Normalised Completion for Stratified Linear Rewriting Systems

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Abstract

We present a stratified normalisation completion procedure for rewriting systems over linear precategories. Our approach relies on a stratification of the set of rewriting rules according to their confluence and termination properties. We introduce stratified termination functions to establish termination for such systems. We illustrate the method on several classes of algebraic structures, including associative and diagrammatic algebras. In these contexts, we show how the stratification of defining rules can be used to compute hom-bases effectively.

1 Introduction

This work is part of a broader program on rewriting methods in algebra. Rewriting theory provides effective tools to address fundamental problems such as the word problem, the computation of linear bases [24] and resolutions [2, 22, 11] of algebraic structures, and the proof of coherence results [23, 8]. These methods apply to a wide range of algebraic structures, including monoids, algebras over operads, and higher-dimensional structures presented by linear operads or monoidal categories [1, 6, 20, 9]. Most approaches rely on transforming systems of algebraic equations into confluent rewriting systems via completion procedures.

In practice, one often needs to work modulo a set of axioms, either because no terminating orientation exists or to perform constructions modulo these axioms (e.g., for computing linear bases). Rewriting modulo has been extensively studied, both in rewriting theory [12, 19] and in the theory of Gröbner–Shirshov bases [5, 21]. Several completion procedures extend the principle of Knuth–Bendix completion [15] to rewriting modulo axioms [13, 18, 14]. A key insight from these approaches is that, in some situations, it is useful to stratify rewriting rules and compute completion layer by layer, taking into account the normalisation of the lower strata. This allows the overall problem to be decomposed into a series of local steps [18, 14].

We present a method for constructing convergent presentations of diagrammatic algebras arising from linear monoidal categories defined by generators and relations. Our goal is to compute linear hom-bases from such convergent presentations [1, 6, 20, 9]. The main difficulty is the complexity of these specifications, which involve many axioms together with intricate interactions between categorical and linear structures. Establishing termination and confluence is particularly challenging due to the two-dimensional combinatorial nature of the rewritten objects, namely 2-cells subject to interchange relations. Our approach is based on decomposing rewriting systems into strata according to termination and confluence properties. We develop techniques adapted to this setting to prove termination and perform completion layer by layer.

The present contribution focuses on the resulting stratified completion procedure. Our constructions are formulated for rewriting systems in linear higher precategories, presented by linear $(n+1)$-prepolygraphs, as recalled in Section 2. This framework encompasses, in particular, associative algebras and diagrammatic algebras. Section 3 introduces a normalised completion procedure based on a stratification of prepolygraphs. It proceeds stratum by stratum: at each step, context-induced reductions are normalised with respect to lower strata and oriented by a monomial order, yielding the next stratum. This normalisation makes the confluence property modular [4, Sec. 9.3]: if each normalised stratum is confluent, then their union is confluent. Finally, we illustrate the effectiveness of the procedure on algebraic examples.

2 Termination of stratified algebraic rewriting systems

Rewriting in precategories. Our results are formulated in the setting of free linear higher precategories [7], a linear version of precategories introduced in [9, 20]. For $n \geq 1$, a *linear n-precategory* is a linear n-category in which the interchange law is not imposed, that is, the $\circ_{n-1}$-composition of n-cells is not required to be functorial. We recall below the notation needed in the sequel and refer to the references above for the definitions.

In the sequel, $\mathcal{C}$ denotes a free linear n-precategory over a field $\Bbbk$ generated by a linear n-prepolygraph. Rewriting is performed modulo the underlying linear structure, in the spirit of Gröbner–Shirshov bases [10]. A *context* of $\mathcal{C}$ is a pair $(\square, C)$ consisting of an $(n-1)$-sphere $\square$ of $\mathcal{C}$ and an n-cell $C[\square]$ that contains exactly one occurrence of $\square$. From [7], every monomial n-cell h in $\mathcal{C}$ admits a unique decomposition $h = C_1[\alpha_1] \circ_{n-1} \cdots \circ_{n-1} C_m[\alpha_m]$, where α_i are generating n-cells in $\mathcal{C}$ and m is the *length of* h, denoted by $|h|$. Any n-cell f can be decomposed as a linear combination $f = \sum \lambda_i m_i$ of monomials m_i, with $\lambda_i \in \Bbbk$. We write $\mathrm{supp}(f)$ for the set of monomials occurring in f with nonzero coefficient. A *(cellular) extension* of $\mathcal{C}$ is a set X equipped with a map $\rho : X \to \mathrm{Sph}_n(\mathcal{C})$, where $\mathrm{Sph}_n(\mathcal{C})$ consists of pairs (f, g) of parallel n-cells. The elements of X are called *X-rewriting rules*, and are denoted by $\alpha : s\alpha \to t\alpha$. Without loss of generality, we may assume that the source of any rule is a monomial. We denote $\partial\alpha := s\alpha - t\alpha$. The *monomial extension of* X is $X^{\mathtt{m}} := \{(s\alpha, v) \mid \alpha \in X, v \in \mathrm{supp}(t\alpha)\}$.

An *X-rewriting step* $u \underset{X}{\to} v$ is an $(n+1)$-cell $\lambda C[\alpha] + 1_f$ in the free $(n+1)$-precategory $\mathcal{C}[X]$ generated by X on $\mathcal{C}$, where $\lambda \in \Bbbk \setminus \{0\}$, $\alpha \in X$, and $sC[\alpha] \notin \mathrm{supp}(f)$, such that $u = \lambda C[s\alpha] + f$ and $v = \lambda C[t\alpha] + f$. An *$X$-rewriting path* $u \underset{X}{\twoheadrightarrow} v$ is an $(n+1)$-cell in the $(n+1)$-precategory $\mathcal{C}[X]$ that is a composite $\sigma = \sigma_1 \circ_n \ldots \circ_n \sigma_p$ of X-rewriting steps such that $s\sigma_1 = u$ and $t\sigma_p = v$.

An n-cell f in $\mathcal{C}$ is in *X-normal form* if it cannot be reduced by any X-rewriting step. We denote by $\mathrm{Nf}(X)$ the set of X-normal forms in $\mathcal{C}$, and by $\mathrm{Nf}_m(X)$ its subset of monomials. An n-cell g is a *normal form of an n-cell f* if $f \underset{X}{\twoheadrightarrow} g$ and g is in X-normal form. In this case, we write $g = f{\downarrow_X}$. If X is *normalising*, an *X-normalisation strategy* is a map $(-){\downarrow_X}$ sending every n-cell f of $\mathcal{C}$ to a normal form $f{\downarrow_X}$. Any context C admits a unique decomposition $f_n \circ_{n-1} \cdots \circ_1 f_1 \circ_0 \square \circ_0 g_1 \circ_1 \cdots \circ_{n-1} g_n$, with i-cells f_i and g_i in $\mathcal{C}$. We write $C \in \mathrm{Nf}(X)$ if $f_n, g_n \in \mathrm{Nf}(X)$ and set the length $|C| := |f_n| + |g_n|$. A context C is *X-admissible* with α if $C \in \mathrm{Nf}(X)$ and $C[s\alpha] \notin \mathrm{Nf}(X)$. We denote the set of such contexts by $\mathbf{Cont}(\alpha, X)$. We define the *context relation* $\sqsubseteq$ on $\mathcal{C}$ by $\beta_1 \sqsubseteq \beta_2$ if $\beta_2 = C[\beta_1]$ for some context C. A *hom-set total order* $\preccurlyeq$ on $\mathcal{C}$ is a *monomial order* if it is monotone and well-founded. We write $\mathrm{lm}_{\preccurlyeq}(f)$ for the $\preccurlyeq$-maximal monomial in $\mathrm{supp}(f)$, and $\mathrm{lc}_{\preccurlyeq}(f)$ for its coefficient.

Stratified rewriting systems. A *filtration of length k* of an extension X on $\mathcal{C}$ is a nested sequence of subextensions $F^*X : F^1X \subseteq \ldots \subseteq F^kX = X$. We set $F^1_sX := F^1X$, and for $i \geq 2$, we set $F^i_sX := F^iX \backslash F^{i-1}X$, called the *$i$-th stratum* of F^*X, so that $X = F^1_sX \sqcup F^2_sX \sqcup \ldots \sqcup F^k_sX$. Such a decomposition is called a *stratification of* X, denoted by F^*_sX. An extension X equipped with F^*_sX is called a *k-stratified rewriting system*, k-StRS for short.

A k-StRS F^*_sX is *normalising* if F^i_sX is normalising for every $1 \leq i \leq k$. A *normalisation strategy* for a normalising k-StRS F^*_sX is a family of normalisation strategies $(-){\downarrow_i}$ on F^i_sX, for $1 \leq i \leq k$, denoted by $(-){\downarrow_*}$.

A *termination function* for a k-StRS $F_s^* X$ is a family of maps $(\delta_i : \mathcal{C}_n \to (Z_i, \preccurlyeq_i))_{1 \leq i \leq k}$, where each $(Z_i, \preccurlyeq_i)$ is a well-founded total preordered set, satisfying the following conditions:

i) $\delta_i(f) \preccurlyeq_i \delta_i(g)$ implies $\delta_i(C[f]) \preccurlyeq_i \delta_i(C[g])$, for all parallel n-cells f, g and context C in $\mathcal{C}$,

ii) $\delta_i(t\alpha) \prec_i \delta_i(s\alpha)$, for all $\alpha \in (F_s^i X)^{\mathfrak{m}}$ and $1 \leq i \leq k$,

iii) $\delta_i(t\beta) \preccurlyeq_i \delta_i(s\beta)$, for all $\beta \in (F^{i-1} X)^{\mathfrak{m}}$ and $2 \leq i \leq k$.

The following theorem gives a sufficient condition under which the termination property of an StRS is modular. Its proof is given in [9].

Theorem 1. *If a k-StRS $F_s^* X$ admits a termination function, then it is terminating.*

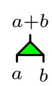
Example 1. Convergent rewriting systems have been widely used in algebra to compute linear bases of algebras [24]. Stratification provides a natural framework for addressing this problem. For instance, consider the associative algebra, seen as a linear 1-category with a single 0-cell, generated by x, y, z and presented by the rules

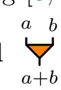
$$\gamma_1 : xy \to yx, \qquad \gamma_2 : yz \to zy, \qquad \gamma_3 : xz \to zx, \qquad \alpha : xyz \to x^2 + y^2 + z^2.$$

The two systems of rules $\{\gamma_1, \gamma_2, \gamma_3\}$ and $\{\alpha\}$ are individually confluent. To construct a convergent presentation from these rules, we consider the 2-StRS $F_s^* X$ generated by the 1-cells x, y, z, with

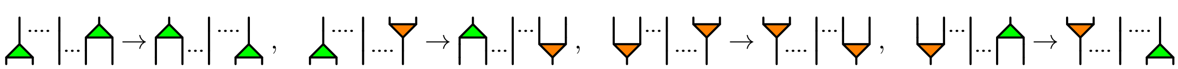
$$F_s^1 X = \{\gamma_1, \gamma_2, \gamma_3\} \qquad \text{and} \qquad F_s^2 X = \{\alpha\}.$$

In the next section, we study the modularity of confluence for this system.

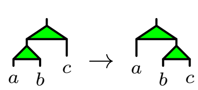
Example 2. We now present an example arising from monoidal categories, formulated in the diagrammatic language of three-dimensional rewriting [3, Chap. 10]. Let $\mathcal{C}$ be the free linear 2-category with a single 0-cell, generated by 

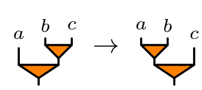
and 

, for all $a, b \in \mathbb{N}^+$, and presented by the following *interchange relations* $\mathbf{Int}(\mathcal{C})$:



We also impose the *associativity* and *coassociativity* relations

$\alpha_{a,b,c}$: 

, and $\beta_{a,b,c}$: 

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for all $a, b, c > 0$. The two systems of rules $\mathbf{Int}(\mathcal{C})$ and $\{\alpha_{a,b,c}, \beta_{a,b,c}\}$ are individually confluent. We therefore consider the linear 2-precategory presented by the 2-StRS $F_s^* X$ with

$$F_s^1 X = \mathbf{Int}(\mathcal{C}) \qquad \text{and} \qquad F_s^2 X = \{\alpha_{a,b,c}, \beta_{a,b,c}\}.$$

In the next section, we compute a linear hom-basis for the category $\mathcal{C}$ by considering the union of these two confluent rewriting extensions.

3 Stratified normalisation completion procedure

In this section, we introduce a normalisation completion procedure for an StRS.

Normalisation. A *normalisation* of a k-StRS F_s^*X, with respect to an X-compatible monomial order $\preccurlyeq$, is a k-StRS $F_s^*\widehat{X}$ defined inductively by the following procedure.

We set $F_s^1\widehat{X} \coloneqq F_s^1 X$. We assume that, for $1 \leq j \leq i$, $F^j\widehat{X}$ has been defined, and we choose a normalisation strategy $(-)\!\downarrow_i$ on $F^i\widehat{X}$. We then define the stratum $F_s^{i+1}\widehat{X}$ as follows.

For every $\alpha \in F_s^{i+1}X$ and context C such that $C[\partial\alpha]\!\downarrow_i \neq 0$, we define the rule $C[\alpha]\!\downarrow_i$ on $F^i\widehat{X}$-normal forms, by setting $sC[\alpha]\!\downarrow_i \coloneqq \mathrm{lm}_{\preccurlyeq}(C[\partial\alpha]\!\downarrow_i)$ and

$$tC[\alpha]\!\downarrow_i \coloneqq \begin{cases} \lambda^{-1}C[\partial\alpha]\!\downarrow_i \, + \omega, & \text{if } \omega \in \mathrm{supp}(C[t\alpha]\!\downarrow_i) \setminus \mathrm{supp}(C[s\alpha]\!\downarrow_i), \\ \lambda^{-1}C[-\partial\alpha]\!\downarrow_i \, + \omega, & \text{otherwise,} \end{cases}$$

where $\omega = sC[\alpha]\!\downarrow_i$ and $\lambda = \mathrm{lc}_{\preccurlyeq}(C[\partial\alpha]\!\downarrow_i)$. We define the following cellular extension on $\mathcal{C}$ consisting of such rules, one for each admissible context of α:

$$[\![\alpha]\!]\!\downarrow_i \coloneqq \{C[\alpha]\!\downarrow_i \mid C \in \mathbf{Cont}(\alpha, F^i\widehat{X})\}.$$

The rules in $[\![\alpha]\!]\!\downarrow_i$ are called the F^iX*-normal reductions* of α. Let $\min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_i$ denote the set of minimal rules in $[\![\alpha]\!]\!\downarrow_i$ with respect to the context relation $\sqsubseteq$. We then define

$$F_s^{i+1}\widehat{X} \coloneqq \bigsqcup_{\alpha \in F_s^{i+1}X} \min\nolimits_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_i \, .$$

We call this procedure the *stratified normalisation procedure*. Given a k-StRS F_s^*X equipped with a well-founded order $\preccurlyeq$, we denote by $\mathtt{SN}(F_s^*X, \preccurlyeq, \downarrow_*)$ the normalisation $F_s^*\widehat{X}$ obtained by applying this procedure to F_s^*X, with respect to a normalisation strategy $(-)\!\downarrow_*$ on $F_s^*\widehat{X}$.

Lemma 1. *For every $F_s^i\widehat{X}$-rewriting step $\sigma = \mu C[\alpha] + 1_f$, with $i > 1$, if $C[s\alpha]$ is $F^{i-1}\widehat{X}$-reducible and $F_s^i\widehat{X}$ is confluent, there exist two $F_s^i\widehat{X}$-rewriting paths τ and ρ (possibly identities) as in the side diagram.*

$$\begin{tikzcd} s\sigma \arrow[r, twoheadrightarrow, "F^{i-1}\widehat{X}"'] \arrow[d, "\sigma", "F_s^i\widehat{X}"'] & (s\sigma)\!\downarrow_{i-1} \arrow[dr, dotted, twoheadrightarrow, "\tau", "F_s^i\widehat{X}"'] & \\ & & e \\ t\sigma \arrow[r, twoheadrightarrow, "F^{i-1}\widehat{X}"'] & (t\sigma)\!\downarrow_{i-1} \arrow[ur, dotted, twoheadrightarrow, "\rho", "F_s^i\widehat{X}"'] & \end{tikzcd}$$

For the proof, see [9]. The following lemma shows the modularity of the confluence property of the k-StRS F_s^*X. Its proof is given in [9].

Lemma 2. *Let $1 \leq j \leq k$. If $F_s^i\widehat{X}$ is confluent for every $i \leq j$, then $F^j\widehat{X}$ is confluent.*

In particular, the extension $\widehat{X} = F^k\widehat{X}$ is confluent. Moreover, since any normalisation $F^*\widehat{X}$ is Tietze equivalent to F^*X, this procedure can be used to compute linear bases. The following theorem provides an instance of this approach for associative algebras and linear monoidal categories.

Theorem 2. *Let X be an StRS that presents an associative algebra (resp. a linear monoidal category) $\mathcal{C}$. If there exists an X-compatible monomial order $\preccurlyeq$ and a stratification F_s^*X such that each stratum of the normalisation $F_s^*\widehat{X}$ is confluent, then the set $\mathrm{Nf}_m(\widehat{X})$ forms a linear basis (resp. linear hom-basis) of $\mathcal{C}$.*

The stratified normalisation procedure does not terminate in general, due to the need to explore an infinite set of admissible contexts. For string rewriting, this set can be constructed recursively, as illustrated below. For diagrammatic rewriting, however, this becomes more difficult. We therefore introduce an effective way to explore this set. We decompose $\min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_i$ by the length of admissible contexts and define $F^n \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_i$ as the $\sqsubseteq$-minimal subset of

$$\{C[\alpha]\!\downarrow_i \mid\ C \in \mathbf{Cont}(\alpha, F^i\widehat{X}) \text{ and } |C| \leq n\}.$$

We have the following result. For the proof, see [9].

Proposition 1. *Let F_s^*X be an StRS and $\alpha \in F_s^iX$. Assume that $|s\beta| = |t\beta|$ for every $\beta \in F^{i-1}\widehat{X}$. If there exists a natural number n such that*

$$n \ \geq \max_{\beta \in F^{i-1}\widehat{X}} \{|s\beta| - |\alpha|\} \quad \text{and} \quad F^n \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_{i-1} = \ F^{n+1} \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_{i-1},$$

then we have $\min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_{i-1} = F^n \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_{i-1}$.

Stratified normalisation completion procedure. Combining stratified normalisation with the *Knuth–Bendix completion procedure* yields the following completion procedure for stratified rewriting systems.

Input:
A k-StRS F_s^*X ;
An X-compatible monomial order $\preccurlyeq$.
Output: A confluent stratified completion of F_s^*X.
$F_s^*\widehat{X} \leftarrow \mathtt{SN}(F_s^*X, \preccurlyeq, \downarrow_*)$
while $F_s^*\widehat{X}$ *is not confluent* **do**
  $I \leftarrow \{j \in \{1, \ldots, k\} \mid F_s^j\widehat{X} \text{ is not confluent}\}$
  for $j \in I$ **do**
    $F_s^j\widehat{X} \leftarrow \mathtt{KB}(F_s^j\widehat{X}, \preccurlyeq, \downarrow_j)$
  $F_s^*\widehat{X} \leftarrow \mathtt{SN}(F_s^*\widehat{X}, \preccurlyeq, \downarrow_*)$
return $F_s^*\widehat{X}$

By Theorem 2, whenever this procedure terminates, it returns a confluent StRS $F_s^*\widehat{X}$. We now illustrate it through several applications.

Example 3. Let F_s^*X be the 2-StRS introduced in Example 1, and let $\preccurlyeq$ be the lexicographic order on the alphabet $\{x, y, z\}$ with $z \prec y \prec x$. The normalisation $F_s^*\widehat{X}$ is obtained by setting $F_s^1\widehat{X} := F_s^1X$. As $\mathrm{Nf}(F^1\widehat{X}) = \{z^iy^jx^\ell \mid i, j, \ell \geq 0\}$, every admissible context is of the form $C = z^{i_1}y^{j_1}x^{\ell_1}\square z^{i_2}y^{j_2}x^{\ell_2}$, with $i_1, j_1, \ell_1, i_2, j_2, \ell_2 \geq 0$. By setting $i = i_1 + i_2$, $j = j_1 + j_2$, and $\ell = \ell_1 + \ell_2$, we deduce that

$$[\![\alpha]\!]\!\downarrow_1 = \ \{z^{i+1}y^{j+1}x^{\ell+1} \to z^iy^jx^{\ell+2} + z^iy^{j+2}x^\ell + z^{i+2}y^jx^\ell \mid i, j, \ell \geq 0\}$$

and we set

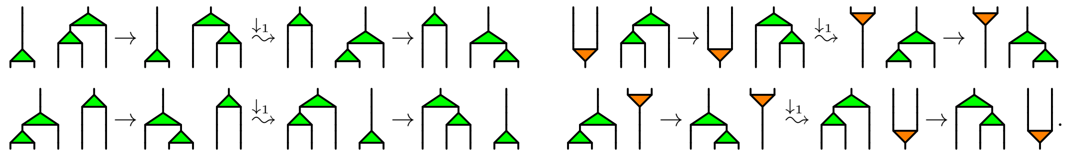
$$F_s^2\widehat{X} \ := \ \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_1 = \ \{\, zy^{j+1}x \to y^jx^2 + y^{j+2} + z^2y^j \mid j \geq 0\,\}.$$

The extension $F_s^1\widehat{X}$ is confluent and $F_s^2\widehat{X}$ has no critical branching, hence is confluent. By Theorem 2, the associative algebra presented by F_s^*X admits the linear basis $\mathrm{Nf}_m(\widehat{X}) = \{\, z^ix^j,\ y^ix^j,\ z^iy^j \mid i, j \geq 0\,\}$.

Example 4. Consider the 2-StRS F_s^*X introduced in Example 2. Let $\preccurlyeq$ be the lexicographic order on the alphabet $\{\ ,\ ,\ \}$ with $\prec \ \prec$ and $\prec$, $\prec$ and

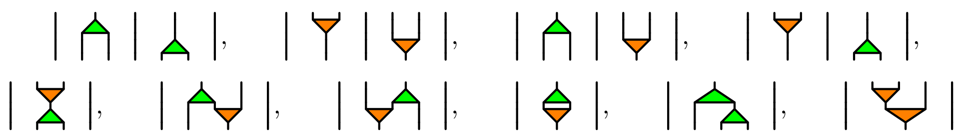
 , for all $a+b<c+d$, or $a+b=c+d$ and $b<d$, and $r_1<r_2$. We compute the normalisation with respect to this monomial order, first on 2-cells of length 1, and then extend it to arbitrary 2-cells [9].

We set $F^1_s\widehat{X} = F^1_s X$. Since $s\alpha \in \mathrm{Nf}(F^1\widehat{X})$, we have $F^0 \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_1 = \{\alpha\}$. For every context C of length 1, $C[s\alpha]$ belongs to $\mathrm{Nf}(F^1\widehat{X})$, except in the following four cases:



In each of these cases, the $\sqsubseteq$-minimal element of the F^1X-normal reduction is α. Thus $F^1 \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_1 = \{\alpha\} = F^0 \min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_1$. Since $|\alpha| = |\beta| = 2$, we obtain $\min_{\sqsubseteq}[\![\alpha]\!]\!\downarrow_1 = \{\alpha\}$ by Proposition 1. Similarly, $\min_{\sqsubseteq}[\![\beta]\!]\!\downarrow_1 = \{\beta\}$. Hence $F^2_s\widehat{X} = F^2_s X$.

Both $F^1_s\widehat{X}$ and $F^2_s\widehat{X}$ are confluent. By Theorem 2, the set $\mathrm{Nf}_m(\widehat{X})$ forms a hom-basis of $\mathcal{C}$; equivalently, every 2-cell admits a unique decomposition $f_1 \circ_1 f_2 \circ_1 \ldots \circ_1 f_n$ into 2-cells in $\mathcal{C}$ of length 1, such that each composite $f_i \circ_1 f_{i+1}$ is one of the following forms:



The two examples above illustrate how the stratified normalisation completion procedure can improve efficiency by decomposing the set of rules and analysing confluence stratum by stratum, thereby avoiding a global analysis of all critical branchings. In fact, the second example corresponds to a subcategory of the q-Schur category. In [9], we construct a 5-StRS presenting the q-Schur category and compute its normalisation and linear hom-basis, illustrating the effectiveness of our approach.

4 Conclusion

We have introduced a stratified normalisation completion procedure for linear rewriting, relying on a monomial order to orient newly generated rules. However, some rewriting systems do not admit any compatible monomial order (e.g. $xyz \to x^3+y^3+z^3$ [10]), while they may still be proved terminating by means of termination functions. This suggests developing a stratified normalisation procedure based on termination functions, independently of monomial orders.

Our constructions, formulated for rewriting in free linear precategories, could also be extended to the non-free setting by working modulo relations. This would require suitable stratifications, normalisation procedures, and termination functions for rewriting modulo.

Finally, the stratified normalisation procedure requires exploring a set of admissible contexts which is, in general, infinite and difficult to describe in higher dimensions. Two directions could be pursued to make the procedure more effective. The first consists of stratifying admissible contexts by their length, thereby yielding finite approximations, as initiated in Proposition 1. The second consists of describing admissible contexts by means of automata, as in the case of operated algebras [16, 17], in order to obtain a more efficient representation. These developments could also provide a basis for future implementations of the stratified normalisation procedure in computer algebra systems. Such implementations would be particularly valuable for diagrammatic algebras, which are often presented by many rewriting rules.

References

[1] Clément Alleaume. Rewriting in higher dimensional linear categories and application to the affine oriented Brauer category. *J. Pure Appl. Algebra*, 222(3):636–673, 2018.

[2] David J. Anick. On the homology of associative algebras. *Trans. Amer. Math. Soc.*, 296(2):641–659, 1986.

[3] Dimitri Ara, Albert Burroni, Yves Guiraud, Philippe Malbos, François Métayer, and Samuel Mimram. *Polygraphs: From Rewriting to Higher Categories*, volume 495 of *London Mathematical Society Lecture Note Series*. 2025. arXiv:2312.00429.

[4] Franz Baader and Tobias Nipkow. *Term rewriting and all that*. Cambridge University Press, 1998.

[5] Bruno Buchberger. History and basic features of the critical-pair/completion procedure. *J. Symbolic Comput.*, 3(1-2):3–38, 1987. Rewriting techniques and applications (Dijon, 1985).

[6] Benjamin Dupont. Rewriting modulo isotopies in Khovanov-Lauda-Rouquier's categorification of quantum groups. *Adv. Math.*, 378:Paper No. 107524, 75, 2021.

[7] Simon Forest and Samuel Mimram. Rewriting in gray categories with applications to coherence. *Mathematical Structures in Computer Science*, 32(5):574–647, 2022.

[8] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos. Coherent presentations of Artin monoids. *Compos. Math.*, 151(5):957–998, 2015.

[9] Stéphane Gaussent, Zuan Liu, and Philippe Malbos. Diagrammatic hom-bases by confluence and normalisation. Preprint, 2026.

[10] Yves Guiraud, Eric Hoffbeck, and Philippe Malbos. Convergent presentations and polygraphic resolutions of associative algebras. *Math. Z.*, 293(1-2):113–179, 2019.

[11] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351, 2012.

[12] Gérard Huet. Confluent reductions: abstract properties and applications to term rewriting systems. *J. Assoc. Comput. Mach.*, 27(4):797–821, 1980.

[13] Jean-Pierre Jouannaud and Helene Kirchner. Completion of a set of rules modulo a set of equations. In *Proceedings of the 11th ACM SIGACT-SIGPLAN Symposium on Principles of Programming Languages*, POPL '84, pages 83–92, New York, NY, USA, 1984. ACM.

[14] Jean-Pierre Jouannaud and Jianqi Li. Church-Rosser properties of normal rewriting. In *Computer science logic 2012*, volume 16 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages 350–365. 2012.

[15] Donald Knuth and Peter Bendix. Simple word problems in universal algebras. In *Computational Problems in Abstract Algebra*, pages 263–297. Pergamon, Oxford, 1970.

[16] Zuan Liu and Philippe Malbos. Polygraphic resolutions for operated algebras. Preprint arXiv:2502.16304, 2025.

[17] Zuan Liu, Zihao Qi, Yufei Qin, and Guodong Zhou. Gröbner-shirshov bases for free multi-operated algebras over algebras. *Journal of Symbolic Computation*, 133:102489, 2026.

[18] Claude Marché. Normalized rewriting: an alternative to rewriting modulo a set of equations. *J. Symbolic Comput.*, 21(3):253–288, 1996.

[19] Gerald E. Peterson and Mark E. Stickel. Complete sets of reductions for some equational theories. *J. Assoc. Comput. Mach.*, 28(2):233–264, 1981.

[20] Léo Schelstraete. Rewriting modulo in diagrammatic algebras and application to categorification. Preprint, arXiv:2502.03028, 2025.

[21] Anatoly I. Shirshov. Some algorithmic problems for Lie algebras. *Sib. Mat. Zh.*, 3:292–296, 1962.

[22] Craig C. Squier. Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, 49(1-2):201–217, 1987.

[23] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

[24] Victor A. Ufnarovskij. Combinatorial and asymptotic methods in algebra. In *Algebra, VI*, volume 57 of *Encyclopaedia Math. Sci.*, pages 1–196. Springer, Berlin, 1995.