Minimal representations of $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$

Joshua Wiscons

California State University, Sacramento

Groupes rangés : le retour

Institut Camille Jordan

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Joint work with T. Altınel (Lyon), B. Chin (Sacramento), L.J. Corredor (Bogotá), A. Deloro (Paris), A. Yu (Sacramento)

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Outline: minimal representations of $\mathsf{Sym}(n)$ and $\mathsf{Alt}(n)$

1. Questions and motivation
2. The abelian case
   - Context (new and general)
   - Results
3. The nonsolvable $2^\perp$ case
   - Context (finite Morley rank)
   - Results

Questions and motivation
(and digressions)

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Generic n-transitivity

Definition

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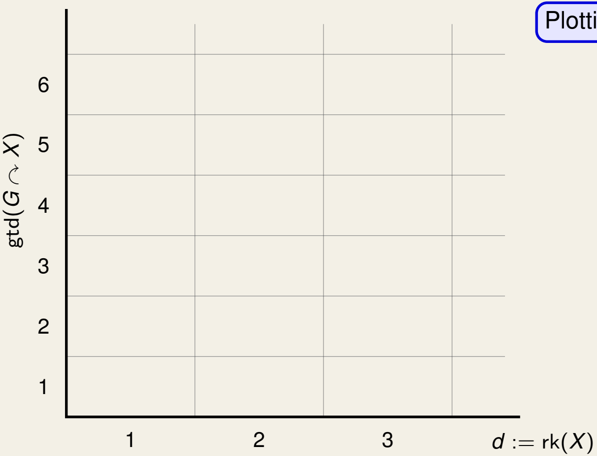
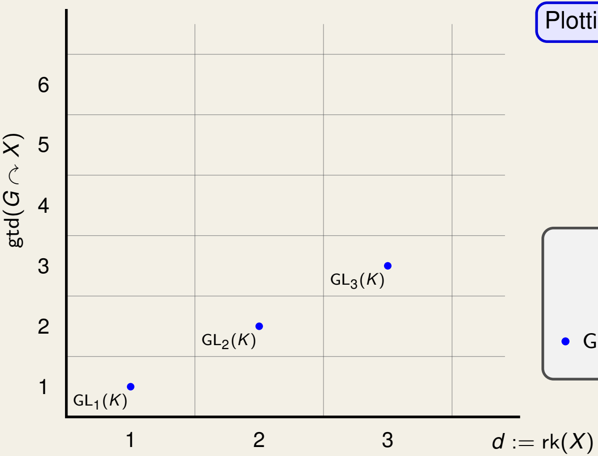
Bounding gtd



Plotting $G \curvearrowright X$ of fMr

Bounding gtd



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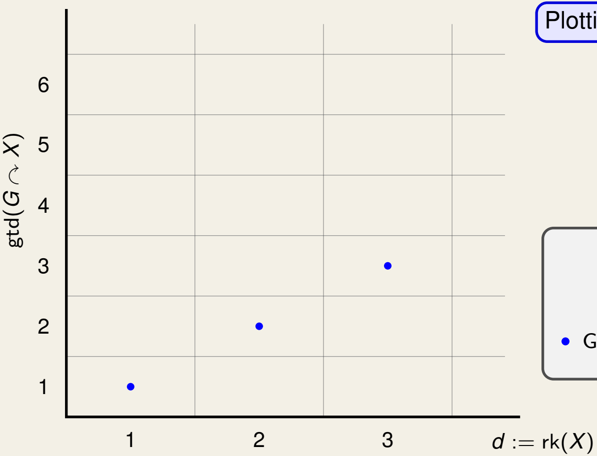

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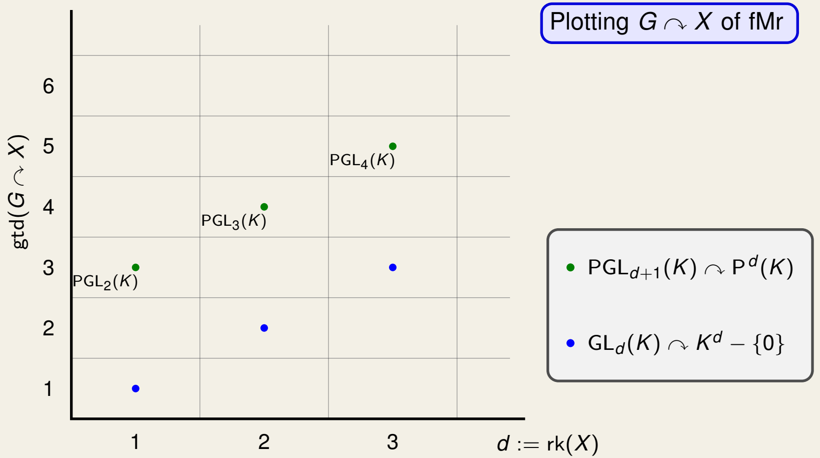

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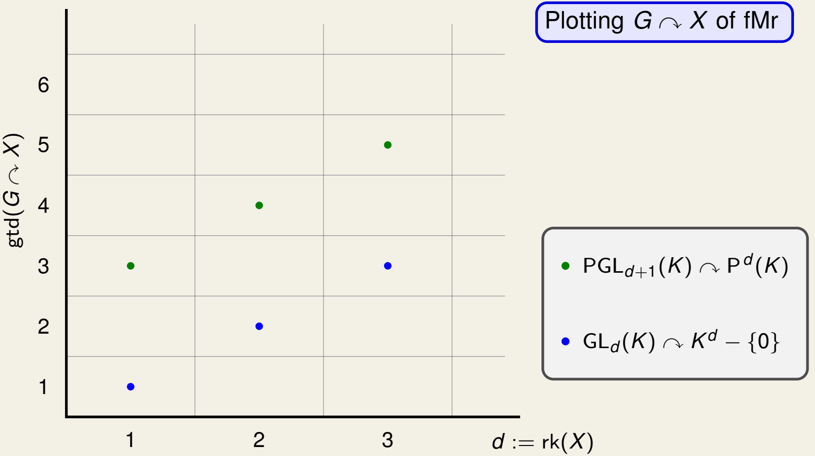

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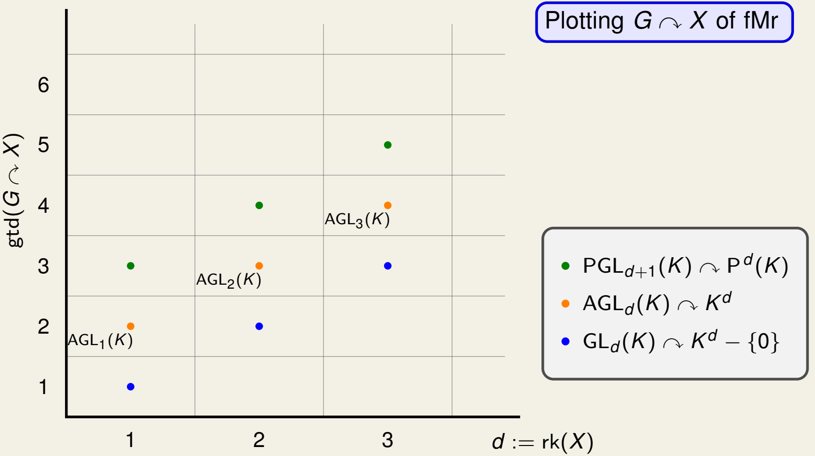

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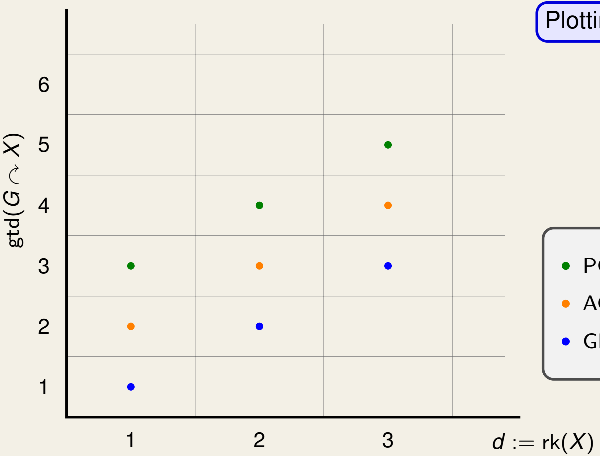

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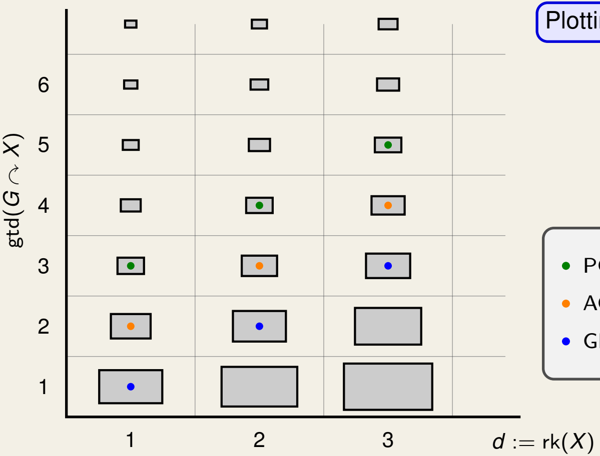

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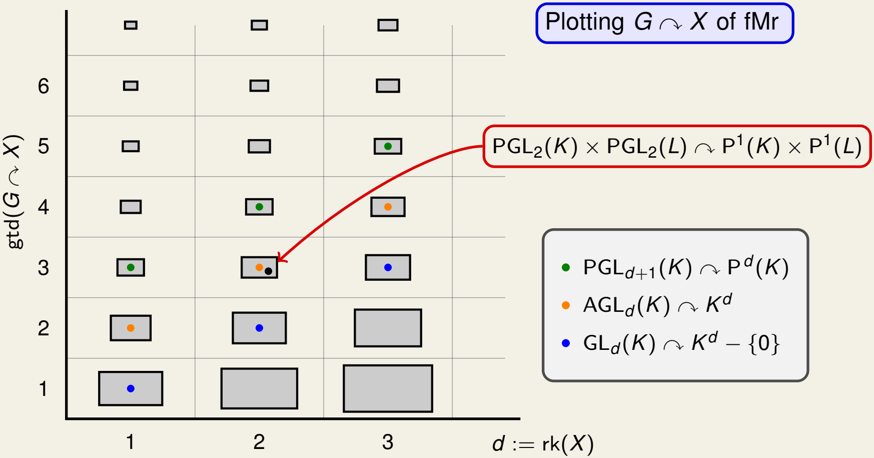

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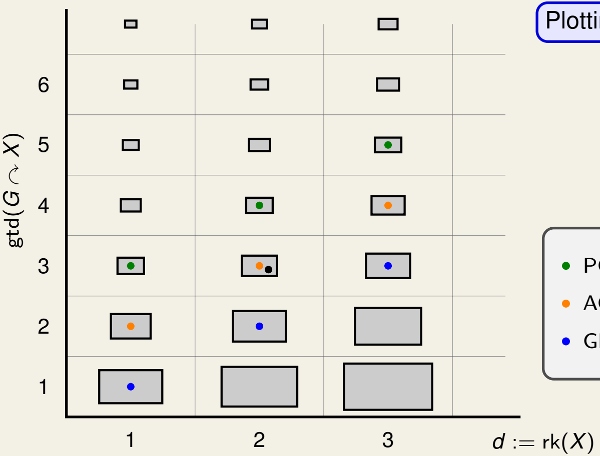

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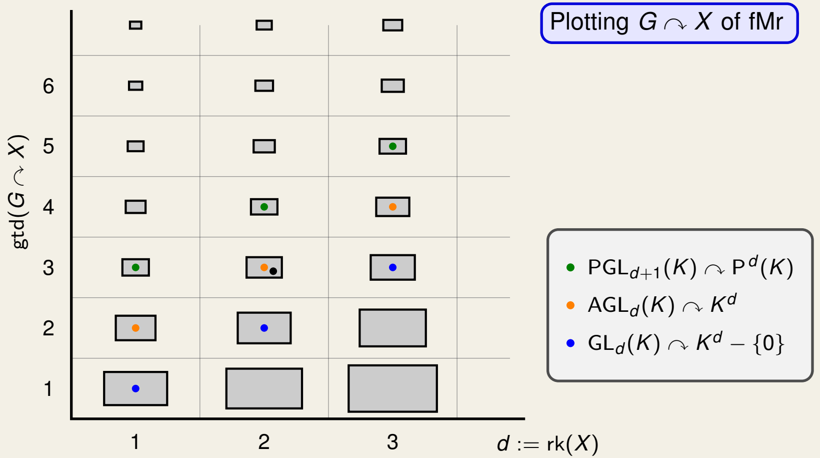

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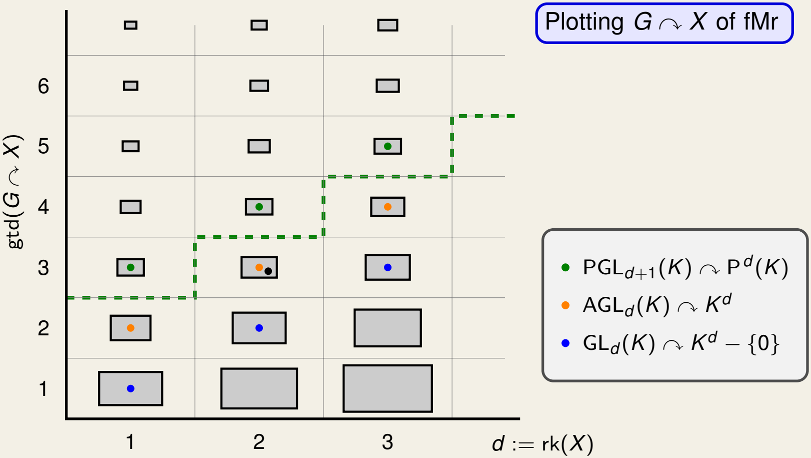

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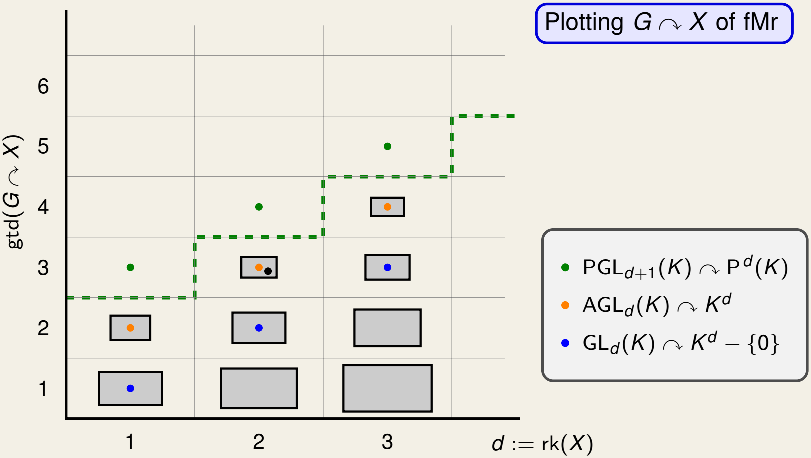

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So we turn to the study of $\mathrm{Sym}(n)$-representations.

The abelian case

Context: modules with an additive dimension

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- $\mathcal{U}_{\mathsf{D}}(V)$ is the collection of all sets definable/interpretable from V.

We need $\mathcal{U}_{\mathsf{A,D}}(V) := \mathcal{U}_{\mathsf{A}}(V) \cap \mathcal{U}_{\mathsf{D}}(V)$ (or less) to carry a dimension.

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Remark

If V is an abelian group (possibly with extra structure), then $\mathcal{U}_{\mathrm{A,D}}(V) = \mathcal{U}_{\mathrm{A}}(V) \cap \mathcal{U}_{\mathrm{D}}(V)$ (with definable morphisms) is a modular universe.

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An additive dimension on a modular universe $\mathcal{U}$ is a function $\mathtt{dim} : \mathrm{Ob}(\mathcal{U}) \to \mathbb{N}$ such that if $f : A \to B$ is in $\mathrm{Ar}(\mathcal{U})$ (i.e. f is a compatible morphism), then

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- We say nothing about chain conditions.
- We also say nothing about elementary extensions.

Examples of groups with an additive dimension

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Dc-irreducible modules always have a characteristic.

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The abelian case

Results

The standard module for Sym(n)

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- $\overline{\mathrm{std}}(n, L) = \mathrm{std}(n, L)/(\mathrm{std}(n, L) \cap \mathrm{triv}(n, L))$

The standard module for $\mathsf{Sym}(n)$

Definition (Standard Module)

Let $\mathsf{perm}(n) = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n$ be the $\mathsf{Sym}(n)$-module where the e_i are permuted naturally. There are two obvious submodules:

- $\mathsf{triv}(n) = \{ce_1 + \cdots + ce_n\}$
- $\mathsf{std}(n) = \{c_1 e_1 + \cdots + c_n e_n \mid \sum c_i = 0\}$

For any abelian group L, we define:

- $\mathsf{perm}(n, L) = \mathsf{perm}(n) \otimes_{\mathbb{Z}} L = \{e_1 \otimes a_1 + \cdots + e_n \otimes a_n\}$
- $\mathsf{triv}(n, L) = \mathsf{triv}(n) \otimes_{\mathbb{Z}} L = \{e_1 \otimes a + \cdots + e_n \otimes a\}$
- $\mathsf{std}(n, L) = \mathsf{std}(n) \otimes_{\mathbb{Z}} L = \{e_1 \otimes a_1 + \cdots + e_n \otimes a_n \mid \sum a_i = 0\}$
- $\overline{\mathsf{std}}(n, L) = \mathsf{std}(n, L)/(\mathsf{std}(n, L) \cap \mathsf{triv}(n, L))$

Remark

Notice that $\mathsf{std}(n, L) \neq \overline{\mathsf{std}}(n, L) \iff \Omega_n(L) \neq 0$.

The standard module for Sym(n)

Remarks

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1. If L is a module with an additive dimension, then $\mathrm{perm}(n, L)$, $\mathrm{std}(n, L)$, and $\overline{\mathrm{std}}(n, L)$ are $\mathrm{Sym}(n)$-modules in the same modular universe.

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Theorem (Corredor-Deloro-W 2018–2021)

Let $n \geq 7$. Suppose V is a faithful, dc-irreducible Sym(n)*-module with an additive dimension. Set $q :=$ char V and $d :=$ dim V.*

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The same is true for Alt(n)*-modules provided $n \geq 10$ when $q = 2$.*

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- One may take $L = B_{(12)}$, placing all relevant objects (including φ) in the modular universe of V.
- We say nothing about the dimension of V.

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We again say nothing about the dimension of V.

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Remark

The proof of the main theorem is then more-or-less assembled as:

$$\text{Geometrization} \to \text{Extension} \to \text{Recognition}$$

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2. The Theorem assumes $d < n$; can this be relaxed? One expects to not encounter the "second smallest" modules until $d \approx \binom{n}{2}$.

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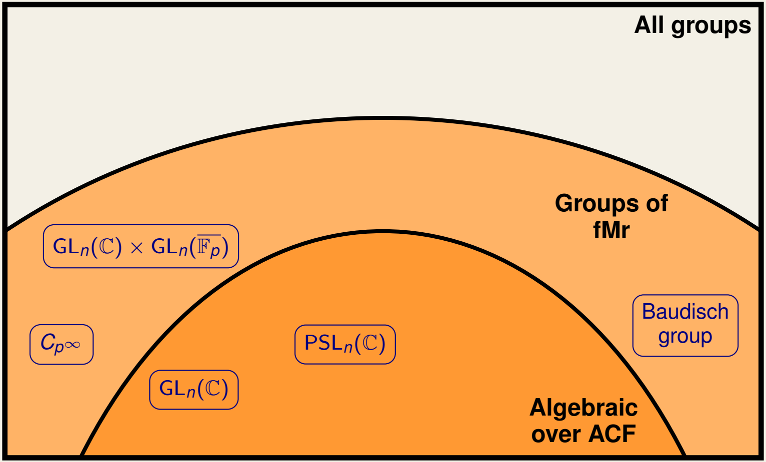
The nonsolvable $2^\perp$ case

Context: groups of finite Morley rank

Groups of finite Morley rank



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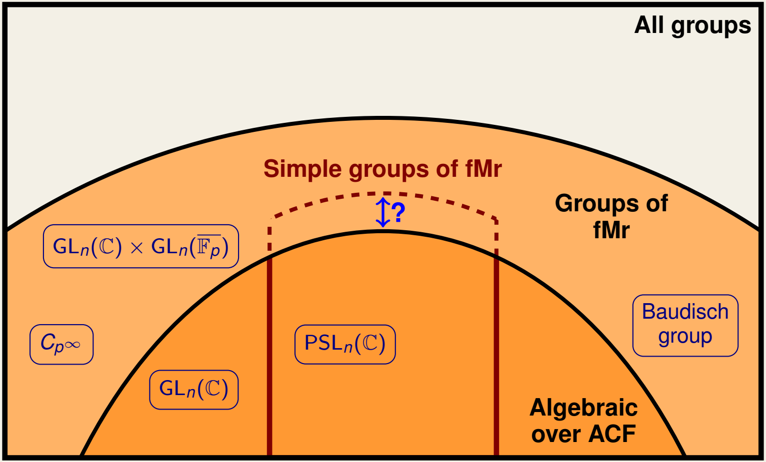

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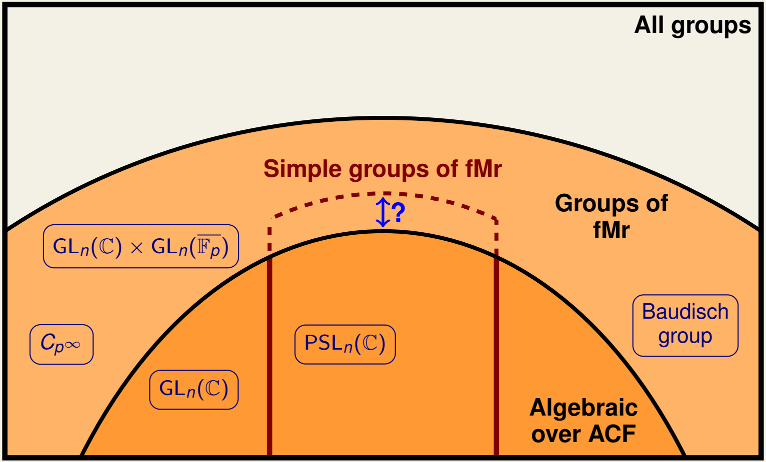

Algebraicity Conjecture:

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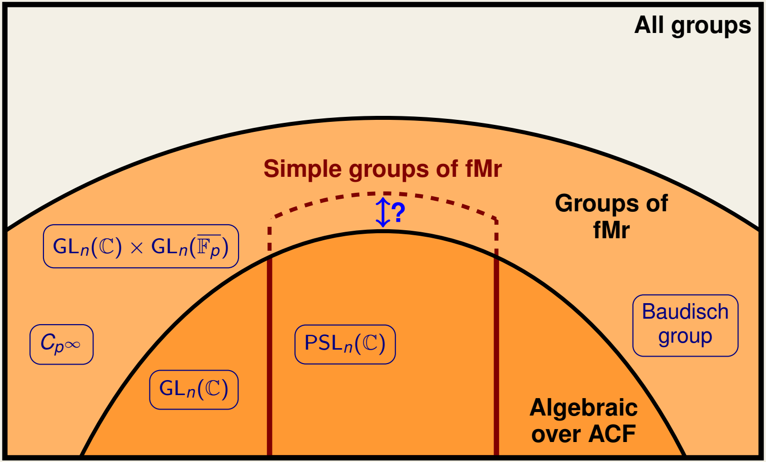

Algebraicity Conjecture: the gap, $\updownarrow$, does not exist.

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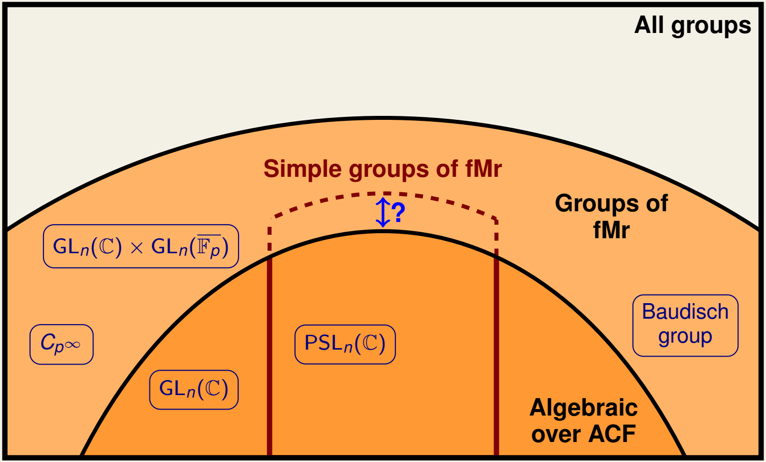

Algebraicity Conjecture: every simple group of fMr is algebraic over an ACF.

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This provides another point of view on our work exploring representations of $\mathrm{Sym}(n)$ and $\mathrm{Alt}(n)$ on nonsolvable $2^\perp$-groups of fMr.

The nonsolvable $2^{\perp}$ case

Results (in progress)

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In both cases, the abelian case (including finite covers) seems relevant.

Le retour