Harmonious groups of finite Morley rank
(first part: Lascar analysis of a group of finite Morley rank revisited)

A work joint with Rachad Bentbib (Université de Poitiers)

Ranked groups - The return

Conference to celebrate the return in France of

Tuna Altınel
Main objective

Better know the structure of $\aleph_1$-categorical groups.

Definition: a theory $T$ is $\kappa$-categorical for an infinite cardinal $\kappa$ if any two models of $T$ of cardinality $\kappa$ are isomorphic.

For instance, we would like to obtain a result of the style:

*Let $G$ be a group with no abelian subgroup of finite index. Then $G$ is $\aleph_1$-categorical if and only if it is of finite Morley rank and no normal subgroup $N$ of finite index in $G$ is a central product $N = HK$ with $H \cap K$ finite and $H$ and $K$ infinite and normal in $G$.***

Another objective:

What are the groups interpretable in a strongly minimal structure?

Definition: a theory $T$ is strongly minimal if, for any model $M = (M; \cdot \cdot \cdot)$ of $T$, any definable subset of $M$ is finite or cofinite.
Groups of finite Morley rank and $\aleph_1$-categoricity

Let $T$ be a complete theory in a countable language.

Morley’s Categoricity Theorem (1965)

If $T$ is $\kappa$-categorical for some uncountable $\kappa$, then $T$ is $\kappa$-categorical for every uncountable $\kappa$.

Example: if $T$ is strongly minimal, then $T$ is $\aleph_1$-categorical.

Theorem (Baldwin, 1973)

If $T$ is $\aleph_1$-categorical, then it is of finite Morley rank.

Theorem (Zilber, 1977)

The theory of an infinite simple group of finite Morley rank is $\aleph_1$-categorical.

Remark: a direct product of two nonisomorphic infinite structures of groups of finite Morley rank is not $\aleph_1$-categorical.
Groups of finite Morley rank: examples

- The main example of a group of finite Morley rank is the group $G(K)$ of rational points of an algebraic group $G$ defined over an algebraically closed field $K$, where the language is the one of groups with a unary predicate for each constructible subset of $G(K)^n$ (Zilber, early 1970s).

**Remark:** such a structure is bi-interpretable with a strongly minimal structure, so it is $\kappa$-categorical for each cardinal $\kappa > |K|$.

- The pure group $(\mathbb{Q}, +)$ is a group of Morley rank 1, and it is not algebraic.
- The **Baudisch group** is a non-algebraic non-abelian nilpotent group of Morley rank 2.

**Remark:** $(\mathbb{Q}, +)$ is a strongly minimal structure, so it is $\aleph_1$-categorical, and the Baudisch group is of Morley rank 2 and it is not abelian-by-finite, so it is $\aleph_1$-categorical.

**Theorem (Tanaka, 1988)**

Let $G$ be a group of Morley rank 2. If $G$ is not abelian-by-finite, then $G$ is $\aleph_1$-categorical.

**Theorem (Tsuboi, 1988)**

Let $G$ be an infinite solvable group of finite Morley rank. If $G$ has no abelian definable subgroup of Morley rank $\geq 2$, then $G$ is $\aleph_1$-categorical.
What about \( \omega \)-categorical groups?

**Theorem** (proved independently by Engeler, Ryll-Nardzewski and Svenonius in 1959)

A countable structure \( \mathcal{M} = (M; \cdots) \) is \( \omega \)-categorical iff \( \text{Aut}(\mathcal{M}) \) has only finitely many orbits in its action on \( \mathcal{M}^n \) for each \( n > 0 \).

**Corollary**

\( \omega \)-categorical groups are uniformly locally finite groups (*i.e.* for each integer \( k \), there is an integer \( n \) such that any subset of cardinal \( k \) generates a subgroup of order at most \( n \)); in particular, they have bounded exponent.

**Theorems**

- (Wilson, 1981) \( \omega \)-categorical groups are locally-solvable-by-residually-finite.
- (Apps, 1982) Let \( G \) be an \( \omega \)-categorical group. Then \( G \) has a finite characteristic series \( 1 = G_0 < G_1 < \cdots < G_n = G \) with \( G_{i+1}/G_i \) an \( \omega \)-categorical characteristically simple group for each \( i \).

**Conjectures**

- (stated independently by Apps and Wilson in 1981) \( \omega \)-categorical characteristically simple locally nilpotent groups are abelian.
- (Wilson, 1981) \( \omega \)-categorical groups are nilpotent-by-residually-finite.
What about totally categorical groups?

Definition: a theory $T$ is said to be totally categorical if it has exactly one model in each infinite power.

Theorem (Baur-Cherlin-Macintyre, 1977)
Totally categorical groups are abelian-by-finite and have bounded exponent.

Theorem (Baur-Cherlin-Macintyre, 1977)
Let $G$ be a locally finite group of bounded exponent and let $A$ be an abelian normal subgroup of finite index in $G$. Then $G$ is totally categorical iff $A$ is a direct sum of finite normal subgroups of $G$ of bounded order.
A particular case: torsion-free nilpotent groups

**Theorem (Zilber, 1982)**

Let $G$ be a nonabelian torsion-free nilpotent group. The theory of the pure group $(G; \cdot)$ is \( \aleph_1 \)-categorical if and only if $G$ satisfies the following two conditions:

- $G$ is isomorphic to a unipotent group over an algebraically closed field of characteristic 0;
- $G$ cannot be decomposed into a direct product of two nontrivial subgroups.

**Theorem (Altınel-Wilson, 2008)**

Every torsion-free nilpotent group $G$ of finite Morley rank has a faithful linear representation over a field of characteristic zero.

**Theorem (Myasnikov-Sohrabi, 2018)**

A torsion-free nilpotent group $G$ has finite Morley rank in the language of groups iff $G \cong G_0 \times G_1 \times \cdots G_n$ for a divisible abelian group $G_0$ and finitely many unipotent algebraic groups $G_1, \ldots, G_n$ over characteristic zero algebraically closed fields.

Remark: these theorems are proved by methods based on the study of nilpotent Lie algebras.
A torsion-free nilpotent group $G$ of finite Morley rank is said to be **homogeneous** if either it is trivial or there is an interpretable algebraically closed field $K$ such that, each nontrivial interpretable quotient group $G/H$ of $G$ contains a copy of $K$.

**Theorem (follows from works by Burdges and F., ~2006)**

Let $G$ be a torsion-free nilpotent group. Then $G/Z(G)$ and $G'$ are the direct products of homogeneous subgroups.
II - Harmonious structures

Groups of finite Morley rank: Lascar analysis

Definition

Let $T$ be an $\omega$-stable theory and, let $\varphi(x)$ and $\phi(x)$ be two formulas. We say that $\varphi$ and $\phi$ are $\sim$-equivalent if, for every model $M$ containing the parameters of $\varphi$ and $\phi$, we have $|\varphi(M)| = |\phi(M)|$.

Theorem (Lascar, 1985)

Let $T$ be the theory of a group $G$ of finite Morley rank. We can find strongly minimal formulas $f_1(x), \ldots, f_n(x)$, possibly imaginary, with parameters in the prime model, such that

- for all the infinite cardinalities $\kappa_1, \ldots, \kappa_n$, there exists a model $\mathcal{M} = (M; \cdots)$ of $T$ such that $|f_i(M)| = \kappa_i$ for each $i$;
- every strongly minimal formula $\varphi(x)$ is $\sim$-equivalent to one of these formulas.
- every elementary extension of a model of that theory augments at least one of the formulas.
Corollary

A group $G$ of finite Morley rank is $\aleph_1$-categorical if and only if all the strongly minimal formulas, with parameters in the prime model, are $\sim$-equivalent (i.e. $n = 1$ in the previous theorem).

Theorem (Lascar, 1985)

Let $G$ be a connected group of finite Morley rank with no infinite normal abelian definable subgroup. Then there are $\aleph_1$-categorical groups $H_1, \ldots, H_n$ such that $G$ is isomorphic to $(H_1 \times \cdots \times H_n)/F$ for a finite subgroup $F$.

We would like delete the hypothesis with no infinite normal abelian definable subgroup.
Definition

Let $\mathcal{M} = (M; \cdots)$ be a structure of finite Morley rank. Two interpretable strongly minimal sets $X$ and $Y$ are said to be analogous if there is a strongly minimal set $U$ and two interpretable maps $f_X : U \rightarrow X$ and $f_Y : U \rightarrow Y$ with cofinite images. The structure $\mathcal{M} = (M; \cdots)$ is harmonious of type $X$ if each interpretable strongly minimal set is analogous to $X$.

Remarks:
- If two interpretable strongly minimal sets $X$ and $Y$ are analogous, their canonical extensions in any elementary extension of $\mathcal{M}$ are analogous too.
- There are integers $m$ and $n$ such that $|f_X^{-1}(\{x\})| \leq m$ and $|f_Y^{-1}(\{y\})| \leq n$ for each $x \in X$ and each $y \in Y$.
- In particular, we have $|X| = |U| = |Y|$.

Questions
- If two strongly minimal sets $X$ and $Y$ are analogous, are $X$ and $Y$ $\sim$-equivalent? We will show this property for groups, thus any harmonious group of finite Morley rank is $\aleph_1$-categorical.
- If two strongly minimal sets $X$ and $Y$ are $\sim$-equivalent, are $X$ and $Y$ analogous?
- In other words, is any $\aleph_1$-categorical group harmonious?
- If $\mathcal{M}$ is an harmonious structure of finite Morley rank, are its elementary extensions harmonious too? We will show this property for groups.
The finite cover property

**Definition (Keisler, 1967)**
Let $\mathcal{M} = (M; \cdots)$ be a model of a complete theory $T$. The structure $\mathcal{M}$ has the **finite cover property** if a formula $\varphi(\bar{x}, \bar{a})$ has the following property:
- for every integer $n$, there is an inconsistent set of at least $n$ formulas $\{\varphi(\bar{x}, \bar{a}_i) ; i \in I\}$ with parameters in $M$ such that every subset of it is consistent.

**Facts**
- If $\mathcal{M} \equiv \mathcal{N}$ and $\mathcal{M}$ has the finite cover property, then $\mathcal{N}$ has also this property.
- If a structure $\mathcal{M}$ does not have the finite cover property, then no structure interpretable in $\mathcal{M}$ has the finite cover property.

- Any unstable structure has the finite cover property.
- No $\aleph_1$-categorical structure has the finite cover property.
- No group of finite Morley rank has the finite cover property.

$\mathcal{M} = (M; \cdots)$ is a structure of finite Morley rank without the finite cover property.

**Facts**
- Any infinite interpretable set contains a strongly minimal set.
Lemma

The relation “to be analogous to” is an equivalence relation on the family of strongly minimal sets.

Proof: for the transitivity, let $Y$ be a strongly minimal set analogous to $X$ and $Z$. Let $V$ and $W$ be strongly minimal sets and let $f_X : V \to X$, $f_Y : V \to Y$, $g_Y : W \to Y$ and $g_Z : W \to Z$ be interpretable maps with cofinite images. Then $f_Y(V) \cap g_Y(W)$ is cofinite in $Y$, so the set $\{(v, w) \in V \times W \mid f_Y(v) = g_Y(w)\}$ is infinite and contains a strongly minimal set $U$. Let $p_V : U \to V$ and $p_W : U \to W$ be the canonical projection maps. Then we have $f_Y \circ p_V = g_Y \circ p_W$ and the maps $f_X \circ p_V : U \to X$, $f_Y \circ p_V : U \to Y$ and $g_Z \circ p_W : U \to Z$ have cofinite images. □

Corollary

If $(X_i)_{i=1, \ldots, n}$ is a finite family of analogous strongly minimal sets, there exists a strongly minimal set $U \subseteq X_1 \times \cdots \times X_n$ and surjective interpretable maps $f_i : U \to X_i$.

Questions

If $(X_i)_{i \in I}$ is an infinite family of analogous strongly minimal sets,

- are there a strongly minimal set $U$ and interpretable maps $f_i : U \to X_i$ with cofinite images?
- are there a strongly minimal set $V$ and interpretable maps $g_i : X_i \to V$ with cofinite images?
Proposition

Any strongly minimal set is analogous to a strongly minimal set of the form $E/R$ for a definable subset $E$ of $M$ and an equivalence relation $R$ over $E$.

Corollary

Any strongly minimal structure is harmonious.

Remarks:

- Let $\mathcal{K} = (K; +, \cdot)$ be an algebraically closed field. Then $\mathcal{K}$ is strongly minimal, so it is harmonious. However, the $\mathcal{K}$-interpretable structure $\mathcal{G} = (K \times K; +, K \times \{0\})$ is not harmonious.

- The last result implies that any structure bi-interpretable with a strongly minimal structure is harmonious.

- The pure group $(\mathbb{Z}/4\mathbb{Z})^\omega$ is harmonious, but it is not bi-interpretable with a strongly minimal structure.
Question

If $M$ is harmonious of type $X$, is $M$ interpretable in a strongly minimal structure?

Remark: Assaf Hasson proved that, for a theory of finite Morley rank with definable MR, the DMP (definable multiplicity property) is a sufficient condition for such an interpretation. Furthermore, he proved that not to have the finite cover property is a necessary condition, but not a sufficient condition, for a structure to be interpreted in a strongly minimal structure.

Definition: Let $T$ be a theory of finite Morley rank with definable MR. Then $T$ has the DMP if for every model $M$, formula $\varphi(\bar{x}, \bar{y})$ and integers $k$ and $n$, the set of tuples $\bar{a}$ in $M$ such that $\varphi(M, \bar{a})$ has Morley rank $k$ and Morley degree $n$ is definable.
We fix a structure $G = (G; \cdot, \cdots)$ of group of finite Morley rank. In particular, no $G$-interpretable structure has the finite cover property, so each $G$-interpretable set contains a strongly minimal set.

**Definition**

An interpretable set $N$ is said to be **harmonious of type $X$** if, for any interpretable subset $E$ of $N^n$ for an integer $n$ and any interpretable equivalence relation $R$ over $E$ such that $E/R$ is strongly minimal, the set $E/R$ is analogous to $X$.

**Remarks:**

- The structure $G$ is harmonious iff the set $G$ is harmonious.
- If a structure $M = (M; \cdots)$ is interpretable in $G$, then it is harmonious iff the set $M$ is harmonious.

**Lemma**

Let $X$ be a strongly minimal set. If $A_1, \ldots, A_n$ are harmonious sets of type $X$, then $A_1 \times \cdots \times A_n$ is harmonious of type $X$. 
**Proposition**

If \( G = X_1 \cdots X_n \) is a finite product of analogous strongly minimal subsets, then \( G \) is bi-interpretable with a strongly minimal structure. In particular, it is harmonious.

**Proof:** there is an interpretable strongly minimal set \( U \) and, for each \( i \), a surjective interpretable map \( f_i : U \to X_i \). We consider the surjective interpretable map \( \gamma : U^n \to G \) defined by \( \gamma(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n) \). We can interpret a strongly minimal structure \( \mathcal{U} \) of base \( U \) such that \( G \) and \( \mathcal{U} \) are bi-interpretable. \( \Box \)

**Corollary**

Let \( K \triangleleft H \) be normal definable subgroups of \( G \) such that \( H/K \) is infinite and has no nontrivial proper \( G \)-normal definable subgroup. Then \( H/K \) is harmonious. Moreover, if \( X \) is the type of \( H/K \), then \( |H/K| = |X| \).

**Proof:** it follows from the Zilber Indecomposability theorem that such a quotient group is generated by a finite product of the \( G \)-conjugates of any strongly minimal subset. \( \Box \)
Proposition

Let \( X \) be a \( G \)-interpretable strongly minimal set. We consider finitely many definable subgroups \( 1 = H_0 \leq \cdots \leq H_m = G \). Then there is an interpretable subset \( E \) of \( H_{i+1}/H_i \) for some \( i \) and an interpretable equivalence relation \( R \) over \( E \) such that \( E/R \) is strongly minimal and analogous to \( X \).

Theorem (Lascar analysis of a group of finite Morley rank revisited)

There are finitely many strongly minimal sets \( X_1, \ldots, X_n \) such that each interpretable strongly minimal set \( X \) of \( G \), is analogous to some \( X_i \).

Proof: we may suppose that, in the proposition above, the groups \( H_i \) are normal in \( G \) and the quotient groups \( H_{i+1}/H_i \) are infinite and minimal for this condition. In particular, \( H_{i+1}/H_i \) is harmonious for each \( i \). \( \square \)

Remark: at this stage, it is not clear that the integer \( n \) is preserved by elementary equivalence.
IV - Elementary extensions

\( \mathcal{M} = (M; \cdot, \cdots) \) is a structure of finite Morley rank without the finite cover property.

**Lemma**

Let \( \mathcal{M}^* = (M^*; \cdot, \cdots) \) be an elementary extension of \( \mathcal{M} \). Let \( X \) and \( Y \) be two interpretable strongly minimal sets in \( \mathcal{M} \) and let \( X^* \) and \( Y^* \) be their canonical extensions in \( \mathcal{M}^* \). Then \( X \) and \( Y \) are analogous iff \( X^* \) and \( Y^* \) are analogous.

**Proof** (for the case where \( X \) and \( Y \) are subsets of \( M \)): We may assume that \( X^* \) and \( Y^* \) are analogous. There is a strongly minimal subset \( V^* \subseteq X^* \times Y^* \) such that the canonical projection maps \( p : V^* \to X^* \) and \( q : V^* \to Y^* \) are sujective. We consider integers \( k \) and \( l \) such that \( |p^{-1}(x)| \leq k \) and \( |q^{-1}(y)| \leq l \) for each \( x \in X^* \) and each \( y \in Y^* \). Let \( \varphi(x, y, \bar{a}) \) be a formula defining \( V^* \) with parameters \( \bar{a} \) in \( M^* \). The set \( V^* \) defined in \( M^* \) by the formula \( \varphi(x, y, \bar{a}) \) has the following properties:

- it is contained in \( X^* \times Y^* \);
- for any \( x \in X^* \), \( \{ y \in Y^* \mid (x, y) \in V^* \} \) is nonempty and has at most \( k \) elements;
- for any \( y \in Y^* \), \( \{ x \in X^* \mid (x, y) \in V^* \} \) is nonempty and has at most \( l \) elements.

Consequently, there exist parameters \( \bar{b} \) in \( M \) such that the set \( V \) defined by \( \varphi(x, y, \bar{b}) \) has the following properties:

- it is contained in \( X \times Y \);
- for any \( x \in X \), \( \{ y \in Y \mid (x, y) \in V \} \) is nonempty and has at most \( k \) elements;
- for any \( y \in Y \), \( \{ x \in X \mid (x, y) \in V \} \) is nonempty and has at most \( l \) elements.

In particular, the set \( V \) is infinite and contains a strongly minimal subset \( \mathcal{W} \), and the last two conditions imply that the canonical projection maps from \( \mathcal{W} \) to \( X \) and \( Y \) have cofinite images. Therefore \( X \) and \( Y \) are analogous. \( \square \)
If two strongly minimal sets $X$ and $Y$ are analogous, they are $\sim$-equivalent.

Let $\mathcal{G} = (G; \cdot, \cdots)$ be a group of finite Morley rank. We recall the existence of finitely many strongly minimal sets $X_1, \ldots, X_n$ such that each interpretable strongly minimal set $X$ of $G$, is analogous to some $X_i$. We may suppose that the sets $X_i$ and $X_j$ are not analogous for $i \neq j$.

Proposition

Let $\mathcal{G}^* = (G^*; \cdot, \cdots)$ be an elementary extension of $\mathcal{G}$ and let $X_1^*, \ldots, X_n^*$ be the canonical extensions of $X_1, \ldots, X_n$ in $\mathcal{G}^*$. Then each interpretable strongly minimal set of $G^*$, is analogous to some $X_i^*$ and the sets $X_i^*$ and $X_j^*$ are not analogous for $i \neq j$.

Corollary

We may assume that $X_1, \ldots, X_n$ are the canonical extensions of strongly minimal sets interpretable in the prime model of $\mathcal{G}$.

In particular, the integer $n$ is preserved by elementary equivalence.

Corollaries

- Any harmonious group of finite Morley rank is $\aleph_1$-categorical.
- If the structure $\mathcal{G}$ is harmonious, then any structure elementary equivalent to $\mathcal{G}$ is harmonious as well.
Theorem

For each strongly minimal subset $X$ of $G$, there is a connected definable subgroup $M_G(X)$ satisfying the following properties:

- $M_G(X)$ is harmonious of type $X$;
- every harmonious subset of type $X$ is contained in finitely many cosets of $M_G(X)$.

In particular,

- $M_G(X)$ contains all the connected harmonious subgroups of $G$ of type $X$;
- $M_G(X)$ is characteristic in $G$;
- the cardinal of $M_G(X)$ is $|X|$.
Corollary

The sugroup of $G$ generated by its harmonious subgroups is the central product $M_G = M_G(X_1) \cdots M_G(X_n)$ and the intersection $M_G(X_i) \cap \prod_{j \neq i} M_G(X_j)$ is finite for each $i$.
Furthermore, each harmonious subset $E$ of $G$ is contained in finitely many cosets of $M_G$.

Questions

If $G$ is connected,
- is $G'$ contained in $M_G$?
- is $M_G/Z(G) = G/Z(G)$?

A positive answer would provide a similar result to the torsion-free nilpotent case.

Theorem (Burdges, F.)

If $G$ is a torsion-free nilpotent group, then $G/Z(G)$ and $G'$ are the direct products of homogeneous subgroups.
Questions

- If the structure $\mathcal{M}$ is harmonious, is its theory $\aleph_1$-categorical?
- If two strongly minimal sets $X$ and $Y$ are $\sim$-equivalent, are $X$ and $Y$ analogous? *A positive answer would imply that any $\aleph_1$-categorical group is harmonious.*
- Characterize $\aleph_1$-categorical pure groups.

Conjecture 1

Any connected group $\mathcal{G} = (G; \cdot)$ of finite Morley rank interprets $\aleph_1$-categorical groups $G_1, \ldots, G_n$ such that $G \cong (G_1 \times \cdots \times G_n)/F$ for a normal finite subgroup $F$.

*Rachad Bentbib studies this problem.*

Remark: if $\mathcal{G} = (G; \cdot, \cdots)$ is a group of finite Morley rank, then $\mathcal{G}$ is bi-interpretable with a structure $\mathcal{G}^\circ = (G^\circ; \cdot, \cdots)$ of base $G^\circ$. 
Conjecture 2

If the theory of a group \((G; \cdot, \cdots)\) is \(\aleph_1\)-categorical then all of its models are interpretable in a strongly minimal structure.

Remark: if all the models of a theory are interpretable in a strongly minimal structure, then the theory has no Vaught pair so it is \(\aleph_1\)-categorical.

Conjecture 3

If the theory of a group \((G; \cdot, \cdots)\) is of finite Morley rank then one of its models is interpretable in a strongly minimal structure.

Remark: if a model of a theory is interpretable in a strongly minimal structure, then the theory is of finite Morley rank.

*The second part of this talk will be given by Rachad Bentbib.*
Bon retour en France, Tuna !