Abstract strategies and coherence

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Abstract. Normalisation strategies give a categorical interpretation of the notion of contracting homotopy via confluent and terminating rewriting. This approach relates standardisation to coherence results in the context of higher-dimensional rewriting systems. On the other hand, globular 2-Kleene algebras provide a formal setting for reasoning about coherence proofs in abstract rewriting systems. In this setting, we formalise the notion of normalisation strategy and we prove a formal coherence theorem for convergent abstract rewriting systems.

Keywords: Normalisation strategies · Kleene algebras · Formalisation · Coherence · Higher-dimensional rewriting.

1 Introduction

As pointed out in [5, 29] a central difficulty in formal mathematics is in balancing readability of specifications and proficient automated proof search. Capturing intuitions while remaining formally rigorous constitutes a first stumbling block, which ideally should result in a setting that provides correct, automated proofs which are readable and even illuminating. A powerful formalisation of abstract rewriting theory may be found in the theory of Kleene algebras. Algebraic abstraction allows for simple proofs in which deduction is replaced by calculation [29]. Proofs in this setting reconstruct intuitive proofs by *diagrammatic reasoning*, making Kleene algebras a formal setting well suited to capture abstract rewriting results. Modal Kleene algebras (MKAs) formalise abstract rewriting systems (ARS), abstractions of graphs of (1-dimensional) transitions, especially with respect to termination and normalisation properties [5, 29]. This setting does not suffice to formalise more subtle properties of normalisation strategies [24], such as standardisation properties, nor for dealing with inherently higher-dimensional transition systems. Indeed, these need a formalisation of equivalences between paths. This line of work started in [20, 12], culminating in the introduction of a specific axiomatics on a 2-dimensional refinement of ARSs.

In this work, we are going one step further by giving a formalisation of a *coherent* extension of diagrammatic reasoning in the algebraic style of MKAs, inspired by coherent presentations in categorical algebra [23], or in algebra [10], and using a rewriting approach in the line of [27]. In a higher categorical structure, certain algebraic properties, *e.g.* associativity of composition, may only hold up to the existence of higher-dimensional morphisms. Given a collection of such higher morphisms, *coherence* is the requirement that the whole structure is contractible, *i.e.* all parallel morphisms are linked by higher morphisms. A *coherence theorem* states that, given a (generating) collection of such morphisms, coherence is satisfied. The objective is to obtain a minimal collection of generating higher morphisms. Graph-theoretical methods on string rewriting systems (SRS) were initiated by Squier in [27] to study coherence problems for monoids, a two dimensional word problem. The main point is to compute extensions of a SRS by *homotopy generators* which take the relations amongst the rewriting paths into account. That is, every pair of zig-zag sequences of rewriting paths with same source and same target can be paved by compositions of these generators. In Squier's approach, when the SRS is convergent, the homotopy generators are defined by the confluence diagrams of the critical branchings of the SRS. This rewriting method for coherence was applied to solve coherence problems in algebra [10, 17, 4], and for monoidal categories [14]. Thereby, the homotopy generators constitute the bottom part of a cofibrant replacement of the monoid presented by the SRS [10, 15]. Squier's constructions were formulated in the categorical language of polygraphs in [16] for monoids and in [13] for higher categories.

In this work, we consider the case of ARS. The extension to the case of SRS will be done in a subsequent work because requires a formalisation of algebraic contexts and of the critical branching lemma, which constitutes a further development of the theory presented here. An ARS is represented by a quiver Φ, aka a 1-*polygraph*, see Section 2. Parallel zig-zag sequences of rewriting paths are pairs of 1-cells in the free groupoid $\Phi^\top$ on Φ with same source and same target. Homotopical generators for the ARS consist of such pairs and form a *cellular extension* X of $\Phi^\top$, see Section 2. The *coherence theorem for* (Φ, X) states that all parallel 1-cells in $\Phi^\top$ are equal modulo X. When Φ is convergent and X is the set of confluence diagrams of (critical) branchings, Squier's method gives a proof of the coherence theorem for Φ. It is exactly this proof that we formalise in this article.

This work uses the algebraic setting of a 2-dimensional (globular) version of MKAs, which model relation algebras and relations among relations, introduced in [3]. Interestingly enough, these 2-dimensional MKAs are close to Concurrent Kleene Algebras (CKAs), which introduce an extra algebraic operation modelling parallel composition, hence equivalences between (1-dimensional) paths.

Structure of the article, and main results. This article is about formalising normalisation strategies and coherence properties in view of automating proofs. In Section 2, we present the categorical formulation of relations among relations in terms of cellular tilings, and based on Squier's coherence result. We then recap the MKA approach for ARS in Section 3. Coherent rewriting in globular modal 2-Kleene algebra, which we introduced in [3], is recalled in Sections 4 and 5. Sections 6 and 7 form the core of our new results, where we first model normalisation strategies in 2-MKAs, and prove abstract coherence properties therein. Our first result, Theorem 1, gives a formalisation of a coherent normalising Newman's lemma. We thereby deduce our main result, Theorem 2, which formalises a proof of contractibility via normalisation strategies.

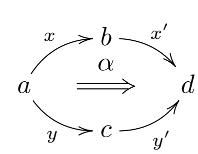
2 Squier's theorem for ARS

We consider an ARS as a *quiver*, *i.e.* a directed graph with parallel and looping transitions, which we call a 1-*polygraph* from the terminology of higher-dimensional rewriting [2, 28]. Denote by $\Phi := (\Phi_0, \Phi_1)$ a 1-polygraph with source and target maps $s_0, t_0 : \Phi_1 \to \Phi_0$. We model the *reflexive, transitive closure* of Φ by the *free* 1-*category* Φ^* generated by Φ, the underlying graph of which consists of the directed paths in Φ. Similarly, the *symmetric, reflexive, transitive closure of* Φ is modelled by the *free* 1-*groupoid* $\Phi^\top$ generated by Φ, its underlying graph consisting of undirected paths. In both cases the source and target maps are obtained by naturally extending those of Φ. The vertices (resp. edges) of such structures will henceforth be referred to as 0-cells (resp. 1-cells), and the set of i-cells of Φ^* (resp. $\Phi^\top$) will be denoted by Φ_i^* (resp. $\Phi_i^\top$). The 0-*composition* of 1-cells x, y is defined when $t_0(x) = s_0(y)$ and is denoted by $x \star_0 y$. The identity 1-cell on $a \in \Phi_0$ is denoted by 1_a and the inverse of a 1-cell x is denoted by x^-. Two 1-cells are *parallel* when they have the same 0-source and 0-target. Directed paths correspond to compositions $x_1 \star_0 \cdots \star_0 x_k$, with $x_i \in \Phi_1$. Similarly, undirected paths correspond to finite compositions of elements of Φ_1 and their formal inverses, quotiented by the relations $x \star_0 x^- \sim 1_{s_0(x)}$, for $x \in \Phi_1$.

A *cellular extension* X of $\Phi^\top$ is a quiver on the edges of $\Phi^\top$, *i.e.* a pair $(\Phi_1^\top, X)$ with source (resp. target) map s_1 (resp. t_1), such that the *globular relations* $t_0 \circ s_1 = t_0 \circ t_1$ and $s_0 \circ s_1 = s_0 \circ t_1$ are satisfied. The elements of X are called *generating* 2-*cells* and may be thought of as (directed) tiles filling the space between parallel 1-cells. The pair (Φ, X) is called a $(2,0)$-*polygraph*.

Recall that the 2-cells in a 2-category may be composed in two different ways. The 0-composition of $\gamma : x \Rightarrow y$ and $\delta : x' \Rightarrow y'$, where $x, y : a \to b$ and $x', y' : b \to c$ are pairs of parallel 1-cells, is a 2-cell $\gamma \star_0 \delta : x \star_0 x' \Rightarrow y \star_0 y'$. The 1-composition of 2-cells $\alpha : x \Rightarrow y$ and $\beta : y \Rightarrow z$, where x, y, z are parallel 1-cells, is a 2-cell $\alpha \star_1 \beta : x \Rightarrow z$. A 2-*groupoid* is a 2-category in which all 1- and 2-cells are invertible for 0- and 1-composition, respectively. Given a $(2,0)$-polygraph (Φ, X), we consider the *free* 2-*groupoid generated by* (Φ, X), denoted by $X^\top$, which has $\Phi^\top$ as its underlying 1-groupoid and containing all finite 0- and 1-compositions of the generating 2-cells in X and their inverses, as well as 0-compositions with 1-cells of $\Phi^\top$.

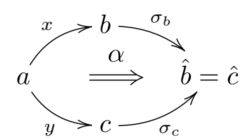
The confluence properties of an ARS Φ can be stated with respect to a cellular extension X of $\Phi^\top$. This approach first appeared in [20] under the terminology of *commuting diagrams*. A local branching (x, y) of Φ is X-*confluent* if there exist 1-cells x', y' in Φ_1^*, and a 2-cell α in the free 2-groupoid $X^\top$ as in the adjacent diagram. The ARS Φ is *locally* X-*confluent* when



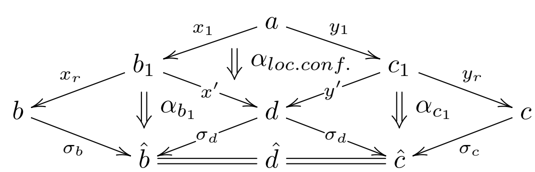
every local branching of Φ is X-confluent. We say that parallel 1-cells f and g of $\Phi^\top$ are *X-congruent* if there exists a 2-cell $\alpha : f \Rightarrow g$ in $X^\top$, and that (Φ, X) is *acyclic* if all parallel 1-cells of $\Phi^\top$ are X-congruent. The ARS Φ *terminates* if it contains no infinite directed paths.

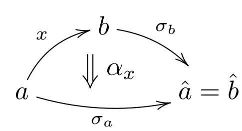
Let us recall the proof that a terminating, locally X-confluent $(2,0)$-polygraph (Φ, X) is acyclic. Firstly note that if an ARS Φ is locally X-confluent then it is locally confluent so, under the hypothesis of termination, is confluent by Newman's lemma. In this case, from every 0-cell a, a *normal form*, *i.e.* a 0-cell irreducible by Φ, may be reached in a finite number of steps. Since Φ is confluent, the normal form of a is unique; we denote it by $\hat{a}$.

By local X-confluence and termination, we may therefore choose, for every 0-cell a of Φ, a 1-cell $\sigma_a : a \to \hat{a}$ in Φ_1^*. A *normalisation strategy* σ is a function $\Phi_0 \to \Phi_1^\top$ which assigns such a σ_a to every 0-cell a, under the condition that $\sigma_b = 1_b$ for any normal form b. Just as normal forms provide a representative 0-cell for connected components in $\Phi^\top$, a normalisation strategy is the given of a representative 1-cell in $\Phi^\top$ among parallel reductions to normal forms.

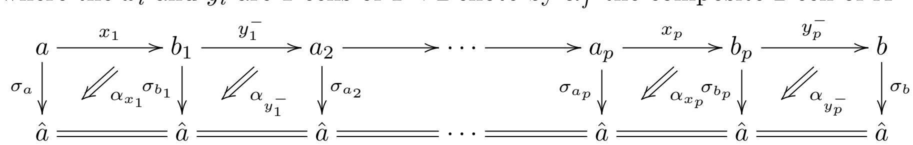


Now that we are equipped with a normalisation strategy σ, we prove by Noetherian induction on the distance from a normal form that for any branching (x, y) of Φ^*, there exists a 2-cell α as in the adjacent diagram. When $s_0(x) = s_0(y)$ is a normal form, we can simply use identity 1- and 2-cells to obtain the desired diagram. For the induction step, we observe that we can write x as $x = x_1 \star_0 x_r$, where x_1 is a 1-cell of Φ and x_r is one step closer to a normal form, and similarly for y. By the hypothesis of local confluence and the Noetherian induction hypothesis, we obtain the result by composing the 2-cells in the diagram on the left below:





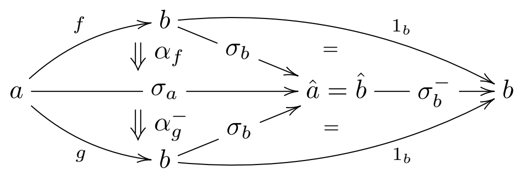
Let $x : a \to b$ be a 1-cell of Φ^*, consider the branching $(x \star_0 \sigma_b, \sigma_a)$ of Φ^*. Since we cannot reduce any further than normal forms, by the above result, as well as a rotation of the 2-cell by properties of 2-groupoids, we obtain a 2-cell α_x as pictured above on the right. A similar 2-cell for all inverses of 1-cells may be found, again using properties of 2-groupoids which we will not develop here. Note that every 1-cell $f : a \to b$ of $\Phi^\top$ can be factorised as $f = x_1 \star_0 y_1^- \star_0 \cdots \star_0 x_p \star_0 y_p^-$, where the x_i and y_i are 1-cells of Φ^*. Denote by α_f the composite 2-cell of $X^\top$:



Compiling all of the above, we obtain the *coherence theorem for ARS*:

Theorem A. *Let Φ be a terminating ARS and X be a cellular extension of $\Phi^\top$. If Φ is locally X-confluent, then for every 1-cell $f : a \to b$ of $\Phi^\top$, there exists a 2-cell $\alpha_f : f \star_0 \sigma_b \Rightarrow \sigma_a$ in the free 2-groupoid generated by (Φ, X).*

Squier's theorem [27] is deduced from the above result. Indeed, we prove that for all parallel 1-cells $f, g : a \to b$ of $\Phi^\top$, the composite 2-cell



in $X^\top$ has source f and target g. This proves that the pair (Φ, X) is acyclic.

Theorem B. *Let Φ be a terminating ARS and X be a cellular extension of $\Phi^\top$. If Φ is locally X-confluent, then (Φ, X) is acyclic.*

This is Squier's formulation of the coherence theorem for ARSs, and is an immediate consequence of Theorem A, relying solely on the definitions of acyclicity and of 2-groupoids.

3 Modal 1-Kleene algebras

In order to fix notation, we recall the definitions of Boolean modal Kleene algebras from [6, 5] and of converse from [1]. We adapt one of the converse axioms in order to establish a natural relationship between domain and conversion akin to that of inverse semigroups, see e.g. [22].

Semirings. A *semiring* is a structure $(S, +, 0, \cdot, 1)$ such that $(S, +, 0)$ is a commutative monoid, $(S, \cdot, 1)$ is a monoid whose *multiplication* $\cdot$ (often denoted by juxtaposition) distributes on the left and the right over the *addition* $+$, and 0 is a left and right annihilator for $\cdot$. A *dioid* is a semiring in which addition is idempotent. In this case, the relation defined by $x \le y \Leftrightarrow x + y = y$, for all $x, y \in S$, is a partial order on S, with respect to which addition and multiplication are monotone, and for which 0 is the minimum.

(Boolean) domain semirings. A *domain semiring* is a dioid S equipped with a *domain operation* $d : S \to S$ satisfying the following five axioms for all $x, y \in S$:

$$x \le d(x)x, \quad d(xy) = d(xd(y)), \quad d(x) \le 1, \quad d(0) = 0, \quad d(x+y) = d(x)+d(y).$$

The set S_d of fixpoints of d forms a distributive lattice with $+$ as join and $\cdot$ as meet, bounded by 0 and 1. We write $p, q, r, \ldots$ for elements of S_d and refer to S_d as the *domain algebra* of S. A *Boolean domain semiring* is a dioid S equipped with an *antidomain operation* $ad : S \to S$ satisfying the following three axioms:

$$ad(x)x = 0, \qquad ad(xy) \le ad(x\, ad^2(y)), \qquad ad^2(x) + ad(x) = 1,$$

for all $x, y \in S$. Setting $d = ad^2$, we recover a domain semiring. In the presence of an antidomain, $S_d = ad(S)$ and ad acts as Boolean complementation on S_d. We denote the restriction of ad to S_d by $\neg$.

Modal semirings. We denote by S^{op} the *opposite* of a dioid S, in which the order of multiplication has been reversed. A *codomain* (resp. *Boolean codomain*) *semiring* is a dioid equipped with a map $r : S \to S$ (resp. $ar : S \to S$) such that (S^{op}, r) (resp. (S^{op}, ar)) is a domain (resp. Boolean domain) semiring. A *modal semiring* is a dioid S which is both a domain and codomain semiring, and satisfies for every $x \in S$, $d(r(x)) = r(x)$ and $r(d(x)) = d(x)$.

Modal Kleene algebras. A *Kleene algebra* is a dioid K equipped with an operation $(-)^* : K \to K$ called the *Kleene star*, satisfying the following axioms:

i) $1 + xx^* \leq x^*$ and $1 + x^*x \leq x^*$ (*unfold axioms*),
ii) $z + xy \leq y \Rightarrow x^*z \leq y$ and $z + yx \leq y \Rightarrow zx^* \leq y$ (*induction axioms*),

for all $x, y, z \in K$. The *Kleene plus* is defined by $x^+ = xx^*$. (Anti-)domain and (anti-)codomain operations extend to Kleene algebras without additional axioms. We thus define a *(Boolean) modal Kleene algebra*, or (Boolean) MKA for short, as a Kleene algebra that is also a (Boolean) modal semiring.

Converse. A *Kleene algebra with converse* [1] is a Kleene algebra K equipped with an involution $\overline{(-)} : K \to K$ that satisfies, for all $x, y \in K$,

$$\overline{(x+y)} = \overline{x} + \overline{y}, \qquad \overline{(x \cdot y)} = \overline{y} \cdot \overline{x}, \qquad \overline{(x^*)} = (\overline{x})^*, \qquad \overline{(\overline{x})} = x, \tag{1}$$

and the inequality $x \leq x\overline{x}x$. In this work, we alter the final axiom in order to relate conversion to the domain operation. We consider an involution $\overline{(-)} : K \to K$ satisfying axioms (1) and

$$x\overline{x} \geq d(x), \tag{2}$$

a similar axiom to that found in inverse semigroups [22]. We observe that such a converse operation exchanges domain and codomain, *i.e.* $d(\overline{x}) = r(x)$ and $r(\overline{x}) = d(x)$, and that for $p \in K_d$, $\overline{p} = p$. A (Boolean) MKA with converse is a (Boolean) MKA equipped with such a converse operation.

Modalities in dimension one. Let K be a MKA. For $x \in K$ and $p \in K_d$, we define *modal forward and backward diamond operators*:

$$|x\rangle p = d(xp), \qquad \langle x|p = r(px). \tag{3}$$

When a statement holds for both forward and backward diamonds, we will write $\langle x \rangle$. Note that by monotonicity of domain, the assignment $x \mapsto \langle x \rangle$ is monotone for the point-wise order on operators. When K is a Boolean MKA, we additionally define *modal box operators*:

$$|x]p = \neg|x\rangle(\neg p), \qquad [x|p = \neg\langle x|(\neg p).$$

These are modal operators in the sense of Boolean algebras with operators [21]. For K with converse, we have $|\overline{x}\rangle = \langle x|$ and $\langle\overline{x}| = |x\rangle$, and similarly for boxes. Boxes and diamonds form a Galois connection, *i.e.*

$$|x\rangle p \leq q \Leftrightarrow p \leq [x|q \qquad \text{and} \qquad \langle x|p \leq q \Leftrightarrow p \leq |x]q. \tag{4}$$

We have $|xy\rangle = |x\rangle \circ |y\rangle$, $\langle xy| = \langle y| \circ \langle x|$, $|xy] = |x] \circ |y]$ and $[xy| = [y| \circ [x|$ for all $x, y \in K$; in what follows we will denote functional composition of modal operators simply by juxtaposition. Finally, star unfold and induction axioms lift to modalities:

$$|1\rangle + |x\rangle|x^*\rangle = |x^*\rangle, \qquad |1\rangle + |x\rangle|x^*\rangle = |x^*\rangle, \tag{5}$$

$$|y\rangle + |x\rangle|z\rangle \leq |z\rangle \Rightarrow |x^*\rangle|y\rangle \leq |z\rangle, \tag{6}$$

where the addition is the point-wise lifting of that in K_d.

Rewriting and modal Kleene algebras. We recall from [5] formalised properties of ARS expressed in MKA. An element $x \in K$ *terminates*, or is *Noetherian*, provided that for all $p \in K_d$ the implication $p \leq |x\rangle p \Rightarrow p = 0$ holds. The set of Noetherian elements of K is denoted by $\mathcal{N}(K)$. The Galois connections (4) yield the following equivalent characterisation of termination:

$$\forall p \in K_d, \quad |x]p \leq p \Rightarrow p = 1.$$

The *exhaustion* of an element $x \in K$, denoted by $exh(x)$, is defined by

$$exh(x) := x^* \cdot \neg d(x). \tag{7}$$

The *normal forms element* of $x \in K$, denoted by nf_x, is defined by

$$\mathrm{nf}_x := r(exh(x)) \in K_d. \tag{8}$$

Confluence properties are captured in MKA by semi-commutation. Given $x, y \in K$, we say that the ordered pair (x, y) *semi-commutes locally* if $xy \leq y^*x^*$, *semi-commutes* if $x^*y^* \leq y^*x^*$, and has the *Church-Rosser property* if $(x+y)^* \leq y^*x^*$. An element $x \in K$ is (*locally*) *confluent* (resp. *Church-Rosser*) if the pair $(\overline{x}, x)$ semi-commutes (resp. has the Church-Rosser property). We say that x is *convergent* if it is both terminating and confluent. These properties are related to exhaustion as follows:

Lemma 1 ([5]). *Let K be a Boolean modal Kleene algebra and $x \in K$. If x terminates, then $d(exh(x)) = 1$. If x is confluent, then $exh(x)$ is deterministic,* i.e. $\langle exh(x)|\ |exh(x)\rangle \leq \langle 1\rangle$.

4 Globular 2-Kleene algebras

In [3], the notion of p-Boolean globular n-Kleene algebra was introduced as a higher-dimensional extension of MKAs. Here we briefly recall the case of $p = 0$ and $n = 2$, and append the notion of converse.

A *modal 2-Kleene algebra* is a structure $(K, +, 0, \odot_i, 1_i, d_i, r_i, (-)^{*_i})_{i=0,1}$, such that for each $i \in \{0, 1\}$, K is a MKA with respect to i-operations, and in which the following additional axioms hold:

i) (*2-dioid axioms*) The *lax interchange law*: for all $A, A', B, B' \in K$,

$$(A \odot_1 A') \odot_0 (B \odot_1 B') \leq (A \odot_0 B) \odot_1 (A' \odot_0 B'),$$

and the 1-unit is an idempotent for 0-multiplication, *i.e.* $1_1 \odot_0 1_1 = 1_1$. Note that these correspond to the standard concurrent semiring axioms [18], except that the equality $1_0 = 1_1$ is normally assumed in this case.

ii) (*Domain 2-semiring axioms*) The (co-)domain operations satisfy absorption axioms $d_1 \circ d_0 = d_0$ and $r_1 \circ r_0 = r_0$. The set K_{d_i} is called the *i-dimensional domain algebra*, and is denoted by K_i.

iii) (*Kleene star axioms*) The 1-star $(-)^{*_1}$ is a *lax morphism* with respect to 0-multiplication of 1-dimensional elements on the right (resp. left), *i.e.* for all $A \in K$ and $\phi \in K_1$,

$$\phi \odot_0 A^{*_1} \leq (\phi \odot_0 A)^{*_1}, \qquad \text{(resp. } A^{*_1} \odot_0 \phi \leq (A \odot_0 \phi)^{*_1}).$$

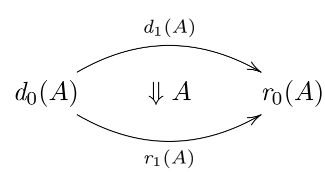
For more details, see [3]. In order to distinguish elements of distinct dimensions, we denote elements of K_0 by $p, q, r, \ldots$, elements of K_1 by $\phi, \psi, \xi, \ldots$, and general elements of K by $A, B, C, \ldots$.

As additional conditions, we may ask that a modal 2-Kleene algebra be globular, Boolean or equipped with converses. These notions are recalled below.

Globular axioms. A modal 2-Kleene algebra K is *globular* if the following *globular relations* hold for all $A, B \in K$:

$$d_0 \circ d_1 = d_0 \ \text{ and } \ d_0 \circ r_1 = d_0, \qquad d_1(A \odot_0 B) = d_1(A) \odot_0 d_1(B),$$
$$r_0 \circ d_1 = r_0, \ \text{ and } \ r_0 \circ r_1 = r_0, \qquad r_1(A \odot_0 B) = r_1(A) \odot_0 r_1(B).$$

As a consequence of the rightmost axioms, K_1 is a MKA with respect to 0-operations. An element A of K will be represented graphically by the adjacent diagram with respect to its 0- and 1-domains and codomains.



Boolean axioms. A modal 2-Kleene algebra is *Boolean* if it is augmented with maps $ad_0 : K \to K$ and $ar_0 : K \to K$, such that $(K, +, 0, \odot_0, 1_0, ad_0, ar_0)$ is a Boolean MKA, *i.e.* ad_0 (resp. ar_0) satisfies the antidomain (resp. anticodomain) axioms and $d_0 = ad_0^2$ (resp. $r_0 = ar_0^2$). The domain algebra K_0 is thus a Boolean algebra whose complementation, denoted by $\neg$, is given by the restriction of ad_0 (and ar_0) to K_0.

Converses. We will consider modal 2-Kleene algebras with *0-converses*, *i.e.* equipped with an operation $\overline{(-)} : K_1 \to K_1$ such that $(K_1, +, 0, \odot_0, 1_0, (-)^{*_0}, \overline{(-)})$ is a MKA with converse. For a more general notion of converse in higher-dimensional Kleene algebra, we refer the reader to [3].

Modalities in 2-semirings. Recall from [3], that the *i-diamond operators* of a modal 2-Kleene algebra K are defined via the (co-)domain operators in each dimension. For $i \in \{0, 1\}$, $A \in K$ and $\phi \in K_i$,

$$|A\rangle_i(\phi) = d_i(A \odot_i \phi), \quad \text{and} \quad \langle A|_i(\phi) = r_i(\phi \odot_i A).$$

These modal operators have all of the properties recalled in Section 3 with respect to i-operations and elements of K_i. Since we are considering Boolean modal 2-Kleene algebras we may additionally define 0-boxes.

Polygraphic model. Let (Φ, X) be a $(2, 0)$-polygraph. We define $K(\Phi, X)$, the *full 2-path algebra over* (Φ, X) as follows. Let $X_2^\top$ denote the set of 2-cells in $X^\top$. The carrier set of $K(\Phi, X)$ is the power set $\mathcal{P}(X_2^\top)$, whose elements, denoted by $A, B, C \ldots$ are sets of 2-cells, which in turn are denoted by $\alpha, \beta, \gamma \ldots$ Recall that for each 1-cell x of $X^\top$, there exists a unique 2-cell 1_x, its identity 2-cell, and similarly, for each 0-cell a there exists a unique 2-cell 1_{1_a}, the identity 2-cell on its identity 1-cell. For $i \in \{0, 1\}$, the i-composition, i-source and i-target maps are thereby defined for cells of any dimension.

For $i \in \{0, 1\}$, the multiplication $\odot_i$ on $K(\Phi, X)$ is the lifting of the composition operations of $X^\top$ to the power-set, *i.e.* for any $A, B \in K(\Phi, X)$,

$$A \odot_i B := \{\alpha \star_i \beta \mid \alpha \in A \wedge \beta \in B \wedge t_i(\alpha) = s_i(\beta)\}.$$

The units are the sets $\mathbb{1}_0 = \{1_{1_a} \mid a \in \Phi_0\}$, and $\mathbb{1}_1 = \{1_x \mid x \in \Phi_1^\top\}$. The addition in $K(\Phi, X)$ is given by set union; the ordering is therefore given by set inclusion. The domain and codomain maps are defined by

$$d_0(A) := \{1_{1_{s_0(\alpha)}} \mid \alpha \in A\}, \qquad r_0(A) := \{1_{1_{t_0(\alpha)}} \mid \alpha \in A\},$$
$$d_1(A) := \{1_{s_1(\alpha)} \mid \alpha \in A\}, \qquad \text{and} \qquad r_1(A) := \{1_{t_1(\alpha)} \mid \alpha \in A\},$$

and are thus given by lifting the source and target maps of $X^\top$ to the power set. The i-antidomain and i-anticodomain maps are then given by complementation with respect to the set of i-cells. The i-star is given by $A^{*_i} = \bigcup_{k \in \mathbb{N}} A^{k_i}$, where in the above, $A^{0_i} := \mathbb{1}_i$ and $A^{k_i} := A \odot_i A^{(k-1)_i}$. For $\psi \in K(\Phi, X)_1$, the converse is given by $\overline{\psi} := \{1_{x^-} \mid 1_x \in \psi\}$.

Proposition 1 ([3]). *Let (Φ, X) be a $(2, 0)$-polygraph. Then, $K(\Phi, X)$ is a globular Boolean modal 2-Kleene algebra.*

5 Coherent rewriting and modal 2-Kleene algebras

We fix K a globular 2-Kleene algebra. Given $A \in K$ and $\phi, \phi' \in K_1$, $|A\rangle_1(\phi) \geq \phi'$ is equivalent to $d_1(A \odot_1 \phi) \geq \phi'$ by definition. In terms of quantification over collections of cells, this means that *for every* u in ϕ', *there exist* v in ϕ and α in A such that the 1-source (resp. 1-target) of α is u (resp. v). This observation motivates the following definitions from [3]. For ϕ, ψ in K_1, an element A in K is a *local confluence filler* for (ϕ, ψ) if $|A\rangle_1(\psi^{*_0} \odot_0 \phi^{*_0}) \geq \phi \odot_0 \psi$, is a *confluence filler* for (ϕ, ψ) if $|A\rangle_1(\psi^{*_0} \odot_0 \phi^{*_0}) \geq \phi^{*_0} \odot_0 \psi^{*_0}$, and is a *Church-Rosser filler* for (ϕ, ψ) if $|A\rangle_1(\psi^{*_0} \odot_0 \phi^{*_0}) \geq (\psi + \phi)^{*_0}$.

The *right* (resp. *left*) *whiskering* of an element $A \in K$ by $\phi \in K_1$ is the element $A \odot_0 \phi$ (resp. $\phi \odot_0 A$). Recall from [3] that whiskering commutes with 1-diamonds, that is, for all $A \in K$ and $\phi, \psi, \phi', \psi', \gamma \in K_1$ such that $\phi' \leq \phi$, $\psi' \leq \psi$, and $d_1(A) \leq \gamma$, we have:

$$\phi' \odot_0 |A\rangle_1(\gamma) \odot_0 \psi' = |\phi' \odot_0 A \odot_0 \psi'\rangle_1(\phi \odot_0 \gamma \odot_0 \psi). \tag{9}$$

Fix a (local) confluence filler A of a pair (ϕ, ψ) of elements in K_1. The *total whiskering of* A, denoted by $\hat{A}$, is the following element of K:

$$\hat{A} := (\phi + \psi)^{*_0} \odot_0 A \odot_0 (\phi + \psi)^{*_0}. \tag{10}$$

The 1-star of $\hat{A}$ is called the *completion* of A. Note that this element *absorbs whiskers*, that is, for every $\xi \leq (\phi + \psi)^{*_0}$,

$$\xi \odot_0 \hat{A}^{*_1} \leq \hat{A}^{*_1} \qquad \text{and} \qquad \hat{A}^{*_1} \odot_0 \xi \leq \hat{A}^{*_1}. \tag{11}$$

6 Formalisation of normalisation strategies

In this section, we formalise the notion of normalisation strategy, introduced in [15]. We first define notions of section, skeleton and strategy in one-dimensional Kleene algebras and show properties thereof. In what follows, we consider a Boolean MKA K with converse and an element $x \in K$.

i) The *equivalence* generated by x is the element $x^\top := (x + \overline{x})^*$. For $p \in K_d$, the *x-saturation* of p is the element $|x^\top\rangle(p) \in K_d$.
ii) A *covering set* for x is an element $q \in K_d$ such that $|x^\top\rangle(q) \geq 1$, *i.e.* whose x-saturation is total. A *section* of x is a minimal covering set.
iii) A *wide sub* of x is an element $w \leq x$ such that $|w\rangle = |x\rangle$ and $\langle w| = \langle x|$. A *skeleton* of x is a minimal wide sub.
iv) Given a section s_0 of x, a *strategy for x relative to s_0* is a skeleton σ of $x^\top s_0$ such that $s_0 \sigma \leq s_0$.

Note that when (Φ, X) is a $(2,0)$-polygraph, we describe Φ in $K(\Phi, X)$ as the element $\phi := \{1_x \,|\, x \in \Phi_1\} \cup \{1_{1_a} \,|\, a \in \Phi_0\}$. In $K(\Phi, X)_1$, which we recall is a Boolean MKA for 0-operations, the equivalence generated by ϕ corresponds to the 1-groupoid $\Phi^\top$, and a section corresponds to a choice of a representative 0-cell for each connected component in $\Phi^\top$. A wide sub of ϕ is a subset ψ such that for any 1-cell $x : a \to b \in \Phi_1$, there exists some parallel 1-cell $x' : a \to b \in \Phi_1$ such that $1_{x'} \in \psi$. A skeleton of ϕ therefore corresponds to the choice of a single 1-cell amongst the sets of parallel 1-cells in Φ; it is thus not unique and does not coincide with ϕ in general. When Φ is convergent and $\{\sigma_a\}_{a\in\phi_0}$ is a strategy in the sense of Section 2, then $\sigma = \{1_{\sigma_a} | a \in \phi_0\}$ is a strategy for ϕ in $K(\phi, X)$ with respect to nf_ϕ. This result is proved for any convergent element of a MKA in Proposition 2.

By definition, a strategy σ satisfies $d(\sigma) = d(x^\top s_0) = 1$, and $r(\sigma) = r(x^\top s_0) = s_0$. The following lemma states that a strategy contains the associated section:

Lemma 2. *Given a section s_0 of x and a strategy σ for x relative to s_0, we have $s_0\sigma = s_0$ and $s_0 \leq \sigma$.*

Proof. By hypothesis we have $s_0\sigma \leq s_0$. Showing that $s_0\sigma$ is a covering set allows us to deduce by minimality of s_0 that $s_0 \leq s_0\sigma \leq \sigma$, which gives both desired conclusions. Since σ is a strategy relative to x, we know that $\langle x^\top s_0| = \langle\sigma|$. We calculate the saturation of $s_0\sigma$

$$\langle x^\top|(s_0\sigma) = r(s_0\sigma x^\top) = \langle x^\top|\langle\sigma|(s_0) = \langle x^\top|\langle x^\top s_0|(s_0) \geq \langle x^\top|(s_0) \geq 1,$$

where we used properties of modalities for the first two steps, then the hypothesis that σ is a strategy. To conclude, we used that $\langle x^\top s_0|(s_0) \geq \langle s_0|(s_0) = s_0$ and that s_0 is a covering set. □

By conversion, we also get $\overline{\sigma}s_0 = s_0$ and $s_0 \leq \overline{\sigma}$. This immediately gives the following properties of a strategy σ relative to a section s_0:

$$\sigma \cdot \sigma = \sigma, \quad \overline{\sigma} \cdot \overline{\sigma} = \overline{\sigma}, \quad \sigma \leq \sigma \cdot \overline{\sigma} \quad \text{and} \quad \overline{\sigma} \leq \sigma \cdot \overline{\sigma}. \tag{12}$$

Indeed, $\sigma\sigma = \sigma s_0\sigma = \sigma s_0 = \sigma$ by the fact that $r(\sigma) = s_0$ and Lemma 2, the case of $\overline{\sigma}$ follows by conversion. Additionally, $s_0 \leq \overline{\sigma}$ so $\sigma = \sigma s_0 \leq \sigma\overline{\sigma}$ and symmetrically for $\overline{\sigma}$.

Next, we will show that the normal forms and exhaustive iteration of a convergent element give us a section and a strategy, respectively. First, we show:

Lemma 3. *Let K a Boolean MKA. For a convergent element $x \in K$, we have $|x^\top\rangle = |exh(x)\rangle\langle exh(x)|$.*

Proof. One direction holds since $exh(x)\overline{exh(x)} \leq x^*\overline{x}^* \leq x^\top$ so by monotonicity of taking diamonds and reversal of diamonds by conversion, we get $|x^\top\rangle \geq |exh(x)\rangle\langle exh(x)|$. The other inequality is obtained via the star induction law for modalities (6). Indeed, it suffices to prove that

$$|1\rangle + |x + \overline{x}\rangle|exh(x)\overline{exh(x)}\rangle \leq |exh(x)\overline{exh(x)}\rangle.$$

We prove the inequality for each of the summands. We treat the case of $|1\rangle$ first: by definition,

$$|exh(x)\overline{exh(x)}\rangle(p) = d(x^*\neg d(x)r(px^*)) = d(x^*r(px^*)\neg d(x)),$$

where we used the so-called *import-export law* [5] $r(yp) = r(y)p$ for codomains and that multiplication is commutative in K_d. Since $p \leq 1$ we have

$$px^* r(px^*)\neg d(x) \leq x^* r(px^*)\neg d(x),$$

and since $(px^*)r(px^*) = px^*$, applying domain on both sides yields

$$|exh(x)\overline{exh(x)}\rangle(p) \geq d(px^*\neg d(x)) = pd(exh(x)) = p,$$

where we used the import-export law for domains $d(py) = pd(y)$ and Lemma 1. Thus $|exh(x)\overline{exh(x)}\rangle \geq |1\rangle$. The case of $|x\rangle$ follows by the star unfold axiom:

$$|x\rangle|x^*\neg d(x)\overline{x}^*\rangle = |xx^*\neg d(x)\overline{x}^*\rangle \leq |x^*\neg d(x)\overline{x}^*\rangle.$$

The final case follows by the hypothesis of confluence:

$$\begin{aligned} |\overline{x}\rangle|x^*\neg d(x)\overline{x}^*\rangle = \langle x||x^*\rangle\langle exh(x)| &\leq \langle x^*||x^*\rangle\langle exh(x)| \\ &\leq |x^*\rangle\langle x^*|\langle exh(x)| \\ &\leq |x^*\rangle\langle exh(x)x^*| = |x^*\neg d(x)\overline{x}^*\rangle, \end{aligned}$$

where we also used $exh(x)x^* = exh(x)$. Applying the star induction axiom for modalities, we obtain the result. □

Now we are ready to relate exhaustion and normal forms to strategies and sections, respectively:

Proposition 2. *If x is convergent, then nf_x is a section of x. Furthermore, any skeleton σ of $exh(x)$ is a strategy for x with respect to nf_x, and we have*

$$\sigma \leq nf_x + x^+, \qquad \overline{\sigma} \leq nf_x + \overline{x}^+ \qquad \textit{and} \quad \overline{\sigma}\sigma = nf_x$$

Proof. First we show that nf_x is a section. It is a covering set since

$$|x^\top\rangle(\mathrm{nf}_x) \geq |exh(x)\rangle(\mathrm{nf}_x) = d(exh(x)) = 1$$

where the last step is by Lemma 1. Suppose now there is some $s \in K_d$ such that $s \leq \mathrm{nf}_x$ and s is a covering set. Since $s \leq \mathrm{nf}_x \leq \neg d(x)$, the star unfold and antidomain axioms give $s \cdot exh(x) = s$, so $\langle exh(x)|(s) = s$.

Therefore $1 = |x^\top\rangle(s) = |exh(x)\rangle\langle exh(x)|(s) = |exh(x)\rangle(s)$, where we used Lemma 3. This means that

$$s \geq \langle exh(x)||exh(x)\rangle(s) = \langle exh(x)|(1) = r(exh(a)) = \mathrm{nf}_x,$$

where the first inequality is by Lemma 1, so we may conclude $\mathrm{nf}_x = s$, *i.e.* nf_x is minimal.

Now we show that a skeleton σ of $exh(x)$ is a strategy for x relative to nf_x. Note that $|x^\top\mathrm{nf}_x\rangle = |x^\top\rangle\langle\mathrm{nf}_x\rangle$ and $\langle x^\top\mathrm{nf}_x| = \langle\mathrm{nf}_x\rangle\langle x^\top|$. By Lemma 3,

$$|x^\top\mathrm{nf}_x\rangle = |exh(x)\rangle\langle exh(x)|\langle\mathrm{nf}_x\rangle = |exh(x)\rangle\langle\mathrm{nf}_x\rangle = |exh(x)\rangle,$$

since $\mathrm{nf}_x exh(x) = \mathrm{nf}_x$, and $exh(x)\mathrm{nf}_x = exh(x)$. A symmetric proof gives $\langle x^\top \mathrm{nf}_x| = \langle exh(x)|$. Since σ is a skeleton of $exh(x)$, its diamonds coincide with those of $exh(x)$ and so, by what precedes, also with those of $x^\top \mathrm{nf}_x$. Since $exh(x) \leq x^\top \mathrm{nf}_x$, σ is a wide sub of $x^\top \mathrm{nf}_x$. Minimality of σ as a wide sub follows from that same inequality plus the hypothesis that it is a skeleton of $exh(x)$. To conclude, note that $\mathrm{nf}_x \sigma \leq \mathrm{nf}_x exh(x) = \mathrm{nf}_x$. The first inequality follows from

$$\sigma \leq exh(x) = x^* \mathrm{nf}_x = (1 + xx^*)\mathrm{nf}_x \leq \mathrm{nf}_x + xx^* = \mathrm{nf}_x + x^+,$$

where we used the definition of $exh(x)$, the left star unfold axiom, $\mathrm{nf}_x \leq 1$ and the definition of the Kleene plus. The inequality for $\overline{\sigma}$ is then obtained by conversion. Finally, since $\sigma \leq exh(x)$ and x is confluent, we get

$$\overline{\sigma}\sigma \leq \overline{exh(x)}exh(x) = \mathrm{nf}_x \overline{x}^* x^* \mathrm{nf}_x \leq \mathrm{nf}_x x^* \overline{x}^* \mathrm{nf}_x = \mathrm{nf}_x,$$

where we also used that $\mathrm{nf}_x \leq \neg d(x) = \neg r(\overline{x})$. □

7 Abstract coherence in 2-MKA

Here we state and prove a formalisation of Theorem A in the context of globular modal 2-Kleene algebras. First we prove the main result of this paper:

Theorem 1 (Coherent normalising Newman's lemma). *Let K be a Boolean globular 2-Kleene algebra such that*

- **i)** $(K_0, +, 0, \odot_0, 1_0, \neg_0)$ *is a complete Boolean algebra,*
- **ii)** K_1 *is continuous with respect to 0-restriction, that is for all $\psi, \psi' \in K_1$ and $(p_\alpha)_\alpha \subseteq K_0$ we have $\psi \odot_0 \sup p_\alpha \odot_0 \psi' = \sup(\psi \odot_0 p_\alpha \odot_0 \psi')$.*

Let $\phi \in K_1$ be convergent and σ be a skeleton of $exh(\phi)$. If A is a local confluence filler for $(\overline{\phi}, \phi)$, then $|\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}) \geq \overline{\phi}^{*_0} \odot_0 \phi^{*_0}$.

Proof. We denote 0-multiplication by juxtaposition. First, we define a predicate RNP expressing restricted normalised paving. Given $p \in K_0$, let

$$RNP(p) \quad \Leftrightarrow \quad |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \geq \overline{\phi}^{*_0} p \phi^{*_0}.$$

By completeness of K_0, we set $r := sup\{p \,|\, RNP(p)\}$ and by continuity of restriction we may infer $RNP(r)$. Furthermore, by downward closure of RNP, we have $RNP(p)$ if, and only if, $p \leq r$. We thereby deduce:

$$\begin{aligned} \forall p.\ (RNP(\langle\phi|_0 p) \Rightarrow RNP(p)) &\Leftrightarrow \forall p.\ (\langle\phi|_0 p \leq r \Rightarrow p \leq r) \\ &\Leftrightarrow \forall p.\ (p \leq |\phi]_0 r \Rightarrow p \leq r) \\ &\Leftrightarrow |\phi]_0 r \leq r \end{aligned}$$

where we used the Galois connection (4). Thus, it suffices to show that

$$\forall p.\ (RNP(\langle\phi|_0 p) \Rightarrow RNP(p))$$

in order to conclude that $r = 1_0$, by Noethericity of ϕ. This method constitutes formalised Noetherian induction for Boolean MKA.
Given $p \in K_0$, we denote by p_ϕ the element $\langle\phi|_0(p) = |\overline{\phi}\rangle_0(p)$. We have

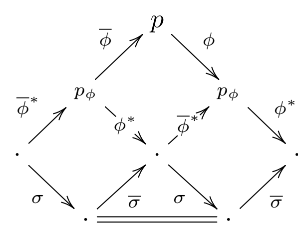
$$p\phi = p\phi r_0(p\phi) = p\phi\langle\phi|_0(p) \leq \phi p_\phi,$$

and similarly $\overline{\phi}p \leq p_\phi\overline{\phi}$. Using the star unfold axioms, we thereby deduce that

$$\overline{\phi}^{*_0} p\phi^{*_0} \leq \overline{\phi}^{*_0} p + \overline{\phi}^{*_0}\overline{\phi}p\phi\phi^{*_0} + p\phi^{*_0} \leq \overline{\phi}^{*_0} p + \overline{\phi}^{*_0} p_\phi\overline{\phi}\phi p_\phi\phi^{*_0} + p\phi^{*_0}.$$

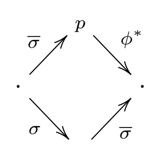
We first examine the middle summand:

$$\begin{aligned}
&\overline{\phi}^{*_0} p_\phi\overline{\phi}\phi p_\phi\phi^{*_0} \\
&\quad\leq \overline{\phi}^{*_0} p_\phi|A\rangle_1(\phi^{*_0}\overline{\phi}^{*_0})p_\phi\phi^{*_0} \\
&\quad\leq |\overline{\phi}^{*_0} p_\phi A p_\phi\phi^{*_0}\rangle_1(\overline{\phi}^{*_0} p_\phi\phi^{*_0}\overline{\phi}^{*_0} p_\phi\phi^{*_0}) \\
&\quad\leq |\hat{A}\rangle_1(\overline{\phi}^{*_0} p_\phi\phi^{*_0}\overline{\phi}^{*_0} p_\phi\phi^{*_0}) \\
&\quad\leq |\hat{A}\rangle_1(|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})\overline{\phi}^{*_0} p_\phi\phi^{*_0}) \\
&\quad\leq |\hat{A}\rangle_1(|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}\overline{\phi}^{*_0} p_\phi\phi^{*_0})) \\
&\quad\leq |\hat{A}\rangle_1(|\hat{A}^{*_1}\rangle_1(|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}\sigma\overline{\sigma}))) \\
&\quad\leq |\hat{A} \odot_1 \hat{A}^{*_1} \odot_1 \hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).
\end{aligned}$$



where we used that A is a local confluence filler for the first step, then commutation of modalities with whiskering (9) and the definition of $\hat{A}$ (10) for the second and third steps. We then use the induction hypothesis $RPN(p_\phi)$ on the left instance of $\overline{\phi}^{*_0} p_\phi\phi^{*_0}$, followed by commutation of modalities with whiskering and whisker absorption (11), and then repeat for the instance on the right. Finally, we used that $\hat{A} \odot_1 \hat{A}^{*_1} \leq \hat{A}^{*_1} \odot_1 \hat{A}^{*_1} \leq \hat{A}^{*_1}$, monotonicity of taking diamonds and $\overline{\sigma}\sigma = \mathrm{nf}_\phi = r(\sigma)$, a consequence of Proposition 2.
It remains to show that $\overline{\phi}^{*_0} p, p\phi^{*_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})$. First, observe that we have

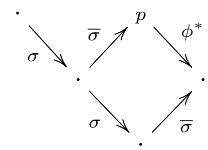
$$\begin{aligned}
\overline{\sigma}p\phi^{*_0} &= \overline{\sigma}p + \overline{\sigma}p\phi^{+_0} \\
&\leq \overline{\sigma} + (\mathrm{nf}_\phi + \overline{\phi}^{+_0})p\phi^{+_0} \\
&= \overline{\sigma} + \overline{\phi}^{+_0} p\phi^{+_0} \leq \sigma\overline{\sigma} + \overline{\phi}^{+_0} p\phi^{+_0} \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).
\end{aligned}$$



The first step is by the unfold axiom, the second uses Proposition 2 to bound $\overline{\sigma}$. The third step uses the fact that nf_ϕ is a left annihilator for ϕ^{+_0} since by definition we have $\mathrm{nf}_\phi \leq \neg d_0(\phi)$. Finally we use the fact that $\overline{\sigma} \leq \sigma\overline{\sigma}$ (12) coupled with $id_{K_1} = |1_1\rangle_1 \leq |\hat{A}^{*_1}\rangle_1$, *i.e.* reflexivity of $\hat{A}^{*_1}$, as well as the bound established by the previous calculation.

For convergent ϕ, we have $d_0(exh(\phi)) = d_0(\phi^{*_0}\neg d_0(\phi)) = 1_0$ by Lemma 1. Since σ is a skeleton of $exh(\phi)$, we have $d_0(\sigma) = 1_0$. By the converse axiom (2), this means that $\sigma\overline{\sigma} \geq 1_0$. Therefore,

$$
\begin{aligned}
p\phi^{*_0} &\leq \sigma\overline{\sigma}p\phi^{*_0}\\
&\leq \sigma|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})\\
&\leq |\hat{A}^{*_1}\rangle_1(\sigma\sigma\overline{\sigma}) = |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}),
\end{aligned}
$$



where we used commutation of whisker with modalities and whisker absorption, as well as $\sigma\sigma = \sigma$ (12). A symmetric argument yields $\overline{\phi}^{*_0}p \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})$, concluding the proof. □

The use of formalised Noetherian induction, as well as the calculation establishing the upper bound for the middle summand, are similar to those in the proof of Newman's lemma in [5]. Due to the fact that our result involves confluences in σ, the bounds for the outer summands require a different approach.

As a direct consequence of Theorem 1, we obtain the following result, which formalises Theorem A. Indeed, if (Φ, X) is a $(2,0)$-polygraph satisfying the corresponding hypotheses, Theorem 2 lifts the result to the power set when applied to $\phi := \{1_x \mid x \in \Phi_1\} \cup \{1_{1_a} \mid a \in \Phi_0\}$ and $A = X$, viewed as elements of $K(\Phi, X)$. Following the argument given in Section 5, the conclusion asserts that *for every* zig-zag sequence $f : a \to b \in \Phi_1^{\top}$, *there exists* a 2-cell $\alpha_f : f \Rightarrow \sigma_a \star_0 \sigma_b^-$ obtained by whiskering and composing elements of X. In a 2-groupoid, this is equivalent to the existence of a 2-cell $f \star_0 \sigma_b \Rightarrow \sigma_a$.

Theorem 2 (Abstract coherence theorem). *Let K be a Boolean globular 2-Kleene algebra satisfying the additional hypotheses in Theorem 1 and $\phi \in K_1$ convergent. Given a normalisation strategy σ and a local confluence filler A for $(\overline{\phi}, \phi)$, we have*

$$|\hat{A}^{*_1}\rangle_1(\sigma \odot_0 \overline{\sigma}) \geq \phi^{\top_0} = (\phi + \overline{\phi})^{*_0}.$$

Proof. We denote 0-multiplication by juxtaposition. As a result of Theorem 1 we have $|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \geq \overline{\phi}^{*_0}\phi^{*_0}$. By the star induction axiom, it suffices to show:

$$1_0 + (\phi + \overline{\phi})|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).$$

By (2) and Proposition 2, we have $\sigma\overline{\sigma} \geq d_0(\sigma) = 1_0$, so by reflexivity of $\hat{A}^{*_1}$, *i.e.* $1_1 \leq \hat{A}^{*_1}$, we have $1_0 \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})$. Furthermore, since $\phi \leq \overline{\phi}^{*_0}\phi^{*_0}$ we have:

$$\phi|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq \overline{\phi}^{*_0}\phi^{*_0}|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma})|\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}) \leq |\hat{A}^{*_1}\rangle_1(\sigma\overline{\sigma}).$$

The case of $\overline{\phi}$ is identical. We conclude via the star induction axiom. □

8 Outlook

In this article, we have introduced a formalisation of the notion of strategy for convergent ARS and thereby obtained an abstract coherence theorem. This constitutes an initial result formalising cofibrant replacements of algebraic structures by rewriting, such as polygraphic resolutions from convergent SRS, [15].

In this perspective, the first step is to formalise the critical branching lemma, a coherent confluence result for SRS. Kleene algebra axioms only allow iteration on the left or right of expressions, but not in context. We expect a formalisation of coherent confluence for SRS using the structure of higher-dimensional quantales [26], similar to higher-dimensional semirings [3] but in which multiplication distributes over arbitrary sums. The second step consists in extending our formalisation of normalisation strategies to higher dimensions, necessary for constructing cofibrant replacements, for example polygraphic resolutions via convergent rewriting systems [15].

Another direction is found in the domain of concurrency theory. Concurrent Kleene algebras (CKA) [19] are a convenient extension of Kleene algebras. While similar to 2-MKAs, these are used to give semantics to concurrent languages and their corresponding proof systems. CKAs enrich classical Kleene algebras with an extra parallel composition operation alongside the classical sequential composition. In particular, CKAs have applications for validation of concurrent programs by formalising Hoare-like proof systems for parallel computations, similarly to MKAs which have applications to verification of hybrid systems [30] and program correctness [11]. We expect that our approach to abstract coherence proofs in 2-Kleene algebras can also find applications to formalisation of proof systems for verifying general concurrent systems, for example based on higher-dimensional trace semantics of Higher-Dimensional Automata [25, 9] (a form of higher-dimensional rewriting system), see e.g. [7, 8].

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