A geometric approximation to the Euler equations : the Vlasov-Monge-Ampère system

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Abstract

This paper studies the Vlasov-Monge-Ampère system (VMA), a fully non-linear version of the Vlasov-Poisson system (VP) where the (real) Monge-Ampère equation $\det \frac{\partial^2 \Psi}{\partial x_i \partial x_j} = \rho$ substitutes for the usual Poisson equation. This system can be derived as a geometric approximation of the Euler equations of incompressible fluid mechanics in the spirit of Arnold and Ebin. Global existence of weak solutions and local existence of smooth solutions are obtained. Links between the VMA system, the VP system and the Euler equations are established through rigorous asymptotic analysis.

1 Introduction

The classical Vlasov-Poisson (VP) system describes the evolution of an electronic cloud in a neutralizing uniform background through the following equations

(1) $$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_\xi f = 0$$

(2) $$\varepsilon^2 \Delta \varphi = \rho - 1,$$

where $f(t,x,\xi) \geq 0$ denotes the electronic density at time $t \geq 0$, point $x \in \mathbb{R}^d$, velocity $\xi \in \mathbb{R}^d$ (usually $d = 3$), $\rho(t,x) \geq 0$ denotes the 'macroscopic' density

(3) $$\rho(t,x) = \int_{\mathbb{R}^d} f(t,x,\xi) d\xi,$$

and $\varphi(t,x)$ denotes the electric potential at time t and point x generated, through the Poisson equation (2), where ε is a coupling constant, by the difference between the electronic density $\rho(t,x)$ and the neutralizing background density, which is supposed to be uniform and normalized to unity. Standard notations $\nabla = (\partial_1, ..., \partial_d)$ and $\Delta = \partial_1^2 + ... + \partial_d^2$ have been used and $\cdot$ stands for the inner product in $\mathbb{R}^d$. The mathematical theory of the VP system is now well understood. In particular, existence of global smooth solutions in three space dimensions has been proved in [24] (see also [18], [26]). In the present paper, a fully nonlinear version of the VP system is addressed :

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_\xi f = 0 \tag{4}$$

$$\det(\mathbb{I} + \varepsilon^2 D^2 \varphi) = \rho, \tag{5}$$

where the (real) Monge-Ampère equation (5) substitutes for the Poisson equation (2). Here, $D^2\varphi(t,x)$ stands for the $d \times d$ symmetric matrix made of all second order $x-$partial derivatives of φ, $\mathbb{I}$ stands for the $d \times d$ identity matrix and det for the determinant of a square matrix. The occurrence of the Monge-Ampère equation in mathematical modeling is not very common. Notice, however, that a very similar system can be found in meteorology with Hoskins' semi-geostrophic equations (cf. [3], [13] and the included references). In a simplified two dimensional setting, the semi-geostrophic equations read

$$\frac{\partial \rho}{\partial t} + \{\varphi, \rho\} = 0 \tag{6}$$

$$\det(\mathbb{I} + \varepsilon^2 D^2 \varphi) = \rho, \tag{7}$$

where $\{\cdot,\cdot\}$ denotes the usual Poisson bracket.
Formally, as the coupling constant ε is small, the VP and VMA equations asymptotically approach each other up to order $O(\varepsilon^4)$. Indeed, linearizing the determinant about the identity matrix leads to

$$\det(\mathbb{I} + \varepsilon^2 D^2 \varphi) = 1 + \varepsilon^2 \Delta\varphi + O(\varepsilon^4). \tag{8}$$

The formal limit, as $\varepsilon = 0$, reads

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f + \nabla_x \varphi \cdot \nabla_\xi f = 0 \tag{9}$$

$$\rho = 1, \tag{10}$$

where constraint (10) substitutes for both the Poisson and the Monge-Ampère equations. The limit system (9,10), that we call constrained Vlasov system, can be seen as a 'kinetic' extension of the Euler equations of classical incompressible fluid mechanics,

$$\partial_t v + (v \cdot \nabla) v = -\nabla p \tag{11}$$

$$\nabla \cdot v = 0, \tag{12}$$

where $v(t,x) \in \mathbb{R}^d$ and $p(t,x) \in \mathbb{R}$ respectively are the velocity and the pressure of the fluid at time t and position x. Indeed, any smooth solution (v,p) provides a 'monokinetic' solution to the constrained Vlasov system (9,10), defined by

$$f(t,x,\xi) = \delta(\xi - v(t,x)), \quad \varphi = -p.$$

Here a monokinetic solution means a delta-valued solution in the ξ variable. In addition, the constrained Vlasov system (9,10) turns out to be a natural extension (or Γ limit) of the Euler equations from both geometrical and variational reasons, as explained in section 2

In a similar way, there is a monokinetic version of the VP system, the so-called (pressureless) Euler-Poisson (EP) system, which reads

$$\partial_t v + (v \cdot \nabla) v = \nabla \varphi \tag{13}$$

$$\partial_t \rho + \nabla \cdot (\rho v) = 0 \tag{14}$$

$$\varepsilon^2 \Delta \varphi = \rho - 1. \tag{15}$$

A rigorous asymptotic analysis of the VMA system as $\varepsilon \to 0$ will be provided (sections 5.1 and 5.2), in the case when the initial electronic density

$$f(t=0,x,\xi) = f^0(x,\xi) \tag{16}$$

is asymptotically monokinetic, namely approaching $\delta(\xi - v_0(x))$, for some smooth divergence free velocity field v_0, as ε tends to zero. Before this asymptotic analysis, we want to explain the geometric origin of the VMA system. It has been known, since Arnold's celebrated work (cf. [2]), that the Euler equations (formally) describe geodesics curves along a suitable group of volume preserving maps, lengths being measured in the L^2 sense. We will show (section 2) that the VMA system just describes approximate geodesics obtained through a very natural penalty method, where ε stands for the penalty parameter. For this geometric interpretation to be valid, the Monge-Ampère equation (5) must be understood in the following weak sense: for each fixed t, $\varphi(t,\cdot)$ is the unique (up to an additive constant) function such that $\Psi(x) = x^2/2 + \varepsilon^2 \varphi(t,x)$ is convex in x and

$$\forall g \in C^0(\mathbb{R}^d), \int_{\mathbb{R}^d} g(\nabla \Psi(x)) \rho(t,x) dx = \int_\Omega g(y) dy, \tag{17}$$

where Ω is a fixed bounded open convex set where the neutralizing background of the electrons is assumed to be located. (This definition is made precise in section 2.3.) Notice that, by construction, $\nabla \Psi$ must be valued in the closure of Ω and, therefore, the potential φ enjoys the following property

$$|x + \varepsilon^2 \nabla_x \varphi(t,x)| \leq \sup_{y \in \Omega} |y| < +\infty.$$

There is no similar bound for the electrostatic potential of the classical VP system. Thus, in some sense, the VMA system can be seen as a nonlinearly saturated version of the VP system.
Beyond the geometric derivation of the VMA system, our main analytic results are as follows:

- The VMA system admits global energy preserving weak solutions.
- The VMA system admits local strong solutions in periodic domains.
- For well prepared, nearly monokinetic initial data, the solutions of the VMA system converge when ε goes to 0 to those of the Euler equations.
- In this asymptotic, the EP system is a higher order approximation of the VMA system.

The paper is organized as follows: in section 2, we first recall the geometric nature of the Euler equations, then we explain why the constrained Vlasov system (9,10) is a natural extension of the Euler equations from a variational point of view, finally we introduce the concept of approximate geodesics for volume preserving maps, and derive the VMA system. Section 3 is devoted to the proof of existence of global energy preserving weak solutions. In section 4, we prove existence of local strong solutions, in the case of a periodic domain. Finally, in section 5, we study the asymptotic behavior of the VMA system as ε goes to 0.

2 The geometric origin of the Vlasov-Monge-Ampère system

2.1 The Euler equations

The motion of an incompressible fluid in a domain $\Omega \subset \mathbb{R}^d$ is classically described by the Euler equations (E):

$$\partial_t v + (v \cdot \nabla) v = -\nabla p \tag{18}$$

$$\nabla \cdot v = 0, \tag{19}$$

with $t \in \mathbb{R}$, $x \in \Omega$, where $v = v(t,x)$ stands for the velocity field and $p = p(t,x)$ for the scalar pressure field. These equations have a nice geometrical interpretation going back to Arnold (see [2]). Introducing $G(\Omega)$ the group of all volume preserving diffeomorphisms of Ω with jacobian determinant equal to 1, and measuring lengths in the L^2 sense, we may define (at least formally) geodesic curves along $G(\Omega)$. It turns out that the Euler equations just describe these curves. For the same reasons, the Euler equations can be seen as the optimality equations for the corresponding minimization problem: given two maps chosen in $G(\Omega)$, find an L^2-shortest path between them along $G(\Omega)$. It was shown by Shnirelman [27] (see also [2] and [28]) that, in the case when Ω is the unit cube in $\mathbb{R}^3$, there are many maps for which there are no such shortest paths. Beyond this negative result, [6] established that minimizing paths are more appropriately described by doubly stochastic measures. These measures (also called polymorphisms) generalize volume preserving maps in the following way: a doubly stochastic measure $\mu(dx,dy)$ is a (Borel) probability measure on $\Omega \times \Omega$ with two projections on each copy of Ω both equal to the (normalized) Lebesgue measure. It is known -see [22], for instance- that any such μ can be weakly approximated by a sequence $\mu_n(dx,dy) = \delta(x - g_n(y))dy$ where each g_n is a volume preserving map of Ω. In [6] it was shown that, in the case considered by Shnirelman for which there is no classical shortest path, minimizing paths along $G(\Omega)$ converge to paths of doubly stochastic measures $t \to \mu(t;dx,dy)$ governed by the following extension of the Euler equations

$$\partial_t \mu + \nabla_x \cdot (\mu v) = 0, \tag{20}$$

$$\partial_t (v\mu) + \nabla_x \cdot (\mu v \otimes v) + \mu \nabla_x p = 0, \tag{21}$$

where $v = v(t;x,y)$ and $p = p(t,x)$ can be respectively seen as the velocity field and the pressure field attached to μ. (Notice that the velocity field v generally depends on the extra variable y and is not a classical but rather a multivalued velocity field.) These equations are just a reformulation of the constrained Vlasov system (9,10). Indeed, it can be checked, under appropriate regularity assumptions, that the kinetic measure f defined by

$$f(t;dx,d\xi) = \int_{y\in\Omega} \delta(\xi - v(t;x,y))\mu(t;dx,dy) \tag{22}$$

solves (9,10) when (μ,v,p) solves (20,21). Thus we conclude that the constrained Vlasov system (9,10) is a natural variational extension of the Euler equations.

2.2 Approximate geodesics

A general strategy to define approximate geodesics along a manifold M (in our case $M = G(\Omega)$) embedded in a Hilbert space H (here $H = L^2(\Omega,\mathbb{R}^d)$) is to introduce a penalty parameter $\varepsilon > 0$ and the following *unconstrained* dynamical system in H

$$\partial_{tt}X + \frac{1}{2\varepsilon^2}\nabla_X\left(d^2(X,M)\right) = 0. \tag{23}$$

In this equation, the unknown $t \to X(t)$ is a curve in H, $d(X,M)$ is the distance (in H) of X to the manifold M, i.e. in our case as $M = G(\Omega)$,

$$d(X,G(\Omega)) = \inf_{g\in G(\Omega)} \|X-g\|_H, \tag{24}$$

and, finally, ∇_X denotes the gradient operator in H. This penalty approach has been used for the Euler equations by the first author in [7]. It is similar-but not identical- to Ebin's slightly compressible flow theory [15], and is a natural extension of the theory of constrained finite dimensional mechanical systems [25]. The penalized system is formally hamiltonian in variables $(X,\partial_t X)$ with Hamiltonian (or energy) given by:

$$E = \frac{1}{2}\|\partial_t X\|_H^2 + \frac{1}{2\varepsilon^2}d^2(X,G(\Omega)).$$

(Multiplying equation (23) by $\partial_t X$, we formally get that the energy is conserved.) Therefore it is plausible that the map $X(t)$ will remain close to $G(\Omega)$ if properly initialized at $t=0$. A formal computation shows that, given a point X for which there is a unique closest point π_X to X in the H closure of $G(\Omega)$, we have:

$$\nabla_X\left(d(X,G)\right) = \frac{1}{d(X,G)}(X-\pi_X). \tag{25}$$

Thus the equation (23) formally becomes:

$$\partial_{tt}X + \frac{1}{\varepsilon^2}(X-\pi_X) = 0. \tag{26}$$

To understand why solutions to such a system may approach geodesics along $G(\Omega)$ as ε goes to 0, just recall that, in the simple framework of a surface S embedded in the 3 dimensional Euclidean space, a geodesic $t \to s(t)$ along S is characterized by the fact that for every t, the plane defined by $\{\dot{s}(t),\ddot{s}(t)\}$ is orthogonal to S. In our case, $\partial_{tt}X(t)$ is nearly orthogonal to $G(\Omega)$ thanks to (26), meanwhile $X(t)$ remains close to $G(\Omega)$.
The approximate geodesic equation was introduced in [7] in order to allow a spatial approximation of $G(\Omega)$ by the group of permutations of N points A_j chosen to form a discrete grid on Ω. On such a discrete group, the concept of geodesics becomes unclear meanwhile approximate geodesics still make sense. They can be interpreted as trajectories of a cloud of N particles X_i moving in the Euclidean

space $\mathbb{R}^{dN}$, which substitutes for H. These particles solve the following coupled system of harmonic oscillators

$$\varepsilon^2 \frac{d^2 X_i}{dt^2} + X_i - A_{\sigma_i} = 0,$$

where σ is a time dependent permutation minimizing, at each fixed time t, $\Sigma \left| X_i - A_{\sigma(i)} \right|^2$ among all other permutations of the first N integers. The convergence of this discrete model to the incompressible Euler equations for well prepared initial data was proved in [7]. In order to study the continuous version (26), a specific study of the projection problem (24) is needed.

2.3 The polar decomposition Theorem

Let us first recall a general measure theoretic definition:

Definition 2.1 *Let A and B be two topological spaces, let ρ be a Borel finite measure of A and X a Borel map $A \to B$, we call the push-forward of ρ by X and note $X\#d\rho$ the Borel measure η on B defined by*

$$\forall f \in C^0(B),\ \int_B f(y) d\eta(y) = \int_A f(X(x)) d\rho(x).$$

Let us now consider the case of a bounded open subset Ω of the Euclidean space $\mathbb{R}^d$ equipped with the Lebesgue measure that we denote dx. We say that a Borel map $s : \overline{\Omega} \to \overline{\Omega}$ is volume (or Lebesgue measure) preserving if $s\#dx = dx$, i.e. if for all $g \in C^0(\overline{\Omega})$ one has $\int_\Omega g(x)dx = \int_\Omega g(s(x))dx$, or equivalently, for any Borel subset B of $\overline{\Omega}$ one has $|s^{-1}(B)| = |B|$. The set of all measure preserving maps of Ω is a closed subset of the Hilbert space $H = L^2(\Omega, \mathbb{R}^d)$ and will be denoted by $S(\Omega)$. Notice that $S(\Omega)$ is only a semi-group for the composition rule and contains the group of volume preserving diffeomorphisms $G(\Omega)$. It is known [23] that, at least in the case when Ω is convex and $d \geq 2$, $S(\Omega)$ is exactly the closure of $G(\Omega)$ in $L^2(\Omega, \mathbb{R}^d)$, which implies $d(., G(\Omega)) = d(., S(\Omega))$.
The polar decomposition Theorem for maps [5] (extended to Riemannian manifolds in [21]) will be crucial for our analysis of the VMA system:

Theorem 2.2 *Let Ω be a bounded convex open subset of $\mathbb{R}^d$, let $X \in L^2(\Omega; \mathbb{R}^d)$ and $\rho_X = X\#dx$, where dx is the Lebesgue measure on Ω. Assume ρ_X to be a Lebesgue integrable function, or, equivalently, X to satisfy the non-degeneracy condition:*

(27) $$\forall E \subset \mathbb{R}^d \text{ Borel },\ |E| = 0 \Rightarrow |X^{-1}(E)| = 0.$$

Then there exists a unique pair $(\nabla\Phi_X, \pi_X)$ where Φ_X is a convex function and $\pi_X \in S(\Omega)$, such that

$$X = \nabla\Phi_X \circ \pi_X. \tag{28}$$

In this 'polar decomposition', π_X is also characterized as the unique closest point to X on $S(\Omega)$ in the L^2 sense and Φ_X is characterized to be (up to an additive constant) the unique convex function on Ω satisfying

$$\int_{\mathbb{R}^d} g(x)d\rho_X = \int_\Omega g(X(y))dy = \int_\Omega g(\nabla\Phi_X(y))dy, \tag{29}$$

for any $g \in C^0(\mathbb{R}^d)$ such that $|g(x)| \leq C(1+|x|^2)$.
In addition, the Legendre-Fenchel transform Ψ_X of Φ_X defined by

$$\Psi_X(x) = \sup_{y\in\Omega}\{x \cdot y - \Phi_X(y)\} \tag{30}$$

is Lipschitz continuous on $\mathbb{R}^d$, with Lipschitz constant bounded by $\sup_{x\in\Omega}|x|$ and has the following properties :
$\nabla\Psi_X(x) \in \Omega$ holds true for ρ_X a.e. x,

$$\int_{\mathbb{R}^d} g(\nabla\Psi_X)\rho_X(x)dx = \int_\Omega g(\nabla\Psi_X(X(x)))dx = \int_\Omega g(x)dx \tag{31}$$

for any $g \in C^0(\overline{\Omega})$, and

$$\nabla\Phi_X(\nabla\Psi_X(x)) = x \quad \rho_X(x)dx \; a.e, \tag{32}$$

$$\nabla\Psi_X(\nabla\Phi_X(y)) = y \quad dy \; a.e, \tag{33}$$

$$\pi_X(y) = \nabla\Psi_X(X(y)) \quad dy \; a.e. \tag{34}$$

We make here several remarks on Theorem 2.2:

Link with the Monge-Ampère equation We can interpret (29) as a weak version of the Monge-Ampère equation:

$$\rho_X(\nabla\Phi)\det D^2\Phi = 1$$

and (31) can be seen as a weak version of another Monge-Ampère equation:

$$\det D^2\Psi = \rho_X$$
$$\nabla\Psi \text{ maps } \mathrm{supp}(\rho_X) \text{ in } \Omega.$$

The pair (Φ_X, Ψ_X) depends in fact only of Ω and the measure $\rho_X = X\#dx$, and if condition (27) fails, then existence and uniqueness of the projection π_X may fail, but existence and uniqueness of $\nabla\Phi_X$ remain true.

Theorem 2.2 and the subsequent remarks allow us to introduce the following notation that will be used throughout the paper:

Definition 2.3 *Let Ω be a fixed bounded convex open set of $\mathbb{R}^d$, let ρ be a positive measure on $\mathbb{R}^d$ of total mass $|\Omega|$, absolutely continuous w.r.t the Lebesgue measure and such that $\int (1+|x|^2)d\rho(x) < +\infty$. We call $\Phi[\Omega,\rho]$, or, in short, $\Phi[\rho]$, the unique up to a constant convex function on Ω satisfying*

$$(35) \quad \forall g \in C^0(\mathbb{R}^d) \cap L^1(d\rho), \int_{\mathbb{R}^d} g(x)d\rho(x) = \int_\Omega g(\nabla\Phi[\Omega,\rho](y))dy.$$

We call $\Psi[\Omega,\rho]$ its Legendre-Fenchel transform satisfying

$$(36) \quad \forall g \in C^0(\mathbb{R}^d) \cap L^1(\Omega, dy), \int_{\mathbb{R}^d} g(\nabla\Psi[\Omega,\rho](x))d\rho(x) = \int_\Omega g(y)dy.$$

If no confusion is possible we may write Φ (resp. Ψ) instead of $\Phi[\Omega,\rho]$ (resp. $\Psi[\Omega,\rho]$).

We will use some additional results from [5]. The first one establishes the continuity of the polar decomposition:

Theorem 2.4 *Let ρ be a Lebesgue integrable positive measure on $\mathbb{R}^d$, with total mass Ω, such that $\int (1+|x|^2)d\rho < +\infty$. Let ρ_n be a sequence of Lebesgue integrable positive measures on $\mathbb{R}^d$, with total mass Ω, such that $\forall n$, $\int (1+|x|^2)d\rho_n < +\infty$. Let $\Phi_n = \Phi[\Omega,\rho_n]$ and $\Psi_n = \Psi[\Omega,\rho_n]$ be as in Definition 2.3. If for any $f \in C^0(\mathbb{R}^d)$ such that $|f(x)| \le C(1+|x|^2)$, $\int f\, d\rho_n$ converges to $\int f\, d\rho$, then*

- *Φ_n converges to $\Phi[\Omega,\rho]$ uniformly on each compact set of Ω and strongly in $W^{1,1}(\Omega)$,*
- *Ψ_n converges to $\Psi[\Omega,\rho]$ uniformly on each compact set of $\mathbb{R}^d$ and strongly in $W^{1,1}(K)$ for every K compact in $\mathbb{R}^d$.*

The second one provides a 'dual' definition of the distance between a map X and the semi-group $S(\Omega)$:

Theorem 2.5 *Let $X \in L^2(\Omega;\mathbb{R}^d)$ and $\rho = X\#dx$, where dx is the Lebesgue measure on Ω. Assume ρ to be a Lebesgue integrable function. Then*

$$\frac{1}{2}d^2(X,S(\Omega)) = \int \left(|x|^2/2 - \Psi[\Omega,\rho](x)\right)\rho(x)dx + \int_\Omega \left(|y|^2/2 - \Phi[\Omega,\rho](y)\right)dy$$

$$= \sup_{u,v} \int \left(|x|^2/2 - u(x)\right)\rho(x)dx + \int_\Omega \left(|y|^2/2 - v(y)\right)dy,$$

where the supremum if performed over all pairs (u,v) of continuous functions on $\mathbb{R}^d$ such that $u(x)+v(y) \ge x\cdot y$ pointwise.

2.4 The Vlasov-Monge-Ampère system

Let us now derive the *VMA* system as the kinetic formulation of the approximate geodesic equation (26). First, from the polar decomposition Theorem 2.2, equation (26) reads

$$\partial_{tt}X(t,x) = \nabla\varphi(t,X(t,x)), \tag{37}$$

where

$$\nabla\varphi(t,x) = \frac{\nabla\Psi[\Omega,\rho(t,\cdot)](x) - x}{\varepsilon^2} \tag{38}$$

and $\Psi[\Omega,\rho]$ is as in Definition (2.3). This means that $\nabla\varphi$ satisfies (5) in a weak form with the additional condition that the range of $x \to x + \varepsilon^2\nabla\varphi(t,x)$ is contained in $\overline{\Omega}$.

Next, let $f^0 \geq 0$ be a given initial density function, that we assume to be in $L^\infty(\mathbb{R}^d \times \mathbb{R}^d)$, compactly supported and satisfying the compatibility condition

$$\int f^0(x,\xi)dxd\xi = |\Omega|. \tag{39}$$

For each $t \geq 0$, let us define $(x,\xi) \to f(t,x,\xi)$ to be f^0 pushed forward by the following ODE

$$\partial_t X(t,x,\xi) = \Xi(t,x,\xi) \tag{40}$$

$$\partial_t \Xi(t,x,\xi) = (\nabla\varphi)(X(t,x,\xi)) \tag{41}$$

$$(X,\Xi)(t=0,x,\xi) = (x,\xi). \tag{42}$$

Then f satisfies the following kinetic (or Liouville) equation

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (\nabla\varphi f) = 0 \tag{43}$$

$$f(0,\cdot,\cdot) = f^0, \tag{44}$$

which must be understood in the following weak sense

$$\begin{aligned} &\forall g \in C_c^\infty([0,+\infty) \times \mathbb{R}^d \times \mathbb{R}^d), \\ &\int_0^\infty dt \int_{\mathbb{R}^d\times\mathbb{R}^d} \left(\frac{\partial g}{\partial t} + \xi\cdot\nabla_x g + \nabla\varphi\cdot\nabla_\xi g\right) f dxd\xi \\ &= -\int_{\mathbb{R}^d\times\mathbb{R}^d} f_0(x,\xi)g(t=0,x,\xi)dxd\xi. \end{aligned} \tag{45}$$

This linear Liouville equation is nonlinearly coupled to equation (38), where ρ is linked to f by equation (3). Finally, we have defined, through (38,43,44), the weak formulation of the *VMA* initial value problem.

The energy of the system is defined by

$$E(t) = \frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi)|\xi|^2 dx d\xi$$
$$+ \frac{1}{2\varepsilon^2}\int_{\mathbb{R}^d} \rho(t,x)\,|\nabla\Psi[\Omega,\rho](t,x)-x|^2\,dx. \tag{46}$$

3 Existence of global renormalized weak solutions

The main result of this section is as follows:

Theorem 3.1 *Let $(x,\xi)\to f^0(x,\xi)\geq 0$ be in $L^\infty(\mathbb{R}^d\times\mathbb{R}^d)$, with compact support in both x and ξ, satisfying condition (39).*
Then the VMA system (38,43,44) admits a global weak solution (f,ρ,Ψ), with $f\in L^\infty(\mathbb{R}^+\times\mathbb{R}^d\times\mathbb{R}^d)$ and $(\rho,\nabla\psi)\in L^\infty([0,T]\times\mathbb{R}^d)$ for all $T>0$. In addition, each such weak solution enjoys the following properties:

- *f is a continuous function of t, valued in $L^p(\mathbb{R}^d\times\mathbb{R}^d)$, for every $1\leq p<\infty$*
- *the density ρ is a continuous function of t, valued in $L^p(\mathbb{R}^d)$, for every $1\leq p<\infty$,*
- *the support of $f(t,\cdot,\cdot)$ in (x,ξ) is compact, with a diameter growing no more than linearly in t.*
- *the total energy defined by (46) is conserved,*
- *the 'renormalization' property (in the sense of [14])*

$$\frac{\partial g(f)}{\partial t}+\nabla_x\cdot(\xi g(f))+\nabla_\xi\cdot(\nabla\varphi g(f))=0$$

  holds true for all $g\in C^1(\mathbb{R})$,
- *the trajectories of (41,42) are uniquely defined for almost every initial condition (x,ξ),*
- *$t\to f(t,\cdot,\cdot)$ is just f^0 pushed forward along the trajectories of (41,42).*

Proof of Theorem 3.1:
We build a sequence of approximate solutions $(f_h,\Psi_h)_{h>0}$ by time discretization and let the time step h go to zero. To handle the limiting process, the non-linear terms will be treated with the help of Theorem 2.4. More precisely if one can extract a subsequence such that, for every t, $f_h(t,\cdot,\cdot)$ converges weakly, then we can deduce from Theorem 2.4 that the corresponding sequence $\nabla\Psi_h(t,\cdot)$ will converge strongly, and this will allow us pass to the limit in the nonlinear term.

3.1 Construction of a sequence of approximate solutions

We consider $\eta \in C_c^\infty(\mathbb{R}^d)$ such that $\eta \geq 0$, $\int_{\mathbb{R}^d} \eta = 1$ and $\eta_h = \frac{1}{h^d}\eta(\frac{\cdot}{h})$. We then seek approximate solutions as solutions of the approximate problem

$$\frac{\partial f_h}{\partial t} + \xi \cdot \nabla_x f_h + \frac{\nabla \Psi_h(x) - x}{\varepsilon^2} \cdot \nabla_\xi f_h = 0 \tag{47}$$

$$f_h(0,x,\xi) = f_h^0(x,\xi) = f_0 *_{x,\xi} \eta_h \otimes \eta_h \tag{48}$$

$$\Psi_h(t) = \eta_h * \Psi[\Omega, \rho(t = nh)] \text{ for } t \in [nh, (n+1)h[. \tag{49}$$

$\nabla \Psi_h$ being a smooth function of space this regularized equation admits a unique solution that one builds by the method of characteristics. Since the flow is divergence-free in the phase space, the solution f_h satisfies

$$\forall p \in [1, +\infty], \ \| f_h(t) \|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)} = \| f_h(0) \|_{L^p(\mathbb{R}^d \times \mathbb{R}^d)}. \tag{50}$$

By construction (through Theorem 2.2), $\nabla \Psi_h$ is valued in the convex bounded set $\overline{\Omega}$. Suppose that $f^0(x,\xi)$ vanishes outside of the set $\{x^2 + \varepsilon^2 \xi^2 \leq C^2\}$ for some constant $C > 0$ fixed and denote $R = \sup_{y \in \Omega} |y|$. Then we have

Lemma 3.2 *$\forall t \geq 0,\ f_h(t,\cdot,\cdot)$ is supported in $\{\sqrt{x^2 + \varepsilon^2 \xi^2} \leq C + Rt/\varepsilon\}$.*

Proof : We just write

$$\varepsilon^2 \partial_{tt} X + X = \nabla \Psi_h(X)$$

in complex notation $-i\varepsilon \partial_t Z + Z = F$, where $Z = X + i\varepsilon \partial_t X$ and $F = \nabla \Psi_h(X)$, which is bounded by R. This leads to

$$Z(t) = Z(0) \exp(-it/\varepsilon) + i\varepsilon^{-1} \int_0^t \exp(-i(t-s)/\varepsilon) F(s) ds$$

ant the desired bound easily follows. Notice here a sharp contrast with the classical VP system, for which the $\xi-$support of the solutions cannot be controlled so easily (except in the one dimensional case). □

Convergence of the sequence of approximate solutions

Using (50) and Lemma 3.2 there exists, for any $1 < p < \infty$, up to the extraction of a subsequence, $f \in L^p([0,T] \times \mathbb{R}^d \times \mathbb{R}^d)$ such that f_h converges weakly to f as $h \to 0$.

It remains to show that the product $f_h \nabla \Psi_h$ converges to the good limit. For this we need strong convergence of $\nabla \Psi_h$. We already know that $\nabla \Psi_h \in L^\infty([0,T] \times \mathbb{R}^d)$. We claim that for all $t > 0$, $\nabla \Psi_h(t,\cdot)$ converges strongly to $\nabla \Psi(t,\cdot)$ in $L^q_{loc}(\mathbb{R}^d)$, $\forall q \in [1, +\infty[$. Indeed, such a strong convergence of $\nabla \Psi_h$ follows from Theorem 2.4 provided that we have for all $t > 0$,

$$\int_{\mathbb{R}^d} g(x) \rho_h(t,x) dx \to \int_{\mathbb{R}^d} g(x) \rho(t,x) dx, \tag{51}$$

for any $g \in C^0(\mathbb{R}^d)$ such that $\int (1+|x|^2) g(x) dx < +\infty$. Note first that from Lemma 3.2, we can restrict ourselves here to test functions g that are compactly supported. Then we show that the sequence ρ_h is relatively compact in $C([0,T], L^p(\mathbb{R}^d) - w)$. This is done by the following lemma:

Lemma 3.3 *For all $T > 0$, for all p with $1 \leq p < \infty$ the sequence f_h (resp. ρ_h) satisfies*

- *f_h (resp. ρ_h) is a bounded sequence in $L^\infty([0,T]; L^p(\mathbb{R}^d \times \mathbb{R}^d))$ (resp. in $L^\infty([0,T]; L^p(\mathbb{R}^d))$,*
- *$\partial_t f_h$ (resp. $\partial_t \rho_h$) is a bounded sequence in $L^\infty([0,T]; W^{-1,p}(\mathbb{R}^d \times \mathbb{R}^d)))$, (resp. in $L^\infty([0,T]; W^{-1,p}(\mathbb{R}^d))$,*

and one can extract from f_h (resp. from ρ_h) a subsequence converging in $C([0,T], L^p(\mathbb{R}^d \times \mathbb{R}^d) - w)$ (resp. in $C([0,T], L^p(\mathbb{R}^d) - w)$).

Proof: the first point uses equation (50) and Lemma 3.2. The second point uses equation (43) and the identity:

$$\partial_t \rho_h = -\nabla_x \cdot \int_{\mathbb{R}^d} \xi f_h d\xi,$$

with the fact that the f_h are uniformly compactly supported in x and ξ (Lemma 3.2); the last point is a classical result of functional analysis (see [17] for example). □

This lemma and Lemma 3.2 yield (51). Then using Theorem 2.4, with ρ the limit of a subsequence of ρ_h, we have convergence of the sequence $\nabla\Psi_h$ to $\nabla\Psi[\Omega, \rho]$ in $C([0,T], L^p(\mathbb{R}^d))$. We have extracted a subsequence f_h such that

- f_h converges in $C([0,T], L^p(\mathbb{R}^d \times \mathbb{R}^d) - w)$ for every $1 \leq p < \infty$.
- ρ_h converges in $C([0,T], L^p(\mathbb{R}^d) - w)$ for every $1 \leq p < \infty$.
- $\nabla\Psi_h(t, \cdot)$ converges in $L^p(\mathbb{R}^d)$ for every t and for every $1 \leq p < \infty$.

Thus the limit $(f, \nabla\Psi)$ satisfies equations (43-44) and the first part of Theorem 3.1 is proved.

3.2 Conservation of energy

We now give a rigorous proof of the conservation of energy following an argument going back to F. Otto (in an unpublished work on the semi-geostrophic equations). We recall the definition of the energy as

$$E(t)=\frac{1}{2}\int_{\mathbb{R}^d\times\mathbb{R}^d} f(t,x,\xi)|\xi|^2dxd\xi+\frac{1}{2\varepsilon^2}\int_{\mathbb{R}^d}\rho(t,x)|\nabla\Psi(t,x)-x|^2dx.$$

We call the first term the kinetic energy E_c and the second term, multiplied by ε^2, the (normalized) potential energy E_p. We have

Proposition 3.4 *Let f be any solution of (43) such that on every interval $[0,T]$, $f(t,\cdot,\cdot)$ is uniformly compactly supported in $|x|,|\xi|\leq R(T)$ for some function $R(T)$. Then the energy of the solution f is conserved.*

Proof: From Theorem 2.5, we know that

$$E_p(t)=\int\left(|x|^2/2-\Psi(t,x)\right)\rho(t,x)dx+\int_\Omega\left(|y|^2/2-\Phi(t,y)\right)dy$$

$$=\sup_{u,v}\int\left(|x|^2/2-u(x)\right)\rho(t,x)dx+\int_\Omega\left(|y|^2/2-v(y)\right)dy,$$

where the supremum if performed over all pairs (u,v) of continuous functions on $\mathbb{R}^d$ such that $u(x)+v(y)\geq x\cdot y$ pointwise. Thus for each $t,t_0\in\mathbb{R}+$, we have

$$E_p(t)\qquad\geq\int\left(|x|^2/2-\Psi(t_0,x)\right)\rho(t,x)dx+\int_\Omega\left(|y|^2/2-\Phi(t_0,y)\right)dy,$$

and this implies

$$\begin{aligned}E_p(t)-E_p(t_0)\qquad&\geq\int_{\mathbb{R}^d}\left(|x|^2/2-\Psi(t_0,x)\right)(\rho(t,x)-\rho(t_0,x))dx\\&=\int_{t_0}^t\int_{\mathbb{R}^d}\partial_t\rho(s,x)\left(|x|^2/2-\Psi(t_0,x)\right)dxds\\&=\int_{t_0}^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\xi f(s,x,\xi)\left(x-\nabla\Psi(t_0,x)\right)dxd\xi ds.\end{aligned}$$

Notice that the product in the second line is licit since $\partial_t\rho$ is in $W^{-1,p}$ for any $1\leq p<\infty$, $f(t,\cdot,\cdot)$ and therefore $\rho(t,\cdot)$ are compactly supported in space uniformly on $[0,T]$, and $\Psi-|x|^2/2$ is in $W^{1,\infty}_{loc}$. Exchanging t_0 and t we would have found

$$E_p(t_0)-E_p(t)\geq\int_t^{t_0}\int_{\mathbb{R}^d\times\mathbb{R}^d}\xi f(s,x,\xi)\left(x-\nabla\Psi(t,x)\right)dxd\xi ds,$$

moreover we have for the kinetic energy

$$\varepsilon^2(E_c(t)-E_c(t_0))=\int_{t_0}^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\xi f(t,x,\xi)\cdot(\nabla\Psi(s,x)-x)dxd\xi ds.$$

Dividing by $t-t_0, t>t_0$ we find

$$\begin{aligned}&\varepsilon^2\frac{E(t)-E(t_0)}{t-t_0}\\&\geq\frac{1}{t-t_0}\int_{t_0}^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\xi f(s,x,\xi)\cdot(\nabla\Psi(s,x)-\nabla\Psi(t_0,x))dxd\xi ds\end{aligned}$$

and

$$\begin{aligned}&\varepsilon^2\frac{E(t)-E(t_0)}{t-t_0}\\&\leq\frac{1}{t-t_0}\int_{t_0}^t\int_{\mathbb{R}^d\times\mathbb{R}^d}\xi f(s,x,\xi)\cdot(\nabla\Psi(t,x)-\nabla\Psi(s,x))dxd\xi ds.\end{aligned}$$

We know from 3.1 that $\nabla\Psi(t,.)$ converges strongly in $L^p_{loc}(\mathbb{R}^d), 1\leq p<\infty$ to $\nabla\Psi(t_0,.)$ as t goes to t_0, and so the right hand sides of the above inequalities converges to 0 and we conclude that

$$\lim_{t>t_0}\frac{E(t)-E(t_0)}{t-t_0}=0.$$

We could take $t<t_0$ and find the same result. Finally we conclude that

$$\frac{dE}{dt}\equiv 0.$$

□

3.3 Renormalized solutions and existence of characteristics

The study of renormalized solutions for transport equations has been introduced in [14] for vector fields in $W^{1,1}$ with bounded divergence. These results have been extended by Bouchut [4] to the case of Vlasov-type equations with acceleration field in BV (A recent result of L. Ambrosio, [1], has extended the existence of renormalized solutions to transport equations with vector fields in BV and with bounded divergence). The fact that solutions of (43, 44) are renormalized solutions is an immediate consequence of the following theorem:

Theorem 3.5 (F. Bouchut)
Let $f \in L^\infty(]0,T[,L^\infty_{loc}(\mathbb{R}^d \times \mathbb{R}^d))$ *satisfy*

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x) f) = 0,$$

with $E(t,x) \in L^1(]0,T[;L^1_{loc}(\mathbb{R}^d)) \cap L^1(]0,T[;BV_{loc}(\mathbb{R}^d))$,
then, for any $g \in C^1(\mathbb{R})$,

$$\frac{\partial g(f)}{\partial t} + \nabla_x \cdot (\xi g(f)) + \nabla_\xi \cdot (E(t,x) g(f)) = 0,$$

and for every $1 \le p < \infty$, f *belongs to* $C(]0,T[,L^p_{loc}(\mathbb{R}^d \times \mathbb{R}^d))$.

In our case the BV bound on the acceleration $\nabla\Psi$ is a direct consequence of the fact that Ψ is a globally Lipschitz convex function. This result implies the strong time continuity results for f and ρ in Theorem 3.1. Finally, as in [14], it can be deduced from the renormalization property that
1) for almost every initial condition (x,ξ), there is a unique trajectory solving (41,42),
2) $t \to f(t)$ is just f^0 pushed forward along these trajectories.

A complete proof is given in appendix.

Remark: From the renormalization property it follows that, once the potential $\Psi(t,x)$ is known, there exists a unique solution to (43) in $L^\infty_{t,x,\xi}$. Of course, this does not imply at all the uniqueness of weak solutions to the Vlasov-Monge-Ampère system! This paragraph ends the proof of Theorem 3.1.

4 Strong solutions

In this section we show existence of strong solutions over a finite time intervall. To do so, we need regularity estimates for solutions of Monge-Ampère equation. We will get rid of the difficulties that may arise at the free boundary of the set $\{\rho > 0\}$ by considering the periodic case. Note that for the Vlasov-Poisson system, existence of global smooth solutions has been proved (see [24]); in the present case, due to the non-linearity of the Monge-Ampère equation, we were only able to obtain a result for finite time.

4.1 The periodic Vlasov-Monge-Ampère system

Polar factorization of maps in a periodic domain

The polar decomposition Theorem has been generalized in [21] to general Riemannian manifolds, while the particular case of the flat torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ had been addressed in [11].

Definition 4.1 *We say that a mapping $Y : \mathbb{R}^d \to \mathbb{R}^d$ is $\mathbb{Z}^d$ additive if the mapping $x \to Y(x) - x$ is $\mathbb{Z}^d$ periodic. The set of all measurable $\mathbb{Z}^d$ additive mappings is denoted by $\mathcal{P}$. For each $x \in \mathbb{R}^d$ we call $\hat{x}$ the class of x in $\mathbb{R}^d/\mathbb{Z}^d$, and for any $X \in \mathcal{P}$, $\hat{X}$ the mapping of $\mathbb{T}^d$ into itself defined by*

$$\forall x \in \mathbb{R}^d, \hat{X}(\hat{x}) = \widehat{X(x)}.$$

We may say if no confusion is possible additive instead of $\mathbb{Z}^d$ additive. Then the following theorem can be deduced from the results of [11] and [21]:

Theorem 4.2 *Let $X : \mathbb{R}^d \to \mathbb{R}^d$ be additive and assume that $\rho_X = X\#dx$ has a density in $L^1([0,1]^d)$. Then there exists a unique pair $(\nabla\Phi_X, \pi_X)$ such that*

$$X = \nabla\Phi_X \circ \pi_X$$

where Φ_X is a convex function and $\Phi_X(x) - |x|^2/2$ is $\mathbb{Z}^d$ periodic, $\pi_X : \mathbb{R}^d \to \mathbb{R}^d$ is additive and $\hat{\pi}_X$ is Lebesgue measure preserving in $\mathbb{T}^d$. Moreover we have

$$\|X - \pi_X\|_{L^2([0,1]^d)} = \|\hat{X} - \hat{\pi}_X\|_{L^2(\mathbb{T}^d)}$$

and, Ψ_X denoting the Legendre transform of Φ_X, we have

$$\pi_X = \nabla\Psi_X \circ X.$$

Remark : The pair (Φ_X, Ψ_X) is uniquely defined by the density $\rho_X = X\#dx$.

Notice that the periodicity of $\Phi_X(x) - |x|^2/2$ implies that $\nabla\Phi_X$ and $\nabla\Psi_X$ are $\mathbb{Z}^d$ additive, and that $\Psi_X - |x|^2/2$ is also $\mathbb{Z}^d$ periodic. As in the previous case, we introduce the following notation:

Definition 4.3 *Let ρ be a probability measure on $\mathbb{T}^d$, with density in $L^1(\mathbb{T}^d)$. We denote $\Phi[\rho]$ (resp. $\Psi[\rho]$) the unique up to a constant convex function such that*

(52) $$\Phi[\rho] - |x|^2/2 \text{ is } \mathbb{Z}^d \text{ periodic },$$

(53) $$\forall f \in C^0(\mathbb{T}^d), \int_{\mathbb{T}^d} f(\widehat{\nabla\Phi[\rho](x)})dx = \int f d\rho$$

(resp. its Legendre fenchel transform).

$\Psi[\rho]$ will thus be a generalized solution of the Monge-Ampère equation $\det D^2\Psi = \rho$.
Next the results of Caffarelli ([8], [9], [10]) on the regularity of solutions of the Monge-Ampère equation yield the following theorem:

Theorem 4.4 *Let $\rho > 0$ be a $C^\alpha(\mathbb{T}^d)$ probability density on $\mathbb{T}^d$, for some $\alpha \in]0,1[$. Then $\Psi = \Psi[\rho]$ (see Definition 4.3) is a classical solution of*

$$\det D^2\Psi = \rho$$

and satisfies:

$$\|\nabla\Psi(x) - x\|_{L^\infty} \leq C(d) = \sqrt{d}/2$$
$$\|D^2\Psi\|_{C^\alpha} \leq K(m, M, \|\rho\|_{C^\alpha})$$

where $m = \inf\rho$ and $M = \sup\rho$.

This theorem is an adaptation of the regularity results stated above, whose complete proof is given in appendix.

The periodic Vlasov-Monge-Ampère system

We now seek $f : (t,x,\xi) \in (\mathbb{T}^d \times \mathbb{R}^d \times [0,T]) \to f(t,x,\xi) \in \mathbb{R}^+$, for some $T > 0$, solution of the initial value problem for the periodic Vlasov-Monge-Ampère (VMA_p) system

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \frac{1}{\varepsilon^2}\nabla_\xi \cdot ((\nabla\Psi[\rho](x) - x) f) = 0 \tag{54}$$

$$f(0,\cdot,\cdot) = f^0, \tag{55}$$

for a given f^0 satisfying the compatibility condition

$$\int f^0(x,\xi)dxd\xi = 1. \tag{56}$$

The macroscopic density ρ is still related to f by equation (3), and $\Psi[\rho]$ is as in Definition 4.3.

4.2 Existence of local strong solutions

We mention first that the proof of existence of global weak solutions adapts with minor changes to the periodic case, and that the obtain for the periodic (VMA_p) system the same result as Theorem 3.1.

Our result in this section is the following:

Theorem 4.5 *Let $f_0 \in W^{1,\infty}(\mathbb{T}^d \times \mathbb{R}^d)$, be such that:*

$$\exists C_0 > 0 : \; f_0 \equiv 0 \text{ for } |\xi| \geq C_0, \tag{57}$$

$$\exists m > 0 : \; \rho_0(x) = \int_{\mathbb{R}^d} f_0(x,\xi)d\xi \geq m \; \forall x \in \mathbb{T}^d, \tag{58}$$

then there exists $T > 0$ and a solution f to the VMA_p system (54,55), in the space $W^{1,\infty}([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$.

Proof of Theorem 4.5: First we deduce from Theorem 4.4:

Corollary 4.6 *Let* $\rho, \Psi = \Psi[\rho]$ *be as in Theorem 4.4. Then, we have*

$$\|D^2\Psi\|_{L^\infty(\mathbb{T}^d)} \le C(m, M, \|\nabla\rho\|_{L^\infty(\mathbb{T}^d)}),$$

and we can define

$$K(m,M,l) = \sup\{\|D^2\Psi[\rho]\|_{L^\infty(\mathbb{T}^d)};\ \ \|\nabla\rho\|_{L^\infty(\mathbb{T}^d)} \le l, \quad m \le \rho \le M\} < \infty.$$

We see that in order to use Theorem 4.4 we need ρ to be bounded away from 0. In the following lemma, we show that under suitable assumptions on the initial data, it is possible to enforce locally in time the condition $0 < m \le \rho \le M$.

Lemma 4.7 *Let* $f \in L^\infty([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$ *satisfy*

$$\begin{aligned} &\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x) f) = 0 \\ &f(0,.,.) = f^0, \end{aligned}$$

with $E \in L^1([0,T], BV(\mathbb{T}^d))$ *and*

$$\|E\|_{L^\infty([0,T]\times\mathbb{T}^d)} \le F,$$

let the initial condition f_0 *be such that*

$$a(x,\xi) \le f(0,x,\xi) \le b(x,\xi),$$

with $\rho_a(x) = \int a(x,\xi) d\xi \ge m > 0$ *and* $\rho_b(x) = \int b(x,\xi) d\xi \le M < \infty$ *and* a, b *satisfying*

$$|\nabla_{x,\xi}(a,b)| \le \frac{c}{1+|\xi|^{d+2}}.$$

Then there exists a constant $R > 0$ *depending on* m, M, c, F*, such that*

$$(\rho_a(x) - Rt) \le \rho(t,x) \le (\rho_b(x) + Rt).$$

The proof of the lemma is given in appendix.

4.2.1 Construction of approximate solutions

Let us consider $(t,x) \to E(t,x)$ a smooth vector-field on $\mathbb{T}^d$, and write

$$T_E(f) = \frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E\, f)\,.$$

If f satisfies $T_E(f) = 0$, we have

$$\begin{aligned} T_E \nabla_x f &= -(\nabla_x E) \cdot \nabla_\xi f \\ T_E \nabla_\xi f &= -\nabla_x f \\ T_E \partial_t f &= -\partial_t E \cdot \nabla_\xi f, \end{aligned}$$

and therefore

$$\frac{d}{dt} \|\nabla_{x,\xi} f\|_{L^\infty} \leq \|\nabla_{x,\xi} f\|_{L^\infty}(1 + \|\nabla_x E\|_{L^\infty}) \tag{59}$$

which implies

$$\|\nabla_{x,\xi} f(t)\|_{L^\infty} \leq \|\nabla_{x,\xi} f(t=0)\|_{L^\infty} \exp\left(\int_0^t (1 + \|\nabla_x E(s)\|_{L^\infty}) ds\right).$$

Now let f_0 be given as in Theorem 4.5, satisfying (57,58). Thanks to Lemma 4.7 it is possible to find t_1, m, M such that for any f satisfying

$$\begin{aligned} T_E(f) &= 0 \\ f(t=0) &= f_0, \end{aligned}$$

for any field $E \in L^1([0,t_1], BV(\mathbb{T}^d))$ satisfying $\|E\|_{L^\infty([0,t_1]\times\mathbb{T}^d)} \leq \sqrt{d}/(2\varepsilon^2)$, we have

$$m \leq \rho(t,\cdot) \leq M, \ \ \forall t \in [0,t_1] \tag{60}$$

$$|\xi|_{max} \leq C_1 = 10 C_0, \tag{61}$$

with f supported in $\{|\xi| \leq |\xi|_{max}\}$ and with C_0 as in Theorem 4.5, so that we have for $0 \leq t \leq t_1$:

$$\|\nabla \rho\|_{L^\infty} \leq \omega_d C_1^d \|\nabla_x f\|_{L^\infty}, \tag{62}$$

ω_d being the volume of the unit ball of $\mathbb{R}^d$. Then we construct a family of approximate solutions (f_h, Ψ_h) to (54), in the same spirit as we did for weak solutions, by solving

$$\begin{aligned} &\frac{\partial f_h}{\partial t} + \xi \cdot \nabla_x f_h + \frac{\nabla \Psi_h(x) - x}{\varepsilon^2} \cdot \nabla_\xi f_h = 0 \\ &f_h(t=0) = f_0 \\ &\Psi_h(t) = \Psi[\rho(t = nh)] \text{ for } t \in [nh, (n+1)h[. \end{aligned}$$

Note that we have neither mollified the term $\nabla\Psi$ nor the initial condition and that $\|\nabla\Psi_h - x\|_{L^\infty} \leq C(d) = \sqrt{d}/2$. Now choose $l = 10\|\nabla_{x,\xi} f_0\|_{L^\infty}\omega_d C_1^d$, if for some $t = nh \leq t_1 - h$ we have

$$\|\nabla_{x,\xi} f^h(t = nh)\|_{L^\infty} \leq \frac{l}{\omega_d C_1^d}$$

this implies, thanks to (62), that

$$\|\nabla_x \rho^h(t = nh)\|_{L^\infty} \leq l,$$

and conditions (60,61) are satisfied because $t \leq t_1$. Then if we denote $K = K(m, M, l)$ as in Corollary 4.6, we have for $nh \leq t < nh + h$,

$$\frac{d}{dt}\|\nabla_{x,\xi} f^h\|_{L^\infty} \leq (K+1)\|\nabla_{x,\xi} f^h\|_{L^\infty},$$

and then

$$\|\nabla_{x,\xi} f^h(t = nh+h)\|_{L^\infty} \leq \|\nabla_{x,\xi} f^h(t = nh+h)\|_{L^\infty} \exp(K+1)h.$$

So if we define T as

$$T = \min\{t_1, t_2\},$$

with $\exp((K+1)t_2) = 10$, we have for $0 \leq t \leq T$,

$$\begin{aligned}
&\|\nabla_{x,\xi} f^h\|_{L^\infty} \leq 10\|\nabla_{x,\xi} f_0\|_{L^\infty} \\
&\|\nabla\rho^h\|_{L^\infty} \leq l \\
&m \leq \rho \leq M \\
&\|D^2\Psi^h\|_{L^\infty} \leq K.
\end{aligned}$$

Thus we can extract a subsequence converging to a strong solution of (54,55). Then we argue as in section 2 to show that all terms converge to the correct limit. This ends the proof of Theorem 4.5.

□

5 Asymptotic analysis

5.1 Convergence to the Euler equation

In this section we justify that the Vlasov-Monge-Ampère system describes approximate geodesics on volume preserving transformations: indeed we will show that weak solutions of this system converge to a solution of the incompressible Euler equations (E) as the parameter ε goes to 0, at least for well prepared initial data. We restrict ourselves to the space periodic case, the macroscopic density ρ is still defined by (3) and the convex potentials $\Phi[\rho], \Psi[\rho]$ are still as in Definition 4.3.

For sake of simplicity, we slightly modify our notations and introduce the following rescaled potentials

$$\tilde{\varphi}[\rho] = \frac{|x|^2/2 - \Psi[\rho]}{\varepsilon},$$
$$\varphi[\rho] = \frac{\Phi[\rho] - |x|^2/2}{\varepsilon},$$

so that

$$\nabla\varphi[\rho] = \nabla\tilde{\varphi}[\rho] \circ \nabla\Phi[\rho],$$

and the VMA_p system takes the following form:

$$\frac{\partial f}{\partial t} + \xi \cdot \nabla_x f - \frac{\nabla\tilde{\varphi}[\rho]}{\varepsilon} \cdot \nabla_\xi f = 0 \tag{63}$$
$$f(0,\cdot,\cdot) = f_0. \tag{64}$$

The energy is given by

$$\begin{aligned} E(t) &= \frac{1}{2}\int f(t,x,\xi)|\xi|^2 dx d\xi + \frac{1}{2}\int |\nabla\varphi|^2 dx \\ &= \frac{1}{2}\int f(t,x,\xi)|\xi|^2 dx d\xi + \frac{1}{2}\int \rho|\nabla\tilde{\varphi}|^2 dx. \end{aligned} \tag{65}$$

It has been shown in section 3.2 that the energy is conserved. The Euler equations for incompressible fluids (E) reads:

$$\partial_t v + v \cdot \nabla v = -\nabla p \tag{66}$$
$$\nabla \cdot v = 0. \tag{67}$$

We shall here consider a smooth solution of E and a weak solution of VMA_p, with 'well prepared initial data', meaning that the initial data of both systems are close a time 0. Then we will show that as time evolves, both solutions stay close to each other.

Theorem 5.1 *Let f be a weak solution of (63, 64) with finite energy, let $(t,x) \to v(t,x)$ be a smooth $C^2([0,T] \times \mathbb{T}^d)$ solution of (66,67) for $t \in [0,T]$, and $p(t,x)$ the corresponding pressure, let*

$$H_\varepsilon(t) = \frac{1}{2}\int f(t,x,\xi)|\xi - v(t,x)|^2 dx d\xi + \frac{1}{2}\int |\nabla\varphi(t,x)|^2 dx,$$

then

$$H_\varepsilon(t) \le C\exp(Ct)(H_\varepsilon(0)+\varepsilon^2),\ \forall t\in[0,T].$$

C depends only on $T, \sup_{0\le s\le T}\left\{\|v(s,.),p(s,.),\partial_t p(s,.),\nabla p(s,.)\|_{W^{1,\infty}(\mathbb{T}^d)}\right\}$.

Remark 1: This estimate is enough to compare the weak solutions f to the VMA_p system (for well prepared initial data) and the smooth solutions v of the Euler equations. For instance, $\int f(t=0,x,\xi)d\xi \equiv 1$ implies $\varphi(t=0,x)\equiv 0$ and therefore,

$$\int |\xi - v(t=0,x)|^2 f(t=0,x,\xi)dxd\xi \le C_0\varepsilon^2$$

implies

$$\sup_{t\in[0,T]}\int |\xi - v(t,x)|^2 f(t,x,\xi)dxd\xi \le C_T\varepsilon^2,$$

where C_T depends only on C_0, T and v.

Remark 2: We see that we consider nearly monokinetic initial data for the VMA_p system.

Proof of Theorem 5.1

We shall show that

$$\begin{aligned}\frac{d}{dt}H_\varepsilon = \quad & -\int f(t,x,\xi)(\xi - v)\nabla v(\xi - v)\\ & +\int f(t,x,\xi)\frac{1}{\varepsilon}v\cdot\nabla\tilde{\varphi}\\ & -\int f(t,x,\xi)(v-\xi)\cdot\nabla p,\end{aligned} \tag{68}$$

where we will use the notation

$$u\ \nabla v\ w = \sum_{i,j=1}^{d} u^i\partial_i v^j w^j.$$

The proof of this identity is postponed to the end of the section.

Now we look at all terms of the right hand side. All the constants that we denote by C are controlled as in Theorem 5.1. We set

$$\begin{aligned}T_1 &= -\int f(t,x,\xi)(\xi - v)\nabla v(\xi - v),\\ T_2 &= \int f(t,x,\xi)\frac{1}{\varepsilon}v\cdot\nabla\tilde{\varphi},\\ T_3 &= -\int f(t,x,\xi)(v-\xi)\cdot\nabla p.\end{aligned}$$

First we have $T_1 \leq CH_\varepsilon$. For T_2 we have

$$\begin{aligned} T_2 = \frac{1}{\varepsilon}\int \rho v \cdot \nabla\tilde{\varphi} = & \quad \frac{1}{\varepsilon}\int v(\nabla\Phi[\rho]) \cdot \nabla\tilde{\varphi}(\nabla\Phi[\rho]) \\ = & \quad \frac{1}{\varepsilon}\int v(x+\varepsilon\nabla\varphi)\cdot\nabla\varphi \\ = & \quad \frac{1}{\varepsilon}\int v\cdot\nabla\varphi + (v(x+\varepsilon\nabla\varphi)-v(x))\cdot\nabla\varphi \\ \leq & \quad 0 + C\int |\nabla\varphi|^2 \leq CH_\varepsilon, \end{aligned}$$

we have used that v is divergence-free thus its integral against any gradient is zero. Next we have the following lemma:

Lemma 5.2 *Let $G : \mathbb{T}^d \to \mathbb{R}$ be Lipschitz continuous such that $\int_{\mathbb{T}^d} G = 0$, then for all $R > 0$, one has*

$$|\int \rho G| \leq \frac{1}{2}\|\nabla G\|_{L^\infty}(\frac{1}{R}\varepsilon^2 + RH_\varepsilon).$$

Proof: We just write a Taylor expansion of G:

$$\begin{aligned} & \left|\int(\rho-1)G\right| = \left|\int (G(x+\varepsilon\nabla\varphi)-G(x)\right| \\ \leq & \quad \varepsilon\|\nabla G\|_{L^\infty}\|\nabla\varphi\|_{L^1} \leq \varepsilon\|\nabla G\|_{L^\infty}H_\varepsilon^{1/2} \leq \frac{1}{2}\|\nabla G\|_{L^\infty}(\frac{1}{R}\varepsilon^2 + RH_\varepsilon). \end{aligned}$$

□

Again, since v is divergence-free, $\int v\cdot\nabla p = 0$, thus from Lemma 5.2 we have

$$-\int \rho v\cdot\nabla p \leq C(\varepsilon^2 + H_\varepsilon).$$

We remind that

$$\partial_t\rho(t,x) = -\nabla_x\cdot\int f(t,x,\xi)\xi d\xi.$$

Since it costs no generality to suppose that for all $t \in [0,T]$, $\int p(t,x)dx \equiv 0$, we obtain that

$$\begin{aligned} \int f(t,x,\xi)\xi\cdot\nabla p &= \int \frac{\partial\rho}{\partial t}p \\ &= \frac{d}{dt}\int \rho p - \int \rho\frac{\partial p}{\partial t} \\ &\leq C(\varepsilon^2 + H_\varepsilon) - \frac{dQ}{dt} \end{aligned}$$

again using Lemma 5.2, where $Q(t) = -\int \rho p$. Thus

$$T_3 \leq C(H_\varepsilon + \varepsilon^2) - \frac{dQ}{dt}$$

and we have the following inequality:

$$\frac{d}{dt}(H_\varepsilon + Q) \leq CH_\varepsilon + O(\varepsilon^2). \tag{69}$$

Moreover, using Lemma 5.2,

$$|Q(t)| \leq C\varepsilon^2 + H_\varepsilon(t)/2, \tag{70}$$

thus

$$H_\varepsilon + Q \geq H_\varepsilon/2 - C\varepsilon^2, \tag{71}$$

and we can transform (69) in

$$\frac{d}{dt}(H_\varepsilon + Q) \leq C(H_\varepsilon + Q) + C\varepsilon^2. \tag{72}$$

Gronwall's lemma then yields

$$H_\varepsilon(t) + Q(t) \leq (H_\varepsilon(0) + Q(0) + Ct\varepsilon^2)\exp(Ct).$$

Using again (70) we obtain

$$H_\varepsilon(t) \leq C(H_\varepsilon(0) + \varepsilon^2)\exp(Ct), \tag{73}$$

which achieves the proof of Theorem 5.1.

□

Proof of identity (68):
We first notice that, for all $g \in C^1(\mathbb{R} \times \mathbb{T}^d)$, we have:

$$\frac{d}{dt}\int \rho(t,x)g(t,x)dx = \int\int f(t,x,\xi)(\partial_t g(t,x) + \xi \cdot \nabla g(t,x))d\xi dx.$$

We also use the conservation of energy defined by (65). Then we get

$$\begin{aligned}
\frac{d}{dt}H_\varepsilon = \quad & \frac{d}{dt}\frac{1}{2}\int f(t,x,\xi)(|v|^2 - 2\xi \cdot v)dxd\xi \\
= \quad & \int f(t,x,\xi)(\partial_t v \cdot v - \partial_t v \cdot \xi) - \frac{1}{2}\int \nabla_x \cdot (f(t,x,\xi)\xi)(|v|^2 - 2\xi \cdot v) \\
& + \frac{1}{2}\int \nabla_\xi \cdot (\frac{1}{\varepsilon}\nabla\tilde{\varphi} f(t,x,\xi))(|v|^2 - 2\xi \cdot v).
\end{aligned}$$

Integrating by part, we get

$$\frac{d}{dt}H_\varepsilon = \int f(t,x,\xi)(\partial_t v\cdot v - \partial_t v\cdot\xi) + \int f(t,x,\xi)\xi\nabla v(v-\xi) + \int f(t,x,\xi)\frac{1}{\varepsilon}\nabla\tilde{\varphi}\cdot v.$$

The first two terms can be rewritten as

$$\begin{aligned} & \int f(t,x,\xi)(\partial_t v\cdot v - \partial_t v\cdot\xi) + \int f(t,x,\xi)\xi\nabla v(v-\xi) \\ = & -\int f(t,x,\xi)(v-\xi)\nabla v(v-\xi) + \int f(t,x,\xi)\partial_t v\cdot(v-\xi) \\ & + \int f(t,x,\xi)v\nabla v(v-\xi) \\ = & -\int f(t,x,\xi)(v-\xi)\nabla v(v-\xi) + \int f(t,x,\xi)(v-\xi)\cdot(\partial_t v + v\cdot\nabla v), \end{aligned}$$

and finally using equation (66) we conclude.

□

5.2 Comparison with the Euler-Poisson system

Here we show that, as mentioned in the introduction, the Euler-Poisson (EP) system is a more accurate approximation to the Vlasov Monge-Ampère system than the Euler equations, as ε goes to zero.

The EP system Let us recall that the (pressureless) Euler-Poisson system describes the motion of a continuum of electrons on a neutralizing background of ions through electrostatic interaction. Let $\bar{v}$ and $\bar{\rho}$ be the velocity and density of electrons. Let $\bar{\varphi}$ be the (rescaled) electric potential. Under proper scaling, these functions of $x \in \mathbb{R}^d$ and $t > 0$ satisfy the Euler-Poisson system:

$$\partial_t \bar{v} + \bar{v}\cdot\nabla\bar{v} = \frac{1}{\varepsilon}\nabla\bar{\varphi} \tag{74}$$

$$\partial_t\bar{\rho} + \nabla\cdot(\bar{\rho}\bar{v}) = 0 \tag{75}$$

$$1 - \varepsilon\Delta\bar{\varphi} = \bar{\rho}. \tag{76}$$

The so-called 'quasi-neutral' limit $\varepsilon \to 0$ of similar systems has been studied for example in [16] and [12], and convergence results have been established using pseudo-differentials energy estimates. For well-prepared initial data, solutions of EP are expected to behave as solutions of Euler incompressible equations. This fact is proved by the second author in his PhD thesis ([20], Chap 2), see also [19]. We give here the complete result that we will use herafter. We will denote by $\bar{v}^\varepsilon$ (resp. f^ε) the solutions of the EP (resp. VMA_p) system with parameter ε.

Theorem 5.3 *Let v be a solution of (66,67) on $[0,T]\times\mathbb{T}^d$, with initial data v_0, and satisfying $v\in L^\infty([0,T],H^s(\mathbb{T}^d))$ for some $s\geq s_0(d)$. There for some $s'>0$, $s'<s$, if $(\overline{v}_0^\varepsilon,\overline{\rho}_0^\varepsilon)$ is such that the sequences*

$$\frac{\overline{v}_0^\varepsilon-v_0}{\varepsilon},\qquad \frac{\overline{\rho}_0^\varepsilon-1}{\varepsilon^2}$$

are bounded in $H^{s'}(\mathbb{T}^d)$, then there exists $T_\varepsilon>0$ with $\liminf_{\varepsilon\to 0}T_\varepsilon\geq T$ and a sequence $(\overline{v}^\varepsilon,\overline{\rho}^\varepsilon)$ of solutions to the EP system on $[0,T_\varepsilon[$ with initial data $(\overline{v}_0^\varepsilon,\overline{\rho}_0^\varepsilon)$, belonging to $L^\infty([0,T_\varepsilon],H^{s'}(\mathbb{T}^d))$. Moreover, for ε small enough, the sequences

$$\frac{\overline{v}^\varepsilon-v}{\varepsilon},\qquad \frac{\overline{\rho}^\varepsilon-1}{\varepsilon^2}$$

are bounded in $L^\infty([0,T],H^{s'}(\mathbb{T}^d))$. Finally, s' goes to $+\infty$ as s goes to $+\infty$.

Assumptions

Here we consider v a solution to E (66, 67) with initial data v_0, a sequence f^ε of solutions of VMA_p (63,64) with initial data f_0^ε, and a sequence $(\overline{v}^\varepsilon,\overline{\rho}^\varepsilon)$ solutions of EP (74, 75, 76) with initial data $(\overline{v}_0^\varepsilon,\overline{\rho}_0^\varepsilon)$. We still define H_ε as in Theorem 5.1:

$$H_\varepsilon(t)=\frac{1}{2}\int f^\varepsilon(t,x,\xi)|\xi-v(t,x)|^2dxd\xi+\frac{1}{2}\int|\nabla\varphi^\varepsilon|^2dx.$$

We introduce the following assumptions:

- **H0** v solution of E satisfies, for some $C_0>0$, $\|v\|_{L^\infty([0,T]H^s(\mathbb{T}^d))}\leq C_0$, and s is large enough so that s' in Theorem 5.3 satisfies $s\geq s'>[\frac{d}{2}]+2$.
- **H1** The sequence $(\overline{v}_0^\varepsilon,\overline{\rho}_0^\varepsilon)$ of initial data of EP is such that, for some $C_1>0$,

$$\sup_{\varepsilon>0}\left\{\frac{1}{\varepsilon}\|\overline{v}_0^\varepsilon-v\|_{H^{s'}(\mathbb{T}^d)},\ \frac{1}{\varepsilon^2}\|\overline{\rho}^\varepsilon-1\|_{H^{s'}(\mathbb{T}^d)}\right\}\leq C_1.$$

- **H2** The sequence f_0^ε satisfies $H_\varepsilon(0)\leq C_2\varepsilon^2$ for some $C_2>0$.

H0, **H1**, **H2** imply that

1. There exists $\tilde{C}_0$ such that

(77) $$\|v\|_{L^\infty([0,T],W^{2,\infty}(\mathbb{T}^d))}\leq\tilde{C}_0.$$

2. From Theorem 5.1, there exists $\tilde{C}_1$ such that

$$H_\varepsilon(t) \le \tilde{C}_1 \varepsilon^2 \text{ for } t \in [0,T]. \tag{78}$$

3. From Theorem 5.4 and Sobolev imbeddings, there exists $\tilde{C}_2$ such that

$$\sup_{\varepsilon<\varepsilon_0} \left\{ \left\| \frac{\bar{v}^\varepsilon - v}{\varepsilon}, \quad \frac{\bar{\rho}^\varepsilon - 1}{\varepsilon^2} \right\|_{L^\infty([0,T],W^{2,\infty}(\mathbb{T}^d))} \right\} \le \tilde{C}_2. \tag{79}$$

We are now ready to prove the following result:

Theorem 5.4 *Let f_0^ε, $\bar{v}_0^\varepsilon, \bar{\rho}_0^\varepsilon, v, T$ be as above, satisfying assumptions* **H0**, **H1**, **H2**. *Define*

$$G_\varepsilon(t) = \frac{1}{2}\int f^\varepsilon(t,x,\xi)|\xi - \bar{v}^\varepsilon(x)|^2 dx d\xi + \frac{1}{2}\int |\nabla\varphi^\varepsilon - \nabla\bar{\varphi}^\varepsilon|^2 dx.$$

Then there exists $C > 0$ such that

$$G_\varepsilon(t) \le C\exp(Ct)(G_\varepsilon(0) + \varepsilon^3), \ \forall t \in [0,T]$$

where C depends on s', C_0, C_1, C_2, T.

Remark: the theorem shows that the distance between solutions of the (EP) system and the VMA_p system measured with G_ε is like $O(\varepsilon^3)$ whereas Theorem 5.1 showed that the distance between the solution of the Euler equation and the VMA_p system was like $O(\varepsilon^2)$. Note also that G_ε and H_ε can both be interpreted as the square of a distance.

Proof of Theorem 5.4: For notational simplicity, we drop most ε's. Proceeding as in (68) and noticing that:

$$\frac{d}{dt}\int_{\mathbb{T}^d} |\nabla\bar{\varphi}|^2 = \frac{1}{\varepsilon}\int_{\mathbb{T}^d} \bar{\rho}\,\bar{v}\cdot\nabla\bar{\varphi}$$

we obtain the following identity:

$$\begin{aligned} \frac{d}{dt}G_\varepsilon = \quad & -\int f(t,x,\xi)(\xi - \bar{v})\nabla\bar{v}(\xi - \bar{v}) \\ & + \int f(t,x,\xi)\frac{1}{\varepsilon}\bar{v}\cdot\nabla\varphi - \int f(t,x,\xi)\frac{1}{\varepsilon}\bar{v}\cdot\nabla\bar{\varphi} \\ & + \int f(t,x,\xi)\frac{1}{\varepsilon}\xi\cdot\nabla\bar{\varphi} + \int \frac{1}{\varepsilon}\bar{\rho}\,\bar{v}\cdot\nabla\bar{\varphi} \\ & - \frac{d}{dt}\int \nabla\bar{\varphi}\cdot\nabla\varphi. \end{aligned} \tag{80}$$

Then we notice

$$\int f(t,x,\xi)\frac{1}{\varepsilon}\xi\cdot\nabla\bar{\varphi} = \frac{d}{dt}\left(\int \frac{1}{\varepsilon}\rho\bar{\varphi}\right) - \frac{1}{\varepsilon}\int \rho\partial_t\bar{\varphi}.$$

Next, we have the following lemma:

Lemma 5.5 *Define for any* $\theta \in C^2(\mathbb{T}^d)$

$$< \nabla\theta > (x) = \int_0^1 \nabla\theta(x + s\varepsilon\nabla\varphi(x))ds,$$
$$< \nabla^2\theta > (x) = \int_0^1 (1-s)\nabla^2\theta(x + s\varepsilon\nabla\varphi(x))ds.$$

Then

$$\begin{aligned}\int \rho\theta &= \int \theta + \varepsilon \int < \nabla\theta > \cdot\nabla\varphi \\ &= \int \theta + \varepsilon\int \nabla\theta\cdot\nabla\varphi + \varepsilon^2\int < \nabla^2\theta > \nabla\varphi\nabla\varphi.\end{aligned}$$

Proof: The proof just uses the Taylor expansion and the identity
$\int \rho\theta = \int \theta(x + \varepsilon\nabla\varphi)$.

□

Using Lemma 5.5, we get

$$\begin{aligned}&\frac{1}{\varepsilon}\int \rho\partial_t\bar{\varphi} \\ = \quad &\frac{1}{\varepsilon}\int \partial_t\bar{\varphi} + \int \partial_t\nabla\bar{\varphi}\cdot\nabla\varphi + \varepsilon\int < \partial_t\nabla^2\bar{\varphi} > \nabla\varphi\nabla\varphi.\end{aligned}$$

We claim that, under our assumptions, we have

$$\|\partial_t\nabla^2\bar{\varphi}\|_{L^\infty([0,T']\times\mathbb{T}^d)} \le C.$$

Proof: from (75), we have

$$\partial_t\bar{\rho} = -\bar{\rho}\nabla\cdot\bar{v} - \bar{v}\cdot\nabla\bar{\rho}.$$

Using (79), we obtain that $\|\partial_t\bar{\rho}\|_{H^{s'-1}} \le C\varepsilon$. Since $H^{s'}(\mathbb{T}^d)$ is continuously embedded in $W^{2,\infty}(\mathbb{T}^d)$, $H^{s'-1}(\mathbb{T}^d)$ is continuously embedded in $L^\infty(\mathbb{T}^d)$.
Then, using (76) and classical elliptic regularity, we have

$$\varepsilon\|\partial_t\nabla^2\bar{\varphi}\|_{H^{s'-1}} \le C\|\partial_t\bar{\rho}\|_{H^{s'-1}},$$

and the desired result follows.

□

This implies, using (78), that

$$\left|\varepsilon\int <\partial_t\nabla^2\overline{\varphi}>\nabla\varphi\nabla\varphi\right|\leq C\varepsilon^3.$$

Next,

$$\int\partial_t\nabla\overline{\varphi}\cdot\nabla\varphi=-\int\partial_t\Delta\overline{\varphi}\varphi$$
$$=\frac{1}{\varepsilon}\int\partial_t\overline{\rho}\varphi=\frac{1}{\varepsilon}\int\overline{\rho}\,\overline{v}\cdot\nabla\varphi.$$

Using again Lemma 5.5, we get

$$\begin{aligned}&\frac{d}{dt}\int\nabla\overline{\varphi}\cdot\nabla\varphi\\ =\quad&\frac{1}{\varepsilon}\frac{d}{dt}\left(\int\rho\overline{\varphi}-\varepsilon^2\int <\nabla^2\overline{\varphi}>\nabla\varphi\nabla\varphi\right)\end{aligned}$$

and for the same reasons we have $\|\nabla^2\overline{\varphi}\|_{L^\infty([0,T]\times\mathbb{T}^d))}\leq C\varepsilon$. This yields

$$Q(t)=\varepsilon\int <\nabla^2\overline{\varphi}>\nabla\varphi\nabla\varphi=O(\varepsilon^4).$$

Moreover, it does not cost to set $\int\overline{\varphi}\equiv 0$ and deduce

$$\int f(t,x,\xi)\frac{1}{\varepsilon}\xi\cdot\nabla\overline{\varphi}-\frac{d}{dt}\int\nabla\overline{\varphi}\cdot\nabla\varphi=-\frac{1}{\varepsilon}\int\overline{\rho}\,\overline{v}\cdot\nabla\varphi+O(\varepsilon^3)+\frac{d}{dt}Q.$$

Thus the remaining terms are

$$R=\frac{1}{\varepsilon}\int[\rho\nabla\tilde{\varphi}-\rho\nabla\overline{\varphi}+\overline{\rho}\nabla\overline{\varphi}-\overline{\rho}\nabla\varphi]\cdot\overline{v}.$$

Calculations that we postpone to the end of the proof show that

$$\begin{aligned}R\leq\quad&\int(\nabla\varphi-\nabla\overline{\varphi})\nabla\overline{v}(\nabla\varphi-\nabla\overline{\varphi})+C\int|\nabla\varphi-\nabla\overline{\varphi}|^2\\&-\frac{1}{2}\int\nabla\cdot\overline{v}(|\nabla\overline{\varphi}|^2-2\nabla\varphi\cdot\nabla\overline{\varphi})+C\varepsilon^3.\end{aligned}\tag{81}$$

with C depending on $\|\nabla^2\bar{v}\|_{L^\infty([0,T]\times\mathbb{T}^d)}$ and $\varepsilon^{-1}\|\nabla^3\bar{\varphi}\|_{L^\infty([0,T]\times\mathbb{T}^d)}$, therefore uniformly bounded thanks to (79). Finally we obtain

$$\begin{aligned}\frac{d}{dt}G_\varepsilon \leq \quad & -\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v})+(\nabla\varphi-\nabla\bar{\varphi})\nabla\bar{v}(\nabla\varphi-\nabla\bar{\varphi})\\ & -\frac{1}{2}\int(\nabla\cdot\bar{v})(|\nabla\bar{\varphi}|^2-2\nabla\bar{\varphi}\cdot\nabla\varphi)+C\int|\nabla\varphi-\nabla\bar{\varphi}|^2\\ & +C\varepsilon^3+\frac{d}{dt}Q\end{aligned}$$

with $|Q(t)|\leq C\varepsilon^4$ for $t\in[0,T]$. From (79) we have $\|\nabla\cdot v\|_{L^\infty([0,T]\times\mathbb{T}^d)}\leq C\varepsilon$ and $\|\nabla\bar{\varphi}\|_{L^\infty([0,T]\times\mathbb{T}^d)}\leq C\varepsilon$, whereas (78) yields $\int|\nabla\varphi|^2\leq C\varepsilon^2$. Note that we also have

$$\begin{aligned}&-\int f(t,x,\xi)(\xi-\bar{v})\nabla\bar{v}(\xi-\bar{v})+(\nabla\varphi-\nabla\bar{\varphi})\nabla\bar{v}(\nabla\varphi-\nabla\bar{\varphi})\\ &+C\int|\nabla\varphi-\nabla\bar{\varphi}|^2\leq CG_\varepsilon.\end{aligned}$$

We conclude that

$$\frac{d}{dt}(G_\varepsilon-Q)\leq C((G_\varepsilon-Q)+\varepsilon^3),$$

and the conclusion of Theorem 5.4 follows by Gronwall's lemma.

□

Proof of identity (81): Here we have to compute:

$$R=\frac{1}{\varepsilon}\int\bar{v}(x+\varepsilon\nabla\varphi)\cdot\nabla\varphi-(\bar{v}\bar{\varphi})(x+\varepsilon\nabla\varphi)+(1-\varepsilon\Delta\bar{\varphi})(\bar{v}\cdot\nabla\bar{\varphi}-\bar{v}\cdot\nabla\varphi)$$

Using Lemma 5.5 we have:

$$\begin{aligned}R= \quad & \frac{1}{\varepsilon}\int\bar{v}\cdot\nabla\varphi-\bar{v}\cdot\bar{\varphi}+\bar{v}\cdot\nabla\bar{\varphi}-\bar{v}\cdot\nabla\varphi\\ & +\int\nabla\bar{v}\cdot\nabla\varphi\nabla\varphi-\nabla(\bar{v}\bar{\varphi})\nabla\varphi-\bar{v}\nabla\bar{\varphi}\Delta\bar{\varphi}+\bar{v}\nabla\varphi\Delta\bar{\varphi}\\ & +\int(<\nabla\bar{v}>-\nabla\bar{v})\nabla\varphi\nabla\varphi-\varepsilon<\nabla^2(\bar{v}\nabla\bar{\varphi})>\nabla\varphi\nabla\varphi.\end{aligned}$$

We see that the first line cancels. Then we show that the last line is bounded by $C\varepsilon^3$.

This is obvious for the last term since from (77, 78) we have $\|\bar{v}\|_{W^{2,\infty}}\leq C$, and $\|\nabla\bar{\varphi}\|_{W^{2,\infty}}\leq C\varepsilon$.

Then for the first term we have the following lemma:

Lemma 5.6 *We define*

$$\Delta = \int (< \nabla\bar{v} > (x) - \nabla\bar{v}(x))\nabla\varphi\nabla\varphi dx,$$

then one has:

$$|\Delta| \le C\varepsilon^{10/3} + C\int |\nabla\varphi - \nabla\bar{\varphi}|^2 dx.$$

Proof: First we show that if $\Theta(R) = \int_{\{|\nabla\varphi|\ge R\}} |\nabla\varphi|^2$,

$$\Theta(R) \le C\int |\nabla\varphi - \nabla\bar{\varphi}|^2 + \frac{C\varepsilon^4}{R^2}.$$

Proof: $\int |\nabla\varphi|^2 \le C\varepsilon^2$, implies that

$$\mathrm{meas}\{|\nabla\varphi| \ge R\} \le C(\frac{\varepsilon}{R})^2.$$

Since $|\nabla\bar{\varphi}(t,x)| \le \varepsilon$ for $(t,x) \in [0,T'x] \times \mathbb{T}^d$

$$\begin{aligned}\Theta(R) &\le \int_{\{|\nabla\varphi|\ge R\}} |\nabla\bar{\varphi}|^2 + \int_{\{|\nabla\varphi|\ge R\}} |\nabla\varphi - \nabla\bar{\varphi}|^2 \\ &\le \frac{C\varepsilon^4}{R^2} + \int |\nabla\varphi - \nabla\bar{\varphi}|^2 .\end{aligned}$$

Then we have

$$\begin{aligned} & \Delta \le C\Theta(R) + \int_{|\nabla\varphi|\le R} |< \nabla\bar{v} > (x) - \nabla\bar{v}(x)|\,\nabla\varphi\nabla\varphi \\ \text{with} \quad & |< \nabla\bar{v} > (x) - \nabla\bar{v}(x)| \le C\varepsilon\,|\nabla\varphi| \\ \text{thus} \quad & \Delta \le C\varepsilon\int_{|\nabla\varphi|\le R} |\nabla\varphi|^3 + C\Theta(R) \\ & \le C\left(\varepsilon R\int |\nabla\varphi|^2 + \frac{\varepsilon^4}{R^2} + \int |\nabla\varphi - \nabla\bar{\varphi}|^2\right) \\ & \le C\left(\varepsilon^3 R + + \frac{\varepsilon^4}{R^2} + \int |\nabla\varphi - \nabla\bar{\varphi}|^2\right)\end{aligned}$$

for all R, so for $R = \varepsilon^{(1/3)}$ one obtains:

$$\Delta \le C\varepsilon^{10/3} + C\int |\nabla\varphi - \nabla\bar{\varphi}|^2 .$$

□

Thus we have shown that $R = S + O(\varepsilon^3)$, and $S = \Sigma_{k=1}^{6} T_k$ where each T_k is given by:

$$
\begin{aligned}
T_1 &= \partial_j \bar{v}_i \partial_j \varphi \partial_i \varphi \\
T_2 &= -\partial_j \bar{v}_i \partial_j \varphi \partial_i \bar{\varphi} \\
T_3 &= -\bar{v}_i \partial_{ij} \bar{\varphi} \partial_j \varphi \\
T_4 &= \partial_j \bar{v}_i \partial_j \bar{\varphi} \partial_i \bar{\varphi} \\
T_5 &= \bar{v}_i \partial_{ij} \bar{\varphi} \partial_j \bar{\varphi} \\
T_6 &= \bar{v}_i \partial_{jj} \bar{\varphi} \partial_i \varphi
\end{aligned}
$$

where we have used Einstein's convention for repeated indices. First we have

$$T_5 = -\frac{1}{2} \int (\nabla \cdot \bar{v}) \, |\nabla \bar{\varphi}|^2$$

$$T_1 + T_2 + T_4 = \int \partial_j \bar{v}_i (\partial_j \varphi - \partial_j \bar{\varphi})(\partial_i \varphi - \partial_i \bar{\varphi}) + T_7$$

with $T_7 = \int \partial_j \bar{v}_i \partial_j \bar{\varphi} \partial_i \varphi$.

$$T_6 = -\int \partial_i \bar{v}_i \partial_{jj} \bar{\varphi} \varphi + \bar{v}_i \partial_{ijj} \bar{\varphi} \varphi$$

and

$$-\int \bar{v}_i \partial_{ijj} \bar{\varphi} \varphi = \int \partial_j \bar{v}_i \partial_{ij} \bar{\varphi} \varphi + \bar{v}_i \partial_{ij} \bar{\varphi} \partial_j \varphi$$

thus

$$T_6 = \int -(\nabla \cdot \bar{v}) \Delta \bar{\varphi} \, \varphi + T_8 - T_3$$

with $T_8 = \int \partial_j \bar{v}_i \partial_{ij} \bar{\varphi} \varphi$. Then

$$
\begin{aligned}
T_8 = \quad & -\int \partial_j \bar{v}_i \partial_j \bar{\varphi} \partial_i \varphi + \partial_{ij} \bar{v}_i \partial_j \bar{\varphi} \varphi \\
& = -T_7 + \int \nabla \cdot \bar{v} (\Delta \bar{\varphi} \varphi + \nabla \bar{\varphi} \nabla \varphi)
\end{aligned}
$$

and finally we obtain

$$
\begin{aligned}
S(t) = \quad & \int \nabla \bar{v} (\nabla \bar{\varphi} - \nabla \varphi)(\nabla \bar{\varphi} - \nabla \varphi) - \frac{1}{2} (\nabla \cdot \bar{v}) \, |\nabla \bar{\varphi} - \nabla \varphi|^2 \\
& + \frac{1}{2} \int (\nabla \cdot \bar{v}) \, |\nabla \varphi|^2
\end{aligned}
$$

and the identity (81) is proved.

□

6 Appendix

Existence and uniqueness of solutions to second order ODE's with BV field

In this section we prove existence and a.e. uniqueness for ordinary differential equations of the form:

$$\frac{d}{dt}\begin{pmatrix} X \\ V \end{pmatrix} = \begin{pmatrix} V \\ E(t,X) \end{pmatrix} \tag{82}$$

for $X \in \mathbb{T}^d$, $Y \in \mathbb{T}^d$, and where the field E belongs to $L^\infty(]0,T[\times\mathbb{T}^d)\cap L^1(]0,T[,BV(\mathbb{T}^d))$. We work in the flat torus for simplicity, but our results are still valid in an open subset of $\mathbb{R}^d$. This result is an adaptation of the proof of [14] that uses the result of [4] on renormalized solutions of transport equations.
Remark: After this proof was written, the authors learned of a result by L. Ambrosio ([1]) that extends the results of [14] to transport equations when the vector field is in BV with bounded divergence.

Renormalized solutions for Vlasov equations with BV field

Theorem 3.4 in [4] adapted to the periodic case sates that if $f \in L^\infty(]0,T[\times\mathbb{T}^d\times\mathbb{R}^d)$ satisfies:

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x) f) = 0, \tag{83}$$

with $E(t,x) \in L^1(]0,T[\times\mathbb{T}^d)\cap L^1(]0,T[,BV(\mathbb{T}^d))$, then for all g Lipschitz continuous we have

$$\frac{\partial g(f)}{\partial t} + \nabla_x \cdot (\xi g(f)) + \nabla_\xi \cdot (E(t,x) g(f)) = 0.$$

The property of renormalization implies that

- solutions to (83) with initial data in $L^\infty_{loc}(\mathbb{T}^d\times\mathbb{R}^d)$ belong to $C(]0,T[,L^p_{loc}(\mathbb{T}^d\times\mathbb{R}^d))$ for any $1 \le p < \infty$,
- solutions to (83) with prescribed initial data in $L^\infty(\mathbb{T}^d\times\mathbb{R}^d)$ are a.e. unique,
- if E_n converges to E in $L^1(]0,T[\times\mathbb{T}^d)$ then the solutions of (83) with E_n instead of E converge to the solution of (83).

We notice that equation (54) satisfies the assumptions of the Theorem, and thus will have the renormalization property. This renormalization property was used in [14] to obtain a.e. uniqueness for solutions of the corresponding ODE's. Indeed, the ODE

$$\partial_t X(t,s,x) = b(t,X)$$
$$X(s,s,x) = x$$

is associated to the transport equation:

$$\partial_t u + b(t,x).\nabla u = 0$$

whose solutions satisfy for all $(t,s) \in]0,T[$

$$u(t,X(t,s,x)) = u(s,x).$$

We extend this consequence to the case of second order equations, with BV acceleration field. To the kinetic equation

$$\partial_t f + \xi \cdot \nabla_x f + E(t,x) \cdot \nabla_\xi f = 0 \tag{84}$$

we associate the second order ODE (82) which can rewritten as $\partial_{tt} X = E(t,X)$. The result is then the following:

Theorem 6.1 *Let $E(t,x) \in L^\infty(]0,T[\times\mathbb{T}^d) \cap L^1(]0,T[,BV(\mathbb{T}^d))$, then the ODE*

$$\partial_{tt} X(t,s,x,\xi) = E(t,X) \tag{85}$$
$$(X(s,s,x,\xi), \partial_t X(s,s,x,\xi)) = (x,\xi) \tag{86}$$

admits an a.e. unique solution.

Remark: Here almost everywhere must be understood for the Lebesgue measure of $\mathbb{R}^6$.

Proof of Theorem 6.1: We know that through equation (82) equation (85) can be considered as a first order differential equation. Let us first consider the case where E is smooth. Note $Y \in \mathbb{T}^d \times \mathbb{R}^d$ (resp. y) for (X,V) (resp. for (x,ξ)) and $B \in \mathbb{R}^d \times \mathbb{R}^d$ for (ξ,E). Then for all $s \in]0,T[$, Y solves:

$$\partial_t Y(t,s,y) = B(t,Y(t,s,y)) \tag{87}$$
$$Y(s,s,y) = y \tag{88}$$

Then for all $t, t_1, t_2, t_3 \in]0, T[$ we have the following:

$$
\begin{aligned}
& Y(t_3, t_2, Y(t_2, t_1, y)) = Y(t_3, t_1, y) \\
& Y(t, t, y) = y \\
& Y(t_1, t_2, Y(t_2, t_1, y)) = y.
\end{aligned}
$$

Differentiating the last equation with respect to t_2 yields:

$$
\partial_s Y(t,s,y) + \nabla_y Y(t,s,y) \cdot B(s,y) = 0 \tag{89}
$$

$$
Y(t,t,y) = y. \tag{90}
$$

$Y_t(s,y) = Y(t,s,y)$ thus solves a transport equation which is nothing but equation (84). Using Theorem 3.5 we know that for all $g : \mathbb{R}^{2d} \to \mathbb{R}$ Lipschitz continuous, $g(t,s,y) = g_0(Y(t,s,y))$ is the unique solution of

$$
\partial_s g(t,s,y) + \nabla_y g(t,s,y) \cdot B(s,y) = 0 \tag{91}
$$

$$
g(t,t,y) = g_0(y). \tag{92}
$$

Now we show existence and uniqueness for solutions of (87,88). Let t and s be fixed. Let us consider a regularization E_n of the the field E and set $B_n = (\xi, E_n)$. We have

- $t \to Y_{1,n}(t,s,y)$ that satisfies (87,88)
- $s \to Y_{2,n}(t,s,y)$ that satisfies (89,90).

From the stability Theorem 2.4 in [14] we know that the whole sequence $t \to Y_{2,n}(t,s,.)$ converges in $C(]0,T[, L^p_{loc}(\mathbb{R}^d \times \mathbb{T}^d))$ to $t \to Y_2(t,s,.)$, the unique renormalized solution of (89,90). Thus for fixed t the whole sequence $Y_{2,n}(t,s,.)$ converges strongly in $L^p_{loc}(\mathbb{R}^d \times \mathbb{T}^d)$. Now since for every n we have $Y_{1,n}(t,s,y) = Y_{2,n}(t,s,y)$ the same property holds for $Y_{1,n}(s,t,.)$. Now we can pass to the limit in the term $B_n(t, Y_{1,n}(t,s,y))$. Indeed, by density of C_c^∞ functions in L^1, if we have $E_s \in C_c^\infty$ approximating E then

$$
\begin{aligned}
& \|B(t, Y_n(t,s,y)) - B(t, Y(t,s,y))\|_{L^1} \\
\leq \; & \|B(t, Y_n(t,s,y)) - B_s(t, Y_n(t,s,y))\|_{L^1} \\
+ \; & \|B_s(t, Y_n(t,s,y)) - B_s(t, Y(t,s,y))\|_{L^1} \\
+ \; & \|B(t, Y(t,s,y)) - B_s(t, Y(t,s,y))\|_{L^1}
\end{aligned}
$$

The second term goes to 0 because of the strong convergence of Y_n, the first and the third go to 0 because Y and Y_n are measure preserving mappings, and so for example $\|B(t,Y(t,s,y)) - B_s(t,Y(t,s,y))\|_{L^1} = \|B(t,y) - B_s(t,y)\|_{L^1}$. So finally we have

$$\begin{aligned} & \|B_n(t,Y_n(t,s,y)) - B(t,Y(t,s,y))\|_{L^1} \\ \leq \quad & \|B_n(t,Y_n(t,s,y)) - B(t,Y_n(t,s,y))\|_{L^1} \\ & + \|B(t,Y_n(t,s,y)) - B(t,Y(t,s,y))\|_{L^1} \end{aligned}$$

that goes to 0 and we can pass to the limit in the equation (87,88) and the existence of a solution to (87,88) is proved.

To obtain uniqueness, we argue as in [14]. Any function of the form $g_0(Y(t,s,y))$ is a solution of (91,92), thus by uniqueness of the solution of the transport equation we obtain uniqueness of the ODE.

□

A remark on ODE's of second order

In this section, we want to solve the Cauchy problem for:

$$\begin{aligned} & \partial_{tt} X(t,x) = E(t,X) \\ & (X(0,x)\,, \partial_t X(0,x)) = (x, v(x)) \end{aligned}$$

with E as above. We are thus interested in monokinetic initial data.

Theorem 6.2 *for all $v^0(x)$ vector field on $\mathbb{T}^d$, and for Lebesgue almost every $\delta v \in \mathbb{R}^d$, there exists an a.e. unique solution to*

$$\begin{aligned} & \partial_{tt} X(t,x) = E(t,X(t,x)) \\ & (X(0,x)\,, \partial_t X(0,x)) = (x, v^0(x) + \delta v) \end{aligned}$$

Proof: Let $g(x,\xi)$ be the indicator function of the set of those $(x;\xi)$ such that the trajectory coming from x is not well defined. We just have to prove that for a.e. $\delta v \in \mathbb{R}^d$ we have $\int g(x, v^0(x) + \delta v) dx = 0$, which is true because

$$\int g(x, v^0(x) + \xi) dx d\xi = \int g(x,\xi) dx d\xi = 0.$$

Stability Using the fact that for E_n converging to E in L^1 with
$E \in L^1(]0,T[, BV(\mathbb{T}^d))$, we have $X_n(t,x,v) \to X(t,x,v)$ in $C([0,T],L^p)$, we have then, for all t, for almost every δv, $X_n(t,x,v^0(x)+\delta v) \to X(t,x,v^0(x)+\delta v)$ in L^p. Thus we have

Theorem 6.3 *If E_n converges to E in L^1 let X_n be solution of*

$$\partial_{tt}X_n(t,x) = E_n(t, X_n(t,x))$$
$$(X_n(0,x), \partial_t X_n(0,x)) = (x, v^0(x) + \delta v)$$

then for all t, for almost every δv, X_n converges in $L^p(\mathbb{R}^3) - s$ to a solution (unique for almost every δv) of

$$\partial_{tt}X(t,x) = E(t,X)$$
$$(X(0,x), \partial_t X(0,x)) = (x, v^0(x) + \delta v).$$

Control of macroscopic density in kinetic equations

We prove here Lemma 4.7:

Lemma 6.4 *Let $f \in L^\infty([0,T] \times \mathbb{T}^d \times \mathbb{R}^d)$ satisfy*

$$\frac{\partial f}{\partial t} + \nabla_x \cdot (\xi f) + \nabla_\xi \cdot (E(t,x) f) = 0 \tag{93}$$

$$f(0,.,.) = f^0 \tag{94}$$

with $E \in L^1([0,T]; BV(\mathbb{T}^d))$ and

$$\|E\|_{L^\infty([0,T]\times\mathbb{T}^d)} \leq F. \tag{95}$$

Let the initial condition f_0 be such that:

$$a(x,\xi) \leq f(0,x,\xi) \leq b(x,\xi),$$

with $\rho_a(x) = \int a(x,\xi) d\xi \geq m > 0$ and $\rho_b(x) = \int b(x,\xi) d\xi \leq M < \infty$ and a, b satisfying

$$|\nabla_{x,\xi} a, b| \leq \frac{c}{1 + |\xi|^{d+2}}. \tag{96}$$

Then there exists a constant $R > 0$ such that

$$(\rho_a(x) - Rt) \leq \rho(t,x) \leq (\rho_b(x) + Rt).$$

Proof: First suppose that the force field and the initial data are smooth. For equation (93,94) we can exhibit characteristics $(x,\xi)(t;t_0,x_0,\xi_0)$, giving the evolution of the particles in the phase space. We have $f(t,x,\xi) = f(t_0,x_0,\xi_0)$. Since the initial data is compactly supported and the force field is bounded in the L^∞ norm, we have

$$|\xi - \xi_0| \le F|t - t_0|,$$
$$|x - x_0| \le (|\xi_0| + \frac{F}{2}|t - t_0|)|t - t_0|.$$

If for $t = 0$ we have $a(x,\xi) \le f(0,x,\xi) \le b(x,\xi)$ then

$$\underline{A}(t,x,\xi) \le f(t,x,\xi) \le \overline{B}(t,x,\xi)$$
$$\underline{A}(t,x,\xi) = \inf_{|\sigma_1|,|\sigma_2|\le 1} a(x + |t-t_0|(\xi + \frac{F}{2}|t-t_0|)\sigma_1, \xi + F|t-t_0|\sigma_2)$$
$$\overline{B}(t,x,\xi) = \sup_{|\sigma_1|,|\sigma_2|\le 1} b(x + |t-t_0|(\xi + \frac{F}{2}|t-t_0|)\sigma_1, \xi + F|t-t_0|\sigma_2).$$

Using (96) and integrating in ξ we find thus a constant $R = R(F,C,d)$ such that for $t - t_0 \le 1$ we have:

$$\rho_a(x) - R|t - t_0| \le \rho(t,x) \le \rho_b(x) + R|t - t_0|.$$

Next we need to show that the solution of the regularized equation converges to the solution we are studying: this result comes from the uniqueness of the solution to (93,94) which is a consequence of the renormalization property. Indeed since E is bounded in BV the system (93,94) admits a unique renormalized solution and the sequence of approximate solutions converge in $C([0,T],L^p_{x,\xi})$ for $1 \le p < \infty$ thus the bounds obtained above are preserved.

□

Regularity of the polar factorization on the flat torus

Here we deduce from [21], [11] and [8], [9], [10] the Theorem 4.4.

Theorem 6.5 *If $\rho \in C^\alpha(\mathbb{T}^d)$ with $0 < m \le \rho \le M$ is a probability measure on $\mathbb{T}^d$ then $\Psi = \Psi[\rho]$ (see Definition 4.3) is a classical solution of*

$$\det D^2\Psi = \rho \tag{97}$$

and satisfies:

$$\|\nabla\Psi(x) - x\|_{L^\infty} \le C(d) = \sqrt{d}/2 \tag{98}$$
$$\|D^2\Psi\|_{C^\alpha} \le K(m,M,\|\rho\|_{C^\alpha}) \tag{99}$$

Proof of Theorem 6.5: Consider ρ a $\mathbb{Z}^d$ periodic probability measure, satisfying

$$0 < m \leq \rho \leq M, \tag{100}$$

and $\Phi[\rho]$ as in Definition 4.3. First it is shown in [11] that

$$|\nabla\Phi[\rho](x) - x| \leq C(d). \tag{101}$$

It follows that the strict convexity argument of [8] applies: indeed if $\Phi = \Phi[\rho]$ is not strictly convex its graph contains a line and this contradicts (101). Moreover since $\Phi - |x|^2/2$ is globally Lipschitz and periodic there exists $N(d)$ such that $\|\Phi - |x|^2/2\|_{L^\infty} \leq N(d)$. It follows then that there exists $0 < r(d) \leq R(d)$ and $M(d)$ such that

$$B(r(d)) \subset \{\Phi - \Phi(0) \leq M(d)\} \subset B(R(d)) \tag{102}$$

It remains to show that our solution is a solution in the Aleksandrov sense of the Monge-Ampère equation

$$m \leq \det D^2\Phi \leq M.$$

This is a direct consequence with minor changes (to adapt to the periodic case) of Lemma 2 of [10]. Then, normalizing Φ to $\tilde{\Phi} = \Phi - \Phi(0) - M(d)$ it follows that $\tilde{\Phi}$ is a solution of

$$\begin{aligned} &\rho(\nabla\tilde{\Phi}) \det D^2\tilde{\Phi} = 1 \\ &\tilde{\Phi} = 0 \quad \text{on } \partial\Omega \\ &B(r(d)) \subset \Omega \subset B(R(d)) \end{aligned}$$

Thus the interior regularity results of [9] apply uniformly to all $\Phi[\rho]$ with ρ satisfying (100) and $\|\rho\|_{C^\alpha(\mathbb{T}^d)}$ bounded and Theorem 4.4 follows.

□

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