Harmonious groups of finite Morley rank

(first part: Lascar analysis of a group of finite Morley rank revisited)

A work joint with Rachad Bentbib (Université de Poitiers)

Ranked groups - The return

Conference to celebrate the return in France of

Tuna Altınel

Main objective

Better know the structure of $\aleph_1$-categorical groups.

Definition: a theory T is κ-**categorical** for an infinite cardinal κ if any two models of T of cardinality κ are isomorphic.

For instance, we would like to obtain a result of the style:

Let G be a group with no abelian subgroup of finite index. Then G is $\aleph_1$-categorical if and only if it is of finite Morley rank and no normal subgroup N of finite index in G is a central product $N = HK$ with $H \cap K$ finite and H and K infinite and normal in G.

Another objective:

What are the groups interpretable in a strongly minimal structure?

Definition: a theory T is **strongly minimal** if, for any model $\mathscr{M} = (M; \cdots)$ of T, any definable subset of M is finite or cofinite.

I - Introduction

Groups of finite Morley rank and $\aleph_1$-categoricity

Let T be a complete theory in a countable language.

Morley's Categoricity Theorem (1965)

If T is κ-categorical for some uncountable κ, then T is κ-categorical for every uncountable κ.

Example: if T is strongly minimal, then T is $\aleph_1$-categorical.

Theorem (Baldwin, 1973)

If T is $\aleph_1$-categorical, then it is of finite Morley rank.

Theorem (Zilber, 1977)

The theory of an infinite simple group of finite Morley rank is $\aleph_1$-categorical.

Remark: a direct product of two nonisomorphic infinite structures of groups of finite Morley rank is not $\aleph_1$-categorical.

Groups of finite Morley rank: examples

- The main example of a group of finite Morley rank is the group $\mathcal{G}(K)$ of rational points of an algebraic group $\mathcal{G}$ defined over an algebraically closed field K, where the language is the one of groups with a unary predicate for each constructible subset of $\mathcal{G}(K)^n$ (Zilber, early 1970s).

Remark: such a structure is bi-interpretable with a strongly minimal structure, *so it is κ-categorical for each cardinal $\kappa > |K|$.*

- The pure group $(\mathbb{Q}, +)$ is a group of Morley rank 1, and it is not algebraic.
- The Baudisch group is a non-algebraic non-abelian nilpotent group of Morley rank 2.

Remark: $(\mathbb{Q}, +)$ is a strongly minimal structure, so it is $\aleph_1$-categorical, and the Baudisch group is of Morley rank 2 and it is not abelian-by-finite, so it is $\aleph_1$-categorical.

Theorem (Tanaka, 1988)

Let G be a group of Morley rank 2. If G is not abelian-by finite, then G is $\aleph_1$-categorical.

Theorem (Tsuboi, 1988)

Let G be an infinite solvable group of finite Morley rank. If G has no abelian definable subgroup of Morley rank $\geqslant 2$, then G is $\aleph_1$-categorical.

What about ω-categorical groups?

Theorem (proved independently by Engeler, Ryll-Nardzewski and Svenonius in 1959)

A countable structure $\mathscr{M} = (M; \cdots)$ is ω-categorical iff $\mathrm{Aut}(\mathscr{M})$ has only finitely many orbits in its action on M^n for each $n > 0$.

Corollary

ω-categorical groups are uniformly locally finite groups (*i.e.* for each integer k, there is an integer n such that any subset of cardinal k generates a subgroup of order at most n); in particular, they have bounded exponent.

Theorems

- (Wilson, 1981) ω-categorical groups are locally-solvable-by-residually-finite.
- (Apps, 1982) Let G be an ω-categorical group. Then G has a finite characterstic series $1 = G_0 < G_1 < \cdots < G_n = G$ with G_{i+1}/G_i an ω-categorical characteristically simple group for each i.

Conjectures

- (stated independently by Apps and Wilson in 1981) ω-categorical characteristically simple locally nilpotent groups are abelian.
- (Wilson, 1981) ω-categorical groups are nilpotent-by-residually-finite.

What about totally categorical groups?

Definition: a theory T is said to be **totally categorical** if it has exactly one model in each infinite power.

Theorem (Baur-Cherlin-Macintyre, 1977)

Totally categorical groups are abelian-by-finite and have bounded exponent.

Theorem (Baur-Cherlin-Macintyre, 1977)

Let G be a locally finite group of bounded exponent and let A be an abelian normal subgroup of finite index in G. Then G is totally categorical iff A is a direct sum of finite normal subgroups of G of bounded order.

A particular case: torsion-free nilpotent groups

Theorem (Zilber, 1982)

Let G be a nonabelian torsion-free nilpotent group. The theory of the pure group $(G; \cdot)$ is $\aleph_1$-catogorical if and only if G satisfies the following two conditions:

- G is isomorphic to a unipotent group over an algebraically closed field of characteristic 0;
- G cannot be decomposed into a direct product of two nontrivial subgroups.

Theorem (Altınel-Wilson, 2008)

Every torsion-free nilpotent group G of finite Morley rank has a faithful linear representation over a field of characteristic zero.

Theorem (Myasnikov-Sohrabi, 2018)

A torsion-free nilpotent group G has finite Morley rank in the language of groups iff $G \simeq G_0 \times G_1 \times \cdots G_n$ for a divisible abelian group G_0 and finitely many unipotent algebraic groups $G_1, \ldots, G_n$ over characteristic zero algebraically closed fields.

Remark: these theorems are proved by methods based on the study of nilpotent Lie algebras.

Definition

A torsion-free nilpotent group G of finite Morley rank is said to be **homogeneous** if either it is trivial or there is an interpretable algebraically closed field K such that, each nontrivial interpretable quotient group G/H of G contains a copy of K.

Theorem (follows from works by Burdges and F., ∼2006)

Let G be a torsion-free nilpotent group. Then $G/Z(G)$ and G' are the direct products of homogeneous subgroups.

II - Harmonious structures

Groups of finite Morley rank: Lascar analysis

Definition

Let T be an ω-stable theory and, let $\varphi(x)$ and $\phi(x)$ be two formulas. We say that φ and ϕ are $\sim$-**equivalent** if, for every model M containing the parameters of φ and ϕ, we have $|\varphi(M)| = |\phi(M)|$.

Theorem (Lascar, 1985)

Let T be the theory of a group G of finite Morley rank. We can find strongly minimal formulas $f_1(x), \ldots, f_n(x)$, possibly imaginary, with parameters in the prime model, such that

- for all the infinite cardinalities $\kappa_1, \ldots, \kappa_n$, there exists a model $\mathcal{M} = (M; \cdots)$ of T such that $|f_i(M)| = \kappa_i$ for each i;
- every strongly minimal formula $\varphi(x)$ is $\sim$-equivalent to one of these formulas.
- every elementary extension of a model of that theory augments at least one of the formulas.

Corollary

A group G of finite Morley rank is $\aleph_1$-categorical if and only if all the strongly minimal formulas, with parameters in the prime model, are $\sim$-equivalent (*i.e.* $n = 1$ in the previous theorem).

Theorem (Lascar, 1985)

Let G be a connected group of finite Morley rank with no infinite normal abelian definable subgroup. Then there are $\aleph_1$-categorical groups $H_1, \ldots, H_n$ such that G is isomorphic to $(H_1 \times \cdots \times H_n)/F$ for a finite subgroup F.

We would like delete the hypothesis with no infinite normal abelian definable subgroup*.*

Definition

Let $\mathscr{M} = (M; \cdots)$ be a structure of finite Morley rank. Two interpretable strongly minimal sets X and Y are said to be **analogous** if there is a strongly minimal set U and two interpretable maps $f_X : U \to X$ and $f_Y : U \to Y$ with cofinite images.
The structure $\mathscr{M} = (M; \cdots)$ is **harmonious of type** X if each interpretable strongly minimal set is analogous to X.

Remarks:

- If two interpretable strongly minimal sets X and Y are analogous, their canonical extensions in any elementary extension of $\mathscr{M}$ are analogous too.
- There are integers m and n such that $|f_X^{-1}(\{x\})| \leqslant m$ and $|f_Y^{-1}(\{y\})| \leqslant n$ for each $x \in X$ and each $y \in Y$.
- In particular, we have $|X| = |U| = |Y|$.

Questions

- If two strongly minimal sets X and Y are analogous, are X and Y $\sim$-equivalent? *We will show this property for groups, thus* ***any harmonious group of finite Morley rank is*** $\aleph_1$***-categorical****.*
- If two strongly minimal sets X and Y are $\sim$-equivalent, are X and Y analogous?
- In other words, is any $\aleph_1$-categorical group harmonious?
- If $\mathscr{M}$ is an harmonious structure of finite Morley rank, are its elementary extensions harmonious too? *We will show this property for groups.*

The finite cover property

Definition (Keisler, 1967)

Let $\mathscr{M} = (M; \cdots)$ be a model of a complete theory T. The structure $\mathscr{M}$ has the **finite cover property** if a formula $\varphi(\overline{x}, \overline{a})$ has the following property:

- for every integer n, there is an inconsistent set of at least n formulas $\{\varphi(\overline{x}, \overline{a_i}) \ ; \ i \in I\}$ with parameters in M such that every subset of it is consistent.

Facts

- If $\mathscr{M} \equiv \mathscr{N}$ and $\mathscr{M}$ has the finite cover property, then $\mathscr{N}$ has also this property.
- If a structure $\mathscr{M}$ does not have the finite cover property, then no structure interpretable in $\mathscr{M}$ has the finite cover property.

Facts

- Any unstable structure has the finite cover property.
- No $\aleph_1$-categorical structure has the finite cover property.
- No group of finite Morley rank has the finite cover property.

$\mathscr{M} = (M; \cdots)$ **is a structure of finite Morley rank without the finite cover property.**

Facts

Any infinite interpretable set contains a strongly minimal set.

Lemma

The relation "to be analogous to" is an equivalence relation on the family of strongly minimal sets.

Proof: for the transitivity, let Y be a strongly minimal set analogous to X and Z. Let V and W be strongly minimal sets and let $f_X : V \to X$, $f_Y : V \to Y$ $g_Y : W \to Y$ and $g_Z : W \to Z$ be interpretable maps with cofinite images.
Then $f_Y(V) \cap g_Y(W)$ is cofinite in Y, so the set $\{(v,w) \in V \times W \mid f_Y(v) = g_Y(w)\}$ is infinite and contains a strongly minimal set U.
Let $p_V : U \to V$ and $p_W : U \to W$ be the canonical projection maps. Then we have $f_Y \circ p_V = g_Y \circ p_W$ and the maps $f_X \circ p_V : U \to X$, $f_Y \circ p_V : U \to Y$ and $g_Z \circ p_W : U \to Z$ have cofinite images. □

Corollary

If $(X_i)_{i=1,\dots,n}$ is a finite family of analogous strongly minimal sets, there exists a strongly minimal set $U \subseteq X_1 \times \cdots \times X_n$ and *surjective* interpretable maps $f_i : U \to X_i$.

Questions

If $(X_i)_{i \in I}$ is an *infinite* family of analogous strongly minimal sets,

- are there a strongly minimal set U and interpretable maps $f_i : U \to X_i$ with cofinite images?
- are there a strongly minimal set V and interpretable maps $g_i : X_i \to V$ with cofinite images?

Proposition

Any strongly minimal set is analogous to a strongly minimal set of the form E/R for a definable subset E of M and an equivalence relation R over E.

Corollary

Any strongly minimal structure is harmonious.

Remarks:

- Let $\mathscr{K} = (K; +, \cdot)$ be an algebraically closed field. Then $\mathscr{K}$ is strongly minimal, so it is harmonious. However, the $\mathscr{K}$-interpretable structure $\mathscr{G} = (K \times K; +, K \times \{0\})$ is not harmonious.
- The last result implies that any structure bi-interpretable with a strongly minimal structure is harmonious.
- The pure group $(\mathbb{Z}/4\mathbb{Z})^\omega$ is harmonious, but it is not bi-interpretable with a strongly minimal structure.

Question

If $\mathscr{M}$ is harmonious of type X, is $\mathscr{M}$ interpretable in a strongly minimal structure?

Remark: Assaf Hasson proved that, for a theory of finite Morley rank with definable MR, the DMP (definable multiplicity property) is a sufficient condition for such an interpretation.
Furthermore, he proved that not to have the finite cover property is a necessary condition, but not a sufficient condition, for a structure to be interpreted in a strongly minimal structure.

Definition: Let T be a theory of finite Morley rank with definable MR. Then T **has the DMP** if for every model M, formula $\varphi(\overline{x}, \overline{y})$ and integers k and n, the set of tuples $\overline{a}$ in M such that $\varphi(M, \overline{a})$ has Morley rank k and Morley degree n is definable.

III - Harmonious sets

We fix a structure $\mathscr{G} = (G; \cdot, \cdots)$ of group of finite Morley rank.
In particular, no $\mathscr{G}$-interpretable structure has the finite cover property, so each $\mathscr{G}$-interpretable set contains a strongly minimal set.

Definition

An interpretable set N is said to be **harmonious of type** X if, for any interpretable subset E of N^n for an integer n and any interpretable equivalence relation R over E such that E/R is strongly minimal, the set E/R is analogous to X.

Remarks:

- The structure $\mathscr{G}$ is harmonious iff the set G is harmonious.
- If a structure $\mathscr{M} = (M; \cdots)$ is interpretable in $\mathscr{G}$, then it is harmonious iff the set M is harmonious.

Lemma

Let X be a strongly minimal set. If $A_1, \ldots, A_n$ are harmonious sets of type X, then $A_1 \times \cdots \times A_n$ is harmonious of type X.

Proposition

If $G = X_1 \cdots X_n$ is a finite product of analogous strongly minimal subsets, then $\mathscr{G}$ is bi-interpretable with a strongly minimal structure.
In particular, it is harmonious.

Proof: there is an interpretable strongly minimal set U and, for each i, a surjective interpretable map $f_i : U \to X_i$. We consider the surjective interpretable map $\gamma : U^n \to G$ defined by $\gamma(x_1, \ldots, x_n) = f_1(x_1) \cdots f_n(x_n)$. We can interpret a strongly minimal structure $\mathscr{U}$ of base U such that $\mathscr{G}$ and $\mathscr{U}$ are bi-interpretable. □

Corollary

Let $K \triangleleft H$ be normal definable subgroups of G such that H/K is infinite and has no nontrivial proper G-normal definable subgroup. Then H/K is harmonious.
Moreover, if X is the type of H/K, then $|H/K| = |X|$.

Proof: it follows from the Zilber Indecomposability theorem that such a quotient group is generated by a finite product of the G-conjugates of any strongly minimal subset. □

Proposition

Let X be a $\mathscr{G}$-interpretable strongly minimal set. We consider finitely many definable subgroups $1 = H_0 \leqslant \cdots \leqslant H_m = G$. Then there is an interpretable subset $\overline{E}$ of H_{i+1}/H_i for some i and an interpretable equivalence relation R over $\overline{E}$ such that $\overline{E}/R$ is strongly minimal and analogous to X.

Theorem (Lascar analysis of a group of finite Morley rank revisited)

There are finitely many strongly minimal sets $X_1, \ldots, X_n$ such that each interpretable strongly minimal set X of G, is analogous to some X_i.

Proof: we may suppose that, in the proposition above, the groups H_i are normal in G and the quotient groups H_{i+1}/H_i are infinite and minimal for this condition. In particular, H_{i+1}/H_i is harmonious for each i. $\square$

Remark: at this stage, it is not clear that the integer n is preserved by elementary equivalence.

IV - Elementary extensions

$\mathscr{M} = (M; \cdot, \cdots)$ *is a structure of finite Morley rank without the finite cover property.*

Lemma

Let $\mathscr{M}^* = (M^*; \cdot, \cdots)$ be an elementary extension of $\mathscr{M}$. Let X and Y be two interpretable strongly minimal sets in $\mathscr{M}$ and let X^* and Y^* be their canonical extensions in $\mathscr{M}^*$. Then X and Y are analogous iff X^* and Y^* are analogous.

Proof (for the case where X and Y are subsets of M): We may assume that X^* and Y^* are analogous. There is a strongly minimal subset $V^* \subseteq X^* \times Y^*$ such that the canonical projection maps $p : V^* \to X^*$ and $q : V^* \to Y^*$ are sujective. We consider integers k and l such that $|p^{-1}(x)| \leqslant k$ and $|q^{-1}(y)| \leqslant l$ for each $x \in X^*$ and each $y \in Y^*$. Let $\varphi(x, y, \bar{a})$ be a formula defining V^* with parameters $\bar{a}$ in M^*. The set V^* defined in $\mathscr{M}^*$ by the formula $\varphi(x, y, \bar{a})$ has the following properties:

- it is contained in $X^* \times Y^*$;
- for any $x \in X^*$, $\{y \in Y^* \mid (x, y) \in V^*\}$ is nonempty and has at most k elements;
- for any $y \in Y^*$, $\{x \in X^* \mid (x, y) \in V^*\}$ is nonempty and has at most l elements.

Consequently, there exist parameters $\bar{b}$ in M such that the set V defined by $\varphi(x, y, \bar{b})$ has the following properties:

- it is contained in $X \times Y$;
- for any $x \in X$, $\{y \in Y \mid (x, y) \in V\}$ is nonempty and has at most k elements;
- for any $y \in Y$, $\{x \in X \mid (x, y) \in V\}$ is nonempty and has at most l elements.

In particular, the set V is infinite and contains a strongly minimal subset W, and the last two conditions imply that the canonical projection maps from W to X and Y have cofinite images. Therefore X and Y are analogous. $\square$

Corollary

If two strongly minimal sets X and Y are analogous, they are $\sim$-equivalent.

Let $\mathscr{G} = (G; \cdot, \cdots)$ be a group of finite Morley rank. We recall the existence of finitely many strongly minimal sets $X_1, \ldots, X_n$ such that each interpretable strongly minimal set X of G, is analogous to some X_i. We may suppose that the sets X_i and X_j are not analogous for $i \neq j$.

Proposition

Let $\mathscr{G}^* = (G^*; \cdot, \cdots)$ be an elementary extension of $\mathscr{G}$ and let $X_1^*, \ldots, X_n^*$ be the canonical extensions of $X_1, \ldots, X_n$ in $\mathscr{G}^*$. Then each interpretable strongly minimal set of G^*, is analogous to some X_i^* and the sets X_i^* and X_j^* are not analogous for $i \neq j$.

Corollary

We may assume that $X_1, \ldots, X_n$ are the canonical extensions of strongly minimal sets interpretable in the prime model of $\mathscr{G}$.
In particular, the integer n is preserved by elementary equivalence.

Corollaries

- **Any harmonious group of finite Morley rank is $\aleph_1$-categorical**.
- If the structure $\mathscr{G}$ is harmonious, then any structure elementary equivalent to $\mathscr{G}$ is harmonious as well.

Theorem

For each strongly minimal subset X of G, there is a connected definable subgroup $M_G(X)$ satisfying the following properties:

- $M_G(X)$ is harmonious of type X;
- every harmonious subset of type X is contained in finitely many cosets of $M_G(X)$.

In particular,

- $M_G(X)$ contains all the connected harmonious subgroups of G of type X;
- $M_G(X)$ is characteristic in G;
- the cardinal of $M_G(X)$ is $|X|$.

Corollary

The sugroup of G generated by its harmonious subgroups is the central product $M_G = M_G(X_1) \cdots M_G(X_n)$ and the intersection $M_G(X_i) \cap \prod_{j \neq i} M_G(X_j)$ is finite for each i.
Furthermore, each harmonious subset E of G is contained in finitely many cosets of M_G.

Questions

If G is connected,

- is G' contained in M_G?
- is $M_{G/Z(G)} = G/Z(G)$?

A positive answer would provide a similar result to the torsion-free nilpotent case.

Theorem (Burdges, F.)

If G is a torsion-free nilpotent group, then $G/Z(G)$ and G' are the direct products of homogeneous subgroups.

V - Questions

Questions

- If the structure $\mathscr{M}$ is harmonious, is its theory $\aleph_1$-categorical?
- If two strongly minimal sets X and Y are $\sim$-equivalent, are X and Y analogous?
  A positive answer would imply that any $\aleph_1$-categorical group is harmonious.
- Characterize $\aleph_1$-categorical pure groups.

Conjecture 1

Any connected group $\mathscr{G} = (G; \cdot)$ of finite Morley rank interprets $\aleph_1$-categorical groups $G_1, \ldots, G_n$ such that $G \simeq (G_1 \times \cdots \times G_n)/F$ for a normal finite subgroup F.

Rachad Bentbib studies this problem.

Remark: if $\mathscr{G} = (G; \cdot, \cdots)$ is a group of finite Morley rank, then $\mathscr{G}$ is bi-interpretable with a structure $\mathscr{G}^\circ = (G^\circ; \cdot, \cdots)$ of base G°.

Conjecture 2

If the theory of a group $(G; \cdot, \cdots)$ is $\aleph_1$-categorical then all of its models are interpretable in a strongly minimal structure.

Remark: if all the models of a theory are interpretable in a strongly minimal structure, then the theory has no Vaught pair so it is $\aleph_1$-categorical.

Conjecture 3

If the theory of a group $(G; \cdot, \cdots)$ is of finite Morley rank then one of its models is interpretable in a strongly minimal structure.

Remark: if a model of a theory is interpretable in a strongly minimal structure, then the theory is of finite Morley rank.

The second part of this talk will be given by Rachad Bentbib.

Bon retour en France, Tuna !