Dimensional Quasi-Frobenius Groups

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Motivation : a specific geometry of involutions

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- Work of J.Wiscons and A.Deloro about generalisations of the geometry of involutions in $SO_3(\mathbb{R})$.
- Involutions form a projective plane in $SO_3(\mathbb{R})$ (half-turns) but the axioms are satisfied only "generically" in $PGL_2(\mathbb{C})$.
- In terms of the internal structure, we have a subgroup $C \leq C_G(i)$ for an involution i such that:
  1. $N_G(C) = C \rtimes \langle w \rangle$ for an involution w acting by inversion.
  2. $\bigcup_G C^g$ is "generic" in the ambiant group.

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  1. Let $C < G$ be a pair of (definably) connected groups definable in an o-minimal structure. Suppose G contains involutions and C is of finite index in $N_G(C)$ and intersects trivially any distinct conjugate. If $\bigcup_G C^g$ contains all translations, i.e., products of involutions, is $G \simeq \mathrm{SO}_3(R)$ for a real closed field R ?
  2. Let $C < G$ be a pair of ranked connected groups. Suppose G contains involutions and is $U_2^\perp$, and C is of finite index in $N_G(C)$ and intersects trivially any distinct conjugate. Is $G \simeq \mathrm{PGL}_2(K)$ for an algebraically closed field K ?

Motivation : connected Frobenius group

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- In a ranked universe we conjecture that a Frobenius group $C < G$ splits, i.e., $G = U \rtimes C$ for a definable subgroup U. If not, we will have a counter-example to the Cherlin-Zilber conjecture.
- A ranked sharply 2-transitive group, in particular $K^+ \rtimes K^\times$ for a ranked field K, is a good example of a connected Frobenius group with involutions.

We work with dimensional groups, i.e., groups definable in a structure such that definable subsets (taken in cartesian products) carry a dimension *dim* satisfying the following axioms :

- Let $f : A \to B$ be a definable function between definable sets A, B. Then $\{b \in B : f^{-1}(b) = n\}$ is definable.

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We shall call a definable subset $X \subseteq G$ *(weakly) generic* if $dim(X) = dim(G)$.

- We require also the DCC on definable subgroups in order to define the connected component.
- Groups definable in o-minimal structures and ranked groups are dimensional groups (with DCC).
- We have a form of elimination of imaginaries in groups definable in o-minimal structures. Moreover, if G is a ranked group, then G^{eq} is ranked. We can make use of the dimension function for interpretable subsets.

Definitions

Definition

- Let $C < G$ be a pair of definable dimensional connected groups. We say $C < G$ is a *Quasi-Frobenius group* (QF) if C is of finite index in $N_G(C)$ (almost selfnormalising) and for all $g \notin N_G(C)$, we have $C^g \cap C = \{1\}$ (TI).

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- We speak of connected Frobenius group if $N_G(C)/C$ is trivial, of *Even Quasi-Frobenius (EQF)* if $N_G(C)/C$ is even and of *Odd Quasi-Frobenius* (OQF) if it is odd.

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Identification of $SO_3(R)$

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Proposition

The group $C < G$ is a (EQF), and there exists an involution $i \in N_G(C) \setminus C$.

Proof.

(sketch) Let $x = ij$ be a translation for $i \in C$ and j in $C_j \neq C$.
Then $i, j \in N_G(C_G(x)^\circ) \leq N_G(C_x)$.
In order to obtain an involution, we consider a 2-element α of minimal order such that $1 \neq \alpha^2 \in C$. □

Proposition

G is semi-simple, i.e., $Z(G)$ is finite and there is no abelian normal infinite subgroup.

Proof.

Let A be a normal abelian infinite subgroup; considering $Z(C_G(A))^\circ$, we can suppose it is definable and connected. If there exists $1 \neq c \in A \cap C$, then $A \leq C_G(c)^\circ \leq C$ and so $C = G$, a contradiction.

Let i be an involution acting on A : this action is by inversion (the definable subgroup $\{a \in A : a^i = a^{-1}\}$ is weakly generic so equals A by connectedness). The subgroup A is 2-divisible, so $A \leq I.I$, a contradiction. □

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- We suppose from now on that $C < G$ is definable in an o-minimal structure.

Fact

If G is definably simple, then :

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2. *(Peterzil, Pillay and Starchenko) If a field K is interpretable in G, then G and K are bi-interpretable.*
3. *G is definably isomorphic to the semi-algebraic connected component of a algebraic group defined over a real closed field (bi-interpretable with G) or to the k-rational points of a linear algebraic group defined over an algebraically closed field k (also bi-interpretable with G).*

Bi-interpretability is with respect to the pure algebraic structure.

We suppose now $Z(G) = \{1\}$ and C is nilpotent.

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Proposition

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Using Bachman's theorem, we can define a field k. This field is real closed and bi-interpretable with $C < G$ (at this point, we need the definability of the connected component in the pure group structure, see Frécon).

Theorem

Let $C < G$ be a (QF) definable in an o-minimal structure. Suppose G contains involutions and $\bigcup_G C^g$ contains all 2-elements and translations. Then $C < G$ is semi-simple. If moreover C is nilpotent and $Z(G)$ is trivial, then $G \simeq SO_3(R)$ for a real closed field R.

Analysis of ranked Quasi-Frobenius groups

Two general lemmas :

Lemma

(Conjugation of Quasi-Frobenius complements) Let $C < G$ be a (QF) and H a definable and connected subgroup such that $H \cap C \neq \{1\}$. Then $H \cap C$ is infinite, and $(H \cap C) < H$ is a (QF). Moreover, for a conjugate C' of C, such that $(H \cap C') \neq \{1\}$, there exists $h \in H$ such that $(H \cap C')^h = (H \cap C)$.

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Lemma

Let $C < G$ be a solvable (QF). Then $G' = F(G)^\circ$ and $G = G' \rtimes C$, with $C \leq K^\times$ for an interpretable field K.

Structure of the 2-torsion

We shall now consider ranked (QF) groups with involutions and $U_2^\perp$ (odd type).

Fact

(Borovik-Nesin) Let $C < G$ be a connected Frobenius group of odd type. Then C contains an unique involution.

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Fact

(Deloro-Wiscons) Let $C < G$ be an (EQF) of odd type. Then all 2-elements are in $\bigcup_G C^g$, $N_G(C) = C \rtimes \langle w \rangle$ for w an involution acting by inversion and C contains a unique involution.

Fact

(Deloro-Jaligot) Let G be a connected group of odd type. Suppose the Prüfer rank is equal to one. Let S be a 2-Sylow subgroup. Then one of the following three possibilities holds :

1. $S = S^\circ$.
2. *($PGL_2(\mathbb{C})$ type)* $S = S^\circ \rtimes \langle w \rangle$ *with w an involution acting by inversion.*
3. *($SL_2(\mathbb{C})$ type)* $S = S^\circ \cdot \langle w \rangle$ *with w a 4-element such that* $w^2 = i \in S^\circ$.

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Proposition

Let $C < G$ a connected Frobenius group of odd type. Then its 2-Sylow subgroups are of type $SL_2(\mathbb{C})$, or connected.

Proposition

(Splitting criterion) Let $C < G$ be an (QF) of odd type with an abelian normal infinite subgroup. Then $G = A \rtimes C$ for an abelian normal definable subgroup A.

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Proof.

Replacing A with $d(A)^\circ$, we can suppose A is definable and connected. We see that $A \cap \bigcup_G C^g = \{1\}$ and all involutions act on A by inversion. Replacing A by $C_G(A)^\circ$ (an abelian group) if necessary, we can suppose $A = C_G(A)^\circ$.
For an involution i, we have $[i, G] \leq C_G(A)^\circ = A$. Let $g \in G$, and consider $[i, g] \in A$. By 2-divisibility, there exists $a \in A$ such that $a^2 = [i, g]$:

$$[i, ga^{-1}] = iag^{-1}iga^{-1} = a^i[i, g]a^{-1} = a^{-1}a^2a^{-1} = 1$$

Finally, we obtain $G = A \rtimes C_G(i) = A \rtimes C$. □

Proposition

Let $C < G$ be an (EQF) of odd type and $(C \cap H) < H$ a definable solvable sub-(QF). Then $(C \cap H) < H$ is a connected Frobenius group (and not only a (QF)).

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Corollary

Let $C < G$ be an (EQF) of odd type. Then it is not solvable.

The Weyl group

The group $N_G(C)/C$ can be viewed as a form of Weyl group. Following the methods of (Altinel, Burdges and Frécon), we can obtain the following result (Frattini arguments) :

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Proposition

Let $C < G$ a (QF) of odd type with C solvable. Let T be a maximal decent torus, Q a generous Carter subgroup and S a maximal p-torus. Then
$N_G(C)/C \simeq N_G(T)/C_G(T) \simeq N_G(Q)/Q \simeq N_G(S)/C_G(S)$.
Moreover, a (OQF) of odd type with solvable complement has Prüfer rank at least equal to two.

A standard" Borel

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Definition

Let B be a Borel subgroup of a (QF) $C < G$ of odd type. It is *standard"* if B has infinite intersection with C, it is standard' if it additionally contains an involution, and *standard* if $C < B$.

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Lemma

(adapted from Deloro-Wiscons) Let $C < G$ be a (QF) of odd type. Let x a translation and i an involution inverting x. If C is $\langle x, i\rangle$-invariant then x is contained in $\bigcup_G C^g$.

Theorem

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Proof.

(sketch) Let $x = ij$ be a translation not contained in $\bigcup_G C^g$ and consider $A = C_G(x)^\circ$. Using the preceding lemma and fact, we can see that A is an abelian group inverted by i and j such that $A \cap \bigcup_G C^g = \{1\}$. Taking A maximal for these properties, we can also prove that A is not an almost selfnormalising group, $C_G(A)^\circ = A$ and A is TI (for the last property, we consider $x_0 \in A \cap A^g \leq C_G(x_0)^\circ$).
We can work as before to obtain $N_G(A)^\circ = A \rtimes (A \cap C)$. It suffices now to consider a Borel subgroup containing $N_G(A)^\circ$. □

If we suppose that $N_G(A)^\circ$ contains involutions, we can precisely identify a standard' Borel subgroup.

Proposition

The subgroup $N_G(A)^\circ = B$ is a Borel subgroup. Moreover, for $1 \neq g \in G \setminus N_G(B)$, $(B \cap B^g)^\circ$ is trivial or an infinite torus, i.e., an abelian divisible group.

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- In a connected Frobenius group of odd type, for any translation $x = ij$, we have : $C_G(x)$ is abelian 2-divisible and $(N_G(C_G(x)) \cap C_i)$ is infinite.

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- In a connected Frobenius group of odd type, for any translation $x = ij$, we have : $C_G(x)$ is abelian 2-divisible and $(N_G(C_G(x)) \cap C_i)$ is infinite.
- If we deal with a sharply 2-transitive group of odd type, the situation is even better : $N_G(C_G(x)) = K^+ \rtimes K^\times$.

Borel subgroup and generosity

Lemma

Let $C < G$ be a (QF) such that C is abelian (or nilpotent under some additional technical hypothesis). If a standard" Borel subgroup B is generous then it is standard.

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Proof.

We have $B = B' \rtimes (B \cap C)$ and $(B \cap C) \leq Z(C)$. As $(B \cap C)$ is a (QF)-complement, it is generous in B. But B is generous (in G) so $X = \{b \in B : \text{b is contained in a finite number of conjugates of B}\}$ is generic in B. There exists $1 \neq c \in C \cap X$ (conjugating C if necessary) but $c \in B \cap B^{c'}$ for all $c' \in C$.

Finally, $N_G(B) \cap C$ is generic in C, and $C \leq N_G(B)^\circ = B$. □

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Finally, $N_G(B) \cap C$ is generic in C, and $C \leq N_G(B)^{\circ} = B$. □

Remark : If B is generically disjoint from its conjugates, it is a generous subgroup. Nevertheless, in $\mathrm{PGL}_2(K)$, a Borel subgroup is generous but it is not generically disjoint from its conjugates.

Classification theorems

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Theorem

Let $C < G$ be a connected Frobenius group of odd type such that Borel subgroups are generous (respect. contains a standard Borel subgroup). Suppose also C is nilpotent (with some technical hypothesis). Then G is solvable and splits.

Two key lemmas

Lemma

Let $C < G$ be a (QF) of odd type with C solvable. Let B a standard' Borel subgroup, $B = B' \rtimes (B \cap C_i)$ and $i \in B$. Suppose $B \cap i^G = N_G(B) \cap i^G$. Then one of the following possibilities holds:

- *G is solvable.*
- *$i^G \setminus N_G(B)$ and $K_B = \{k \in i^G \setminus N_G(B) : RM(T_B(k)) \geq RM(B) - RM(C_G(i))\}$ are generic in i^G, where $T_B(k) = \{b \in B : b^k = b^{-1}\}$ for an involution k not normalising B.*

Proof.

(sketch) Suppose $B \cap i^G$ is generic in i^G and consider $N = \bigcap_G B^g$ (a definable subgroup containing all involutions). Using the splitting criterion, we can show that $(C \cap N^\circ) < N^\circ$ is a (QF) group. Combining the conjugation of quasi-Frobenius complements and a Frattini argument, we obtain $G = N^\circ \cdot N_G(C)$. It suffices to note that $N_G(C)$ is solvable. □

Lemma

Let $C < G$ be a (QF) of odd type. Suppose there exists a standard Borel subgroup $B = B' \rtimes C$. Suppose also $i^G \setminus N_G(B)$ and K_B are generic in i^G. Then for any involution k in K_B, the subgroup $I_k^\circ = (B \cap B^k)^\circ$ is a conjugate of C containing an involution j_k.

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Proof.

(sketch) We notice that $T_k(B) \subseteq I_k$ and so $RM(I_k) \geq RM(F(B))$. Moreover, I_k° is abelian inverted by k (conjugation of (QF)-complements) and we can decompose

$$I_k^\circ = d(T) * U_{0,1}(I_K) * \ldots U_0(I_k) * (U_3(I_k) \times U_{pmax}(I_k))$$

If $I_k^\circ \cap \bigcup_G C^g = \{1\}$, by the structure of ρ-Sylow, we can show that $I_k^\circ \leq F(B)^\circ \cap F(B^k)^\circ$ and $F(B)^\circ = F(B^k)^\circ$. Then, $N_G(F(B)^\circ)^\circ$ contains B, B^k and is a (QF) of odd type with a normal abelian group, so this group splits and is solvable, a contradiction. Finally, $I_k^\circ = C_{j_k}$, for an involution j_k. □

Consequences for connected Frobenius groups

Corollary

Let $C < G$ be a connected Frobenius of odd type with a standard Borel subgroup B. Then G is solvable and splits.

Proof.

If G is not solvable then I_k°, for an involution $k \in K_B$, is a conjugate of C. But k normalises I_k° and so $k \in N_G(I_k^\circ) = I_k^\circ$, a contradiction. □

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Proposition

Let $C < G$ be a connected Frobenius group of odd type. Suppose $N_G(A)^\circ = N^\circ$ is a standard' Borel subgroup (in particular, this is true for a sharply 2-transitive group). Then G is solvable or $T_{N^\circ}(k)$ and $B \cap B^k$ are finite, for $k \in K_{N^\circ}$.

Identification of $\mathrm{PGL}_2(K)$

- To identify $\mathrm{PGL}_2(K)$, we try to find a (B, N)-pair (De Medts-Tent, Wiscons).

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Definition

A split (B, N)-pair of rank 1 is a quadruplet of groups B,N, $H = N \cap B$ normal in N and U abelian such that :

- $G = \langle B, N \rangle$.
- $[N : H] = 2$.
- For all $k \in N \setminus H$, $H = B \cap B^k$, $G = B \cup BkB$ and $B \neq B^k$.
- $B = U \rtimes H$.

Final Steps

Lemma

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Remark : Behind the proof lies a rank computation (Deloro-Jaligot) which uses the full strength of the existence of a standard Borel subgroup.

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- How can we prove that our standard" Borel subgroup is standard ?
- How could we analyse ranked (QF) groups without assuming the complement is solvable ?

Thank you for your attention !

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