Harmonious Groups

Part 2: Their structure inside groups of finite Morley rank

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Outline

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Motivation

Theorem (Feferman & Vaught, 1959)

Let L be a first order language, I a nonempty set, and for each $i \in I$, let A_i be an L-structure and B_i be an elementary equivalent to (resp. extension of) A_i. Then $\prod_{i \in I} B_i$ is an elementary equivalent to (resp. extension of) $\prod_{i \in I} A_i$.

Corollary

Let H_1 and H_2 be groups in the language $\{\cdot, e, ^{-1}\}$ and $\tilde{H_1}$ be an elementary extension of H_1. Then $\tilde{H_1} \times H_2$ is an elementary extension of $H_1 \times H_2$.

Motivation

Corollary

Let G be a group and H and K be infinite definable subgroups of G such that $G = H \times K$. Then G is not $\aleph_1$-categorical.

Proof.

Let $\tilde{H}$ be an elementary extension of H having a cardinality larger than $\operatorname{card} K$. Then $\tilde{G} := \tilde{H} \times K$ is an elementary extension of G and $(G, \tilde{G})$ is a Vaughtian pair for K. □

Motivation

Theorem

Let G be a group of finite Morley rank. If $G^\circ = HK$ where:

- *$H, K \trianglelefteq G$ not necessarily definable;*
- *$[H, K] = 1$;*
- *$H \cap K$ is finite;*
- *H is abelian and K is not,*

then G is not $\aleph_1$-categorical.

Recalling some notions

Let G be a group of finite Morley rank.

Proposition (Lascar, 1985)

G doesn't have the finite cover property.

Definition

We say that two strongly miminal sets X and Y are ***analogous*** if there is a strongly minimal set U and two interpretable maps $U \to X$ and $U \to Y$ with cofinite images. This is an equivalence relation.
An interpretable set N is said to be ***harmonious*** of type X if every strongly minimal set interpretable in N is analogous to X.

Outline

A fundamental lemma

Lemma

Let $H_0 = 1 \leq H_1 \leq \cdots \leq H_n = G$ be definable subgroups of G. Then for each interpretable strongly minimal set X, there exists $i \in \{0, \ldots, n-1\}$ such that X is analogous to a strongly minimal set interpretable in H_{i+1}/H_i.

Corollary

If X is an interpretable strongly minimal set and $H_0 = 1 \leq H_1 \leq \cdots \leq H_n = G$ are definable subgroups of G such that H_{i+1}/H_i is harmonious of type X for each $i \in \{0, \ldots, n-1\}$, then G is harmonious of type X.

Maximal harmonious subgroups

Definition

Let X be an interpretable strongly minimal set. We define $M_G(X)$ to be the maximal normal connected definable subgroup of G which is harmonious of type X.

Proposition

- *$M_G(X)$ is definably characteristic in G.*
- $\operatorname{card} M_G(X) \in \{1, \operatorname{card} X\}$.
- *Every harmonious subset $E \subseteq G$ of type X is contained in finitely many cosets of $M_G(X)$.*

Maximal harmonious subgroups

Corollary

There exists a strongly minimal subset X of G such that $M_G(X) \neq 1$.

Proof.

Let X be a strongly minimal subset of G. It is harmonious of type X, thus contained in finitely many cosets of $M_G(X)$. □

Corollary

If E is an infinite subset of G of type X, then $M_G(X) \neq 1$.

Maximal harmonious subgroups

Lemma

Let H be a definable subgroup of finite index in G. If H is harmonious of type X, so is G.

Corollary

Let H be an interpretable group and let M be a normal connected interpretable subgroup of H of infinite index maximal for these conditions. Then H/M is harmonious.

Proof.

Let X be a strongly minimal subset of H/M. Then $M_{H/M}(X)$ is not trivial, and by maximality of M, it has finite index in H/M. Therefore H/M is harmonious of type X. □

Finitely many equivalence classes

Theorem

There are finitely many strongly minimal sets $X_1, \ldots, X_n$ such that every interpretable strongly minimal set X of G is analogous to X_i for some $i \in \{1, \ldots, n\}$.

Proof.

Let $(H_i)_{i \in \{0, \ldots, n\}}$ be definable subgroups of G defined as follows: $H_0 = 1$, $H_n = G$ and H_i is a maximal proper normal definable subgroup of infinite index in H_{i+1} for each $i \in \{1, \ldots, n-1\}$. Each H_i / H_{i-1} is harmonious of type X_i for some X_i. By the fundamental lemma, every interpretable strongly minimal set X is analogous to X_i for some $i \in \{1, \ldots, n\}$. □

Notation

Fix $X_1, \ldots, X_n$ be representatives of the different equivalence classes under the "analogous" relation, and X be an arbitrary interpretable strongly minimal set.

Outline

Defining $W_G(X)$

Definition

An interpretable set N in G is said to be of type $X^\perp$ if none of its interpretable strongly minimal sets is analogous to X.
Let $W_G(X)$ be the intersection of the definable subgroups W of G for which G/W is of type $X^\perp$.

Properties

Proposition

$W_G(X)$ is a definable subgroup definably characteristic in G, and $G/W_G(X)$ is of type $X^\perp$.

Proof.

There exists $W_1, \ldots, W_k$ definable subgroups of G such that $W_G(X) = \bigcap_{i=1}^n W_i$. Define $H_i := \left(\bigcap_{j=1}^{k-i} W_j\right) / W_G(X)$ for each $i \in \{0, \ldots, k-1\}$ and $H_k = G/W_G(X)$. Then H_{i+1}/H_i is of type $X^\perp$ for each $i \in \{0, \ldots, k-1\}$, and so is $G/W_G(X)$ according to the fundamental lemma. □

$W\_(X)$ as an operator

Corollary

$W_{W_G(X)}(X) = W_G(X)$.

Proof.

Let $H_0 := 1$, $H_1 := W_G(X)/W_{W_G(X)}(X)$ and $H_2 := G/W_{W_G(X)}(X)$. Then H_{i+1}/H_i is of type $X^\perp$ for each $i \in \{0, 1\}$, and so is $G/W_{W_G(X)}(X)$. Therefore $W_G(X) \leq W_{W_G(X)}(X)$. □

$W\_(X)$ as an operator

Corollary

If H is a definable subgroup of G, then $W_H(X) \leq W_G(X) \cap H$.

Proof.

The quotient group
$W_H(X)W_G(X)/W_G(X) \cong W_H(X)/(W_H(X) \cap W_G(X))$ is of type $X^{\perp}$, thus $W_H(X) = W_{W_H(X)}(X) \leq W_H(X) \cap W_G(X)$. □

Properties

Corollary

$W_G(X)$ is connected.

Proof.

$W_G(X)/W_G(X)^\circ$ is finite, thus of type $X^\perp$. Therefore $W_G(X) = W_{W_G(X)}(X) \leq W_G(X)^\circ$. □

Relation to harmonious groups

Proposition

$M_G(X) \leq W_G(X)$.

Proof.

$M_G(X)/(M_G(X) \cap W_G(X)) \cong M_G(X)W_G(X)/W_G(X)$ is harmonious of type X and is also of type $X^{\perp}$, thus it is a finite group. Since $M_G(X)$ is connected, we must have $M_G(X) \cap W_G(X) = M_G(X)$. □

Relation to harmonious groups

Lemma

If G is harmonious of type X then $W_G(X) = G^\circ$.

Proof.

Let $W \leq G^\circ$ definable such that G°/W is of type $X^\perp$. We also know that G°/W is harmonious of type X. Thus G°/W is finite, which entails $W = G^\circ$. □

Towards a decomposition

Lemma

Let N be a definable subgroup of infinite index in $W_G(X)$. If $W_G(X)/N$ is harmonious, then it is harmonious of type X.

Proof.

Assume $W_G(X)/N$ is harmonious of type Y. If Y is not analogous to X, then $W_G(X)/N$ is of type $X^\perp$, thus $W_G(X) = W_{W_G(X)}(X) \leq N$. □

Proposition

If X and Y are two nonanalogous interpretable strongly minimal sets of G, then $[W_G(X), W_G(Y)] = 1$.

Proof

We procede by induction on the Morley rank of G. Let N be a minimal infinite definable connected subgroup of G. By minimality of N, it is harmonious of type Z for some Z. By induction hypothesis,
$[W_G(X), W_G(Y)]N/N = [W_{G/N}(X), W_{G/N}(Y)] = 1$, thus
$[W_G(X), W_G(Y)] \leq N$.
Suppose that there exists $u \in W_G(X)$ and $v \in W_G(Y)$ such that $[u, v] \neq 1$. Then the adjoint maps $\mathrm{ad}_u : W_G(Y) \to N$ and $\mathrm{ad}_v : W_G(X) \to N$ induce interpretable embeddings of $W_G(Y)/C_{W_G(Y)}(u)$ and $W_G(X)/C_{W_G(X)}(v)$ into N. These quotient groups are thus harmonious of type Z, and according to the previous lemma, Z is analogous to both X and Y. □

The main theorem

Theorem

The connected component of G is the central product of the $(W_G(X_i))_{i\in\{1,\dots,n\}}$. In particular, we have

$$G^\circ/Z(G^\circ) \cong \prod_{i=1}^{n} \overline{W_G(X_i)}$$

where $\overline{W_G(X_i)} = W_G(X_i)Z(G)/Z(G)$ for each $i \in \{1,\dots,n\}$.

Proof

Since the different $W_G(X_i)$ commute with each other, it's enough to show that they generate G°. Let H be a maximal proper definable normal subgroup of G. Then G/H is harmonious of fingerprint X_i for some $i \in \{1, \ldots, n\}$. Thus $W_G(X_i)H/H = W_{G/H}(X_i) = G/H$, which shows that no proper definable subgroup contains all the $(W_G(X_i))_{i\in\{1,\ldots,n\}}$. Hence G° is their central product.
The remaining follows from the fact that every element of $W_G(X_i)Z(G) \cap \prod_{j\neq i}(W_G(X_j)Z(G)$ is central in G for each $i \in \{1, \ldots, n\}$. □

A harmonious quotient of $W_G(X)$

Proposition

If G is connected, then $W_G(X)Z(G)/Z(G)$ is harmonious of type X.

Proof.

Let Y not analogous to X. Then $[W_G(X), W_{W_G(X)}(Y)] = [W_{W_G(X)}(X), W_{W_G(X)}(Y)] = 1$, thus $W_{W_G(X)}(Y)$ is central in $W_G(X)$, thus $W_G(X)/Z(W_G(X))$ is of type $Y^\perp$, and we have $Z(W_G(X)) = W_G(X) \cap Z(G)$. □

Fact

Harmonious groups are $\aleph_1$-categorical.

About the commutator subgroup

Corollary

If G is connected, then $[G, W_G(X)] \leq M_G(X)$.
In particular, if $M_G(X) = 1$, then $W_G(X)$ is abelian.

Theorem

If G is connected, then G' is contained in the central product of the $(M_G(X_i))_{i \in \{1,\ldots,n\}}$.

Outline

Some terminology

Definition

A definable ***section*** of G is a quotient group U/V for some definable $V \trianglelefteq U \leq G$.
Let H be a (not necessarily definable) subgroup of G. Consider the section $(H \cap U)V/V$. If it is equal to U/V, we say that H ***covers*** U/V. If it is finite, we say that H ***almost avoids*** U/V.
A definable section of type X (resp. $X^\perp$) is called an X***-section*** (resp. $X^\perp$***-section***).

A property of interest

Definition

Let K be an interpretable group in G, and $H \leq K$ not necessarily interpretable. We say that H has the (*) property for X in K if H covers all the definable connected X-sections and almost avoids all the connected $X^{\perp}$-sections.

Lemma

We get an equivalent definition if we consider minimal connected sections.

$W_G(X)$ covers the X-sections

Proposition

$W_G(X)$ covers the X-sections.

Proof.

The quotient $(U/V)/((W_G(X) \cap U)V/V) \cong U/(W_G(X) \cap U)$ is both of type X and $X^\perp$. □

Searching for (*) inside $W_G(X)$

Lemma

Let W be a definable subgroup of G which covers the X-sections of G and H a subgroup of W. Then H has the (*) property for X in G if and only if it has the (*) property for X in W.

A possible decomposition

Conjecture

Assume G is connected. Then, for each $i \in \{1, \ldots, n\}$, there exists $H_i \leq W_G(X_i)$ which has the (*) property for X_i in G. This entails that G is the central product of the $(H_i)_{i \in \{1,\ldots,n\}}$, and that

$$G/F = \prod_{i=1}^{n} H_i F/F$$

for some finite central subgroup F of G.

Questions

- Is X_i interpretable in H_i?
- If this is the case, is H_i harmonious of type X_i?

Thank you for your attention.