Small groups of finite Morley rank with a tight automorphism

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Joint work with Pınar Uğurlu

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The current state of (C-Z) stated in terms of the Sylow 2-subgroups:

In any group G of fRM, the Sylow 2-subgroups are conjugate[Borovik, Poizat 2007] and their structure is well-understood:

$$\overline{\mathrm{Syl}_G}^{\circ} \cap \mathrm{Syl}_G = \mathrm{Syl}_G^{\circ} = U * T,$$

where U is 2-*unipotent* and T is 2-*divisible*. If the ambient group G is infinite simple then:

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where U is 2-*unipotent* and T is 2-*divisible*. If the ambient group G is infinite simple then:

- Either $\mathrm{Syl}_G = 1$ (*degenerated type*) or Syl_G° is infinite.[Borovik, Burdges, Cherlin 2007]
- Either $U = 1$ (odd type) or $T = 1$ (even type). (No *mixed* type groups exist.)[Altınel, Borovik, Cherlin 2008]
- If $1 \neq \mathrm{Syl}_G^{\circ} = U$ then (C-Z) holds. Namely, $G \cong X(K)$ for an a.c. field K of $\mathrm{char}(K) = 2$.[Altınel, Borovik and Cherlin 2008]

Small groups

Let $H = X(K)$ be a Chevalley group for K a.c. with $\mathrm{char}(K) \neq 2$, and T be a maximal algebraic torus of H. The 'size' of H can be described in different ways, e.g. by $\dim_{Zar}(H)$ or by

$$\dim_{Zar}(T) = \mathrm{pr}_2(H) = \#\text{ of copies of } \mathbb{Z}_{2^\infty} := \{x \in \mathbb{C}^\times : x^{2^n} = 1, n \in \mathbb{N}\}.$$

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  - *Minimal simple* groups: every proper definable connected subgroup is solvable.[Jaligot, Cherlin, Deloro, Altinel, Frécon, Burdges...]

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  - *Minimal simple* groups: every proper definable connected subgroup is solvable.[Jaligot, Cherlin, Deloro, Altinel, Frécon, Burdges...]
  - The presence of a *tight* automorphism whose fixed-point subgroup is pseudofinite—this is our framework.

Pseudofinite fields and simple pseudofinite groups

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- $(\mathbb{Q}, +) \equiv \prod_{p \in P} C_p/\mathcal{U}$ is pseudofinite.
- A (twisted) Chevalley group $X(F)$ is pseudofinite iff F is pseudofinite.
- Algebraically closed fields are not pseudofinite.
- $F \equiv \prod_{p \in P} \mathbb{F}_p/\mathcal{U}$ is pseudofinite of $\mathrm{char}(F) = 0$.

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Theorem (Ax 1968)

An infinite field is pseudofinite iff it is perfect, quasi-finite and PAC.

Theorem (Wilson 1995 and Ryten 2007)

A simple group is pseudofinite iff it is isomorphic to a (twisted) Chevalley group over a pseudofinite field.

The Principal conjecture

The theory ACFA of algebraically closed fields with generic automorphism is axiomatised[Chatzidakis, Hrushovski 1999] as follows: $(K, \sigma) \models \mathrm{ACFA}$ iff:

- $K \models \mathrm{ACF}$ and $\sigma \in \mathrm{Aut}(F)$.
- Let V be an irreducible variety and let S be an irreducible subvariety of $V \times \sigma(V)$ s.t. both $\pi_1 : S \to V$ and $\pi_2 : S \to \sigma(V)$ are dominant. Then there exists $a \in V(K)$ s.t. $(a, \sigma(a)) \in S$.

If $(K, \sigma) \models \mathrm{ACFA}$ then $\mathrm{Fix}_K(\sigma)$ is pseudofinite.[Macintyre1997/Chatzidakis, Hrushovski 1999]

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Fixed point subgroups of generic automorphisms of ‘structures with certain nice model-theoretic properties’ resemble pseudofinite groups.[Hrushovski 2002]

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Let G be an infinite simple group of finite Morley rank with a generic automorphism α. Then the fixed point subgroup $C_G(\alpha)$ is pseudofinite.

- (C-Z) $\Rightarrow$ (PC).[Chatzidkis and Hrushovski 1999]
- We aim to prove that (PC) $\Rightarrow$ (C-Z).

A tight automorphism α

From now on:

1. Groups (resp. fields) are considered in pure group (resp. field) language.
2. Given a subset X of a group of fRM G, $\overline{X}$ is the definable closure of X in G.

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An automorphism α of an infinite simple group of fRM G is called *tight* if, for any connected definable and α-invariant subgroup $H \leqslant G$, $\overline{C_H(\alpha)} = H$.

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Example: We have $(K, \phi_{\mathcal{U}}) \models \mathrm{ACFA}$[Hrushovski 1996], where

$$\phi_{\mathcal{U}} : K = \prod_{p_i \in P} \mathbb{F}_{p_i}^{alg} / \mathcal{U} \longrightarrow \prod_{p_i \in P} \mathbb{F}_{p_i}^{alg} / \mathcal{U}, \quad [x_i]_{\mathcal{U}} \mapsto [x_i^{p_i}]_{\mathcal{U}}$$

is the *non-standard Frobenius automorphism* of K.

Let $G = X(K)$ be a simple Chevalley group and H be a definable, connected and $\phi_{\mathcal{U}}$-invariant subgroup of G. Then $\phi_{\mathcal{U}}$ induces an automorphism on G s.t.

- $X(\mathrm{Fix}_K(\phi_{\mathcal{U}})) = X(\prod_{p_i \in P} \mathbb{F}_{p_i} / \mathcal{U}) \cong \prod_{p_i \in P} X(\mathbb{F}_{p_i}) / \mathcal{U}$ is pseudofinite.
- $C_H(\phi_{\mathcal{U}}) = H(k)$, with k pseudofinite. So, $\overline{C_H(\phi_{\mathcal{U}})}^{Zar} = H$.

Tight α with pseudofinite fixed-point subgroup

The socle $\mathrm{Soc}(H)$ of a group H is the subgroup generated by all minimal normal non-trivial subgroups of a group H.

Theorem (Uğurlu 2009)

Let G be an infinite simple group of fRM and α be a tight automorphism of G s.t. $C_G(\alpha) = P \equiv \prod_{i \in I} P_i/\mathcal{U}$ is pseudofinite. Then there is a definable normal S of P s.t.

$$X(F) \cong \prod_{i \in I} \mathrm{Soc}(P_i)/\mathcal{U} \equiv S \trianglelefteq P \leqslant \mathrm{Aut}(S),$$

where F is a pseudofinite field. Moreover, $\overline{S} = G$.

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Remarks:

1. G has involutions as the simple pseudofinite group S has involutions.
2. For almost all i, The socle $\mathrm{Soc}(P_i)$ is uniformly definable normal subgroup of P_i. So $P/S \equiv \prod_{i\in I}(P_i/\mathrm{Soc}(P_i))/\mathcal{U}$.

Summary: the approach towards (C-Z) $\Leftrightarrow$ (PC)

Let G be an infinite simple group of fRM with a tight automorphism α whose fixed point subgroup $P = C_G(\alpha)$ is pseudofinite. We have $\mathrm{pr}_2(G) = n \geqslant 1$. To prove that (C-Z) $\Leftrightarrow$ (PC) we need to prove the following two steps:

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1. *Algebraic identification step*: We know that there is a pseudofinite (twisted) Chevalley group $S = X(F)$ s.t. $\overline{S} = G$.
   - Show that S is of untwisted Lie type X and of Lie rank n, and, that $\mathrm{char}(F) \neq 2$.
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2. *Model-theoretic step*: Prove that a generic automorphism of G is tight.

Our results (K. and Uğurlu 2021)

From now on, G is an infinite simple group of fRM with $\mathrm{pr}_2(G) = 1$ admitting a tight automorphism α whose fixed-point subgroup $C_G(\alpha) = P \equiv \prod_{i \in I} P_i/\mathcal{U}$ is pseudofinite and $S \cong X(F) \cong \prod_{i \in I} \mathrm{Soc}(P_i)/\mathcal{U}$.

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Almost an theorem

$G \cong \mathrm{PSL}_2(K)$ for K a.c. of $\mathrm{char}(K) \neq 2$.

Sylow 2-subgroups of S and G, $S \cong \mathrm{PSL}_2(F)$

Theorem (Deloro and Jaligot 2010)

Let H be an odd type connected group of fRM with $\mathrm{pr}_2(H) = 1$. Then exactly one of the following holds.

1. $\mathrm{Syl}_H = \mathrm{Syl}_H^\circ \cong \mathbb{Z}_{2^\infty}$.
2. $\mathrm{Syl}_H = \mathrm{Syl}_H^\circ \rtimes \langle \omega \rangle$ *for an involution* ω *which inverts* Syl_H°.
3. $\mathrm{Syl}_H = \mathrm{Syl}_H^\circ \cdot \langle \omega \rangle$ *for an element* ω *of order* 4 *which inverts* Syl_H°.

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Any Sylow 2-subgroup Syl_G of G must be of type (2) as for otherwise S satisfies the FO-expressible statement 'Every subgroup of order 4 is cyclic'.

- Sylow 2-subgroups of S are either conjugate dihedral groups or as Syl_G. In particular, the finite simple groups in the ultraproduct S has dihedral Sylow 2-subgroups.

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Theorem (Gorenstein and Walter 1962)

Let H be a finite simple group with dihedral Sylow 2-subgroups. Then either $H \cong \mathrm{PSL}_2(q)$, $q \geqslant 5$ or $H \cong A_7$.

Structures of $S \cong \mathrm{PSL}_2(F)$, $\mathrm{PGL}_2(F)$ and P

1. $\mathrm{PSL}_2(F) \cong S \leqslant C_G(\alpha) = P \leqslant G$, for F pseudofinite of $\mathrm{char}(F) \neq 2$.
2. $P \leqslant \mathrm{PGL}_2(F) \rtimes \mathrm{Aut}(F)$.

Subgrps of S [images of $\pi: GL_2(F) \to PGL_2(F)$ $\cap\, SL_2(F)$]

$\mathcal{U} \cong F^+$

$T \cong (F^\times)^2$: $i \in T \iff -1$ is a square in $F^\times$

$B = \mathcal{U} \rtimes T = N_S(\mathcal{U})$

$N_S(T) = \langle T, w_0 \rangle$, w_0 an involution inverting T

Subgrps of $PGL_2(F) = H$

$\mathcal{U} \cong F^+$

$\Pi \cong F^\times$: has unique inv. i

$B = \mathcal{U} \rtimes \Pi$

$N_H(\Pi) = N_H(T) = \langle \Pi, w_0 \rangle$

Let $P = \prod_{i \in I} P_i / \mathcal{U}$. Then $P/S \cong \prod_{i \in I} P_i / \mathrm{Soc}(P_i) / \mathcal{U} = \prod_{i \in I} \left(P_i / \mathrm{PSL}_2(q_i) \right) / \mathcal{U}$.

$P_i \hookrightarrow \mathrm{PGL}_2(q_i) \rtimes \mathrm{Aut}(q_i) \Rightarrow P_i / \mathrm{PSL}_2(q_i)$ has an abelian subgp of index 2.

$\Rightarrow P/S$ is abelian-by-finite.

$x \in P \Rightarrow x = sdf$, $s \in S$, $d \in \mathrm{Diag}(S)$, $f \in \mathrm{Aut}(F)$. $\mathrm{Diag}(S) \rtimes \mathrm{Aut}(F)$ leave invariant $\mathcal{U} \cong F^+$ and $T \cong (F^\times)^2$. $\Rightarrow P/S \cong N_P(T)/N_S(T) \cong N_P(\mathcal{U})/B$

How to identify G

Aim: $\overline{S} = G$ is a split Zassenhaus group, acting on the set of left cosets of $\overline{B}$ in G, with a one-point stabiliser $\overline{B}$ and a two-point stabiliser $\overline{T}$. This implies that $G \cong \mathrm{PSL}_2(K) = \mathrm{PGL}_2(K)$ for K a.c. and of $\mathrm{char}(K) \neq 2$. [Delahan, Nesin 1995]

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For above, we need to observe things as

- $G = \overline{B} \sqcup \overline{U}\omega_0\overline{B}$.
- $\overline{B}^g \cap \overline{U} = 1$ for all $g \in G \setminus \overline{B}$ (in particular, $N_G(\overline{U}) = N_G(\overline{B}) = \overline{B}$).
- $C_G^\circ(u) = C_G^\circ(\overline{U}) = \overline{U}$ for all $u \in \overline{U}^*$.
- $N_G(\overline{T}) = C_G(i) = \langle \overline{T}, \omega_0 \rangle$.
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1. Prove that $[P:S]<\infty$ and use that to get information up to connected components:

For $u \in U^*$: $C_{C^\circ_G(u)}(\alpha) \underset{f.i}{\leq} C_{C_G(u)}(\alpha) = C_{C_G(\alpha)}(u) \underset{f.i}{\geq} C_S(u) = U$. $\Rightarrow C^\circ_G(u) = \overline{U}^\circ$.

2. Prove that there is an involution $i \in \overline{T}^\circ$. Then, for $\mathbb{Z}_{2^\infty} \ni i$, $C_G(\mathbb{Z}_{2^\infty}) = \overline{T}$.

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1. $(N_P^\circ(U))' \leqslant U$ and $(N_P^\circ(T))' \leqslant N_S(T)$.
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Sketch of proof of (1).

Clearly $(N_P^\circ(U))' \leqslant B$.
As $\overline{N_P^\circ(U)}$ is connected and solvable, $\overline{N_P^\circ(U)}'$ is nilpotent[Nesin 1990].
As $\overline{B}^\circ \leqslant \overline{N_P^\circ(U)}$ we have $\overline{N_P^\circ(U)}' \leqslant F(\overline{B}^\circ)$.
It can be proven that $F(\overline{B}^\circ) = \overline{U}$ which gives us $(N_P^\circ(U))' \leqslant U$.

$i \in \overline{T}^\circ$

With assumptions in Theorem (Version 2):

- For the unique involution $i \in T$, we have

$$C_G^\circ(i) = \overline{C_{C_G^\circ(i)}(\alpha)}^\circ = \overline{C_{C_G(i)}(\alpha)}^\circ = \overline{C_{C_G^\circ(\alpha)}(i)}^\circ = \overline{C_S(i)}^\circ = \overline{\langle T, \omega_0 \rangle}^\circ = \overline{T}^\circ.$$

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Without extra assumptions:

- Enough to prove that $C_G^\circ(t) = \overline{T}^\circ$ for all $t \in \overline{T}^\circ$: Then $\overline{T}$ is generous in G and so there is $1 \neq x \in \overline{T} \cap C_G(H)$ for some maximal decent torus of H of G. So $\mathbb{Z}_{2^\infty} \leqslant H \leqslant C_G^\circ(x) = \overline{T}^\circ$.

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Idea for doing above: Prove that $\overline{\bigcup_{t\in\overline{T}^*} C_G^\circ(t) \cup \omega_0}^\circ$ is abelian by considering its intersection with the maximal subgroups of S.