From Borovik-Cherlin to bounding nonminimality

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In several recent applications of the model theory of differential fields one has a specific algebraic differential equation

$$P(x, \delta x, \delta^2 x, \delta^3 x) = 0$$

over a differential field (k, δ), and one wants to show that the set of solutions in a model of DCF_0 is strongly minimal.

A key difficulty in verifying strong minimality is that one must consider subsets definable over extensions of the base differential field k.

The goal of this joint work with James Freitag is to restrict how big an extension of k needs to be considered, and thereby make the problem more tractable to algebraic/computational methods in specific low order cases.

An abstract setting for the problem

T a complete totally transcendental theory, $\mathcal{U} \models T$ saturated, $p \in S(A)$ stationary type of finite rank.

Recall that p is *minimal* if $U(p) = 1$.
Equivalently, p is nonalgebraic but every forking extension is algebraic.

Fact

If $U(p) > 1$ then there are $a_1, \ldots, a_d \models p$ and a nonalgebraic forking extension of p to $Aa_1 \cdots a_d$.

Definition

Let $\mathrm{nmdeg}(p)$ be the least such d.

Problem

Can we bound $\mathrm{nmdeg}(p)$ *when p is of finite rank?*

Consequences of nmdeg > 1

If $\mathrm{nmdeg}(p) > 1$ then p has no *proper fibrations*:

$$\text{if } a \models p \text{ and } b \in \mathrm{dcl}(Aa) \setminus \mathrm{acl}(A) \text{ then } a \in \mathrm{acl}(Ab).$$

Indeed, a proper fibration would yield a definable function on p whose fibres are infinite but of lower rank than p.

Fact (M., Pillay 2014)

If p admits no proper fibrations then it is semiminimal. If, moreover, every type interalgebraic with p has no proper fibrations, then p is interalgebraic with a type that is internal to some non locally modular minimal type.

Recall that *internal to X* means that $p(\mathcal{U}) \subseteq \mathrm{dcl}(B, X)$ for some $B \supseteq A$.

As nmdeg is an interalgebraic-invariant, we reduce to studying types that are internal to a non locally modular minimal type.

The internal case: permutation groups arise

In DCF_0 internal to a non locally modular minimal type means internal to the field of constants.

More generally, suppose that p is internal to an A-(type-)definable minimal set X with:

- X_{ind} admits elimination of imaginaries
- $\text{acl}(A) \cap X$ is infinite.

Because of X-internality, the theory of binding groups comes into play:

$$G := \text{Aut}_A(p/X) \text{ acts definably and faithfully on } S := p(\mathcal{U}).$$

No proper fibrations implies p is also *weakly orthogonal to* X, namely if $a \models p$ then $a \mathop{\smile\!\!\!\!\!\!\mid}_A X$. It follows that (G, S) is transitive and p is isolated. So (G, S) is a definable homogeneous space of finite rank!

Proposition

(G, S) *is generically d-transitive, where* $d = \mathrm{nmdeg}(p)$.

Proof.

Let $r > 0$ be least such that $p^{(r)}$ is not weakly orthogonal to X.
Let $\bar{a} \models p^{(r-1)}$ and $q := p \upharpoonright_{A\bar{a}}$.

Non weak orthogonality gives an $A\bar{a}$-definable proper fibration $f : q(\mathcal{U}) \to X$.

As the generic fibre of f is infinite but of lower rank than p, there are cofinitely many such fibres, and hence one over $\mathrm{acl}(A)$. Such a fibre yields a nonalgebraic forking extension of p over $A\bar{a}$.

So $d \leq r - 1$ and $p^{(d)}$ is weakly orthogonal to X.

Hence $G = \mathrm{Aut}_A(p^{(d)}/X)$ acts transitively on $p^{(d)}$, as desired. □

Theorem (Freitag, M.)

If the Borovik-Cherlin conjecture holds in X_{ind} *then* $\text{nmdeg}(p) \leq U(p) + 1$.

Proof.

Suppose $d \geq n + 2$ where $d = \text{nmdeg}(p)$ and $n = U(p) = \text{RM}(S)$.

So (G, S), the binding group action, which is definable in X_{ind}, is generically $(n + 2)$-transitive.

By Borovik-Cherlin, $(G, S) = \big(\text{PGL}_{n+1}(F), \mathbb{P}^n(F)\big)$.

Fixing distinct $a, b \in S$ we have that $H := \text{Stab}_G(a, b)$ must have an infinite orbit $\mathcal{O}$ on S of rank $< n$; namely, either the line on which a, b lie or its complement.

Writing $\mathcal{O} = Hc$ for some $c \in S$ we get a nonalgebraic forking extension of p over $Aabc$ implying $d \leq 3$. This forces $n = 1$ contradicting the fact that p is not minimal. □

The rank 2 case

Altınel-Wiscons have established the BC conjecture in rank 2. So:

Corollary

If T is such that every non locally modular minimal type is nonorthogonal to a 0-(type-)definable minimal set X that admits elimination of imaginaries and has infinitely many algebraic points, then every rank 2 type in T has degree of nonminimality at most 3.

The case of DCF_0.

Let's repeat:

Theorem

If the Borovik-Cherlin conjecture holds in X_{ind} then $\operatorname{nmdeg}(p) \leq U(p) + 1$.

In many theories of interest (DCF_0, $DCF_{0,m}$, CCM) we know that X_{ind} is a pure algebraically closed field.

Hence the truth of Borovik-Cherlin in ACF_0 implies the bound $\operatorname{nmdeg}(p) \leq U(p) + 1$ in these theories.

But the Borovik-Cherlin conjecture is true in ACF_0, by results of Popov on the generic transitivity of simple algebraic group actions, and the work of Macpherson-Pillay reducing to the simple case (a strategy suggested by Borovik and Cherlin). Hence:

Corollary

In DCF_0 *(and in* $\mathrm{DCF}_{0,m}$ *and* CCM*) the nonminimality degree is at most the U-rank plus* 1*.*

Questions:

- Is there an absolute bound on degree of nonminimality? (All examples constructed so far have nmdeg ≤ 2.) A more careful study of Popov's classification suggests that this should be the case.
  This is the subject of ongoing work with Freitag.
- Is the BC conjecture true in DCF_0? Not just for binding groups (which reduce to ACF_0).
  This is the subject of ongoing work with Freitag and Jiminez.
- What about the infinite rank setting. So "degree of nonregularity"?
- What about ACFA, with a view toward algebraic *difference* equations?