Actions of finite groups and simple algebraic groups on abelian groups of finite Morley rank

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Dedicated to Tuna Altınel

In this talk:

Results which emphasise close connections and analogies between

- ω-stable groups of finite Morley rank, and
- finite groups and algebraic groups.

Also:

- various comments on development of the theory of groups of fMr,
- on possible generalisations,
- and some memories of Lyon.

Setup

V : a connected abelian group of finite Morley rank.

G : a finite set of definable isomorphisms $V \longrightarrow V$ closed under composition and inversion.

We say : the group G acts on V **definably**.

Size and complexity of finite groups

H : finite group

p : a prime number.

We introduce:

$d_p(H)$: the minimal degree of a faithful linear representation of H over the a.c. field $\overline{\mathbb{F}_p}$.

$r_p(H)$: the minimal Morley rank of a connected infinite elementary abelian p-group V of finite Morley rank such that H acts on V faithfully and definably.

The first result

$d_p(H)$: the minimal degree of a faithful linear representation of H over the a.c. field $\overline{\mathbb{F}_p}$.

$r_p(H)$: the minimal Morley rank of a connected infinite elementary abelian p-group V of finite Morley rank such that H acts on V faithfully and definably.

Theorem

In this notation,

$$d_p(H) = r_p(H).$$

Corollary

Corollary via Larsen-Pink (2011). There is a function

$$J : \mathbb{N} \longrightarrow \mathbb{N}$$

with the following property:

If H is a finite simple group which acts definably and faithfully on a connected elementary abelian p-group of Morley rank $n > 0$ then either

- $|H| \leqslant J(n)$, *or*
- *H is a group of Lie type in characteristic p.*

Jordan type theorems, history

Camille Jordan, 1878:
There is a function

$$J : \mathbb{N} \longrightarrow \mathbb{N}$$

with the following property: every finite subgroup of GL_n over a field of characteristic zero possesses an abelian normal subgroup of index $\leqslant J(n)$.

Jordan type theorems, today

Guld 2020: an impressive survey of theorems of Jordan type for finite groups in algebraic (in characteristic 0) and differential geometries.

Guld 2020: For every group G of birational automorphisms of a variety over a field of characteristic zero there exists a constant $J \in \mathbb{N}$ such that every finite subgroup X of G contains a normal subgroup Y, s.t.

- Y is nilpotent of class at most two,
- $|X : Y| < J$.

There should be some model theory behind that.

Principal tool: enveloping algebras

V : a connected abelian group of finite Morley rank.

G : a finite set of definable isomorphisms $V \longrightarrow V$ closed under composition and inversion.

$E = E(G)$: the ring generated by G in $\mathrm{End}\, V$, the **enveloping algebra** of G.

Observe

- $E = \{ a_1 g_1 + \cdots a_n g_n, \quad n = |G|, \quad a_i \in \mathbb{F}_p \}$.
- Elements of E act on V **definably**.

Wedderburn-Maltsev Theorem

Let R be a finite-dimensional associative algebra over a finite field $\mathbb{F}_p$ of prime order p and J its radical. Then

(a) $R = J + S$ where S is a semisimple algebra, $J \cap S = 0$ and S is the direct sum of matrix algebras

$$S = S_1 \oplus \cdots \oplus S_k, \quad S_i \simeq M_{d_i \times d_i}(\mathbb{F}_{p^{m_i}}), \quad i = 1, 2, \ldots, k.$$

(b) Let $Q = 1 + J$. Then Q is a normal p-subgroup in the multiplicative group R^* of R. Moreover, R^* is a semidirect product $R^* = Q \rtimes S^*$ of Q and the multiplicative group S^* of S. In particular,

$$S^* \simeq \mathrm{GL}_{d_1}(\mathbb{F}_{p^{m_1}}) \times \cdots \times \mathrm{GL}_{d_k}(\mathbb{F}_{p^{m_k}}).$$

Smoothly irreducible actions

G is **smoothly irreducible** on V if every connected definable G-invariant subgroup of V equals 0 or V.

Theorem

If a finite group G is smoothly irreducible on V then

$$E(G) \simeq M_n(F),$$

the full matrix algebra of $n \times n$ matrices over some finite field F of characteristic p.

Comment

- *The model-theoretic assumptions in the Theorem can perhaps be relaxed.*
- *Motivated by research on permutation groups of finite Morley rank by* **Tuna Altınel, Adrien Deloro, Josh Wiscons**.

Linearisation of actions of simple algebraic groups.

Theorem

Let $G = G_1 \times \cdots \times G_m$ where each G_i is the group of points over some algebraically closed field K_i of characteristic $p > 0$. a simple algebraic group defined over K_i.

Assume that G acts faithfully, definably and irreducibly on a connected elementary abelian p-group V of finite Morley rank.

Then all K_i are definably isomorphic to the same field K and V has a structure of a finite dimensional K-vector space compatible with the action of G, and G is a Zariski closed subgroup of $GL_K(V)$.

This theorem started as Question B.38 in the book with **Ali Nesin** of 1994.

Surprisingly this was not known...

Corollary

G *is a simple algebraic group over an algebraically closed field* K *of characteristic* $p > 0$

V *a unipotent group of exponent* p

$H = V \rtimes G$ *is also an algebraic group over* K

V *does not have closed* G*-invariant subgroups other than* 0 *and* V*.*

Then

V *has a structure of a finite dimensional vector space over* K *invariant under the action of* G*.*

Idea of proof

K_0 algebraic closure of the prime field in K

$K_1 < K_2 < \dots$ finite fields, $\bigcup_{i=1}^{\infty} K_i = K_0$.

$G_i = G(K_i)$ finite groups.

$G_0 = G(K_0)$

$E_i = E(G_i)$ enveloping algebras.

$E = \bigcup_{i=1}^{\infty} E_i$ locally finite algebra.

$E = E(G_0)$

- Analysis of E yields

$$E \simeq M_n(K_0), \text{ the full matrix algebra over } K_0$$

- Then we show that if $Z = Z(E)^*$ then

$$\boldsymbol{M = C_G(Z)) = G}, \text{ that is, } \boldsymbol{[Z, G] = 1}.$$

Last step

Macpherson–Pillay 1995, and Deloro–Wagner 2020:
Let D be the ring of all definable endomorphisms of V and $Z = C_D(G)$. Assume that Z is infinite. Then

1. Z is an a.c. field definable in $V \rtimes G$ and the action of Z on V gives V a structure of a finite dimensional Z-vector space (with a Z-linear action of G).
2. G is a Zariski closed subgroup in $\mathrm{GL}_Z(V)$.
3. The enveloping algebra (over Z) $R = R(G)$ is the full matrix algebra $\mathrm{End}_Z(V)$ and is definable in $V \rtimes G$.

Intermediate subgroups.

Fact:
If in previous notation (G, M) is a structure of finite Morley rank with M a subgroup of G containing G_0 then $M = G$.

Proof follows from **Poizat 2001** and **Mustafin 2004**.

Bruno Poizat

This result plays crucial role in the proof:

Poizat 2001:

V : a finite dimensional vector space over an algebraically closed field K of characteristic $p > 0$

$H < \mathrm{GL}(V)$ an infinite simple group such that the structure $(\mathrm{GL}(V), H)$ has finite Morley rank.

Then H is a simple algebraic group over K.

It is a great pleasure for me to recall:

B 2008, LOGICUM LUGDUNENSIS: Removed the use of Classification of Finite Simple Groups.

Improvement

Theorem (B-Berkman 2021)

Let V be a connected elementary abelian p-group of finite Morley rank and G a connected group of finite Morley which acts on V faithfully, definably, and smoothly irreducibly.

Assume that G contains a normal definable subgroup $L \lhd G$, where L is a simple algebraic group over an algebraically closed field F of characteristic p.

Then V has a structure of a finite dimensional F-vector space, the action of G on V is F-linear, and G is a Zariski closed subgroup in $GL_F(V)$.

Many thanks for your attention!