Generically multiply transitive actions on solvable groups in the finite Morley rank context

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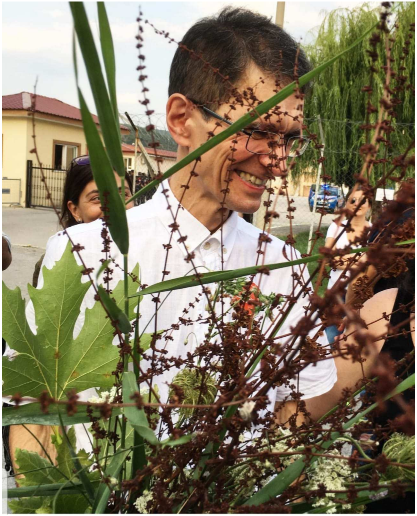
Foreword

- Joint work with Alexandre Borovik.
- Outline of the talk

  1. (Standard) transitive group actions
  2. Generically multiply transitive group actions on elementary abelian p groups, for odd p
     - Generic Identification Theorem for simple groups (2004, 2011)
     - Groups with a pseudoreflection subgroup (2012)
     - Generically sharply multiply transitive actions (2018)
     - Generically multiply transitive group actions (2021)
  3. Actions on solvable groups

- Special thanks to Lütfiye Bozdağ.

Part I: Some Group Theory



Balıkesir, July 2019

Standard Transitivity

Assume that G is a group acting on a set X.

Recall

If for every $x, y \in X$, there exists a (unique) $g \in G$ satisfying $gx = y$, then we say that the action is (sharply) transitive.

Note that G has an induced action on X^n for all $n \geqslant 1$. Set $\Delta = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\}$.

Recall

If for every $(x_1, \ldots, x_n), (y_1, \ldots, y_n) \in X^n \setminus \Delta$, there exists some $g \in G$ satisfying $gx_i = y_i$ for every i, then we say that the action is n-transitive. Sharpness is similarly defined.

Examples of Sharp Transitivity

Examples

- For any field K, $K^* \ltimes K^+$ acts sharply 2-transitively on K.
- S_n acts sharply n and $(n-1)$-transitively, and A_n acts sharply $(n-2)$-transitively on $\{1, 2, \ldots, n\}$.
- $\mathrm{GL}_2(K)$ acts sharply 2-transitively on the set of pairs of linearly independent vectors in $K^2 \times K^2$.
- $\mathrm{PGL}_2(K)$ acts sharply 3-transitively on $\mathcal{P}_1(K)$.

Note that the action $\mathrm{GL}_2(K) \curvearrowright K^2$ respects the group structure of K^2.

Classification

For a group, possesing a large sharp transitivity degree is a highly restrictive condition.

Theorem (Jordan, 1872)

If G is a finite group with a sharply n-transitive action for some $n \geqslant 4$, then the possibilities are as follows.
For $n = 4$; S_4, S_5, A_6, M_{11}.
For $n = 5$; S_5, S_6, A_7, M_{12}.
For $n \geqslant 6$; S_n, S_{n+1}, A_{n+2}.

Theorem (Tits, 1952, and Hall, 1954)

There is no infinite group with a sharp n-transitive action, for $n \geqslant 4$.

Algebraic Actions

Let G be a reductive algebraic group acting algebraically on an irreducible variety V.

Theorem (Knop, 1983)

For $n \geqslant 2$, if G acts n-transitively on V, then either $n = 2$, and the action is PGL_{m+1} on $\mathcal{P}_m$; or $n = 3$ and the action is PGL_2 on $\mathcal{P}_1$.

Note that the action is not assumed to be sharp.
If the induced action of G on V^n is transitive on an open subset, then Popov calles it a generically n-transitive action.

Theorem (Popov, 2007)

If characteristic is 0, among simple algebraic groups, only those of type A_n have generically 5-transitive or higher actions. To be more precise, A_n has $n+2$; E_6 has 4; other types have 2 or 3.

Tea break between Istanbul and Balıkesir, November 2019

Form now on, all structures have finite Morley rank, all subgroups and actions are definable, rk means Morley rank.

Definitions

- A definable subset $A \subseteq X$ is called strongly generic in X, if $\mathrm{rk}(X \setminus A) < \mathrm{rk}(X)$.
- If $\mathrm{rk}(A) = \mathrm{rk}(X)$, then we say A is weakly generic in X.
- When X has Morley degree 1, these two notions coincide.

In this talk, generic means strongly generic.

Some Examples of Generic Subsets

Examples

- All cofinite sets are generic in an infinite set.
- For $n \geqslant 2$, $X^n \setminus \Delta$ is generic in X^n, where $\Delta = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\}$.
- If K is an infinite field, then the set of all linearly independent pairs of vectors is generic in $K^2 \times K^2 = K^4$.
- Similarly, the set of all linearly independent vectors is generic in $\underbrace{K^n \times \cdots \times K^n}_{n \text{ times}} = K^{n^2}$.

Generic Transitivity

Assume that G is a group acting on a set X.

Definition

If G is (sharply) transitive on a generic subset of X, then we say G acts generically (sharply) transitively on X.

Example

Let K be a field, then K^* acts *generically* sharply transitively on K^+, but not sharply transitively. In fact, for all $n \geqslant 1$, $(K^*)^n$ acts generically sharply transitively on $(K^+)^n$.

Generic Multiple Transitivity

Recall

Let G be a group acting on a set X and $n \geqslant 2$. Then G acts n-transitively on X iff G acts transitively on $X^n \setminus \Delta$, where $\Delta = \{(x_1, \ldots, x_n) \in X^n \mid x_i = x_j \text{ for some } i \neq j\}$.

Note that Δ is 'small', compared to X^n. Therefore, the idea is to relax the condition on $X^n \setminus \Delta$ while keeping it large, and obtain new and natural examples.

Definition

If the induced action of G on X^n is generically transitive, then we say G acts generically n-transitively on X. That is, X^n has a generic subset on which G acts transitively.

Generic Multiple Transitivity

Examples

For every $n \geqslant 1$, the natural action of:

- $\mathrm{GL}_n(K)$ on K^n is generically sharply n-transitive.
- $\mathrm{AGL}_n(K)$ on K^n is generically sharply $(n+1)$-transitive.
- $\mathrm{PGL}_{n+1}(K)$ on $\mathcal{P}_n(K)$ is generically sharply $(n+2)$-transitive.

Remark

Only in the first action X has a group structure respected by G.

A Motivating Problem

(Borovik, Cherlin, 2008)

Let G be a connected group acting on a set X definably, faithfully and generically sharply $(n+2)$-transitively, where $n = \mathrm{rk}(X)$. Show that $(G, X) \cong (\mathrm{PGL}_{n+1}(K), \mathcal{P}_n(K))$, for some algebraically closed field K.

Bonn, November 2018

A Partial Answer

Tuna Altınel and Joshua Wiscons solved this problem for $n = 2$ (2018) and gave a partial result for $n \geqslant 3$ (2019).

More Motivating Problems

Problem (Borovik, Cherlin, 2008)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically n-transitively. If $n = \mathrm{rk}(V)$, then is it true that V has a vector space structure of dimension n over an algebraically closed field, and $G \cong \mathrm{GL}(V)$ acts on V naturally?

Problem (Borovik, Cherlin, 2008)

Let G be a connected group of finite Morley rank acting faithfully, definably, and generically m-transitively on an abelian group V of Morley rank n. Then show that $m \leqslant n$.

An Affirmative Answer

Answer (B.– Borovik, 2021)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically m-transitively, where $m \geqslant \mathrm{rk}(V)$. Then $m = \mathrm{rk}(V)$ and $G \curvearrowright V$ is equivalent to $\mathrm{GL}_m(K) \curvearrowright K^m$ for some algebraically closed field K.

Comments

- Connectedness assumptions are not needed.
- V can be solvable.

Corollary

Generically m-transitive and generically sharply m-transitive actions on solvable groups of rank at most m coincide.

Full Statement

Theorem (B.– Borovik, 2021)

Let G be a group acting definably, faithfully and generically m-transitively on a solvable group V where $m \geqslant \mathrm{rk}(V)$. Then $m = \mathrm{rk}(V)$ and $G \curvearrowright V$ is equivalent to $\mathrm{GL}_m(K) \curvearrowright K^m$ for some algebraically closed field K.

Known Cases

Let $n = \mathrm{rk}(V)$.

- $n = 1$ follows from Poizat/Zilber, also from Hrushovski,
- $n = 2$ from Deloro (2009),
- $n = 3$ from Borovik, Deloro (2016), Frécon (2018).

Small Rank

Theorem (Deloro, 2009)

Let G be a connected non-solvable group acting faithfully on a connected abelian group V of rank 2. Then $G \curvearrowright V$ is equivalent to $\mathrm{GL}_2(K) \curvearrowright K^2$ or $\mathrm{SL}_2(K) \curvearrowright K^2$, for some algebraically closed field K.

Theorem (Borovik, Deloro, 2016) + (Frécon, 2018)

Let G be a connected non-solvable group acting faithfully and minimally on an abelian group V of rank 3. Then $G \curvearrowright V$ is equivalent to either the adjoint action $\mathrm{PSL}_2(K) \times Z(G) \curvearrowright K^3$, or the natural action $\mathrm{SL}_3(K) * Z(G) \curvearrowright K^3$ for some algebraically closed field K, ~~or G is a simple bad group of rank 3~~.

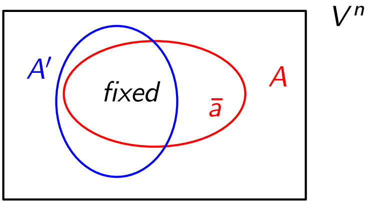
Large Rank

Setting

G is a connected group, V is a connected elementary abelian p-group, where p is an odd prime, G acts on V definably, faithfully and generically n-transitively, and $n \geqslant \mathrm{rk}(V)$.

Observation

Let A be the generic subset of V^n on which G acts transitively. Then A is closed under taking inverses and permuting coordinates.



Hyperoctahedral Group

Observations

- Pick some $\bar{a} = (a_1, \dots, a_n) \in A$. Then $(\pm a_1, \dots, \pm a_n)$ and $\sigma(\bar{a}) = (a_{\sigma(1)}, \dots, a_{\sigma(n)})$ lie in A for $\sigma \in S_n$.
- By transitivity, there exist an element $e_i \in G$ which maps $\bar{a}$ to $(a_1, \dots, -a_i, \dots, a_n)$ and $g_\sigma \in G$ which maps $\bar{a}$ to $\sigma(\bar{a})$ for $\sigma \in S_n$.
- Set H to be the setwise and N be the pointwise stabilizer in G of the set $\{\pm a_1, \dots, \pm a_n\}$. Then $N \trianglelefteq H$ and the images of e_i's and g_σ's in H/N generate a subgroup isomorphic to $S_n \ltimes \mathbb{Z}_2^n$.
- A version of Maschke's Theorem and structural analysis of N shows that it is trivial.

Generically sharply n-transitive actions

Theorem (B.– Borovik, 2018)

Let G be a connected group acting on a connected abelian group V definably, faithfully and generically sharply n-transitively, where $n \geqslant \text{rk}(V)$. If V has no involutions, then $n = \text{rk}(V)$ and $G \curvearrowright V$ is equivalent to $\text{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K.

Sketch of Proof

- In this case, G contains copies of the hyperoctahedral group $S_n \ltimes \mathbb{Z}_2^n$.
- Set $U_i = [V, e_i]$ and assume without loss of generality that $\text{rk}(U_i) = 1$, then $V = \oplus_{i=1}^n U_i$ and hence $\text{rk}(V) = n$.
- Do induction on n; a subgroup of $C_G(e_1)$ acts generically sharply $(n-1)$-transitively on $\oplus_{i=2}^n U_i$ hence we have $\text{GL}_{n-1}(K)$ in G.
- Obtain a torus $(K^*)^n$ of full rank in G where K is an algebraically closed field of odd or zero characteristic.

Pseudoreflection Subgroups

Theorem (B.– Borovik, 2012)

Let G be a connected group acting on a connected abelian group V faithfully and irreducibly. If G contains a pseudoreflection subgroup R such that $\mathrm{rk}[V, R] = 1$, and $\mathrm{psrk}(G) = \mathrm{rk}(V)$, then $G \curvearrowright V$ is equivalent to $\mathrm{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K, where $n = \mathrm{rk}(V)$.

Definitions

A connected definable abelian subgroup R is called a pseudoreflection subgroup if $V = [V, R] \oplus C_V(R)$, and R acts transitively on the nonzero elements of $[V, R]$. Moreover, $\mathrm{psrk}(G)$ is the maximal number of pairwise commuting pseudoreflection subgroups in G.

Pseudoreflection Subgroups (Cont'd)

Theorem (B.– Borovik, 2012)

Let G be a connected group acting on a connected abelian group V faithfully and irreducibly. If G contains a pseudoreflection subgroup R such that $\mathrm{rk}[V, R] = 1$, and $\mathrm{psrk}(G) = \mathrm{rk}(V)$, then $G \curvearrowright V$ is equivalent to $\mathrm{GL}_n(K) \curvearrowright K^n$ for some algebraically closed field K, where $n = \mathrm{rk}(V)$.

Sketch of Proof

Let G be a counter example of minimal Morley rank.

- Centralizers of non-central involutions in G are direct sums of general linear groups.
- G/Z is simple and the Generic Identification Theorem applies, hence G is a Chevalley group of Lie rank at least 3.
- We get a contradiction.

This finishes the proofs of three theorems.

Istanbul, May 2021

Two Useful Observations

For remaining cases, the following is useful.

Observation 1

In the inductive setting, if V has properties that pass to quotients (such as being abelian, solvable, or divisible) then G acts minimally on V; that is, no infinite definable proper subgroup of V is left invariant under the action of G.

Proof

Assume $U < V$ is G-invariant, then G acts generically n-transitively on V/U. By induction hypothesis, $\text{rk}(V/U) = n$. Since $\text{rk}(V) \leqslant n$, we get $\text{rk}(U) = 0$. $\square$

Observation 2

Since G acts generically n-transitively on V, we have $\text{rk}(G) \geqslant \text{rk}(A) = \text{rk}(V^n) = n\,\text{rk}(V)$.

V is Divisible

Theorem (B.– Borovik, 2021)

Let G be a connected group acting on a connected divisible abelian group V definably, faithfully and generically n-transitively, where $n \geqslant \mathrm{rk}(V)$. Then $n = \mathrm{rk}(V)$ and $(G, V) \cong (\mathrm{GL}_n(K), K^n)$ for some algebraically closed field K of characteristic 0.

Sketch of Proof

- The action is G-minimal and $\mathrm{rk}(V) = n$, thanks to the first useful observation.
- Hence Loveys–Wagner (1993) applies and we get $(G, V) \cong (H, K^m)$ and a subgroup $H \leqslant \mathrm{GL}_m(K)$ for some algebraically closed field K of characteristic 0 and $m \leqslant n$.
- Since $\mathrm{rk}(V) = n$, $\mathrm{rk}(G) \geqslant n^2$, necessarily $G \cong \mathrm{GL}_n(K)$. $\square$

V is of Exponent 2

Theorem (B.– Borovik, 2021)

Let G be a connected group acting on a connected elementary abelian 2-group V definably, faithfully and generically n-transitively, where $n \geqslant \mathrm{rk}(V)$. Then $n = \mathrm{rk}(V)$ and $(G, V) \cong (\mathrm{GL}_n(K), K^n)$ for some algebraically closed field K of characteristic 2.

First Observations

- G has involutions.
- G has no non-trivial 2-torus, hence G is of even type; that is, its Sylow$^\circ$ 2-subgroups are definable and 2-unipotent.

Groups of Even Type

Reference

Tuna Altınel, Alexandre Borovik, Gregory Cherlin; Simple Groups of Finite Morley Rank, AMS, 2008. (xx+556 pp.)

Even Type Theorem

Infinite simple groups of finite Morley rank and even type are algebraic groups over algebraically closed fields.

Istanbul Airport, June 2021

Groups of Even Type

Corollary

Let G be a connected group of finite Morley rank containing no nontrivial 2-torus. Then $O_2(G) = O_2^\circ(G)$ and it is a definable unipotent subgroup of G. The quotient $G/O_2^\circ(G)$ has the form $Q * S$ with S a central product of quasisimple algebraic groups over algebraically closed fields of characteristic 2, and Q a connected group without involutions.

Linearization Theorems of Borovik (2021)

Corollary

Let V be a connected elementary abelian p-group and G a connected group which acts on V faithfully, and irreducibly. Assume that G contains a normal subgroup $L \lhd G$, which is a quasisimple algebraic group over an algebraically closed field F of characteristic p. Then V has a structure of a finite dimensional F-vector space, the action of G on V is F-linear, and G is a Zariski closed subgroup in $\mathrm{GL}_F(V)$.

Idea of the Proof

We can express $G = L * C_G(L)$. By Clifford, $V = \bigoplus V_i$, where V_i's are isomorphic irreducible L-modules. Let R be the enveloping algebra of L over V (that is, the subring generated by L in $E = \mathrm{End}(V)$). Then $F = C_E(L)$ is an algebraically closed field, V_i's and hence V are vector spaces over F. Moreover, $R \curvearrowright V_i$ and hence $G \curvearrowright V$ respect the vector space structure. For Zariski closure, use Poizat (2001) and Mustafin (2004).

V is of Exponent 2

Proof of the $\exp(V) = 2$ case

In our case $O_2(G)^\circ = 1$ since otherwise it centralizes an infinite subgroup in V, contradicting the useful observation above. By the Corollary, $G = Q * S$, where $S = S_1 * \cdots * S_k$ a central product of quasisimple algebraic groups. By the first useful observation, G acts irreducibly on $\bar{V} = V/C_G(V)$, and S_1 is a normal subgroup which is quasisimple algebraic. Therefore, Borovik's Linearization Theorem applies, and again by comparing ranks we get the result. $\square$

V is Solvable

Main Theorem (B.– Borovik, 2021)

Let G be a connected group acting on a connected solvable group V definably, faithfully and generically n-transitively, where $n \geqslant \text{rk}(V)$. Then $n = \text{rk}(V)$ and $(G, V) \cong (\text{GL}_n(K), K^n)$ for some algebraically closed field K.

Sketch of Proof

- V is abelian, because otherwise use the first useful observation on V/V'.
- V is divisible or of bounded exponent. Recall $V = D \oplus B$. If V is neither divisible nor of bounded exponent, then use the same trick with V/D.

V is Solvable

Sketch of Proof (Cont'd)

- If V is divisible, we are done by a former theorem.
- If $\exp(V) = k < \infty$, then k is prime. Otherwise the homomorphism $V \to V$, $v \mapsto pv$ has a finite kernel $V_p = \{v \in V \mid pv = 0\}$ and a finite image $V_{k/p}$.
- If V is abelian and $\exp(V) = p$ is prime, then V is an elementary abelian p-group, hence we use the above theorems. $\square$

Connectedness is Superfluous

Sketch of Proof

- When a group acts on a group generically m-transitively, then the set has Morley degree 1. (For $m > 1$, Borovik–Cherlin, 2008; $m = 1$ is easy.)
- When a group G acts on a set of Morley degree 1 generically m-transitively, then G° acts on the same set generically m-transitively. (Altınel–Wiscons, 2018)
- Let G be a group acting on an elementary abelian p-group V of Morley rank n, where p is an odd prime. Assume that there exists an algebraically closed field F such that V is definably isomorphic to the additive group of the F-vector space F^n, $\mathrm{GL}_n(F) \leqslant G$, and the action is the natural action. Then $G = \mathrm{GL}_n(F)$. (B.– Borovik, 2021)

Some Related Questions

Question 1 (B.– Borovik, 2021)

Can the finite Morley rank condition be relaxed in this argument? For example, finite chains of centralizers?

Question 2 (B.– Borovik, 2021)

Let G be a connected group acting on a (not necessarily abelian) connected group H definably, faithfully and generically 2-transitively. Is H abelian? What else can we say about H?

Thanks

Antalya Algebra Days, May 2019

"This photo contains four of Tuna's favourite things: friendship, solidarity, mathematics and freedom" – Evren Altınel