Mock hyperbolic reflection spaces and Frobenius groups of finite Morley rank

Katrin Tent,
joint work with Tim Clausen

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Motivation

A group G action on a set X is called sharply 2-transitive if for any two pairs of distinct elements from X there is a unique element in G taking one pair to the other.

Conjecture

Any sharply 2-transitive group of finite Morley rank is of the form $K \rtimes K^*$ for some algebraically closed field K.

Two parts:

1. Any sharply 2-transitive group of finite Morley rank *splits*, i.e. is of the form $N \rtimes N^*$ for some *nearfield* N.
2. Any nearfield of finite Morley rank is algebraically closed.

The existence of non-split sharply 2-transitive groups was established only recently (Rips, Segev, T.).

The second part is recent a theorem by Altınel, Berkmann and Wagner, leaving part 1.

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Frobenius groups

A group G together with a proper non-trivial subgroup H is a Frobenius group $H < G$ if H is malnormal, i.e. $H \cap H^g = 1$ for all $g \in G \setminus H$.

Theorem (Frobenius, 1901)

If $H < G$ is a finite Frobenius group, then $H < G$ splits, i.e. there is a normal subgroup $N \triangleleft G$ such that $G = N \rtimes H$.

For finite groups this result can be seen as a predecessor for the classification of CA groups, CN groups, and groups of odd order.
There is no known analog for groups of finite Morley rank. However, it is known that the existence of a non-split Frobenius group of finite Morley rank would contradict the Algebraicity Conjecture.

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Frobenius groups of finite Morley rank

Only partial results are known. Let $H < G$ be a Frobenius group of finite Morley rank.

- If G is solvable, then $H < G$ splits.
- If H is finite, then $H < G$ splits.
- If $(G^0 \cap H) < G^0$ splits, then $H < G$ splits.

Here G^0 denotes the connected component of G, i.e. the smallest definable subgroup of finite index. In particular, it suffices to show that connected Frobenius groups split.

Fact

Let G be a connected Frobenius group with (connected) complement H. Then G lies in one of the following mutually exclusive cases:

- H contains a unique involution (G is of odd type);
- G does not contain any involutions (G is of degenerate type);
- $G \setminus \bigcup_{g \in G} H^g$ contains involutions (G is of even type).

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Frobenius groups of small rank

A Frobenius group $H < G$ is *full* if $G = \bigcup_{g \in G} H^g$. A full Frobenius group of finite Morley rank does not contain involutions and cannot split.

Theorem (Frécon, 2016)

There is no simple full Frobenius group of Morley rank 3.

More generally, if $H < G$ is a simple full Frobenius group of Morley rank $n = \mathrm{MR}(G)$ with abelian complement H of rank $k = \mathrm{MR}(H)$, then $n > 2k + 1$ (Wagner).
A similar result for groups with involutions:

Theorem

If G is a non-split sharply 2-transitive group of characteristic $\neq 2$ of Morley rank $2n$ with a maximal near-field of Morley rank k, then $n > 2k + 1$.

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Main results

Both results use geometries with similar properties. We will see that both results can be proved (and generalized) using *mock hyperbolic reflection spaces*, a class of geometries defined on involutions.

Theorem

Let $H < G$ be a connected Frobenius group of finite Morley rank such that H contains an involution and denote the set of involutions by J. Then J forms a mock hyperbolic reflection space and if $\mathrm{MR}(J) \leq 2\mathrm{MR}(\lambda) + 1$ for almost all lines λ, then $H < G$ splits.

If $H < G$ is a connected Frobenius group, H contains an involution, and $\mathrm{MR}(G) \leq 10$, then either $H < G$ splits or G is a simple non-split sharply 2-transitive group of characteristic $\neq 2$ and $\mathrm{MR}(G)$ is either 8 or 10.

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Main results II

Theorem

Let $H < G$ be a connected Frobenius group of Morley rank n without involutions and let H be an abelian Frobenius complement of Morley rank k. Then $n > 2k$. If $n = 2k+1$, then $H < G$ splits as $G = N \rtimes H$ and if N is solvable, then there is an interpretable field K of characteristic $\neq 2$ such that $G = K_+ \rtimes H$, $H \leq K^*$, and H acts on K_+ by multiplication.

Mock hyperbolic reflection spaces

Let G be a group and let J be the set of its involutions. We view involutions as points and the action of an involution (by conjugation) as a point-reflection.

Definition

J forms a *mock hyperbolic reflection space* if the following three axioms are satisfied:

- J forms a linear space such that three distinct involutions are collinear iff their product is an involution. In other words, two involutions $i \neq j$ determine the line
$$\ell_{ij} = \{k \in J : ij \in kJ\}$$
and any two involutions are contained in a unique line.
- Midpoints exist and are unique, i.e. given $i, j \in J$ there is a unique $k \in J$ such that $i^k = j$.
- Given two lines $\lambda \neq \delta$ there is at most one $i \in J$ such that $\lambda^i = \delta$.

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Examples

1. Consider $\mathrm{PSL}_2(\mathbb{R})$, the group of orientation preserving isometries of the real hyperbolic plane. The involutions in $\mathrm{PSL}_2(\mathbb{R})$ can be identified with the hyperbolic plane and hence they form a mock hyperbolic reflection space.
2. There exist non-split sharply 2-transitive groups of characteristic 0. The involutions in the examples constructed by Rips and myself form a mock hyperbolic reflection space.
3. Examples can be constructed from uniquely 2-divisible full Frobenius groups with abelian Frobenius complement.
4. Let A be a uniquely 2-divisible abelian group and let $\epsilon : A \to A$ be defined by $a \mapsto a^{-1}$. Then the involutions in $A \rtimes \langle \epsilon \rangle$ are given by $J = A \times \{\epsilon\}$ and J forms a mock hyperbolic reflection space consisting of a single line.

Fact

Every finite mock hyperbolic reflection space consists of a single line.

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The geometry of mock hyperbolic reflection spaces

Let G be a group such that $J \subset G$ forms a mock hyperbolic reflection space.

Proposition

The following are equivalent:

1. $J^2 = iJ$ for any involution $i \in J$;
2. J^2 is an abelian group;
3. J consists of a single line.

If λ is a line in J, then $N_G(\lambda) \cap J = \lambda$ and hence λ forms a mock hyperbolic reflection space in $N_G(\lambda)$ that consists of a single line.

Proposition

$G = iJ.\mathrm{Cen}(i)$ for any involution $i \in J$, i.e. $g \in G$ can be uniquely written as $g = ijh$ with $h \in \mathrm{Cen}(i)$ and $j \in J$.
In particular, if the geometry is trivial, then $G = iJ \rtimes \mathrm{Cen}(i)$ splits.

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$G = iJ.\mathrm{Cen}(i)$ for any involution $i \in J$, i.e. $g \in G$ can be uniquely written as $g = ijh$ with $h \in \mathrm{Cen}(i)$ and $j \in J$.
In particular, if the geometry is trivial, then $G = iJ \rtimes \mathrm{Cen}(i)$ splits.

The geometry of mock hyperbolic reflection spaces

Let G be a group such that $J \subset G$ forms a mock hyperbolic reflection space.

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The following are equivalent:

1. $J^2 = iJ$ for any involution $i \in J$;
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Generic projective planes

Now assume that G is a group of finite Morley rank. We also assume that $\mathrm{MR}(J) = n$, $\mathrm{MD}(J) = 1$, $\mathrm{MR}(\lambda) = k$, and $\mathrm{MD}(\lambda) = 1$ for all lines λ.

Definition

A definable subset $X \subseteq J$ is a *generic projective plane* if

1. $\mathrm{MR}(X) = 2k$ and $\mathrm{MD}(X) = 1$, and
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where Λ_X is the set of all lines $\lambda \subseteq J$ such that $\mathrm{MR}(\lambda \cap X) = k$.

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There is no generic projective plane $X \subseteq J$.

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Idea: There is a subgroup $H \leq G$ such that $H \cap J = X$. A line $\lambda \subseteq X$ intersects rank $2k$ many lines in X. The family $\{\lambda^i : i \in X \setminus \lambda\}$ consists of rank $2k$ many lines in X that do not intersect λ. This contradicts $\mathrm{MD}(\Lambda_X) = 1$. □

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A rank inequality

Suppose J is a proper mock hyperbolic reflection space, i.e. $n > k$. Then $n > 2k$. ($n = \mathrm{MR}(J)$, $k = \mathrm{MR}(\lambda)$)

Proposition

If $n = 2k + 1$, then G has a definable connected normal subgroup N of rank $2n - k$ such that $J^2 \subseteq N$ is a generic subset of N.

Proof

Assume iJ is indecomposable. Then it suffices to show that $\mathrm{MR}(J^3) = \mathrm{MR}(J^2)$ and we can put $N = \langle iJ \rangle$.
Put $m = \mathrm{MR}(J^3) - \mathrm{MR}(J^2)$. We aim to show $n > 2k + m$.
Let $\mu : J \times J \times J \to J^3$ be the multiplication map. For $\alpha \in J^3$ we set

$$X_\alpha = \{i \in J : \exists r, s : irs = \alpha\}.$$

Since $\mathrm{MR}(J^3) = 2n - k + m$, there must be $\alpha \in J^3 \setminus J$ such that $\mathrm{MR}(\mu^{-1}(\alpha)) \leq n + k - m$. Put $X = X_\alpha$ for such an α. Then $\mathrm{MR}(X) \leq n - m$.

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Proof (cont.)

We now aim to show that $2k < \mathrm{MR}(X)$. If $irs = \alpha$ and $v \in \ell_{rs}$, then $\ell_{iv} \subseteq X$: We have $irs = ivu$ for some $u \in \ell_{rs}$ and moreover for each $p \in \ell_{iv}$ there is some $q \in \ell_{iv}$ such that $pq = iv$ and hence $pqu = ivu = irs = \alpha$. Hence each point in X is contained in Morley rank k many lines which are contained in X. Hence X must have Morley rank at least $2k$.

Now assume $\mathrm{MR}(X) = 2k$. Let Λ be the set of lines obtained as above. For each $x \in X$ the set $\{\lambda \in \Lambda : x \in \lambda\}$ has Morley rank k and Morley degree 1. This allows to extract a generic projective plane and yields a contradiction.

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Frobenius groups with involutions

Let $H < G$ be a connected Frobenius group of finite Morley rank such that H contains an involution.

Proposition

The set of involutions J forms a mock hyperbolic reflection space and all lines in J are infinite.

Proof

Let $i \in H$ be the unique involution. Then $H = \mathrm{Cen}(i)$ and $J = i^G$.
If $a \in J^2 \setminus \{1\}$, then $\mathrm{Cen}(a) \cap \mathrm{Cen}(i) = \{1\}$. In particular, $\mathrm{Cen}(a)$ is uniquely 2-divisible.
If $a = ij$, then $a^i = a^{-1}$ and hence i acts on $\mathrm{Cen}(a)$ as an involutory automorphism without fixed points. Therefore $\mathrm{Cen}(a)$ is abelian and inverted by i. In particular, $\mathrm{Cen}(a) \subseteq iJ$.
This shows that iJ is uniquely 2-divisible and hence J has *unique midpoints* (since $(it)^2 = ii^t$).

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Let $H < G$ be a connected Frobenius group of finite Morley rank such that H contains an involution.

Proposition

The set of involutions J forms a mock hyperbolic reflection space and all lines in J are infinite.

Proof

Let $i \in H$ be the unique involution. Then $H = \mathrm{Cen}(i)$ and $J = i^G$.

If $a \in J^2 \setminus \{1\}$, then $\mathrm{Cen}(a) \cap \mathrm{Cen}(i) = \{1\}$. In particular, $\mathrm{Cen}(a)$ is uniquely 2-divisible.

If $a = ij$, then $a^i = a^{-1}$ and hence i acts on $\mathrm{Cen}(a)$ as an involutory automorphism without fixed points. Therefore $\mathrm{Cen}(a)$ is abelian and inverted by i. In particular, $\mathrm{Cen}(a) \subseteq iJ$.

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Let $n = \mathrm{MR}(J)$. If J consists of a unique line, then G splits as $G = iJ \rtimes H$. Hence we may assume that almost all lines are of rank k for some $1 \leq k < n$.

Theorem

If $n \leq 2k + 1$, then J consists of a unique line and $H < G$ splits.

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Assume that all lines are of rank k. We may assume $n = 2k + 1$. Then G has a connected normal subgroup N of rank $2n - k$ such that $J^2 \subseteq N$ is a generic subset. Since $G = iJ.\mathrm{Cen}(i)$ the subgroup $N \cap \mathrm{Cen}(i)$ must have rank $n - k > 0$. Then $N \cap H < N$ is again a Frobenius group and hence $\bigcup_{g \in G}(N \cap H^g)$ is generic in N. This contradicts $J^2 \cap H = 1$. □

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Frobenius groups without involutions

To apply the previous results to Frobeniusgroups without involutions, we have to extend the group.

A groupoid $(L, \cdot, 1)$ is a *K-loop* if

1. it is a loop, i.e. the equations

$$ax = b \quad \text{and} \quad xa = b$$

   have unique solutions for all $a, b \in L$,

2. it satisfies the Bol condition, i.e.

$$a(b \cdot ac) = (a \cdot ba)c$$

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Given $a \in L$ let $\lambda_a : L \to L$ be defined by $\lambda_a(x) = ax$. Given $a, b \in L$ we define the precession map

$$\delta_{a,b} = \lambda_{ab}^{-1}\lambda_a\lambda_b.$$

These maps are characterized by

$$a \cdot bx = ab \cdot \delta_{a,b}(x) \quad \text{for all } x \in L.$$

If L is a K-loop, then the precession maps are automorphisms and we set

$$\mathcal{D} = \mathcal{D}(L) = \langle \delta_{a,b} : a, b \in L \rangle \leq \mathrm{Aut}(L).$$

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Let L be a K-loop and let $\mathcal{A} \leq \mathrm{Aut}(L)$ a group of automorphisms such that $\mathcal{D}(L) \subseteq \mathcal{A}$. Then the quasidirect product $L \rtimes_Q \mathcal{A}$ given by the set $L \times \mathcal{A}$ together with the multiplication

$$(a, \alpha)(b, \beta) = (a \cdot \alpha(b), \delta_{a,\alpha(b)}\alpha\beta)$$

forms a group with neutral element $(1, \mathrm{id})$. Inverses are given by

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Let G be a uniquely 2-divisible group . Then

$$a \otimes b = a^{1/2} b a^{1/2}$$

makes G into a K-loop $L = (G, \otimes, 1)$ and integer powers of elements in L agree in G and L. Moreover, given $a, b \in G$ the precession map $\delta_{a,b}$ is given by conjugation with

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Now let $H < G$ be a connected Frobenius group of finite Morley rank without involutions such that H is abelian. Set $n = \mathrm{MR}(G)$ and $k = \mathrm{MR}(H)$. Note that G is uniquely 2-divisible and centerless.
Set $L = (G, \otimes, 1)$ and define $\epsilon \in \mathrm{Aut}(L)$ by $\epsilon(x) = x^{-1}$. Note that ϵ is central in $\mathrm{Aut}(L)$. In particular, $G \times \langle \epsilon \rangle \leq \mathrm{Aut}(L)$.
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Proposition

If $H < G$ is full, then J forms a mock hyperbolic reflection space and all lines are of rank k.
In general, J forms a *generic mock hyperbolic reflection space*, i.e. we only require that almost all lines exist.

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Now show that $G = (N \cap G)^0 \rtimes H$. □

Frobenius groups without involutions VI

Theorem

Let $H < G$ be a connected Frobenius group of Morley rank n without involutions and let H be an abelian Frobenius complement of Morley rank k. Then $n > 2k$. If $n = 2k+1$, then $H < G$ splits as $G = N \rtimes H$ and if N is solvable, then there is an interpretable field K of characteristic $\neq 2$ such that $G = K_+ \rtimes H$, $H \leq K^*$, and H acts on K_+ by multiplication.

Proof.

Assume $H < G$ is full. We only show that $H < G$ splits.
Set $\mathcal{G} = L \rtimes_Q (G \times \langle \epsilon \rangle)$ where $L = (G, \otimes, 1)$.
$\mathcal{G}$ has a normal subgroup N of rank $2n - k$ containing J^2.
Fix an involution $i \in J$. Then $\mathrm{Cen}(i) \cong G \times \langle \epsilon \rangle$ and $N \cap \mathrm{Cen}(i)$ must have rank $n - k$ because $\mathcal{G} = iJ\mathrm{Cen}(i)$.
Now show that $G = (N \cap G)^0 \rtimes H$. □