unique solution to the ODE starting at x.

(b) Non negative measure-valued solutions to the PDE with initial data μ^0 concentrated in B are unique.

From this result, we deduce the following corollary:

Corollary 3.3. *The potential Ψ being held fixed, and satisfying $\Delta\Psi \in L^\infty([0,T] \times \mathbb{R}^3)$, for any $f^0 \in \mathcal{M}^+(\mathbb{R}^6)$ there exists a unique weak solution to (1) (i.e. in the sense of (5)) with initial datum f^0 which is given by*

$$f(t) = Y(t,\cdot,\cdot)_\# f^0, \tag{20}$$

where $Y = (X, \Xi)$ solves (16, 17). Note also that we will have

$$\rho(t) = X(t,\cdot,\cdot)_\# f^0. \tag{21}$$

We remind the reader that the measure $f(t) = Y(t,\cdot,\cdot)_\# f^0$ is defined by $f(t)(B) = f^0(Y^{-1}(t)(B))$ for all Borel subsets B of $\mathbb{R}^6$.

Remark. This corollary does not solve the uiqueness problem, but says that when one considers the *linear* problem, there is a unique weak measure-valued solution to the transport equation (1), that we can represent with the help of characteristics.

3.2 Final estimate

Given an initial distribution $f^0(x,\xi) \in \mathcal{M}^+(\mathbb{R}^6)$ with $\int_{\mathbb{R}^6} f^0 = 1$, we take two solutions (f_1, f_2) to (VP) with bounded density ρ_i and initial datum f^0. We have $-\Delta\Psi = \rho_i, i = 1,2$ in the sense of (3). We then consider the associated characteristics Y_1 and Y_2, where for $i = 1,2$, $Y_i = (X_i, \Xi_i)(t,x,\xi)$ and X_i, Ξ_i solve (16, 17) with force field $\nabla\Psi_i$. From Corollary 3.3, we will have $f_i(t) = Y_i(t)_\# f^0, i = 1,2$. We then consider

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^6} f^0(x,\xi)\, |Y_1(t,x,\xi) - Y_2(t,x,\xi)|^2. \tag{22}$$

Remark. Notice that $(Y_1(t), Y_2(t))_\# f^0$ is a probability measure on $\mathbb{R}^6 \times \mathbb{R}^6$, with marginals $f_1(t)$ and $f_2(t)$, hence by Definition 2.1, we have the following important observation:

Lemma 3.4. *Let Q be defined through (22), then*

$$W_2^2(f_1(t), f_2(t)) \leq 2Q(t)$$

and

$$W_2^2(\rho_1(t), \rho_2(t)) \leq 2Q(t).$$

In particular, $Q(t) = 0$ implies $f_1(t) = f_2(t)$.

PROOF. The proof follows immediately from Definition 2.1 and (20, 21) by noticing that $\Pi = (Y_1(t), Y_2(t))_\# f^0$ is a probability measure on $\mathbb{R}^6 \times \mathbb{R}^6$ with marginals $f_1(t)$ and $f_2(t)$ and that $\pi = (X_1(t), X_2(t))_\# f^0$ is a probability measure on $\mathbb{R}^3 \times \mathbb{R}^3$ with marginals $\rho_1(t)$ and $\rho_2(t)$.

□

Our proof will rely on an estimate on the Wasserstein distance between f_1 and f_2, while the proof of [9] was obtained by estimating the H^{-1} norm of $f_1 - f_2$.

Of course $Q(0) = 0$, and since $Y_i, \partial_t Y_i(t, x, \xi)$ belong to $L^\infty([0, T]; C^0(\mathbb{R}^6))$ (see Lemma 3.1), one can differentiate Q with respect to time. We thus have:

$$\begin{aligned}
\frac{d}{dt}Q(t) &= \int_{\mathbb{R}^6} f^0(x,\xi)(Y_1(t,x,\xi) - Y_2(t,x,\xi)) \cdot \partial_t(Y_1(t,x,\xi) - Y_2(t,x,\xi)) \\
&= \int_{\mathbb{R}^6} f^0(x,\xi)\left[(X_1(t,x,\xi) - X_2(t,x,\xi)) \cdot (\Xi_1(t,x,\xi) - \Xi_2(t,x,\xi))\right] \\
&- \int_{\mathbb{R}^6} f^0(x,\xi)\Big[(\Xi_1(t,x,\xi) - \Xi_2(t,x,\xi)) \cdot \\
&\qquad\qquad (\nabla\Psi_1(t, X_1(t,x,\xi)) - \nabla\Psi_2(t, X_2(t,x,\xi)))\Big].
\end{aligned}$$

The second line is bounded by $Q(t)$, and using Cauchy-Schwartz inequality, the third line is bounded by

$$\begin{aligned}
& (2Q)^{1/2}(t)\left(\int_{\mathbb{R}^6} f^0(x,\xi)\,|\nabla\Psi_1(t, X_1(t,x,\xi)) - \nabla\Psi_2(t, X_2(t,x,\xi))|^2\right)^{1/2} \\
\leq\ & (2Q)^{1/2}(t)\left(\int_{\mathbb{R}^6} f^0(x,\xi)\,|\nabla\Psi_2(t, X_1(t,x,\xi)) - \nabla\Psi_2(t, X_2(t,x,\xi))|^2\right)^{1/2} \\
+\ & (2Q)^{1/2}(t)\left(\int_{\mathbb{R}^6} f^0(x,\xi)\,|\nabla\Psi_2(t, X_1(t,x,\xi)) - \nabla\Psi_1(t, X_1(t,x,\xi))|^2\right)^{1/2} \\
=\ & (2Q)^{1/2}(t)\left([T_1(t)]^{1/2} + [T_2(t)]^{1/2}\right),
\end{aligned}$$

where

$$T_1(t) = \int_{\mathbb{R}^6} f^0(x,\xi)\,|\nabla\Psi_2(t,X_1(t,x,\xi)) - \nabla\Psi_2(t,X_2(t,x,\xi))|^2$$
$$T_2(t) = \int_{\mathbb{R}^6} f^0(x,\xi)\,|\nabla\Psi_2(t,X_1(t,x,\xi)) - \nabla\Psi_1(t,X_1(t,x,\xi))|^2\,.$$

Hence we have

$$\frac{d}{dt}Q(t) \le Q(t) + (2Q)^{1/2}(t)\left([T_1(t)]^{1/2} + [T_2(t)]^{1/2}\right), \tag{23}$$

and we will now estimate T_2 and then T_1.

For T_2 we have, using (21) and Theorem 2.7,

$$\begin{aligned} T_2(t) &= \int_{\mathbb{R}^3} \rho_1(t,x)\,|\nabla\Psi_1(t,x) - \nabla\Psi_2(t,x)|^2 \\ &\le \max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\}^2 W_2^2(\rho_1(t),\rho_2(t)), \end{aligned}$$

Hence, in view of Lemma 3.4, we conclude that

$$T_2(t) \le 2\max\{\|\rho_1\|_{L^\infty}, \|\rho_2\|_{L^\infty}\}^2 Q(t).$$

Now, we evaluate T_1 by standard arguments. We first recall Lemma 3.1 to see that $\nabla\Psi_i$ are uniformly bounded in L^∞ by a constant C_Δ that depends on $\|\rho_i\|_{L^\infty} + \|\rho_i\|_{L^1}$. (Of course here we have $\|\rho_i(t)\|_{L^1} \equiv 1$.) Since at time $t=0$ we have $Y_1(t,x,\xi) \equiv Y_2(t,x,\xi) \equiv (x,\xi)$, for any $C>0$, we can take T small enough such that $\|Y_1 - Y_2\|_{L^\infty([0,T]\times\mathbb{R}^6)} \le C$. Thus using again Lemma 3.1, we have, for C_Δ depending on $\|\rho_i\|_{L^\infty}, i=1,2$, and as long as $\|Y_1 - Y_2\|_{L^\infty} \le \frac{1}{2}$,

$$\begin{aligned} T_1 &= \int_{\mathbb{R}^6} f^0(x,\xi)\,|\nabla\Psi_2(t,X_1(t,x,\xi)) - \nabla\Psi_2(t,X_2(t,x,\xi))|^2 \\ &\le C_\Delta^2 \int_{\mathbb{R}^6} f^0(x,\xi)\left(|X_1 - X_2|^2 \log^2\frac{1}{|X_1 - X_2|}\right)(t,x,\xi) \\ &= \frac{C_\Delta^2}{4}\int_{\mathbb{R}^6} f^0(x,\xi)\left(|X_1 - X_2|^2 \log^2(|X_1 - X_2|^2)\right)(t,x,\xi). \end{aligned}$$

Then we use that $x \mapsto x\log^2 x$ is concave for $0 \le x \le 1/e$, and we can assume (taking T small enough) that $\|Y_1 - Y_2\|_{L^\infty([0,T]\times\mathbb{R}^6)} \le 1/e$, therefore by Jensen's inequality we have

$$\begin{aligned} &T_1(t) \\ \le\quad & \frac{C_\Delta^2}{4}\left[\int_{\mathbb{R}^6} f^0(x,\xi)|X_1 - X_2|^2(t,x,\xi)\right] \log^2\left[\int_{\mathbb{R}^6} f^0(x,\xi)|X_1 - X_2|^2(t,x,\xi)\right] \\ =\quad & \frac{C_\Delta^2}{2} Q(t)\log^2(2Q(t)). \end{aligned} \tag{24}$$

Combining all these bounds in (23), we obtain that, as long as $\|Y_1 - Y_2\|_{L^\infty} \leq 1/e$,

$$\frac{d}{dt}Q(t) \leq CQ(t)(1 + \log \frac{1}{Q(t)}),$$

where C depends only on $\|\rho_i\|_{L^\infty([0,T)\times\mathbb{R}^3)}$. We can now conclude by standard arguments that if $Q(0) = 0$, $Q \equiv 0$ on $[0, T)$, which achieves the proof of Theorem 1.2. $\square$

4 Adaptation of this proof to the 2-d Euler incompressible equations

As mentioned above, this proof adapts naturally to the case of the Euler 2-d equation in its vorticity form (see [7] for a complete reference on Euler equations). Hence we consider the following system:

$$\partial_t \omega + \nabla \cdot (\omega \nabla^\perp \Psi) = 0, \tag{25}$$

$$-\Delta \Psi = \omega, \tag{26}$$

$$\omega|_{t=0} = \omega_0, \tag{27}$$

where $\nabla^\perp \Psi$ means $(\partial_{x_2}\phi, -\partial_{x_1}\phi)$, and ω is thus the rotational of the velocity field. The domain considered here will be $\mathbb{R}^2$ (in which case equation (26) is solved in the sense of (3) with $\epsilon = 1$). We will consider the flow $X(t, x)$ associated to the velocity field $\nabla^\perp\Psi$, and in the case where $\omega \in L^\infty \cap L^1$, thanks to Lemma 3.1, $\nabla\Psi$ will be log-Lipschitz, hence $X(t, x)$ will be continuous with respect to (t, x), and we will have $\omega(t) = X(t)_\#\omega_0$.

The main modification is that we deal with a measure ω which is not positive anymore. However, we can make the following observation (see Lemma 4.6): if ω_0 has positive and negative parts ω_0^+, ω_0^-, then for any solution $\omega(t)$, one will have

$$\omega^+(t) = X(t)_\#\omega_0^+,$$
$$\omega^-(t) = X(t)_\#\omega_0^-.$$

In particular $TM(\omega^+(t))$ (resp. $TM(\omega^-(t))$), the total mass of $\omega^+(t)$ (resp. of $\omega^-(t)$) remains equal to $TM(\omega_0^+)$ (resp. $TM(\omega_0^-)$) the total mass of ω_0^+ (resp. of ω_0^-). With a slight generalization, we can define the Wasserstein distance between signed measures of same total mass, without requiring that they are probability measures. The object of the next paragraph is to generalize to the present situation the objects and results of paragraph 2.

4.1 Generalization of Theorem 2.7 to non probabilistic measures

Definition 4.1. *Let ω_1, ω_2 be two positive measures on $\mathbb{R}^d$ of same total mass $TM(\omega_1) = TM(\omega_2) = M$. We define*

$$W_2(\omega_1, \omega_2) = \inf_\gamma \left(\int_{\mathbb{R}^d \times \mathbb{R}^d} |x-y|^2 d\gamma(x,y) \right)^{1/2}$$

where γ runs on all positive measures on $\mathbb{R}^d \times \mathbb{R}^d$ with marginals ω_1 and ω_2.

Remark. One can check that this definition is consitent with the Definition 2.1 when one considers probability measures.

As a trivial adaptation of Theorem 2.3, we have the following

Theorem 4.2. *Assume that in Definition 4.1 ω_1 is absolutely continuous with respect to the Lebesgue measure, the the infimim in Definition 4.1 is reached by $\gamma_{opt} = \omega_1(x)\delta(y = \nabla\phi(x))$, for some convex function ϕ.*

Proof. Apply Theorem 2.3 to ρ_1, ρ_2, where $\rho_i = \omega_i M^{-1}$ are probability measures. $\square$.

We then observe that with this generalized definition of the Wasserstein distance, Theorem 2.7 still holds with no modification:

Theorem 4.3. *Let ω_1, ω_2 be two positive measures on $\mathbb{R}^d$ of same finite total mass M, and with densities in L^∞ with respect to the Lebesgue measure. Let $\Psi_i, i = 1,2$ solve $-\Delta\Psi_i = \omega_i$ in the sense of (3). Then*

$$\|\nabla\Psi_1 - \nabla\Psi_2\|_{L^2(\mathbb{R}^d)} \leq \left[\max\{\|\omega_1\|_{L^\infty}, \|\omega_2\|_{L^\infty}\}\right]^{\frac{1}{2}} W_2(\omega_1, \omega_2), \tag{28}$$

where $W_2(\omega_1, \omega_2)$ is now given by Definition 4.1.

Proof. Consider the probability measures $\rho_i = \omega_i M^{-1}, i = 1, 2$. Apply Theorem 4.3 to ρ_1, ρ_2. Then check that $W_2(\omega_1, \omega_2) = M^{1/2} W_2(\rho_1, \rho_2)$, and conclude. $\square$

We can then use this result to estimate the H^{-1} norm of the difference of two measures on the condition that the total masses of their positive (resp. negative) parts coïncide.

Theorem 4.4. *Let ω_1 and ω_2 be two bounded measures on $\mathbb{R}^d$, with positive (resp. negative) parts $\omega_i^+, i = 1,2$ (resp. $\omega_i^-, i = 1,2$) such that*

$$TM(\omega_1^+) = TM(\omega_2^+),$$
$$TM(\omega_1^-) = TM(\omega_2^-).$$

Assume that ω_1 and ω_2 have densities in L^∞ with respect to the Lebesgue measure, and define Ψ_1, Ψ_2 the solutions of $-\Delta\Psi_i = \omega_i$ in the sense of (3). Then

$$\begin{aligned}\|\nabla\Psi_1 - \nabla\Psi_2\|_{L^2(\mathbb{R}^d)} &\leq \left[\max\{\|\omega_1^+\|_{L^\infty}, \|\omega_2^+\|_{L^\infty}\}\right]^{1/2} W_2(\omega_1^+, \omega_2^+) \\ &+ \left[\max\{\|\omega_1^-\|_{L^\infty}, \|\omega_2^-\|_{L^\infty}\}\right]^{1/2} W_2(\omega_1^-, \omega_2^-),\end{aligned}$$

where W_2 is now given by Definition 4.1.

Proof. We consider, for $i = 1, 2$, Ψ_i^+ (resp. Ψ_i^-) to be solution of $-\Delta\Psi_i^+ = \omega_i^+$ (resp. $-\Delta\Psi_i^- = \omega_i^-$). Of course $\Psi_i = \Psi_i^+ - \Psi_i^-$, and

$$\|\nabla\Psi_1 - \nabla\Psi_2\|_{L^2} \leq \|\nabla\Psi_1^+ - \nabla\Psi_2^+\|_{L^2} + \|\nabla\Psi_1^- - \nabla\Psi_2^-\|_{L^2}.$$

Then, by Theorem 4.3, we have

$$\|\nabla\Psi_1^+ - \nabla\Psi_2^+\|_{L^2} \leq \left[\max\{\|\omega_1^+\|_{L^\infty}, \|\omega_2^+\|_{L^\infty}\}\right]^{1/2} W_2(\omega_1^+, \omega_2^+).$$

Doing the same way for Ψ_i^-, we conclude the proof. $\square$

4.2 Conclusion of the proof

Then one can reprove the following result, due to Youdovich ([11]):

Theorem 4.5. *Let ω_0 belong to $L^1 \cap L^\infty(\mathbb{R}^2)$. There exists a unique solution to (25, 26, 27) in $\mathbb{R}^2$, such that $\omega(t) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$.*

Proof. The proof of this result is a straighforward adaptation of the proof for Vlasov-Poisson. For two solutions (ω_1, ω_2) of (25, 26) with same initial condition $\omega_0 \in L^\infty \cap L^1(\mathbb{R}^2)$, and both satisfying $\omega \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$, we consider the characteristics $X_i, i = 1, 2$ that are solution to

$$\begin{aligned}&\partial_t X_i(t, x) = \nabla^\perp\Psi_i(t, X_i(t, x)), \\ &\Delta\Psi_i = \omega_i, \\ &X(0, x) = x.\end{aligned}$$

Those characteristics are well defined thanks to the $L^\infty \cap L^1$ bound on ω_i that yields a Log-Lipschitz continuity of the velocity fields (see Lemma 3.1 and [7, Chapter 8]). Here we do not consider measure-valued solutions to the transport equation (25), hence we do not need such a sophisticated result as Theorem 3.2. We consider solutions with bounded density, hence from Di-Perna Lions theory (see [3]), it is enough that $\nabla\Psi \in W^{1,1}$ to have uniqueness of L^∞ solutions to the Cauchy problem for the *linear* (i.e. with a given velocity field $\nabla^\perp\Psi$) transport equation (25). This unique solution can be represented as $\omega_i(t) = X_i(t)_\#\omega_0$. Note also that still thanks to the regularity of the velocity field, solutions that satisfy $\omega(t) \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^2)$ will automatically satisfy $\|\omega(t)\|_{L^\infty(\mathbb{R}^2)} \equiv \|\omega_0\|_{L^\infty(\mathbb{R}^2)}$.

We then consider

$$Q(t) = \frac{1}{2}\int_{\mathbb{R}^2} |\omega_0(x)|\,|X_2(t,x) - X_1(t,x)|^2 dx.$$

We first have a straightforward adaptation of Lemma 3.4 to our present case:

Lemma 4.6. *Let $Q, X_1, X_2, \omega_1 = X_{1\#}\omega_0, \omega_2 = X_{2\#}\omega_0$ be defined as above for all $t \in \mathbb{R}^+$. Then, for all t*

$$TM(\omega_1^+) = TM(\omega_2^+),$$
$$TM(\omega_1^-) = TM(\omega_2^-),$$

moreover

$$W_2^2(\omega_1^+(t), \omega_2^+(t)) + W_2^2(\omega_1^-(t), \omega_2^-(t)) \le 2Q(t),$$

where W_2 is given in Definition 4.1.

Proof. The equality of the total masses come from the observation that the mappings X_i are homeomorphism from $\mathbb{R}^2$ to itself for all $t \in \mathbb{R}^+$, hence $\omega_i^\pm = X_i(t)_\#\omega_0^\pm$.

Then since $\gamma^+(t) = (X_1(t), X_2(t))_\#\omega_0^+$ has marginals $\omega_1^+(t), \omega_2^+(t)$, we have by Definition 4.1

$$W_2^2(\omega_1^+(t), \omega_2^+(t)) \le \int \omega_0^+(x)|X_1(t,x) - X_2(t,x)|^2 dx.$$

Doing the same for the negative part, we conclude. □

We now differentiate Q, this yields

$$\frac{dQ}{dt} = \int_{\mathbb{R}^2} |\omega_0(x)|(X_1 - X_2)\cdot(\nabla^\perp\Psi_1(X_1) - \nabla^\perp\Psi_2(X_2))(t,x)dx.$$

Again we bound this integral by the sum of two terms

$$\begin{aligned}\frac{dQ}{dt} &\le \int_{\mathbb{R}^2} |\omega_0(x)||X_1 - X_2||\nabla\Psi_1(t,X_1) - \nabla\Psi_2(t,X_1)|(t,x)dx \\ &+ \int_{\mathbb{R}^2} |\omega_0(x)||X_1 - X_2||\nabla\Psi_2(t,X_1) - \nabla\Psi_2(t,X_2)|(t,x)dx.\end{aligned}$$

By Cauchy-Schwartz's inequality we have

$$\frac{dQ}{dt} \leq [Q(t)]^{1/2}([T_1(t)]^{1/2} + [T_2(t)]^{1/2}),$$

where

$$\begin{aligned} T_2(t) &= \int_{\mathbb{R}^2} |\omega_1(t,x)||\nabla\Psi_1(t,x) - \nabla\Psi_2(t,x)|^2 dx, \\ T_1(t) &= \int_{\mathbb{R}^2} |\omega_0(x)|\,|\nabla\Psi_2(t,X_1(t,x)) - \nabla\Psi_2(t,X_2(t,x))|^2 dx. \end{aligned}$$

The terms T_1, T_2 are then evaluated in the same way as in the proof for Vlasov-Poisson. We get, thanks to Theorem 4.4, and using that $\|\omega_1(t)\|_{L^\infty} \equiv \|\omega_0\|_{L^\infty}$,

$$T_2 \leq \|w_0\|_{L^\infty}^2 \Big(W_2(\omega_1^+,\omega_2^+) + W_2(\omega_1^-,\omega_2^-)\Big)^2.$$

We then use Lemma 4.6, to obtain finally that $T_2 \leq 4\|w_0\|_{L^\infty}^2 Q(t)$.

For T_1, we proceed exactly as for the term T_1 of the proof for Vlasov-Poisson: we take $T > 0$ small enough so that $\|X_1 - X_2\|_{L^\infty([0,T]\times\mathbb{R}^2)} \leq \frac{1}{e}$, and we obtain as in (24) that for $t \in [0,T]$,

$$T_1(t) \leq \frac{C_\Delta^2}{2} Q(t) \log^2(2Q(t)),$$

where C_Δ comes from Lemma 3.1 and depends only on $\|\omega_0\|_{L^\infty} + \|\omega_0\|_{L^1}$. Finally, we obtain that, as long as $\|X_1(t) - X_2(t)\|_{L^\infty(\mathbb{R}^2)} \leq \frac{1}{e}$

$$\frac{d}{dt}Q(t) \leq CQ(t)(1 + \log\frac{1}{Q(t)}),$$

where C depends only on $\|\omega_0\|_{L^\infty} + \|\omega_0\|_{L^1}$.

We can now conclude by standard arguments that if $Q(0) = 0$, $Q \equiv 0$ on $\mathbb{R}^+$, which achieves the proof of Theorem 4.5. $\square$

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