Dimensional Quasi-Frobenius Groups

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Motivation: a specific geometry of involutions

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• Involution forms a projective plane in $SO_3(\mathbb{R})$ (half-turns) but the axioms are satisfied only “generically” in $PGL_2(\mathbb{C})$. 
Motivation: a specific geometry of involutions

- Work of J. Wiscons and A. Deloro about generalisations of the geometry of involutions in $\text{SO}_3(\mathbb{R})$.
- Involutions form a projective plane in $\text{SO}_3(\mathbb{R})$ (half-turns) but the axioms are satisfied only “generically” in $\text{PGL}_2(\mathbb{C})$.
- In terms of the internal structure, we have a subgroup $C \leq C_G(i)$ for an involution $i$ such that:
  1. $N_G(C) = C \rtimes \langle w \rangle$ for an involution $w$ acting by inversion.
  2. $\bigcup_G C^g$ is “generic” in the ambient group.
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  1. Let $C < G$ be a pair of (definably) connected groups definable in an o-minimal structure. Suppose $G$ contains involutions and $C$ is of finite index in $N_G(C)$ and intersects trivially any distinct conjugate. If $\bigcup_G C^g$ contains all translations, i.e., products of involutions, is $G \cong SO_3(R)$ for a real closed field $R$?
• A common geometry of involutions for two distinct model-theoretic settings: ranked universe and o-minimal structure.

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  1. Let \( C < G \) be a pair of (definably) connected groups definable in an o-minimal structure. Suppose \( G \) contains involutions and \( C \) is of finite index in \( N_G(C) \) and intersects trivially any distinct conjugate. If \( \bigcup_G C^g \) contains all translations, i.e., products of involutions, is \( G \cong SO_3(R) \) for a real closed field \( R \)?
  2. Let \( C < G \) be a pair of ranked connected groups. Suppose \( G \) contains involutions and is \( U_2^\perp \), and \( C \) is of finite index in \( N_G(C) \) and intersects trivially any distinct conjugate. Is \( G \cong PGL_2(K) \) for an algebraically closed field \( K \)?
Motivation: connected Frobenius group

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- In a ranked universe we conjecture that a Frobenius group $C < G$ splits, i.e., $G = U \rtimes C$ for a definable subgroup $U$. If not, we will have a counter-example to the Cherlin-Zilber conjecture.
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- In a ranked universe we conjecture that a Frobenius group $C < G$ splits, i.e., $G = U \rtimes C$ for a definable subgroup $U$. If not, we will have a counter-example to the Cherlin-Zilber conjecture.
- A ranked sharply 2-transitive group, in particular $K^+ \rtimes K^\times$ for a ranked field $K$, is a good example of a connected Frobenius group with involutions.
We work with dimensional groups, i.e., groups definable in a structure such that definable subsets (taken in cartesian products) carry a dimension $\dim$ satisfying the following axioms:

- Let $f : A \to B$ be a definable function between definable sets $A, B$. Then $\{b \in B : f^{-1}(b) = n\}$ is definable.
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We shall call a definable subset $X \subseteq G$ (weakly) generic if $\text{dim}(X) = \text{dim}(G)$.
• We require also the DCC on definable subgroups in order to define the connected component.

• Groups definable in o-minimal structures and ranked groups are dimensional groups (with DCC).

• We have a form of elimination of imaginaries in groups definable in o-minimal structures. Moreover, if $G$ is a ranked group, then $G^{eq}$ is ranked. We can make use of the dimension function for interpretable subsets.
Definitions

Definition

- Let $C < G$ be a pair of definable dimensional connected groups. We say $C < G$ is a Quasi-Frobenius group (QF) if $C$ is of finite index in $N_G(C)$ (almost selfnormalising) and for all $g \notin N_G(C)$, we have $C^g \cap C = \{1\}$ (TI).
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• We speak of connected Frobenius group if $N_G(C)/C$ is trivial, of Even Quasi-Frobenius (EQF) if $N_G(C)/C$ is even and of Odd Quasi-Frobenius (OQF) if it is odd.
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We have: $N_G(C) = C$ and $C_G(c) \leq C$ for all $1 \neq c \in C$. 
Identification of $\text{SO}_3(R)$

- From now $C < G$ will be a (QF) containing involutions and we suppose all 2-elements and translations are contained in $\bigcup_G C^g$. 
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- Notation: $I$ refers to the set of involutions and $I,I$ to the set of translations. For $x \in G$, we note $C_x$ the conjugate of $C$ containing $x$ (if it exists).
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Proposition

*The group $C < G$ is a (EQF), and there exists an involution $i \in N_G(C) \setminus C$.*
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Proposition

The group $C < G$ is a (EQF), and there exists an involution $i \in N_G(C) \setminus C$.

Proof.

(sketch) Let $x = ij$ be a translation for $i \in C$ and $j$ in $C_j \neq C$. Then $i, j \in N_G(C_G(x)^\circ) \leq N_G(C_x)$. In order to obtain an involution, we consider a 2-element $\alpha$ of minimal order such that $1 \neq \alpha^2 \in C$. 

□
Proposition

$G$ is semi-simple, i.e., $Z(G)$ is finite and there is no abelian normal infinite subgroup.

Proof.

Let $A$ be a normal abelian infinite subgroup; considering $Z(C_G(A))$, we can suppose it is definable and connected. If there exists $1 \neq c \in A \cap C$, then $A \leq C_G(c)^{\circ} \leq C$ and so $C = G$, a contradiction.

Let $i$ be an involution acting on $A$: this action is by inversion (the definable subgroup $\{a \in A : a^i = a^{-1}\}$ is weakly generic so equals $A$ by connectedness). The subgroup $A$ is 2-divisible, so $A \leq I.I$, a contradiction.
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• We suppose from now on that $C < G$ is definable in an o-minimal structure.

Fact

If $G$ is definably simple, then:

1. (Edmundo, Jaligot and Otero) Carter subgroups, i.e., maximal nilpotent connected almost selfnormalising subgroups, are abelian.
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3. $G$ is definably isomorphic to the semi-algebraic connected component of a algebraic group defined over a real closed field (bi-interpretable with $G$) or to the $k$-rational points of a linear algebraic group defined over an algebraically closed field $k$ (also bi-interpretable with $G$).

Bi-interpretability is with respect to the pure algebraic structure.
We suppose now $Z(G) = \{1\}$ and $C$ is nilpotent.
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**Proposition**

The group $C < G$ is definably simple, $C$ contains an unique involution and all involutions in $N_G(C) \setminus C$ are in the same coset (action by inversion). Moreover, involutions form a projective plane such that $i, j, k$ are collinear iff $ijk$ is an involution.
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Using Bachman’s theorem, we can define a field $k$. This field is real closed and bi-interpretable with $C < G$ (at this point, we need the definability of the connected component in the pure group structure, see Frécon).
Theorem
Let $C < G$ be a (QF) definable in an o-minimal structure. Suppose $G$ contains involutions and $\bigcup_G C^g$ contains all 2-elements and translations. Then $C < G$ is semi-simple. If moreover $C$ is nilpotent and $Z(G)$ is trivial, then $G \cong SO_3(R)$ for a real closed field $R$. 
Two general lemmas:

**Lemma**

*(Conjugation of Quasi-Frobenius complements)* Let $C < G$ be a (QF) and $H$ a definable and connected subgroup such that $H \cap C \neq \{1\}$. Then $H \cap C$ is infinite, and $(H \cap C) < H$ is a (QF). Moreover, for a conjugate $C'$ of $C$, such that $(H \cap C') \neq \{1\}$, there exists $h \in H$ such that $(H \cap C')^h = (H \cap C)$.
Analysis of ranked Quasi-Frobenius groups

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**Lemma**

Let $C < G$ be a solvable (QF). Then $G' = F(G)^\circ$ and $G = G' \rtimes C$, with $C \leq K^\times$ for an interpretable field $K$.
Structure of the 2-torsion

We shall now consider ranked (QF) groups with involutions and $U_2^\perp$ (odd type).

**Fact**

*(Borovik-Nesin)* Let $C < G$ be a connected Frobenius group of odd type. Then $C$ contains an unique involution.
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(Deloro-Wiscons) Let $C < G$ be an (EQF) of odd type. Then all 2-elements are in $\bigcup_G C^g$, $N_G(C) = C \rtimes \langle w \rangle$ for $w$ an involution acting by inversion and $C$ contains a unique involution.
Fact

(Delor-Jaligot) Let $G$ be a connected group of odd type. Suppose the Prüfer rank is equal to one. Let $S$ be a 2-Sylow subgroup. Then one of the following three possibilities holds:

1. $S = S^\circ$.
2. $(\text{PGL}_2(\mathbb{C}) \text{ type}) S = S^\circ \rtimes \langle w \rangle$ with $w$ an involution acting by inversion.
3. $(\text{SL}_2(\mathbb{C}) \text{ type}) S = S^\circ \cdot \langle w \rangle$ with $w$ a 4-element such that $w^2 = i \in S^\circ$. 
Fact

*(Deloro-Jaligot)* Let $G$ be a connected group of odd type. Suppose the Prüfer rank is equal to one. Let $S$ be a 2-Sylow subgroup. Then one of the following three possibilities holds:

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2. *(PGL$_2$(C) type)* $S = S^\circ \rtimes \langle w \rangle$ with $w$ an involution acting by inversion.

3. *(SL$_2$(C) type)* $S = S^\circ \cdot \langle w \rangle$ with $w$ a 4-element such that $w^2 = i \in S^\circ$.

Proposition

Let $C < G$ a connected Frobenius group of odd type. Then its 2-Sylow subgroups are of type SL$_2$(C), or connected.
Proposition

(Splitting criterion) Let $C < G$ be an (QF) of odd type with an abelian normal infinite subgroup. Then $G = A \rtimes C$ for an abelian normal definable subgroup $A$. 

Proof.
Replacing $A$ with $d(A)$, we can suppose $A$ is definable and connected. We see that $A \cap \bigcup G C g = \{1\}$ and all involutions act on $A$ by inversion. Replacing $A$ by $C_G(A) \circ (an\ abelian\ group)$ if necessary, we can suppose $A = C_G(A) \circ$. For an involution $i$, we have $[i, G] \leq C_G(A) \circ = A$. Let $g \in G$, and consider $[i, g] \in A$. By 2-divisibility, there exists $a \in A$ such that $a^2 = [i, g]$: $[i, ga - 1] = iag - 1iga - 1 = a_i [i, g]a - 1 = a - 1 a^2 a - 1 = 1$. Finally, we obtain $G = A \rtimes C_G(i) = A \rtimes C_G$. 


Proposition

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Proof.
Replacing \( A \) with \( d(A)^{\circ} \), we can suppose \( A \) is definable and connected. We see that \( A \cap \bigcup_G C^g = \{1\} \) and all involutions act on \( A \) by inversion. Replacing \( A \) by \( C_G(A)^{\circ} \) (an abelian group) if necessary, we can suppose \( A = C_G(A)^{\circ} \).

For an involution \( i \), we have \( [i, G] \leq C_G(A)^{\circ} = A \). Let \( g \in G \), and consider \( [i, g] \in A \). By 2-divisibility, there exists \( a \in A \) such that \( a^2 = [i, g] : \)

\[
[i, ga^{-1}] = iag^{-1}iga^{-1} = a^i[i, g]a^{-1} = a^{-1}a^2a^{-1} = 1
\]

Finally, we obtain \( G = A \rtimes C_G(i) = A \rtimes C \). \( \qed \)
Proposition

Let \( C < G \) be an (EQF) of odd type and \( (C \cap H) < H \) a definable solvable sub-(QF). Then \( (C \cap H) < H \) is a connected Frobenius group (and not only a (QF)).
Proposition

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Corollary

Let $C < G$ be an (EQF) of odd type. Then it is not solvable.
The Weyl group

The group $N_G(C)/C$ can be viewed as a form of Weyl group. Following the methods of (Altinel, Burdges and Frécon), we can obtain the following result (Frattini arguments):

\[ N_G(C)/C \cong N_G(T)/C_G(T) \cong N_G(Q)/Q \cong N_G(S)/C_G(S). \]
The Weyl group

The group $N_G(C)/C$ can be viewed as a form of Weyl group. Following the methods of (Altinel, Burdges and Frécon), we can obtain the following result (Frattini arguments):

**Proposition**

Let $C < G$ a (QF) of odd type with $C$ solvable. Let $T$ be a maximal decent torus, $Q$ a generous Carter subgroup and $S$ a maximal $p$-torus. Then

$$N_G(C)/C \simeq N_G(T)/C_G(T) \simeq N_G(Q)/Q \simeq N_G(S)/C_G(S).$$

Moreover, a (OQF) of odd type with solvable complement has Prüfer rank at least equal to two.
We try to identify a Borel subgroup whose structure is reminiscent of $K^+ \ltimes K^\times$ (the Borel subgroups of $\text{PGL}_2(K)$ have precisely this structure).
A standard” Borel

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**Definition**

Let $B$ be a Borel subgroup of a (QF) $C < G$ of odd type. It is *standard”* if $B$ has infinite intersection with $C$, it is standard’ if it additionally contains an involution, and *standard* if $C < B$. 
Fact

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Lemma

(adapted from Deloro-Wiscons) Let $C < G$ be a (QF) of odd type. Let $x$ a translation and $i$ an involution inverting $x$. If $C$ is $\langle x, i \rangle$-invariant then $x$ is contained in $\bigcup G C^g$. 
Theorem

Let $C \triangleleft G$ a (QF) of odd type such that $C$ is solvable. Then there is standard” borel $B$. Moreover, if $C \triangleleft G$ is an (EQF), this standard” borel is a connected Frobenius group.
Theorem

Let $C < G$ a (QF) of odd type such that $C$ is solvable. Then there is standard” borel $B$. Moreover, if $C < G$ is an (EQF), this standard” borel is a connected Frobenius group.

Proof.

(sketch) Let $x = ij$ be a translation not contained in $\bigcup G C^g$ and consider $A = C_G(x)$. Using the preceding lemma and fact, we can see that $A$ is an abelian group inverted by $i$ and $j$ such that $A \cap \bigcup G C^g = \{1\}$. Taking $A$ maximal for these properties, we can also prove that $A$ is not an almost selfnormalising group, $C_G(A) = A$ and $A$ is TI (for the last property, we consider $x_0 \in A \cap A^g \leq C_G(x_0)$).

We can work as before to obtain $N_G(A) = A \rtimes (A \cap C)$. It suffices now to consider a Borel subgroup containing $N_G(A)$.

If we suppose that $N_G(A)^\circ$ contains involutions, we can precisely identify a standard' Borel subgroup.

**Proposition**

The subgroup $N_G(A)^\circ = B$ is a Borel subgroup. Moreover, for $1 \neq g \in G \setminus N_G(B)$, $(B \cap B^g)^\circ$ is trivial or an infinite torus, i.e., an abelian divisible group.
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- In a connected Frobenius group of odd type, for any translation $x = ij$, we have: $C_G(x)$ is abelian 2-divisible and $(N_G(C_G(x)) \cap C_i)$ is infinite.
If we suppose that $N_G(A)^\circ$ contains involutions, we can precisely identify a standard’ Borel subgroup.

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- In a connected Frobenius group of odd type, for any translation $x = ij$, we have: $C_G(x)$ is abelian 2-divisible and $(N_G(C_G(x)) \cap C_i)$ is infinite.
- If we deal with a sharply 2-transitive group of odd type, the situation is even better: $N_G(C_G(x)) = K^+ \ltimes K^\times$. 
Lemma

Let $C < G$ be a (QF) such that $C$ is abelian (or nilpotent under some additional technical hypothesis). If a standard” Borel subgroup $B$ is generous then it is standard.
Borel subgroup and generosity

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Proof.
We have $B = B' \rtimes (B \cap C)$ and $(B \cap C) \leq Z(C)$. As $(B \cap C)$ is a (QF)-complement, it is generous in $B$. But $B$ is generous (in $G$) so $X = \{b \in B : b$ is contained in a finite number of conjugates of $B\}$ is generic in $B$. There exists $1 \neq c \in C \cap X$ (conjugating $C$ if necessary) but $c \in B \cap B^{c'}$ for all $c' \in C$.
Finally, $N_G(B) \cap C$ is generic in $C$, and $C \leq N_G(B) = B$. □
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We have $B = B' \rtimes (B \cap C)$ and $(B \cap C) \leq Z(C)$. As $(B \cap C)$ is a (QF)-complement, it is generous in $B$. But $B$ is generous (in $G$) so $X = \{ b \in B : b$ is contained in a finite number of conjugates of $B \}$ is generic in $B$. There exists $1 \neq c \in C \cap X$ (conjugating $C$ if necessary) but $c \in B \cap B^{c'}$ for all $c' \in C$.

Finally, $N_G(B) \cap C$ is generic in $C$, and $C \leq N_G(B)^{\circ} = B$. \qed

Remark : If $B$ is generically disjoint from its conjugates, it is a generous subgroup. Nevertheless, in $\text{PGL}_2(K)$, a Borel subgroup is generous but it is not generically disjoint from its conjugates.
Classification theorems

Two theorems
Classification theorems

Two theorems

Theorem

Let $C < G$ be an (EQF) of odd type such that Borel subgroups are generous (respect. contains a standard Borel subgroup) then $G \simeq \text{PGL}_2(K)$, for an algebraically closed field $K$. 
Classification theorems

Two theorems

Theorem
Let $C < G$ be an (EQF) of odd type such that Borel subgroups are generous (respect. contains a standard Borel subgroup) then $G \cong \text{PGL}_2(K)$, for an algebraically closed field $K$.

Theorem
Let $C < G$ be a connected Frobenius group of odd type such that Borel subgroups are generous (respect. contains a standard Borel subgroup). Suppose also $C$ is nilpotent (with some technical hypothesis). Then $G$ is solvable and splits.
Two key lemmas

Lemma

Let \( C < G \) be a (QF) of odd type with \( C \) solvable. Let \( B \) a standard’ Borel subgroup, \( B = B' \rtimes (B \cap C_i) \) and \( i \in B \). Suppose \( B \cap i^G = N_G(B) \cap i^G \). Then one of the following possibilities holds:

- \( G \) is solvable.
- \( i^G \setminus N_G(B) \) and \( K_B = \{ k \in i^G \setminus N_G(B) : \text{RM}(T_B(k)) \geq \text{RM}(B) - \text{RM}(C_G(i)) \} \) are generic in \( i^G \), where \( T_B(k) = \{ b \in B : b^k = b^{-1} \} \) for an involution \( k \) not normalising \( B \).
Proof.

(sketch) Suppose $B \cap i^G$ is generic in $i^G$ and consider $N = \bigcap_G B^g$ (a definable subgroup containing all involutions). Using the splitting criterion, we can show that $(C \cap N^\circ) < N^\circ$ is a (QF) group. Combining the conjugation of quasi-Frobenius complements and a Frattini argument, we obtain $G = N^\circ \cdot N_G(C)$. It suffices to note that $N_G(C)$ is solvable.
Lemma

Let $C < G$ be a (QF) of odd type. Suppose there exists a standard Borel subgroup $B = B' \rtimes C$. Suppose also $i^G \setminus N_G(B)$ and $K_B$ are generic in $i^G$. Then for any involution $k$ in $K_B$, the subgroup $I_k^\circ = (B \cap B^k)^\circ$ is a conjugate of $C$ containing an involution $j_k$. 

Proof. (sketch) We notice that $T_k(B) \subseteq I_k$ and so $RM(I_k) \geq RM(F(B))$. Moreover, $I_k^\circ$ is abelian inverted by $k$ (conjugation of (QF)-complements) and we can decompose $I_k^\circ$ as $\prod_{0} T^U_0(I_k) \times U_{p_{\text{max}}}(I_k)$. If $I_k^\circ \cap \bigcup G_C = \{1\}$, by the structure of $\rho$-Sylow, we can show that $I_k \leq F(B) \cap F(B^k)$ and $F(B) = F(B^k)$. Then, $N_G(F(B) \cap F(B^k))$ contains $B, B^k$ and is a (QF) of odd type with a normal abelian group, so this group splits and is solvable, a contradiction. Finally, $I_k^\circ = C j_k$, for an involution $j_k$. 


Lemma

Let $C < G$ be a (QF) of odd type. Suppose there exists a standard Borel subgroup $B = B' \times C$. Suppose also $i^G \setminus N_G(B)$ and $K_B$ are generic in $i^G$. Then for any involution $k$ in $K_B$, the subgroup $I_k^o = (B \cap B^k)^\circ$ is a conjugate of $C$ containing an involution $j_k$.

Proof.

(sketch) We notice that $T_k(B) \subseteq I_k$ and so $RM(I_k) \geq RM(F(B))$. Moreover, $I_k^o$ is abelian inverted by $k$ (conjugation of (QF)-complements) and we can decompose

$$I_k^o = d(T) \ast U_{0,1}(I_k) \ast \ldots U_0(I_k) \ast (U_3(I_k) \times U_{pmax}(I_k))$$

If $I_k^o \cap \bigcup_G C^g = \{1\}$, by the structure of $\rho$-Sylow, we can show that $I_k^o \leq F(B)^\circ \cap F(B^k)^\circ$ and $F(B)^\circ = F(B^k)^\circ$. Then, $N_G(F(B)^\circ)^\circ$ contains $B, B^k$ and is a (QF) of odd type with a normal abelian group, so this group splits and is solvable, a contradiction. Finally, $I_k^o = C_{jk}$, for an involution $j_k$. $\square$
Corollary

Let $C < G$ be a connected Frobenius of odd type with a standard Borel subgroup $B$. Then $G$ is solvable and splits.

Proof.

If $G$ is not solvable then $I_k^o$, for an involution $k \in K_B$, is a conjugate of $C$. But $k$ normalises $I_k^o$ and so $k \in N_G(I_k^o) = I_k^o$, a contradiction.
Consequences for connected Frobenius groups

Corollary

Let $C < G$ be a connected Frobenius of odd type with a standard Borel subgroup $B$. Then $G$ is solvable and splits.

Proof.
If $G$ is not solvable then $I^\circ_k$, for an involution $k \in K_B$, is a conjugate of $C$. But $k$ normalises $I^\circ_k$ and so $k \in N_G(I^\circ_k) = I^\circ_k$, a contradiction.

Proposition

Let $C < G$ be a connected Frobenius group of odd type. Suppose $N_G(A)^\circ = N^\circ$ is a standard’ Borel subgroup (in particular, this is true for a sharply 2-transitive group). Then $G$ is solvable or $T_{N^\circ}(k)$ and $B \cap B^k$ are finite, for $k \in K_{N^\circ}$. 
Identification of $\text{PGL}_2(K)$

- To identify $\text{PGL}_2(K)$, we try to find a $(B, N)$-pair (De Medts-Tent, Wiscons).
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Definition

A split $(B, N)$-pair of rank 1 is a quadruplet of groups $B, N$, $H = N \cap B$ normal in $N$ and $U$ abelian such that:

- $G = \langle B, N \rangle$.
- $[N : H] = 2$.
- For all $k \in N \setminus H$, $H = B \cap B^k$, $G = B \cup BkB$ and $B \neq B^k$.
- $B = U \ltimes H$. 
Final Steps

Lemma

Let \( C < G \) be an (EQF) of odd type. Let \( B \) be a standard borel. Then \( B \) is self-normalising.
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Lemma
Let $C < G$ be an (EQF) of odd type. Let $B$ be a standard borel. Suppose $i^G \setminus B$ and $K_B$ are generic in $i^G$ and $l_k = (B \cap B^k)\circ$ is a conjugate of $C$. Then for all $g \in G \setminus B$, we have $G = B \cup BgB$. 
Introduction

Dimensional and o-minimal Quasi-Frobenius groups

Ranked Quasi-Frobenius Groups

Final Steps

Lemma
Let \( C < G \) be an (EQF) of odd type. Let \( B \) be a standard borel. Then \( B \) is self-normalising.

Lemma
Let \( C < G \) be an (EQF) of odd type. Let \( B \) be a standard borel. Suppose \( i^G \setminus B \) and \( K_B \) are generic in \( i^G \) and \( l_k = (B \cap B^k)^\circ \) is a conjugate of \( C \). Then for all \( g \in G \setminus B \), we have \( G = B \cup BgB \).

Remark: Behind the proof lies a rank computation (Deloro-Jaligot) which uses the full strength of the existence of a standard Borel subgroup.
Conclusion

- Is there a better way to characterize involutive definable automorphisms in o-minimal structures?
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Conclusion

• Is there a better way to characterize involutive definable automorphisms in o-minimal structures?
• How can we prove that our standard” Borel subgroup is standard?
• How could we analyse ranked (QF) groups without assuming the complement is solvable?
Thank you for your attention!
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