Cubical coherent confluence, ω-groupoids and the cube equation

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Abstract – We study the confluence property of abstract rewriting systems internal to cubical categories. We introduce cubical contractions, a higher-dimensional generalisation of reductions to normal forms, and employ them to construct cubical polygraphic resolutions of convergent rewriting systems. Within this categorical framework, we establish cubical proofs of fundamental rewriting results – Newman's lemma, the Church–Rosser theorem, and Squier's coherence theorem – via the pasting of cubical coherence cells. We moreover derive, in purely categorical terms, the cube law known from the λ-calculus and Garside theory. As a consequence, we show that every convergent abstract rewriting system freely generates an acyclic cubical groupoid, in which higher-dimensional generators can be replaced by degenerate cells beyond dimension two.

Keywords – Coherence proofs, abstract rewriting systems, higher-dimensional rewriting, cubical categories, cubical contractions, cubical coherent confluence, cube law.

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1. Introduction

This work started from the study of n-branchings of rewriting paths in polygraphic resolutions and homotopical reduction-completion procedures of higher-dimensional rewriting systems [16, 19]. Such branchings can be regarded as computations starting in the same state. An important property of branching computations is confluence, which holds if these computations may eventually join in a common state. Higher-dimensional rewriting is usually based on strict ω-categories [2], wich compose cells of globular shape. Yet it often seems more natural to assemble confluence and other rewriting diagrams into higher-dimensional cubes. So why not use cubical categories instead for rewriting?

The relationship between rewriting theory [38] – a fundamental model of computation with far-reaching applications in mathematics and computer science – and higher globular categories is natural and well studied [2]. We consider it in its purest form through abstract rewriting systems, through (1-poly)graphs $\partial^-, \partial^+ : X_1 \longrightarrow X_0$, where X_0 is a set of 0-cells or vertices, X_1 is a set of 1-cells or directed edges, and ∂^-, ∂^+ are source and target maps relating them. A rewriting path or computation is then a morphism or 1-cell in the (free) path category generated by such a graph. Higher structure emerges in rewriting either through structured objects, or alternatively through relationships between rewriting paths and higher relationships between higher relationships. The free monoid used in string rewriting, for instance, is a category with a single 0-cell; rewriting steps then become 2-cells. Alternatively, in the left square below, the 2-cell A expresses a relationship between the rewriting paths along its faces.

$$\begin{array}{ccc} w & \xrightarrow{f} & x \\ {\scriptstyle g}\downarrow & A & \downarrow{\scriptstyle h} \\ y & \xrightarrow[k]{} & z \end{array} \qquad\qquad \begin{array}{ccc} w & \xrightarrow{f} & x \\ {\scriptstyle g}\downarrow & A(f,g) & \downarrow{\scriptstyle g|f} \\ y & \xrightarrow[f|g]{} & z \end{array}$$

The square on the right expresses confluence of the branching $y \xleftarrow{g} w \xrightarrow{f} x$ more specifically in the sense that the paths f, g can be extended from y and z to some common vertex z, the notation $A(f,g)$ indicating the existential dependency of its faces $f|g$ and $g|f$ on f and g. Likewise, confluences of n-branchings lead naturally to coherence n-cubes, which globular categories obviously model as globes.

Rewriting with higher cells requires higher-dimensional rewriting systems supplying generators, relations and rewriting paths in higher dimensions: so-called computads [36, 37] or polygraphs [10]. Polygraphic resolutions [2, 19, 32] then amount to the construction of higher-dimensional rewriting systems with desirable properties such as confluence and termination guarantees. When rewriting with structured objects, these can be obtained via reduction-completion procedures that resolve obstacles to confluence given by certain n-branchings [16]. These have been developed for resolving algebraic and categorical structures in homological algebra for categories [19, 32], associative algebras [18, 27] and operads [31], as well as for algebraic [16] and categorical [11] coherence proofs.

Proofs about rewriting systems are often presented in semi-formal diagrammatic style. The literature abounds in particular with diagrams gluing cubes [4, 38]. In higher-dimensional rewriting, this amounts to composing higher cells in the underlying categories.

The idea of using cubical categories for higher-dimensional rewriting is not new. A cubical approach has been pioneered by Lucas [28–30], building on Brown and Higgin's cubical categories [1, 8], which in turn add compositions to the cubical sets of Serre [34] and Kan [21]. Lucas has in particular proved the existence of cubical polygraphs, adapting ideas by Batanin [5] and Garner [15]. His polygraphs carry a monoidal structure to capture "string" rewriting with monoid objects. Using this formalism he has verified some standard confluence properties using cubical 2-polygraphs, and studied certain polygraphic resolutions for monoids. Our work is strongly influenced by his. Al-Agl, Brown and Steiner have shown that cubical categories with connection maps are equivalent to globular ones [1], which suggests that one may translate between these two approaches to higher-dimensional rewriting.

Higher confluence properties, in dimension 3 and with emphasis on cubes, have received longstanding interest in the rewriting literature, too. Lévy has derived a cube law in the λ-calculus, showing that all 3-branchings of certain rewriting paths of λ-terms extend around the edges of 3-dimensional confluence cubes [26]. Several sections in Barendregt's monograph on the λ-calculus [4] are devoted to this cube law and a theory of residuals akin to $f|g$ and $g|f$ in the diagram above. A comprehensive survey on the cube law in rewriting has been writen by Endrullis and Klop [14], including work by Klop himself, who has returned to 3-confluences and the cube law several times within four decades. Endrullis and Klop not only open up fascinating relationships with knot and Garside theory [12, 14], they also use the cube law as a hypothesis for a 3-confluence proof. By contrast, van Oostrom has recently sketched a combinatorial bricklaying procedure for 3-confluence proofs that is meant to satisfy the cube law by construction [39].

Here, we combine the two lines of work on cubical higher-dimensional rewriting and higher confluence proofs in the context of polygraphic resolutions of higher-dimensional cubical abstract rewriting systems, which we present as constructions of certain cubical ω-groupoids.

To this end, we first extend the framework of cubical higher-dimensional rewriting with contractions, which are essential for constructing cubical polygraphs with the rewriting properties desired. For this, we work with cubical (ω, p)-categories where cells in dimensions greater than $p+1$ are invertible. Their definitions are recalled in Section 2. Our notion of contraction, introduced in Section 3, is given by a family of lax transformations [1, 28, 29], a generalisation of natural transformations to cubical categories. Intuitively, contractions extend rewriting strategies to higher dimensions. For their definition, we first impose a quotient structure in dimension p on the underlying (ω, p)-category, and then define a section as a choice of a representative, for instance a normal form. Contractions extend this choice function recursively to higher dimensions. This leads to a notion of contracting cubical (ω, p)-category, in which all cells of dimension greater than $p + 1$ can be contracted. The main result in this context, Theorem 3.2.5, shows that every contracting $(\omega, 0)$-category (hence every cubical ω-groupoid) is acyclic, so that all boundaries with a cubical hole can be filled with a cell.

As examples of abstract cubical rewriting, we revisit some classical diagrammatic confluence proofs in higher dimensions as cubical cell compositions in Section 4 , including variants of Newman's lemma and the Church-Rosser theorem in two cubical directions. We also prove a variant of Squier's theorem [35], which requires contractions and can be seen as a low-dimensional version of Theorem 3.2.5 for confluent and terminating rewriting systems. In particular, we present a proof of Newman's lemma in three cubical directions without explicitly use of the cube law, as it is an immediate consequence of the geometry imposed by the axioms of cubical categories. Using contractions, we can even derive the cube law without involving coherence 3-cells. To simplify proofs, we use an internal abstract rewriting system in an (ω, p)-category, which can be seen as a generalisation of a polygraph.

Our final contribution, in Section 5, lies in the study of polygraphic resolutions of cubical categories. More specifically, we construct an acyclic cubical ω-groupoid from an abstract rewrite systems $\partial^-, \partial^+ : X_1 \to X_0$, using a normalisation strategy based on contractions. For this, we first introduce an explicit construction of cubical polygraphs and prove Theorem 5.1.3, a converse of Theorem 3.2.5, showing that free cubical ω-groupoids on polygraphs are acyclic if and only they are contracting. We then turn to polygraphic resolutions of confluent and terminating abstract rewriting systems, extending them recursively to acyclic ω-groupoids in Theorem 5.3, which involves studying their n-branchings. Finally, in Theorem 5.3.2, we refine this construction so that it generates no non-trivial higher cells in dimension greater than 2. This result confirms in a more structural way that the cube law does not require coherence 3-cells in our setting. For abstract rewriting systems, no cubes are needed, because homotopically, all cubes are empty.

In combination, these contributions shed in particular some light on the cube law and address a longstanding question in the rewriting community, which has been asked quite poignantly by Klop [22]: "*One would expect [...] in higher category theory [...] that the Cube Equation [...] would be very much present [...]. But it seems that the contrary is the case: nowhere [...] one encounters the Cube Equation or residual notions. (I would love to be corrected!) How come? [...] Is a fundamental notion as* ***confluence*** *a total stranger in categories?* ".

2. Preliminaries on Cubical Categories

Cubical categories, introduced by Brown and Higgins [7, 9], are cubical sets equipped with partial composition operations along the faces of higher-dimensional cubes, and with identity cells in every dimension. In this section, we adopt the axioms of Al-Agl, Brown and Steiner [1], augmented with the cell-invertibility structure introduced by Lucas [28], and we recall the notion of lax transformations of cubical categories—referred to as 1-fold left homotopies in [1]. Our setting is that of cubical ω-categories, possibly equipped with connections and inverses, as formalised in [17]. For each $n \in \mathbb{N}$, a cubical n-category is defined as the truncation of a cubical ω-category.

2.1. Cubical ω-categories

We henceforth assume that Greek letters α, β occurring as superscripts of operators range over $\{-,+\}$.

2.1.1. A *cubical ω-category* C consists of

- **i)** a family $(C_k)_{0\leqslant k}$ of sets of *k-cells of* C,
- **ii)** *face maps* $\partial^\alpha_{k,i} : C_k \to C_{k-1}$, for $1 \leqslant i \leqslant k$, satisfying the *cubical relations*

$$\partial^\alpha_{k-1,i}\partial^\beta_{k,j} = \partial^\beta_{k-1,j-1}\partial^\alpha_{k,i} \qquad (1 \leq i < j < k), \tag{2.1.2}$$

- **iii)** *degeneracy maps* $\varepsilon_{k,i} : C_{k-1} \to C_k$, for $1 \leqslant i \leqslant k$,
- **iv)** *composition maps* $\circ_{k,i} : C_k \times_{k,i} C_k \to C_k$, for $1 \leqslant i \leqslant k$, defined on the pullback $C_k \times_{k,i} C_k$ of the cospan $C_k \overset{\partial^+_{k,i}}{\to} C_{k-1} \overset{\partial^-_{k,i}}{\leftarrow} C_k$.

These data are subject to the relations listed in Appendix A.1.1. Throughout this paper we consider cubical ω-categories with

- **v)** *connection maps* $\Gamma^\alpha_{k,i} : C_{k-1} \to C_k$, for $1 \leqslant i < k$, satisfying the relations in Appendix A.1.2.

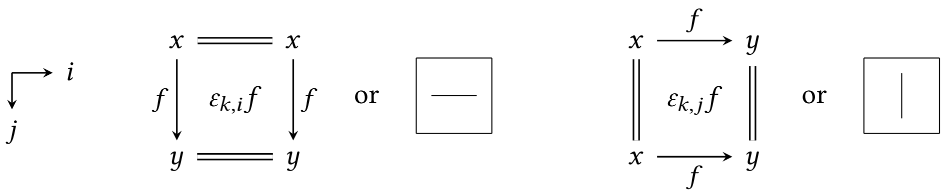
A *functor* $F : C \to \mathcal{D}$ *of cubical ω-categories* is a family of maps $(F_k : C_k \to \mathcal{D}_k)_{0\leqslant k}$ that preserve all face, degeneracy, composition and connection maps, see Appendix A.1.3.

All categories considered are cubical, so we drop this adjective wherever possible.

2.1.3. Any k-cell A and its faces can be represented, for $1 \leqslant i < j \leqslant k$, by the diagram

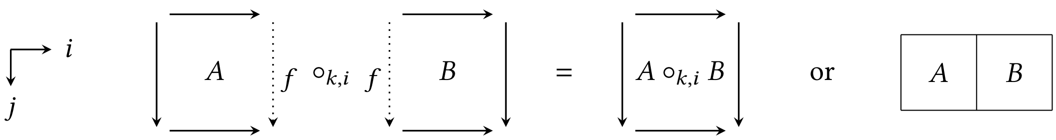
$$\begin{array}{ccc} & \xrightarrow{\; i\;} & \\ \big\downarrow j & & \end{array} \qquad \begin{array}{ccc} \partial^-_{k-1,i}\partial^-_{k,j}A & \xrightarrow{\partial^-_{k,i}A} & \partial^-_{k-1,i}\partial^+_{k,j}A \\ {\scriptstyle \partial^-_{k,j}A}\big\downarrow & A & \big\downarrow{\scriptstyle \partial^+_{k,j}A} \\ \partial^+_{k-1,i}\partial^-_{k,j}A & \xrightarrow[\partial^+_{k,i}A]{} & \partial^+_{k-1,i}\partial^+_{k,j}A \end{array}$$

The arrows on the left indicate the two directions along which the faces of the cell A are drawn. *Degeneracies*, cells in the codomains of degeneracy maps, are illustrated as follows, where boxes as those on the right have been introduced in [1]:



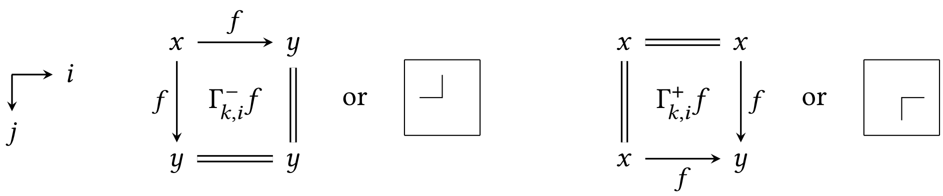
The arrows between the two copies of x or y are drawn as equality arrows to indicate that these faces are themselves degenerate.

The $\circ_{k,i}$-composition of two k-cells A, B in direction i glues these cells along i if the upper faces of the first cell in all other directions match the lower faces in all other directions of the second:

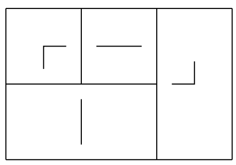


Such diagrams make it easy to check that the degeneracies $\varepsilon_{k,i}$ provide identities for the $\circ_{k,i}$-composition.

Connections are cells in the codomains of the connection maps $\Gamma_{k,i}$. Their diagrams are as follows [1]:



A cell in C is *thin* if it is a composite of degeneracies and connections [6, 7, 9]. An example is



We follow common practice and omit dimension indices k if suitable.

2.2. Cubical (ω, p)-categories and lax transformations

2.2.1. Invertibility. Invertible cubical cells were introduced by Brown and Higgins [9] to define cubical ω-groupoids. Here we start with more general definitions for cubical (ω, p)-categories [28]. A k-cell A of an ω-category C is R_i*-invertible*, for $1 \leqslant i \leqslant k$, if there is a k-cell B such that

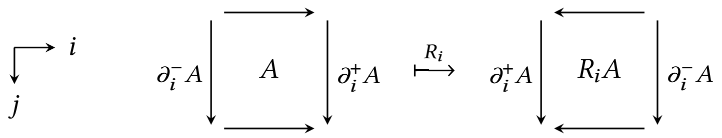
$$A \circ_i B = \varepsilon_i \partial_i^- A \qquad \text{and} \qquad B \circ_i A = \varepsilon_i \partial_i^+ A.$$

The k-cell B is thus uniquely defined and denoted $R_i A$, using the (partial) *inversion* map R_i. A k-cell A has an R_i*-invertible shell*, for $1 \leqslant i \leqslant k$, if

i) the cells $\partial^\alpha_j A$ are R_{i-1}-invertible, for every $1 \leqslant j < i$,

ii) the cells $\partial^\alpha_j A$ are R_i-invertible, for every $i < j \leqslant k$.

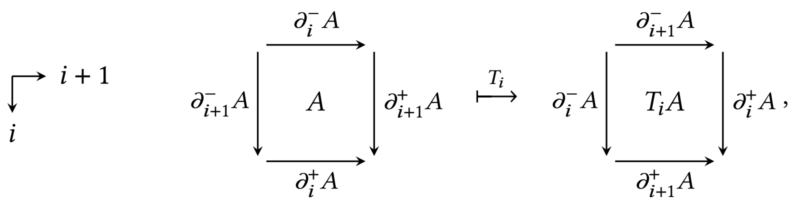
Inverting a k-cell A along direction i swaps the faces $\partial^-_i A$, $\partial^+_i A$ and inverts all other faces:



Using the map R_i, Lucas [28] introduced an alternative inversion map

$$T_iA := \left(\varepsilon_i\partial^-_{i+1}A \circ_{i+1} \Gamma^+_i\partial^+_iA\right) \circ_i \left(R_i\left(\Gamma^+_i\partial^-_{i+1}A \circ_{i+1} a \circ_{i+1} \Gamma^-_i\partial^+_{i+1}A\right)\right) \circ_i \left(\Gamma^-_i\partial^-_iA \circ_{i+1} \varepsilon_i\partial^+_{i+1}A\right),$$

for all $1 \leqslant i < k$ and every k-cell A. The T_i exchange the faces of a cell A between the directions i and $(i+1)$ while applying inversion maps to all other faces:



Additional properties of inversion maps, which are needed later, are listed in Appendix §A.1.4.

2.2.2. (ω, p)-categories and ω-groupoids. An (ω, p)*-category* C is an ω-category in which every k-cell with an R_i-invertible shell is R_i-invertible for all $k > p$ and $1 \leqslant i \leqslant k$. A *functor of* (ω, p)*-categories* is a functor between the underlying ω-categories. An ω*-groupoid* is an $(\omega, 0)$-category.

2.2.3. Lax transformations. We recall Lucas' definition of lax transformations (called *lax* 1*-transfors* by him) [28, 29]. They adapt natural transformations to cubical categories. We use them to define contractions of (ω, p)-categories in Section 3.1.

A *lax transformation* $\eta : F \Rightarrow G$ between (ω, p)-functors $F, G : C \to \mathcal{D}$ is a family of maps that sends each k-cell x in C to a $(k+1)$-cell η_x in $\mathcal{D}$, for every $k \in \mathbb{N}$. It satisfies, for all $1 \leqslant i \leqslant k$ and k-cells x, y in C,

i) if $i \neq 1$ then $\partial^-_1 \eta_x = F(x)$, $\partial^+_1\eta_x = G(x)$ and $\partial^\alpha_i \eta_x = \eta_{\partial^\alpha_{i-1}x}$,

ii) $\eta_{x \circ_i y} = \eta_x \circ_{i+1} \eta_y$ if x and y are i-composable,

iii) $\eta_{\varepsilon_i z} = \varepsilon_{i+1}\eta_z$ if $k < n-1$,

iv) $\eta_{\Gamma^\alpha_i z} = \Gamma^\alpha_{i+1}\eta_z$ if $i < k < n-1$.

Axiom i) indicates that σ_x is a transformation from $F(x)$ to $G(x)$, in the sense that its source and target faces in direction 1 are determined by $F(x)$ and $G(x)$, respectively. Its faces in the other directions are determined by the value of σ at the faces of x, suggesting that σ can be defined recursively in the dimensions. The shape of σ_x is

$$\begin{array}{ccc} F(\partial^-_{i-1}x) & \xrightarrow{F(x)} & F(\partial^+_{i-1}x) \\ {\scriptstyle \sigma_{\partial^-_{i-1}x}}\downarrow & \Downarrow \sigma_x & \downarrow{\scriptstyle \sigma_{\partial^+_{i-1}x}} \\ G(\partial^-_{i-1}x) & \xrightarrow[G(x)]{} & G(\partial^+_{i-1}x) \end{array} \qquad (\text{directions: } i \rightarrow,\ 1 \downarrow)$$

3. Cubical contractions and acyclicity

In this section, we introduce contractions for cubical categories, extending the corresponding notion for globular categories [19], and generalising the normalisation strategies of rewriting theory to higher dimensions. The main result in this section, Theorem 3.2.5, shows that contracting ω-groupoids are acyclic, providing a constructive method for proving acyclicity.

3.1. Contractions

Defining contractions for an (ω, p)-category C requires a notion of section, and in turn the construction of quotient p-categories on (ω, p)-categories.

3.1.1. The face maps in the coequaliser

$$C_{p+1} \underset{\partial^+_1}{\overset{\partial^-_1}{\rightrightarrows}} C_p \xrightarrow{\pi} \overline{C}_p$$

in the category Set compare the two faces of a $(p+1)$-cell in direction 1. We could have chosen any other direction i instead to construct $\overline{C}_p$, as the following lemma shows.

3.1.2. Lemma. *In every cubical (ω, p)-category C, the following coequalisers are equal for $2 \leqslant j \leqslant p+1$:*

$$C_{p+1} \underset{\partial^+_1}{\overset{\partial^-_1}{\rightrightarrows}} C_p \xrightarrow{\pi} \overline{C_p} \qquad \textit{and} \qquad C_{p+1} \underset{\partial^+_j}{\overset{\partial^-_j}{\rightrightarrows}} C_p \xrightarrow{\pi} \overline{C_p}.$$

Proof. Two p-cells f, g in C are in the same equivalence class of the second coequaliser if and only if there is a $(p+1)$-cell A in C such that, for all $1 \leqslant i \leqslant p+1$ such that $i \neq j$,

$$\partial^-_j A = f, \qquad \partial^+_j A = g, \qquad \partial^\alpha_i A = \begin{cases} \varepsilon_{j-1}\partial^\alpha_i f & \text{if } i < j, \\ \varepsilon_j \partial^\alpha_{i-1} f & \text{if } i > j. \end{cases}$$

These identities assemble to the diagram

$$\begin{array}{ccc} x & \xrightarrow{f} & y \\ \| & A & \| \\ x & \xrightarrow[g]{} & y \end{array} \qquad (\text{directions: } i \rightarrow,\ j \downarrow)$$

Let A be such a cell and define $B = R_1 T_2 \dots T_{j-1}\left(\Gamma^+_{(1\ j)} f \circ_j A \circ_j \Gamma^-_{(1\ j)} g\right)$ where $\Gamma^+_{(1\ j)}$ and $\Gamma^-_{(1\ j)}$ are extended connections defined as $\Gamma^\alpha_{(l\ m)} = T_{m-1} \dots T_{l+1} \Gamma^\alpha_l$, for all $l < m$. The faces of B in direction 1 are equal to f and g; all others are degenerate. Thus f and g are in the same equivalence class for the first coequaliser. The reverse direction is similar. □

3.1.3. Quotient category $\overline{C}_p$. We equip the set $\overline{C}_p$ with face, composition, degeneracy and connection maps. For the composition map $\circ_i$, for $i < p$, we write $X \times_{C_i} X$ for the pullback of $X \xrightarrow{\partial_i^-} C_i \xleftarrow{\partial_i^+} X$ for any set X. We use the coequaliser

$$C_{p+1} \times_{C_i} C_{p+1} \underset{\partial_1^+ \times \partial_1^+}{\overset{\partial_1^- \times \partial_1^-}{\rightrightarrows}} C_p \times_{C_i} C_p \longrightarrow \overline{C_p \times_{C_i} C_p} \simeq \overline{C}_p \times_{C_i} \overline{C}_p$$

($\simeq$ is unique because coequalisers and pullbacks commute in Set) to define $\circ_i : \overline{C}_p \times_{C_i} \overline{C}_p \to \overline{C}_p$ as the unique map for which the diagram

$$\begin{array}{ccccc} C_{p+1} \times_{C_i} C_{p+1} & \rightrightarrows & C_p \times_{C_i} C_p & \longrightarrow & \overline{C}_p \times_{C_i} \overline{C}_p \\ \downarrow \circ_i & & \downarrow \circ_i & & \vdots \circ_i \\ C_{p+1} & \rightrightarrows & C_p & \longrightarrow & \overline{C}_p \end{array}$$

commutes. Face, degeneracy and connection maps are defined likewise, using the universal property of the coequaliser $\overline{C}_p$. This extends C_{p-1} to an (ω, p)-category, also denoted $\overline{C}_p$. Its p-cells are equivalence classes modulo C_{p+1}, and it has degenerate and connection cells in dimensions higher than p.

3.1.4. Unital sections. The canonical projection (ω, p)-functor $\pi : C \to \overline{C}_p$ is an identity on k-cells for $k < p$. It sends p-cells to their equivalence classes in $\overline{C}_p$ and k-cells of dimension $k > p$ to degenerate cells. The *fibre* of π over a p-cell u in $\overline{C}_p$ extends to the $(\omega, 0)$-category C_u defined as follows:

i) its 0-cells are the p-cells x in C such that $\pi(x) = u$,

ii) its k-cells are the $(p+k)$-cells f in C such that $\partial^{\alpha_1}_{p+1,1} \partial^{\alpha_2}_{p+2,1} \dots \partial^{\alpha_k}_{p+k,1} f \in u$, for every $k \geqslant 1$,

iii) its face maps $\partial'^{\alpha}_{k,i}$ on C_u are the $\partial^\alpha_{p+k,i}$, for all $1 \leqslant i \leqslant k$,

iv) likewise for the degeneracy, connection and composition maps.

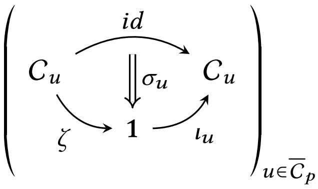
A *section* of the projection $\pi : C \to \overline{C}_p$ is a family

$$\iota = (\iota_u : \mathbf{1} \to C_u)_{u \in \overline{C}_p}$$

of $(\omega, 0)$-functors, where $\mathbf{1}$ is the terminal category in C. We only consider *unital* sections, which satisfy $\iota_{\pi(t)} = t$ for every thin p-cell t in C and for all $p \geqslant 1$, but usually omit this adjective.

The section ι sends each p-cell u in $\overline{C}_p$ to a functor ι_u with the representative p-cell of u in C in its image, while leaving all thin cells unchanged. We write ι_u for this representative of u as well. Moreover, for every k-cell f of C with $p \leqslant k$ we write $\widehat{f}$ for the image of $\iota_{\pi(f)}$ in $C_{\pi(f)}$ by abuse of notation,. Example diagrams for sections are given in §3.1.8.

3.1.5. Contractions. Let ι be a section of the projection $\pi : C \to \overline{C}_p$. A *$\iota$-contraction* of C is a family σ of lax transformations



where ζ is the unique $(\omega, 0)$-functor into $\mathbf{1}$, such that

$$\sigma_{\iota_u} = \varepsilon_1 \iota_u \qquad \text{and} \qquad \sigma_{\sigma_f} = \Gamma_1^- \sigma_f, \tag{3.1.6}$$

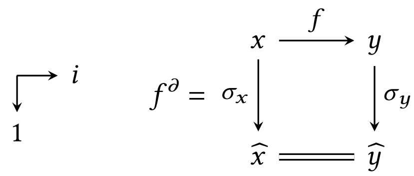
for each u in $\overline{C}_p$ and f in C_k with $p \leqslant k$, and where σ_g stands for $(\sigma_{\pi(g)})_g$ for each cell g in C_ℓ for $p \leqslant \ell$.

Expanding this definition, a ι-contraction σ is a family of maps $(C_k \to C_{k+1})_{k \geq p}$ such that for each k-cells f, g in C and every i with $p + 1 \leqslant i \leqslant k$, the conditions **i)-iv)** from §2.2.3 hold:

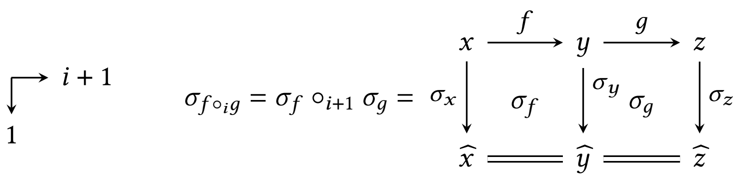
i) The *boundary* $\partial(\sigma_f)$ is the $(k-1)$-square f^∂ defined by

$$\partial_1^- f^\partial = f, \qquad \partial_1^+ f^\partial = \varepsilon_k \ldots \varepsilon_{p+1} \widehat{x}, \qquad \partial_i^\alpha f^\partial = \sigma_{\partial_{i-1}^\alpha f},$$

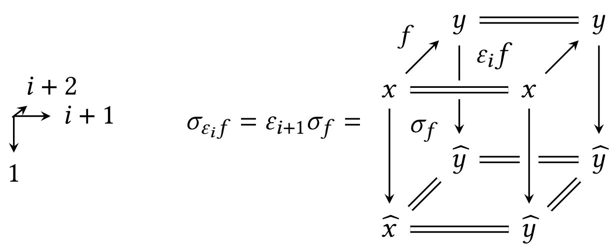
which yields the diagram



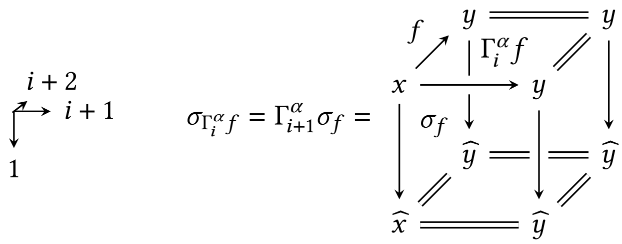
ii) If f and g are $\circ_i$-composable, then



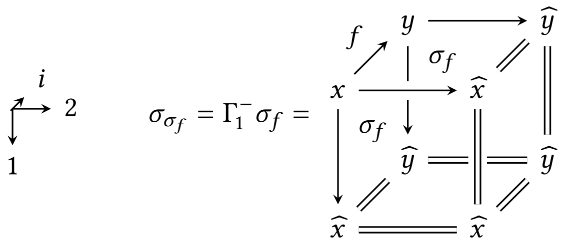
iii)



iv) If $i < k$, then



In addition, the second condition in (3.1.6) expands as follows: σ_{σ_f} is the thin cell



The first condition in (3.1.6) is equivalent to $\sigma_{\widehat{x}} = \varepsilon_1 \widehat{x}$ for each p-cell x in C:

$$\widehat{x} \overset{\sigma_{\widehat{x}}}{=\!=\!=} \widehat{x}$$

Examples of contractions in low dimensions are given in §3.1.8. Contractions, understood as families of lax transformations, can be computed recursively across all dimensions, starting from a chosen section. They are also compatible with inverses, as stated in the following lemma.

3.1.7. Lemma. *For every k-cell f with $p \leqslant k$, and for all $1 \leqslant i \leqslant k$ and $1 \leqslant j < k$,*

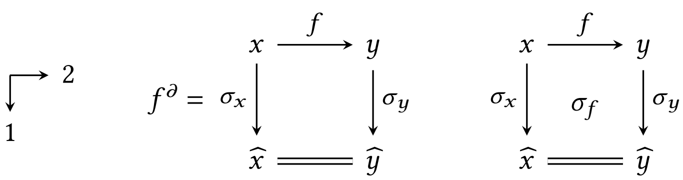
$$\sigma_{R_i f} = R_{i+1}\sigma_f \qquad \text{and} \qquad \sigma_{T_j f} = T_{j+1}\sigma_f.$$

Proof. For $\sigma_{R_i f} = R_{i+1}\sigma_f$, we check that $\sigma_{R_i f} \circ_{i+1} \sigma_f$ and $\sigma_f \circ_{i+1} \sigma_{R_i f}$ are thin cells. The claim then holds because thin cells with the same boundaries are equal [28]. The proof of $\sigma_{T_j f} = T_{j+1}\sigma_f$ is similar. □

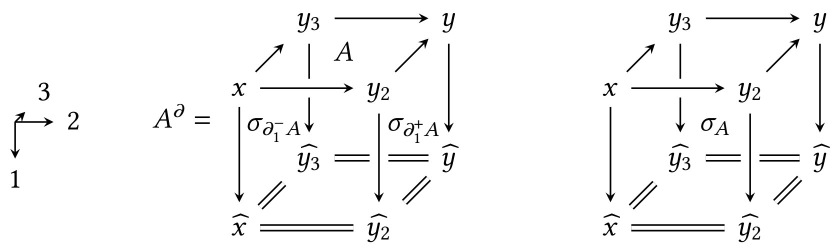
3.1.8. We present example diagrams for f^∂ and σ_f for a cell f of low dimension in an ω-groupoid C.

i) If $x \in C_0$, then x^∂ is the 0-square $(x, \widehat{x})$ and $\sigma_x : x \to \widehat{x}$ the 1-cell filling it.

ii) If $f \in C_1$, then f^∂ is an 1-square and σ_f a 2-cell filling it:



iii) If $A \in C_2$, then A^∂ is a 2-square and σ_A a 3-cell filling it:



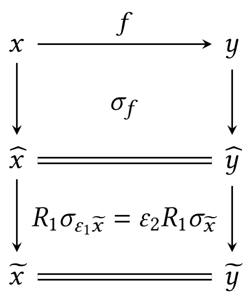
3.1.9. An (ω, p)-category is *contracting* if it admits a contraction. This property does not depend on particular choices of sections. For each $\widehat{(-)}$-contraction σ, we can define a $\widetilde{(-)}$-contraction τ such that, for every k-cell f, with $p \leqslant k < n$, the $(k+1)$-cell τ_f is the composition

$$\tau_f = \sigma_f \circ_1 R_1 \sigma_{\varepsilon_k \ldots \varepsilon_{p+1} \widetilde{x}} = \sigma_f \circ_1 \varepsilon_{k+1} \ldots \varepsilon_{p+2} R_1 \sigma_{\widetilde{x}},$$

where $x = \partial^-_{p+1} \ldots \partial^-_k f$. For $p = 0$ and $x \in C_0$, for instance, τ_x is the $\circ_1$-composition

$$x \xrightarrow{\sigma_x} \widehat{x} \xrightarrow{R_1 \sigma_{\widetilde{x}}} \widetilde{x}$$

and for $f \in C_1$, τ_f is the $\circ_1$-composition



3.2. Acyclic ω-groupoids

We now show that acyclicity of ω-groupoids can be obtained by constructing contractions. Our proof unfolds cubes into cubes with degenerate faces in each direction $i \geqslant 2$ using folding and unfolding maps [1, Def. 3.1].

3.2.1. Acyclicity. Defining acyclicity for a cubical (ω, p)-category C requires three further notions:

i) A k-*square* of C, for $k \geqslant 0$, is a family $(f_i^\alpha)_{1 \leqslant i \leqslant k+1, \alpha}$ of k-cells in C such that

$$\partial_i^\alpha f_j^\beta = \partial_{j-1}^\beta f_i^\alpha, \tag{3.2.2}$$

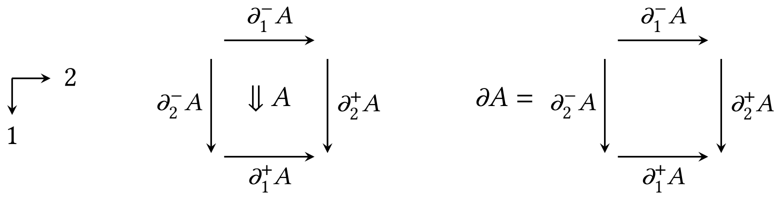
for all $1 \leqslant i < j \leqslant k+1$. We write $\mathrm{Sq}_k(C)$ for the set of k-squares of C.

ii) The *boundary* ∂A of a k-cell A in C, for $k \geqslant 1$, is the $(k-1)$-square $(\partial_i^\alpha A)_{1\leqslant i\leqslant k,\alpha}$.

iii) A *filler* of a k-square S is a $(k+1)$-cell A such that $\partial A = S$.

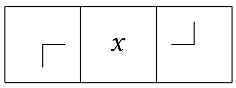
An (ω, p)-category C is *acyclic* if, for $k \geqslant p$, every k-square of C has a filler.

The following diagrams show a 2-cell A and its boundary 1-square:



3.2.3. Folding and unfolding. Let C be an ω-category. The *folding maps* $\psi_i, \Psi_j, \Phi_k : C_m \to C_m$ are defined, for $1 \leqslant i \leqslant m-1$, $1 \leqslant j \leqslant m$ and $0 \leqslant k \leqslant m$ as

$$\psi_i(x) = \Gamma_i^+ \partial_{i+1}^- x \circ_{i+1} x \circ_{i+1} \Gamma_i^- \partial_{i+1}^+ x =$$



,

$$\Psi_j = \begin{cases} id & \text{if } j = 1, \\ \psi_{j-1}\Psi_{j-1} & \text{otherwise} \end{cases} = \psi_{j-1}\psi_{j-2}\dots\psi_1,$$

$$\Phi_k = \begin{cases} id & \text{if } k = 0, \\ \Phi_{k-1}\Psi_k & \text{otherwise} \end{cases} = \Psi_1\Psi_2\dots\Psi_k = \psi_1(\psi_2\psi_1)\dots(\psi_{k-1}\dots\psi_1).$$

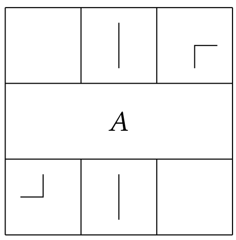
They extend to maps from $(m-1)$-squares to $(m-1)$-squares [1, Prop. 8.5].

Consider the sets

$$\mathsf{SqF}_{m-1}^{\varphi} = \{(S, A) \in \mathsf{Sq}_{m-1}(C) \times C_m \mid \partial A = \varphi(S)\}$$

of squares with corresponding fillers, for $\varphi \in \{\psi_i, \Psi_j, \Phi_k\}$. The *unfolding maps* $\overline{\psi}_i : \mathsf{SqF}_{m-1}^{\psi_i} \to C_m$, $\overline{\Psi}_j : \mathsf{SqF}_{m-1}^{\Psi_j} \to C_m$ and $\overline{\Phi}_k : \mathsf{SqF}_{m-1}^{\Phi_k} \to C_m$ are defined as

$$\overline{\psi}_i(S, A) = (\varepsilon_i S_i^- \circ_{i+1} \Gamma_i^+ S_{i+1}^+) \circ_i A \circ_i (\Gamma_i^- S_{i+1}^- \circ_{i+1} \varepsilon_i S_i^+) =$$



$$\overline{\Psi}_j(S, A) = \begin{cases} A & \text{if } j = 1, \\ \overline{\Psi}_{j-1}(S, \overline{\psi}_{j-1}(\Psi_{j-1}(S), A)) & \text{otherwise,} \end{cases}$$

$$\overline{\Phi}_k(S, A) = \begin{cases} A & \text{if } k = 0, \\ \overline{\Psi}_k(S, \overline{\Phi}_{k-1}(\Psi_k(S), A)) & \text{otherwise.} \end{cases}$$

3.2.4. Lemma. *Every folding or unfolding map $\overline{\varphi} \in \{\overline{\psi}_i, \overline{\Psi}_j, \overline{\Phi}_k\}$ satisfies $\partial\overline{\varphi}(S, A) = S$, for every $(m-1)$-square S and m-cell A such that $\partial A = \varphi(S)$.*

Proof. The proof of $\overline{\psi}_i$ is straightforward. Those for $\overline{\Psi}_j$ and $\overline{\Phi}_k$ follow by induction. □

We are now prepared for the main result of this section.

3.2.5. Theorem. *Every contracting ω-groupoid is acyclic.*

Proof. Suppose C is an ω-groupoid with a section $\widehat{(-)}$ of the projection $\pi : C \to \overline{C}_0$ and a contraction σ. For $m \geqslant 2$, let S be an $(m-1)$-square. We set $T = \Phi_m(S)$, $g^\alpha = T_1^\alpha$ and $A = \sigma_{g^-} \circ_1 R_1\sigma_{g^+}$. Then

$$T_k^\alpha = \varepsilon_1 \partial_1^- T_k^\alpha = \varepsilon_1 \partial_1^+ T_k^\alpha = \varepsilon_1 \partial_k^\alpha g^- = \varepsilon_1 \partial_k^\alpha g^+$$

for every $1 < k \leqslant m$ by [1, Prop. 3.6]. It follows that $\partial_k^\alpha g^- = \partial_k^\alpha g^+$, for every $1 < k \leqslant m$. Hence A is a filler of T, because, for $1 < k \leqslant m$,

$$\partial_k^\alpha A = \partial_k^\alpha \sigma_{g^-} \circ_1 R_1 \partial_k^\alpha \sigma_{g^+} = \sigma_{\partial_{k-1}^\alpha g^-} \circ_1 R_1 \sigma_{\partial_{k-1}^\alpha g^+} = \sigma_{\partial_{k-1}^\alpha g^-} \circ_1 R_1 \sigma_{\partial_{k-1}^\alpha g^-} = \varepsilon_1 \partial_1^- \sigma_{\partial_{k-1}^\alpha g^-} = T_k^\alpha,$$

and the case $k = 1$ is obvious. Finally, set $B = \overline{\Phi}_m(S, A)$. By the above calculation and Lemma 3.2.4, $\partial B = S$, that is, B is a filler of S and acyclicity of C follows. □

3.2.6. The case $n = 2$. Theorem 3.2.5 remains valid for n-groupoids with $n \geqslant 2$. First, the definitions of sections and contraction in §3.1.4 and §3.1.5 extend to n-groupoids, forgetting all cells of dimension greater than n. The proof replays that for ω-groupoids, except that only $(m-1)$-squares with $2 \leqslant m \leqslant n$ require consideration. As an example, we show that every contracting 2-groupoid C is acyclic. Suppose C has a $\widehat{(-)}$-contraction σ. We start with a 1-square

$$S = \begin{array}{ccc} a & \xrightarrow{S_1^-} & b \\ {\scriptstyle S_2^-}\big\downarrow & & \big\downarrow{\scriptstyle S_2^+} \\ c & \xrightarrow[S_1^+]{} & d \end{array} .$$

The folding maps yield the 1-square

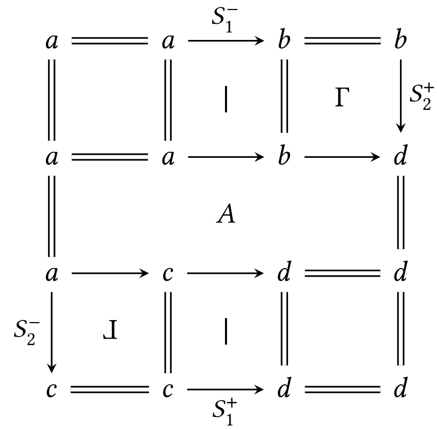
$$T = \Phi_2(S) = \Psi_2(S) = \psi_1(S) = \begin{array}{ccccccc} a & = & a & \xrightarrow{S_1^-} & b & \xrightarrow{S_2^+} & d \\ \Vert & \Gamma & \big\downarrow & S & \big\downarrow & \lrcorner & \Vert \\ a & \xrightarrow[S_2^-]{} & c & \xrightarrow[S_1^+]{} & d & = & d \end{array} = \begin{array}{ccccc} a & \xrightarrow{S_1^-} & b & \xrightarrow{S_2^+} & d \\ \Vert & & & & \Vert \\ a & \xrightarrow[S_2^-]{} & c & \xrightarrow[S_1^+]{} & d \end{array} .$$

The contraction σ fills the 1-square T with

$$A = \begin{array}{ccc} a & \xrightarrow{T_1^- = S_1^- \circ_1 S_2^+} & d \\ {\scriptstyle \sigma_x}\big\downarrow & \sigma_{T_1^-} & \big\downarrow{\scriptstyle \sigma_{x'}} \\ \widehat{a} & = & \widehat{d} \\ {\scriptstyle \sigma_x}\big\uparrow & R_1\sigma_{T_1^+} & \big\uparrow{\scriptstyle \sigma_{x'}} \\ a & \xrightarrow[T_1^+ = S_2^- \circ_1 S_1^+]{} & d \end{array} = \begin{array}{ccc} a & \xrightarrow{T_1^-} & d \\ \Vert & \sigma_{T_1^-} \circ_1 R_1\sigma_{T_1^+} & \Vert \\ a & \xrightarrow[T_1^+]{} & d \end{array} .$$

The unfolding maps then allow us to construct the following filler of S, showing that C is acyclic:

$$B = \overline{\Phi}_2(S,A) = \overline{\Psi}_2(S,A) = \overline{\psi}_1(S,A) =$$



Lucas has established a variant of Theorem 3.2.5 for cubical monoidal $(2,0)$-polygraphs [30]. Instead of using folding and unfolding maps, he rotates cells with the same shape as contractions with the inversion maps R_i and T_i, and then glues them using connection maps. Folding and unfolding maps seem to make the proof for cubical ω-groupoids easier. These maps rotate all the faces of cubes in direction 1, so that the proof does not become more difficult with increasing dimension.

4. Cubical coherent confluence

We now use the cubical machinery introduced in the previous section to establish confluence properties of abstract rewriting systems (ARS) in cubical $(p+2)$- or $(p+3)$-categories, for any $p \in \mathbb{N}$. Although cubes have $(p+2)$ dimensions, we restricted rewriting relations in two or three fixed directions. Apart from coherent versions of Newman's lemma and the Church-Rosser theorem in two directions, we also prove Newman's lemma also in three directions, for which additional structure was present or a specific cube law had to be imposed previously [4, 14, 24, 26]. The coherence in these results expresses the way to tile confluence or local confluence diagrams by pasting a given set of higher-dimensional witnesses. Finally, as a special case of Theorem 3.2.5, we derive a cubical version of Squier's theorem, using normalisation strategies as special kinds of sections and contractions. We assume familiarity with the basics of classical rewriting [3, 13, 23, 38].

4.1. Confluence fillers

4.1.1. Abstract rewriting in cubical categories. Let C be a $(p+2)$-category for some $p \in \mathbb{N}$. We fix an integer i such that $1 \leqslant i \leqslant p-1$, representing a choice of direction. A *p-ARS in C* is a subset $\mathcal{X}_C$ of C_{p+1}, whose elements are non-degenerate in direction i. We write $\mathcal{X}_C^{\circ_i}$ (resp. $\mathcal{X}_C^{\top_i}$) for the smallest subsets of C_{p+1} that contain $\mathcal{X}_C$ and are stable under $\circ_i$-compositions (resp. $\circ_i$-compositions and inversions). The elements of $\mathcal{X}_C^{\circ_i}$ are sequences $(f_1, \ldots, f_k)$ of $\circ_i$-composable $(p+1)$-cells in C, called *rewriting paths of length k*, which we identify with their composite $f_1 \circ_i \ldots \circ_i f_k$ in C. The elements of $\mathcal{X}_C^{\top_i}$ are sequences of $\circ_i$-composable $(p+1)$-cells in C and their R_i-inverses, called *rewriting zigzags.*

The p-ARS $\mathcal{X}_C$ is *Noetherian* (in direction i) if it admits no rewriting path of infinite length. This property is needed for proofs by induction on rewriting paths.

A *branching (in direction i)* of $\mathcal{X}_C$ is a pair (f_1, f_2) of $(p+1)$-cells in $\mathcal{X}_C^{\circ_i}$ such that $\partial_i^- f_1 = \partial_i^- f_2$. It is *local* if $f_1, f_2 \in \mathcal{X}_C$. We denote by $\mathsf{B}(\mathcal{X}_C)$ (resp. $\mathsf{LB}(\mathcal{X}_C)$) the set of branchings (resp. local branchings) of $\mathcal{X}_C$. The p-ARS $\mathcal{X}_C$ is *(locally) confluent (in direction i)* if for every (local) branching (f_1, f_2) of $\mathcal{X}_C$, there are $g_1, g_2 \in \mathcal{X}^{\circ_i}$ such that

$$\partial_i^+ f_1 = \partial_i^- g_1, \qquad \partial_i^+ f_2 = \partial_i^- g_2, \qquad \partial_i^+ g_1 = \partial_i^+ g_2.$$

These identities determine the *confluence diagram*

$$\begin{array}{ccc} x & \xrightarrow{f_2} & y_2 \\ {\scriptstyle f_1}\big\downarrow & & \big\downarrow{\scriptstyle g_2} \\ y_1 & \xrightarrow[g_1]{} & z \end{array} \qquad (i \downarrow,\ i+1 \rightarrow) \tag{4.1.2}$$

The p-ARS $\mathcal{X}_C$ is *convergent* if it is confluent and Noetherian.

4.1.3. Confluence fillers. A *(local) confluence filler (in direction i)* of a (local) branching (f_1, f_2) of $\mathcal{X}_C$ is a $(p+2)$-cell $A_2(f_1, f_2)$ in C such that

$$\partial_i^- A_2(f_1, f_2) = f_2, \qquad \partial_{i+1}^- A_2(f_1, f_2) = f_1, \qquad \partial_i^+ A_2(f_1, f_2), \partial_{i+1}^+ A_2(f_1, f_2) \in \mathcal{X}_C^{\circ_i}.$$

This determines a (local) confluence diagram similar to (4.1.2):

$$\begin{array}{ccc} x & \xrightarrow{f_2} & y_2 \\ {\scriptstyle f_1}\big\downarrow & A_2(f_1, f_2) & \big\downarrow{\scriptstyle \partial_{i+1}^+ A_2(f_1,f_2)} \\ y_1 & \xrightarrow[\partial_i^+ A_2(f_1,f_2)]{} & z \end{array} \qquad (i \downarrow,\ i+1 \rightarrow)$$

We write $\mathsf{LCf}(\mathcal{X}_C)$ (resp. $\mathsf{Cf}(\mathcal{X}_C)$) for the subset of C_{p+2} of cells with the shape of a local confluence filler (resp. confluence filler), that is, $(p+2)$-cells A such that

$$\partial_i^+(A), \partial_{i+1}^+(A) \in \mathcal{X}_C^{\circ_i}, \qquad \partial_i^-(A), \partial_{i+1}^-(A) \in \mathcal{X}_C, \quad (\text{resp. } \partial_i^-(A), \partial_{i+1}^-(A) \in \mathcal{X}_C^{\circ_i}).$$

Therefore, A_2 defines a map $A_2 : \mathsf{LB}(\mathcal{X}_C) \to \mathsf{LCf}(\mathcal{X}_C)$ (resp. $A_2 : \mathsf{B}(\mathcal{X}_C) \to \mathsf{Cf}(\mathcal{X}_C)$).

We can now state and prove a coherent cubical version of Newman's lemma.

4.1.4. Proposition. *For a Noetherian p-ARS $\mathcal{X}_C$, each map A_2 extends from $\mathsf{LB}(\mathcal{X}_C) \to \mathsf{LCf}(\mathcal{X}_C)$ to $\mathsf{B}(\mathcal{X}_C) \to \mathsf{Cf}(\mathcal{X}_C)$.*

Proof. We extend the map A_2 by Noetherian induction in direction i on the source of branchings. We order p-cells by the relation $\preccurlyeq$ generated by $\mathcal{X}_C$, defined by $x \preccurlyeq y$ if there is a rewriting path f such that $\partial_i^- f = y$ and $\partial_i^+ f = x$.

The base case is trivial. For the induction step, le (f_1, f_2) be a branching. If f_1 is a degeneracy in direction i, the result is trivial, as the map A_2 is extended by the formula $A_2(f_1, f_2) = \varepsilon_i f_2$. The case where f_2 is a degeneracy in direction i is similar. In the other cases, we decompose $f_1 = g_1 \circ_i h_1$ and $f_2 = g_2 \circ_i h_2$, with $g_1, g_2 \in \mathcal{X}_C$ and $h_1, h_2 \in \mathcal{X}_C^{\circ_i}$ and extend A_2 recursively as

$$A_2(f_1, f_2) = \big(A_2(g_1, g_2) \circ_{i+1} A_2(\partial_i^+ A_2(g_1, g_2), h_2)\big) \circ_i A_2(h_1, \partial_i^+ \big(A_2(g_1, g_2) \circ_{i+1} A_2(\partial_i^+ A_2(g_1, g_2), h_2)\big)).$$

This pasting of cubes ressembles the classical diagrammatic proof of Newman's lemma:

$$\begin{array}{ccccc} x & \xrightarrow{\;g_2\;} & x' & \xrightarrow{\;h_2\;} & x'' \\ {\scriptstyle g_1}\downarrow & A_2(g_1, g_2) & \downarrow & A_2(\partial_i^+ A_2(g_1, g_2), h_2) & \downarrow \\ y & \longrightarrow & y' & \longrightarrow & y'' \\ {\scriptstyle h_1}\downarrow & \multicolumn{3}{c}{A_2(h_1, \partial_i^+ \big(A_2(g_1, g_2) \circ_{i+1} A_2(\partial_i^+ A_2(g_1, g_2), h_2)\big))} & \downarrow \\ z & \multicolumn{3}{c}{\longrightarrow} & z' \end{array} \qquad (i \downarrow,\ i+1 \rightarrow) \tag{4.1.5}$$

□

4.1.6. Church-Rosser fillers. A *Church-Rosser filler (in direction i)* of a cell f in $\mathcal{X}_C^{\top_i}$ is a $(p+2)$-cell $B(f)$ in C such that

$$\partial_i^- B(f) = f, \qquad \partial_i^+ B(f) = \varepsilon_i \partial_i^+ \partial_{i+1}^+ B(f), \qquad \partial_{i+1}^- B(f), \partial_{i+1}^+ B(f) \in \mathcal{X}^{\circ_i},$$

which determines the *Church-Rosser diagram*

$$(i \downarrow,\ i+1 \rightarrow) \qquad \begin{array}{ccc} x & \xrightarrow{\;f\;} & y \\ {\scriptstyle \partial_{i+1}^- B(f)}\downarrow & B(f) & \downarrow{\scriptstyle \partial_{i+1}^+ B(f)} \\ z & = & z \end{array}$$

Once again this correspondence defines a map $B : \mathcal{X}_C^{\top_i} \to \mathsf{CR}(\mathcal{X}_C)$, were $\mathsf{CR}(\mathcal{X}_C)$ denotes the subset of C_{p+2} of cells of the shape of a Church-Rosser fillers.

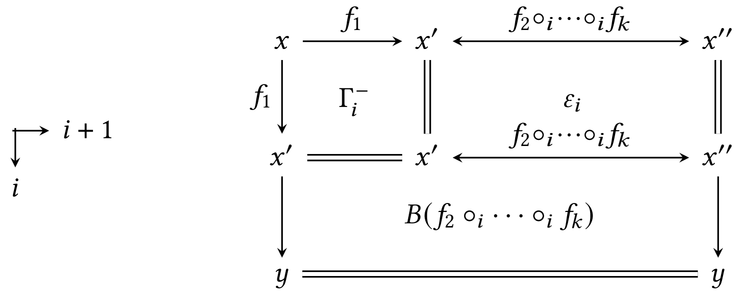
With these definitions, we prove a coherent cubical version of the Church-Rosser theorem.

4.1.7. Proposition. *For a p-ARS $\mathcal{X}_C$ in a $(p+2, p+1)$-category C, each map $A_2 : \mathsf{B}(\mathcal{X}_C) \to \mathsf{Cf}(\mathcal{X}_C)$ induces a map $B : \mathcal{X}_C^{\top_i} \to \mathsf{CR}(\mathcal{X}_C)$.*

Proof. Every cell f in $\mathcal{X}_C^{\top_i}$ is an zigzag $f_1 \circ_i \cdots \circ_i f_k$ of minimal length k of non-$\circ_i$-identity cells in $\mathcal{X}_C^{\circ_i}$ and of R_i-inverses of such cells. We define the map B on cells of $\mathcal{X}_C^{\top_i}$ by induction on their length k. The base case $k = 1$ is trivial. For the induction step, for $k \geqslant 2$ and $f_1 \in \mathcal{X}_C^{\circ_i}$, we extend B recursively:

$$B(f) = \big(\Gamma_i^- f_1 \circ_{i+1} \varepsilon_i(f_2 \circ_i \cdots \circ_i f_k)\big) \circ_i B(f_2 \circ_i \cdots \circ_i f_k),$$

which corresponds to the diagram

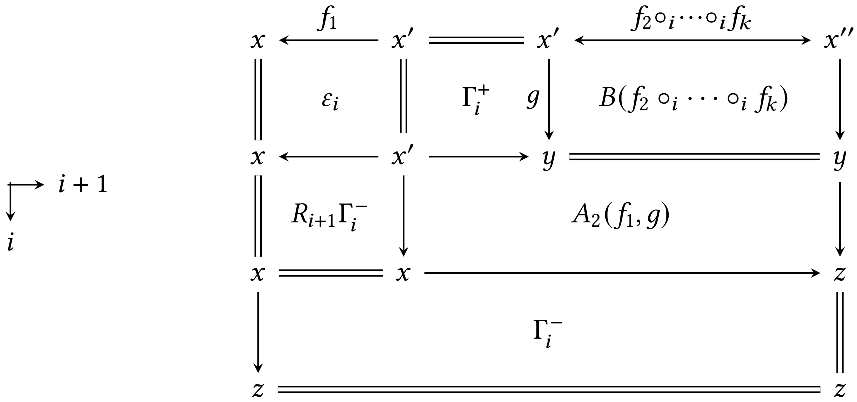


(4.1.8)

Otherwise, for $k \geqslant 2$ and $R_i f_1 \in \mathcal{X}_C^{\circ_i}$, we extend B recursively using the map A_2:

$$B(f) = \big(\varepsilon_i f_1 \circ_{i+1} \Gamma_i^+ f_1 \circ_{i+1} B(f_2 \circ_i \cdots \circ_i f_k)\big) \circ_i \big(R_{i+1}\Gamma_i^- f_1 \circ_{i+1} A_2(f_1, g)\big) \circ_i \Gamma_i^- \partial_i^+ A_2(f_1, g),$$

where g denotes the $(p+1)$-cell $\partial_{i+1}^- B(f_2 \circ_i \cdots \circ_i f_k)$. This corresponds to the diagram



(4.1.9)

□

The diagrams in the proof of the coherent Church-Rosser theorem reduce to the familiar triangular shapes in the classical diagrammatic Church-Rosser proof once degeneracies are collapsed and the corresponding p-cells are identified.

4.2. 3-Confluence and the cube law

4.2.1. 3-confluence fillers. A 3-*branching (in direction i)* of a p-ARS $\mathcal{X}_C$ in a $(p+3)$-category C is a triple (f_1, f_2, f_3) of $(p+1)$-cells in $\mathcal{X}_C^{\circ_i}$ such that $\partial_i^- f_1 = \partial_i^- f_2 = \partial_i^- f_3$. It is *local* if $f_1, f_2, f_3 \in \mathcal{X}_C$. We denote by $\mathsf{B}_3(\mathcal{X}_C)$ (resp. $\mathsf{LB}_3(\mathcal{X}_C)$) the set of 3-branchings (resp. local 3-branchings) of $\mathcal{X}_C$.

A *(local)* 3-*confluence filler* with respect to a map $A_2 : \mathsf{B}(\mathcal{X}_C) \to \mathsf{Cf}(\mathcal{X}_C)$ of a (local) 3-branching (f_1, f_2, f_3) is a $(p+3)$-cell $A_3(f_1, f_2, f_3)$ in C with faces

$$\partial_i^- A_3(f_1, f_2, f_3) = A_2(f_2, f_3), \qquad \partial_i^+ A_3(f_1, f_2, f_3) = A_2(\partial_i^+ A_2(f_1, f_2), \partial_i^+ A_2(f_1, f_3)),$$

$$\partial_{i+1}^- A_3(f_1, f_2, f_3) = A_2(f_1, f_3), \qquad \partial_{i+1}^+ A_3(f_1, f_2, f_3) = A_2(\partial_{i+1}^+ A_2(f_1, f_2), \partial_i^+ A_2(f_2, f_3)),$$

$$\partial_{i+2}^- A_3(f_1, f_2, f_3) = A_2(f_1, f_2), \qquad \partial_{i+2}^+ A_3(f_1, f_2, f_3) = A_2(\partial_{i+1}^+ A_2(f_1, f_3), \partial_{i+1}^+ A_2(f_2, f_3)).$$

We write $\mathsf{Cf}_3(\mathcal{X}_C, A_2)$ (resp. $\mathsf{LCf}_3(\mathcal{X}_C, A_2)$) for the set of confluence fillers (resp. local confluence fillers) with respect to the map A_2.

These definitions allow us to prove a coherent cubical Newman's lemma in three directions, and thus in three dimensions.

4.2.2. Proposition. *Let $\mathcal{X}_C$ be a Noetherian p-ARS in a $(p+3)$-category C with a map $A_2 : \mathsf{LB}(\mathcal{X}_C) \to \mathsf{LCf}(\mathcal{X}_C)$. Then each map A_3 extends from $\mathsf{LB}_3(\mathcal{X}_C) \to \mathsf{LCf}_3(\mathcal{X}_C, A_2)$ to $\mathsf{B}_3(\mathcal{X}_C) \to \mathsf{Cf}_3(\mathcal{X}_C, A_2)$.*

Proof. By Proposition 4.1.4, the map A_2 extends from local to arbitrary branchings and confluences. We extend the map A_3 by induction in direction i on the source of the 3-branchings.

The base case is trivial. Let (f_1, f_2, f_3) be a 3-branching with source x and suppose that the map A_3 extends to 3-branchings with source a p-cell reduced from x. For each $1 \leqslant i \leqslant 3$, we write $f_i = f_i' \circ_i f_i''$, where f_i' belongs to $\mathcal{X}_C$. By assumption, the local 3-branching (f_1', f_2', f_3') is filled by the 3-confluence filler $B = A_3(f_1', f_2', f_3')$. Then, using the induction hypothesis,

– the 3-branching $\left(\partial_{i+1}^- \partial_{i+2}^+ B, \partial_i^- \partial_{i+2}^+ B, f_3''\right)$ is filled by the 3-confluence filler

$$C \;=\; A_3(\partial_{i+1}^- \partial_{i+2}^+ B, \partial_i^- \partial_{i+2}^+ B, f_3''),$$

– the 3-branching $\left(\partial_{i+1}^- \partial_{i+1}^+ B, f_2'', \partial_i^- \partial_{i+1}^+ (B \circ_{i+2} C)\right)$ is filled by the 3-confluence filler

$$D \;=\; A_3(\partial_{i+1}^- \partial_{i+1}^+ B, f_2'', \partial_i^- \partial_{i+1}^+ (B \circ_{i+2} C)),$$

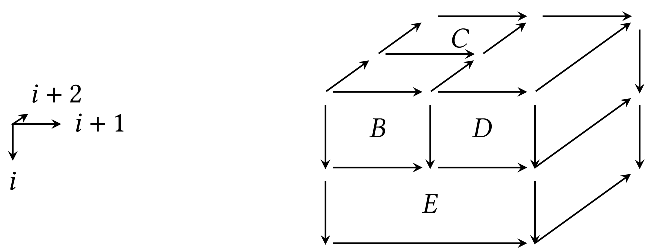
– the 3-branching $\left(f_1'', \partial_{i+1}^- \partial_i^+ ((B \circ_{i+2} C) \circ_{i+1} D), \partial_i^- \partial_i^+ ((B \circ_{i+2} C) \circ_{i+1} D)\right)$ is filled by the 3-confluence filler

$$E \;=\; A_3(f_1'', \partial_{i+1}^- \partial_i^+ ((B \circ_{i+2} C) \circ_{i+1} D), \partial_i^- \partial_i^+ ((B \circ_{i+2} C) \circ_{i+1} D)).$$

We then extend the map A_3 inductively, setting

$$A_3(f_1, f_2, f_3) \;=\; ((B \circ_{i+2} C) \circ_{i+1} D) \circ_i E.$$

This construction corresponds to the diagram

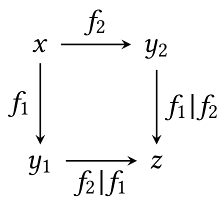


□

4.2.3. The cube law. Our functional approach to confluence fillers admits an interpretation in terms of residual paths and of the cube law. Indeed, the map A_2 allows defining a *residuation* operation

$$f_1 \mid f_2 \;\coloneqq\; \partial^+_{i+1} A_2(f_1, f_2),$$

for every branching (f_1, f_2). This operation is well-known from the λ-calculus [4, 26]. It gives rise to the confluence diagram

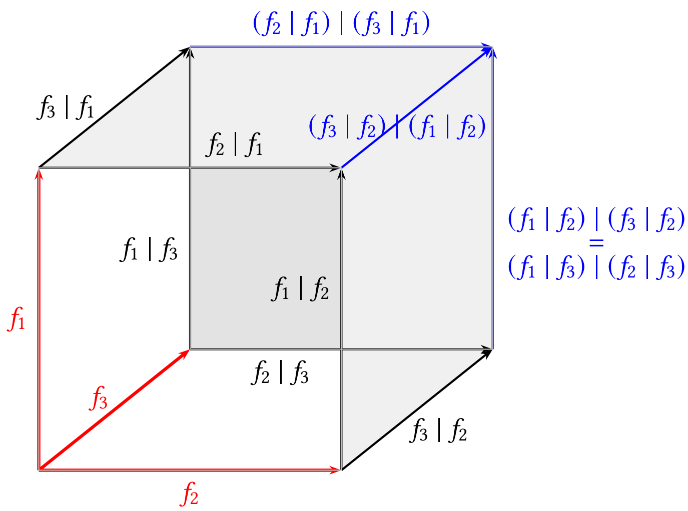


To work with residuals, it helps memorising $f_1 \,|\, f_2$ as the translation of f_1 along f_2 in the square spanned by f_1 and f_2, and $f_2 \mid f_1$ as the translation of f_2 along f_1.

Lévy has shown that residuation satisfies the cube law in λ-calculus [26, Lemma 2.2.1], see also [4, Lemma 12.2.6] and [12, Def. 4.49], which is often presented as a single cube law up to permutation of indices. For a 3-branching (f_1, f_2, f_3), the *cube law* state that

$$(f_i \mid f_j)(f_k \mid f_j) \;=\; (f_i \mid f_k)(f_j \mid f_k),$$

for all pairwise distinct i, j, k in $\{1, 2, 3\}$. Geometrically, this law assembles rewriting paths around the following cube spanned by the 3-branching:



In this cube, the residual path $f_1 \,|\, f_2$ is obtained by translating f_1 along f_2 in the front square and the residual path $f_3 \,|\, f_2$ by translating f_3 along f_2 in the bottom square, so that $(f_1 \,|\, f_2) \,|(f_3 \,|\, f_2)$ is the residual path of these two residual paths on the back face of the cube. Similar translations show that $(f_1 \mid f_3) \,|(f_2 \mid f_3)$ represents the same arrow. The other instantiations of f_1, f_2, f_3 in the cube law produce the remaining blue arrows and thus assemble all arrows around the cube.

The cube law follows from the cubical relations (2.1.2) applied to the cube $A_3(f_1, f_2, f_3)$, for instance,

$$(f_1 \mid f_2) \mid (f_3 \mid f_2) = \partial^+_{i+1}\partial^+_{i+1}A_3(f_1, f_2, f_3) = \partial^+_{i+1}\partial^+_{i+2}A_3(f_1, f_2, f_3) = (f_1 \mid f_3) \mid (f_2 \mid f_3).$$

They are thus a natural and immediate consequence of the way faces are attached to cells of cubical sets, hence of the geometry of cubes that emerges somewhat accidentally from the laws of λ-calculus. The cube law has appeared more recently as a postulate in 3-confluence proofs in classical abstract rewriting [14, 24].

4.3. Normalising confluence

Next we bring the sections and contractions from Section 4 into play and prove normalising variants of Newman's lemma and the Church-Rosser theorem. We also state and prove a cubical version of Squier's theorem, which requires normalisation.

4.3.1. Normal forms and contractions. Let $\mathcal{X}_C$ be a convergent p-ARS in a $(p+2)$-category C. A cell $x \in C_p$ is a *normal form (in direction i)* if there are no cells $f \in \mathcal{X}_C$ for which $\partial^-_i f = x$. By convergence, any rewriting path that starts from any $x \in C_p$ terminates in a unique normal form $\widehat{x}$. This determines a section $\widehat{(-)}$ of the projection $\pi : C \to \overline{C}_p$, as defined in §3.1.4. For every $x \in C_p$ we choose a $(p+1)$-cell $\sigma_x \in \mathcal{X}^{\circ_i}_C$ such that $\sigma : C_p \to C_{p+1}$ is a contraction in the $(p+1, p)$-category generated by C_{p+1}, as defined in §3.1.5.

4.3.2. Normalising fillers. A *normalising (local) confluence filler (in direction i)* of a (local) branching (f_1, f_2) of $\mathcal{X}_C$ is a $(p+2)$-cell $A_2(f_1, f_2)$ in C such that

$$\partial^-_i A_2(f_1, f_2) = f_2, \qquad \partial^-_{i+1}A_2(f_1, f_2) = f_1, \qquad \partial^+_i A_2(f_1, f_2) = \sigma_{\partial^+_i f_1}, \qquad \partial^+_{i+1}A_2(f_1, f_2) = \sigma_{\partial^+_i f_2}.$$

These identities assemble to a (local) confluence diagram

$$\begin{array}{ccc} & & x & \xrightarrow{f_2} & y_2 \\ \ulcorner\!\rightarrow i+1 & & {\scriptstyle f_1}\downarrow & A_2(f_1,f_2) & \downarrow{\scriptstyle \sigma_{y_2}} \\ \downarrow i & & y_1 & \xrightarrow[\sigma_{y_1}]{} & \widehat{x} \end{array} \tag{4.3.3}$$

A *normalising Church-Rosser filler (in direction i)* of a cell f in $\mathcal{X}^{\top_i}_C$ is a $(p+2)$-cell $B(f)$ in C of shape

$$\begin{array}{ccc} & & x & \xrightarrow{f} & y \\ \ulcorner\!\rightarrow i+1 & & {\scriptstyle \sigma_x}\downarrow & B(f) & \downarrow{\scriptstyle \sigma_y} \\ \downarrow i & & \widehat{x} & = & \widehat{x} \end{array}$$

We write $\mathsf{NCf}_3(\mathcal{X}_C, A_2)$ (resp. $\mathsf{NLCf}_3(\mathcal{X}_C, A_2)$) for the set of normalising confluence fillers (resp. normalising local confluence fillers) with respect to A_2 and $\mathsf{NCR}(\mathcal{X}_C)$ the set of normalising Church-Rosser fillers of $\mathcal{X}_C$.

These notions allow us to prove normalising variants of Newman's lemma and the Church-Rosser theorem with the same diagrams as before, but with normal forms and degeneracies in suitable places.

4.3.4. Lemma. *For a Noetherian p-ARS $\mathcal{X}_C$, each map A_2 extends from* $\mathsf{LB}(\mathcal{X}_C) \to \mathsf{NLCf}(\mathcal{X}_C)$ *to* $\mathsf{B}(\mathcal{X}_C) \to \mathsf{NCf}(\mathcal{X}_C)$.

Proof. The proof is similar to that of Proposition 4.1.4, but confluence fillers are now normalising. In Diagram (4.1.5) we thus replace y', y'' and z' by $\widehat{x}$ and arrows between the $\widehat{x}$ by degeneracies. □

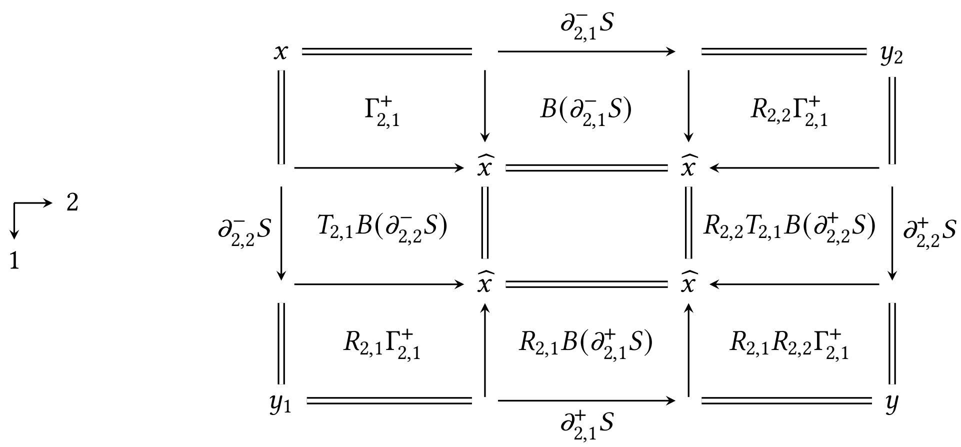
4.3.5. Lemma. *For a p-ARS $\mathcal{X}_C$ in a $(p+2, p+1)$-category C, each map $A_2 : \mathsf{B}(\mathcal{X}_C) \to \mathsf{NCf}(\mathcal{X}_C)$ induces a map $B : \mathcal{X}_C^{\top_i} \to \mathsf{NCR}(\mathcal{X}_C)$.*

Proof. By the obvious replacements in the diagrams in the proof of Proposition 4.1.7. □

Finally, we state a cubical version of Squier's theorem [35] for 1-groupoids. Its formulation motivates the extension of the notion of cubical normalization strategies to higher dimensions, which is the subject of the next section.

4.3.6. Proposition. *For a convergent ARS $\mathcal{X}_C$ each map $A_2 : \mathsf{LB}(\mathcal{X}_C) \to \mathsf{NLCf}(\mathcal{X}_C)$ extends to a witness 2-cell for a proof of acyclicity of the groupoid ${\mathcal{X}_C}^{\top_1}$.*

Proof. Lemmas 4.3.4 and 4.3.5 allow us to construct a map B from zigzags to normalising Church-Rosser fillers. Every square S is then obtained by the following composition of cubes:



□

4.3.7. Remark. Proposition is a low-dimensional version of Theorem 3.2.5, proved without using folding maps. The same method has been used by Lucas [30], rotating and gluing confluence fillers to fill a square. Yet extending to higher dimensions as in Theorem 3.2.5 seems combinatorially difficult, as it requires rotating and gluing all confluence fillers of the faces of a k-square.

For a converse of Squier's theorem for cubical ω-groupoids freely generated by $(\omega, 0)$-polygraphs see Theorem 5.1.3 below.

4.3.8. The cube law revisited. Contractions allow defining $(f \mid g) = \sigma_{\partial_i^+(g)}$ for any branching (f, g). For 3-branchings (f_1, f_2, f_3), we can then derive the cube law,

$$(f_1 \mid f_2) \mid (f_3 \mid f_2) = \sigma_{\partial_i^+(\sigma_{\partial_i^+(f_2)})} = \varepsilon_i \widehat{\partial_i^+(f_2)} = \varepsilon_i \widehat{\partial_i^+(f_3)} = \sigma_{\partial_i^+(\sigma_{\partial_i^+(f_3)})} = (f_1 \mid f_3) \mid (f_2 \mid f_3),$$

without using 3-confluence fillers explicitly. Contractions also allow constructing 3-confluence fillers more easily, and extending the techniques in this section to higher dimensions. In Section 5 we formalise higher-dimensional versions of normalising confluence diagrams, generated from the confluence of n-branchings, in cubical n-polygraphs and for $n \geqslant 2$. We use them further to construct ω-groupoids on convergent ARS.

5. Cubical groupoids in abstract rewriting

In this section, we present extensions and applications of Theorem 3.2.5 to cubical polygraphs, after briefly recalling their structure in Subsection 5.1. Theorem 5.1.3 shows that a free ω-groupoid on a polygraph is acyclic if and only if it is contracting. In Subsection 5.2, we construct free ω-groupoids extending convergent ARS, defining for each $k \geqslant 2$ a map A_k from local k-branchings to local k-confluence fillers and thereby accounting for the confluence of higher-dimensional branchings. Finally, in Theorem 5.3.2, we show that a suitable choice of 2-cells for A_2 refines this construction so that all k-cells are thin for $k \geqslant 2$. This shows that abstract rewriting with normalisation strategies does not require the generation of coherence cells in dimensions higher than 2. Together, these two results provide cubical analogues of Squier's theorem for ARSs.

5.1. Cubical polygraphs, contractions and acyclicity of cubical groupoids

First we recall the notion of *cubical polygraph.* The existence of this structure was originally established by Lucas in the context of Gray categories [28] . Yet the explicit construction of the free category generated by a cubical polygraph was not made explicit therein. For completeness, we provide such a construction while deferring a detailed development to Appendix A.2, including a proof of existence of the free (cubical) $(n-1)$-groupoid $X_{n-1}^{\top}$.

5.1.1. Cubical polygraphs. A *cubical* $(1,0)$*-polygraph* (a 1*-polygraph*) (X_0, X_1) consists of a set X_0 of 0*-generators* and a set X_1 of 1*-generators* or *rewriting steps*, equipped with *source* and *target* maps $\partial_{1,1}^{\alpha} : X_1 \to X_0$. It freely generates a 1-category X^*, as well as a 1-groupoid $X^{\top}$. A *cubical cellular extension* of a cubical $(n-1,0)$-category C is a set X_n of n*-generators* and face maps $\partial_{n,i}^{\alpha} : X_n \to C_{n-1}$ for $1 \leqslant i \leqslant n$ which satisfy the cubical relations (2.1.2).

A *cubical* $(n,0)$*-polygraph* $X = (X_0, \dots, X_n)$ is formed by a cubical $(n-1,0)$-polygraph $(X_0, \dots, X_{n-1})$ and a cubical cellular extension X_n of the free (cubical) $(n-1)$-groupoid $X_{n-1}^{\top}$. A *cubical* $(\omega,0)$*-polygraph* is obtained by a colimit construction; it consists of a family of sets $X = (X_0, X_1, \dots)$ such that every subfamily $(X_0, \dots, X_n)$ is a cubical $(n,0)$-polygraph. A polygraph is acyclic if and only if the associated free groupoid is acyclic.

All polygraphs considered in the sequel are cubical.

In order to construct acyclic polygraphs that extend a convergent ARS in Sections 5.2 and 5.3, we characterise acyclicity via the existence of contractions in Theorem 5.1.3, adapting a similar result for globular polygraphs [19]. We start with the following characterisation of contractions.

5.1.2. Lemma. *Let X be an $(\omega, 0)$-polygraph and $\widehat{(-)}$ a section of the projection $\pi : X^\top \to \overline{X^\top}_0$. The contractions of $X^\top$ are in bijective correspondence with the following data:*

i) *a family of 1-cells σ_x in $X_1^\top$ with boundary $x^\partial = (x, \widehat{x})$, for every 0-cell x in X_0 such that $x \neq \widehat{x}$,*

ii) *a family of $(k+1)$-cells σ_f in $X_{k+1}^\top$, for every $k > 0$, with boundary f^∂, for every k-cell f in X_k that is not of the form σ_g for some g in $X_{k-1}^\top$.*

Here, f^∂ is defined recursively with respect to the dimension of k-cells f in $X_k^\top$, as in Section 3.1.5.

Proof. A contraction has fixed values on thin cells, R-inverses, compositions and on elements of the form $\widehat{x}$ for $x \in X_0$ or σ_g for some g in $X_{k-1}^\top$. So the values of σ_f for f in $X_k^\top$ are uniquely and completely determined by its values on generators of the form given in the lemma. A construction of the free groupoid $X^\top$ can be found in Appendix A.2. □

We can now prove the converse direction to Theorem 3.2.5 for polygraphs.

5.1.3. Theorem. *The free ω-groupoid generated by an $(\omega, 0)$-polygraph is acyclic if and only if it is contracting.*

Proof. Let X be an acyclic cubical $(\omega, 0)$-polygraph. We construct a contraction σ recursively in the dimension of cells. This yields a contraction of the cubical $(k+1, 0)$-polygraph $(X_0, \ldots, X_{k+1})$ for each $0 \leqslant k < n$. For $k = 0$ and every 0-cell $x \in X_0$ such that $x \neq \widehat{x}$, we choose $\sigma_x : x \to \widehat{x}$ in $X_1^\top$, which exists by definition. This yields a contraction of (X_0, X_1). For $k > 0$, suppose σ is a contraction of $(X_0, \ldots, X_k)$ and take a k-cell $f \in X_k$ which is not of the form σ_g for some $g \in X_{k-1}^\top$. By acyclicity, the k-square f^∂ admits a filler A in $X_{k+1}^\top$, and we set $\sigma_f := A$. By Lemma 5.1.2, σ extends to a contraction of the $(k+1, 0)$-polygraph $(X_0, \ldots, X_{k+1})$. Taking the colimit yields a contraction of X.

The reverse implication follows from Theorem 3.2.5, considering the ω-groupoid $C = X^\top$ freely generated by X. □

We use Theorem 5.1.3 in Theorems 5.3 and 5.3.2 below to calculate acyclic extensions of ARS.

5.2. An acyclic ω-groupoid from convergence

Next we describe the construction that extends a convergent ARS into an acyclic ω-groupoid generated by its higher-order branchings.

5.2.1. Abstract rewriting systems. For a 1-polygraph X, we consider the cellular extension X_1 as an ARS on 0-cells in the free category X^*, as defined in §4.1.1, and a section $\widehat{(-)}$ defined by the normal forms in §3.1.5. A *normalisation strategy* for X is a contraction $\sigma : X_0 \to X_1^*$ with respect to $\widehat{(-)}$ defined, for each $x \in X_0$, as

$$\sigma_x \;=\; \eta_x \circ_1 \sigma_{\partial_1^+(\eta_x)},$$

where $\eta_x \in X_1$ is the first rewriting step of σ_x.

5.2.2. The polygraph $\mathrm{C}_\omega(X)$. Let X be a convergent 1-polygraph and σ a normalisation strategy for X. For every $x \in X_0$, we fix a strict order $<$ on the set $\{f \in X_1 \mid \partial_1^- f = x\}$, making η_x the least element. We construct an $(\omega, 0)$-polygraph involving higher-order branchings and their confluences by transfinite recursion, defining a sequence of cellular extensions $(\mathrm{C}_k(X))_{k\geqslant 0}$ by

i) $\mathrm{C}_0(X) := X_0$ and $\mathrm{C}_1(X) := X_1$,

ii) For $k = 2$, $\mathrm{C}_2(X) := \{A_2(f_1, f_2) \mid f_1, f_2 \in X_1^*,\ f_1 < f_2,\ \partial_1^-(f_1) = \partial_1^-(f_2)\}$, whose face maps of the 2-cell $A_2(f_1, f_2)$, drawn in (4.3.3), are defined by, for $1 \leqslant i \leqslant 2$,

iii) For $k \geqslant 3$, $\mathrm{C}_k(X) := \{A_k(f_1, \dots, f_k) \mid f_i \in X_1^*,\ f_i < f_{i+1},\ \partial_1^-(f_i) = \partial_1^-(f_{i+1})$ for $1 \leqslant i \leqslant k-1\}$, whose face maps of $A_k(f_1, \dots, f_k)$ are defined by, for $1 \leqslant i \leqslant k$,

$$\partial_i^- A_k(f_1, \dots, f_k) = A_{k-1}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_k), \qquad \partial_i^+ A_k(f_1, \dots, f_k) = \Gamma_{k-2}^- \dots \Gamma_1^- \sigma_{\partial_1^+(f_i)}.$$

The $(\omega, 0)$-polygraph $\mathrm{C}_\omega(X)$ is the colimit of this construction.

The following lemma shows that the k-generators $A_k(f_1, \dots, f_k)$ are well-defined.

5.2.3. Lemma. *For every $k \geqslant 2$, $C_k(X)$ defined in §5.2.2 is a cubical cellular extension of $\mathrm{C}_{k-1}(X)^\top$.*

Proof. We need to check the square equations (3.2.2). For $k = 2$,

$$\begin{aligned}
\partial_1^-\partial_1^- A_2(f_1, f_2) &= x = \partial_1^-\partial_2^- A_2(f_1, f_2),\\
\partial_1^-\partial_1^+ A_2(f_1, f_2) &= y_2 = \partial_1^+\partial_2^- A_2(f_1, f_2),\\
\partial_1^+\partial_1^- A_2(f_1, f_2) &= y_1 = \partial_1^-\partial_2^+ A_2(f_1, f_2),\\
\partial_1^+\partial_1^+ A_2(f_1, f_2) &= \widehat{x} = \partial_1^+\partial_2^+ A_2(f_1, f_2),
\end{aligned}$$

which shows that $A_2(f_1, f_2)$ forms a 2-cell. For $k \geqslant 3$ and $1 \leqslant i < j \leqslant k$,

$$\begin{aligned}
\partial_i^-\partial_j^- A_k(f_1, \dots, f_k) &= A_{k-2}(f_1, \dots, f_{i-1}, f_{i+1}, \dots, f_{j-1}, f_{j+1}, \dots, f_k) = \partial_{j-1}^-\partial_i^- A_k(f_1, \dots, f_k),\\
\partial_i^-\partial_j^+ A_k(f_1, \dots, f_k) &= \Gamma_{k-3}^- \dots \Gamma_1^- \sigma_{t_0(f_j)} = \partial_{j-1}^+\partial_i^- A_k(f_1, \dots, f_k),\\
\partial_i^+\partial_j^- A_k(f_1, \dots, f_k) &= \Gamma_{k-3}^- \dots \Gamma_1^- \sigma_{t_0(f_i)} = \partial_{j-1}^-\partial_i^+ A_k(f_1, \dots, f_k),\\
\partial_i^+\partial_j^+ A_k(f_1, \dots, f_k) &= \Gamma_{k-3}^- \dots \Gamma_i^- \partial_i^+ \Gamma_i^- \dots \Gamma_1^- \sigma_{t_0(f_j)}\\
&= \Gamma_{k-3}^- \dots \Gamma_i^- \varepsilon_i \dots \varepsilon_1 \partial_1^+ \sigma_{t_0(f_j)} = \varepsilon_1 \dots \varepsilon_1 \widehat{x}\\
&= \Gamma_{k-3}^- \dots \Gamma_{j-1}^- \varepsilon_{j-1} \dots \varepsilon_1 \partial_1^+ \sigma_{t_0(f_i)}\\
&= \Gamma_{k-3}^- \dots \Gamma_{j-1}^- \partial_{j-1}^+ \Gamma_{j-1}^- \dots \Gamma_1^- \sigma_{t_0(f_i)}\\
&= \partial_{j-1}^+\partial_i^+ A_k(f_1, \dots, f_k).
\end{aligned}$$

Thus $A_k(f_1, \dots, f_k)$ is a $(k-1)$-square. □

5.2.4. Extending σ to a contraction of $\mathrm{C}_\omega(X)^\top$. We further extend σ to a contraction of the ω-groupoid $\mathrm{C}_\omega(X)^\top$. By Lemma 5.1.2, it suffices to define a $(k+1)$-cell σ_f, for each $k \geqslant 1$, only for those k-generators f that are not of the form σ_g for some $g \in \mathrm{C}_{k-1}(X)^\top$.

For each $f \in X_1$ not of the form σ_z for some $z \in X_0$, we define the following 2-cell σ_f in $\mathrm{C}_\omega(X)^\top$ that fills the 1-square:

$$f^\partial = \begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle \sigma_x}\downarrow & & \downarrow{\scriptstyle \sigma_y} \\ \widehat{x} & = & \widehat{x} \end{array}$$

If $f \neq \eta_x$ with $x = \partial_1^-(f)$, and $x' = \partial_1^+(\eta_x)$, then we set

$$\sigma_f := A_2(\eta_x, f) \circ_1 \Gamma_1^- \sigma_{x'} = \begin{array}{ccc} x & \xrightarrow{f} & y \\ {\scriptstyle \eta_x}\downarrow & & \downarrow{\scriptstyle \sigma_y} \\ x' & \xrightarrow[\sigma_{x'}]{} & \widehat{x} \\ {\scriptstyle \sigma_{x'}}\downarrow & & \| \\ \widehat{x} & = & \widehat{x} \end{array}$$

where $\sigma_x = \eta_x \circ_1 \sigma_{x'}$ and $\eta_x \in X_1$. Otherwise, if $f = \eta_x$, we set

$$\sigma_{\eta_x} := \Gamma_1^- \eta_x \circ_1 \varepsilon_2 \sigma_{x'} = \begin{array}{ccc} x & \xrightarrow{\eta_x} & x' \\ {\scriptstyle \eta_x}\downarrow & & \| \\ x' & = & x' \\ {\scriptstyle \sigma_{x'}}\downarrow & & \downarrow{\scriptstyle \sigma_{x'}} \\ \widehat{x} & = & \widehat{x} \end{array}$$

For $k \geqslant 2$, let A be a k-generator in $\mathrm{C}_k(X)$ that is not of the form σ_g for some $g \in \mathrm{C}_{k-1}(X)^\top$. Then $A = A_k(f_1, \ldots, f_k)$ and we define a $(k+1)$-cell σ_A in $\mathrm{C}_\omega(X)^\top$ that fills the k-square A^∂. If $f_i \neq \eta_x$ for all i, where $x = \partial_1^-(f_i)$ and $x' = \partial_1^+(\eta_x)$, then we set

$$\sigma_A := A_{k+1}(\eta_x, f_1, \ldots, f_k) \circ_1 \Gamma_k^- \ldots \Gamma_1^- \sigma_{x'}. \tag{5.2.5}$$

If $f_1 = \eta_x$, then we set

$$\sigma_A := \Gamma_1^- A \circ_1 \varepsilon_2 \Gamma_{k-1}^- \ldots \Gamma_1^- \sigma_{x'}. \tag{5.2.6}$$

5.2.7. Lemma. *Each σ_A defined as above is well-defined and a filler of A^∂.*

Proof. In the case (5.2.5), the formula is well-defined because

$$\partial_1^+ A_{k+1}(\eta_x, f_1, \ldots, f_k) = \Gamma_{k-1}^- \ldots \Gamma_1^- \sigma_{x'} = \partial_1^- \Gamma_k^- \ldots \Gamma_1^- \sigma_{x'}.$$

Also, σ_A fills A^∂ because,

$$\begin{aligned} \partial_1^- \sigma_A &= \partial_1^- A_{k+1}(\eta_x, f_1, \ldots, f_k) = A_k(f_1, \ldots, f_k) = A, \\ \partial_1^+ \sigma_A &= \partial_1^+ \Gamma_1^- \ldots \Gamma_1^- \sigma_{x'} = \varepsilon_1 \ldots \varepsilon_1 \partial_1^+ \sigma_{x'} = \varepsilon_{k+1} \ldots \varepsilon_1 \widehat{x}, \end{aligned}$$

and, for $j > 1$,

$$\begin{aligned}\partial_j^- \sigma_A &= \partial_j^- A_{k+1}(\eta_x, f_1, \dots, f_k) \circ_1 \partial_j^- \Gamma_k^- \dots \Gamma_1^- \sigma_{x'} \\ &= A_k(\eta_x, f_1, \dots, f_{j-2}, f_j, \dots, f_k) \circ_1 \Gamma_{k-1}^- \dots \Gamma_j^- \partial_j^- \Gamma_j^- \dots \Gamma_1^- \sigma_{x'} \\ &= A_k(\eta_x, f_1, \dots, f_{j-2}, f_j, \dots, f_k) \circ_1 \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \sigma_{A_{k-1}(f_1, \dots, f_{j-2}, f_j, \dots, f_k)} = \sigma_{\partial_{j-1}^- A},\end{aligned}$$

$$\begin{aligned}\partial_j^+ \sigma_A &= \partial_j^+ A_{k+1}(\eta_x, f_1, \dots, f_k) \circ_1 \partial_j^+ \Gamma_k^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{i-1})} \circ_1 \Gamma_{k-1}^- \dots \Gamma_j^- \varepsilon_j \partial_j^+ \Gamma_{j-1}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{i-1})} \circ_1 \varepsilon_{k-1} \dots \varepsilon_j \partial_j^+ \Gamma_{j-1}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{i-1})} \circ_1 \varepsilon_{k-1} \dots \varepsilon_1 \partial_1^+ \sigma_{x'} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{i-1})} = \Gamma_{k-1}^- \dots \Gamma_2^- \sigma_{\sigma_{t_0(f_{i-1})}} \\ &= \sigma_{\Gamma_{k-2}^- \dots \Gamma_1^- \sigma_{t_0(f_{i-1})}} = \sigma_{\partial_{j-1}^+ A}.\end{aligned}$$

In the case (5.2.6), the formula is well-defined because

$$\partial_1^+ \Gamma_1^- A = \varepsilon_1 \partial_1^+ A = \varepsilon_1 \Gamma_{k-2}^- \dots \Gamma_1^- \sigma_{x'} = \partial_1^- \varepsilon_2 \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{x'},$$

and σ_A fills A^∂ because, we have $\partial_1^- \sigma_A = \partial_1^- \Gamma_1^- A = A$, and

$$\partial_1^+ \sigma_A = \partial_1^+ \varepsilon_2 \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{x'} = \varepsilon_1 \partial_1^+ \Gamma_1^- \dots \Gamma_1^- \sigma_{x'} = \varepsilon_1 \dots \varepsilon_1 \partial_1^+ \sigma_{x'} = \varepsilon_{k+1} \dots \varepsilon_1 \widehat{x},$$

and, for $j > 1$,

$$\begin{aligned}\partial_j^- \sigma_A &= \partial_j^- \Gamma_1^- A \circ_1 \partial_j^- \varepsilon_2 \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_1^- \partial_{j-1}^- A \circ_1 \varepsilon_2 \Gamma_{k-2}^- \dots \Gamma_{j-1}^- \partial_{j-1}^- \Gamma_{j-1}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_1^- A_{k-1}(f_1, \dots, f_{j-2}, f_j, \dots, f_k) \circ_1 \varepsilon_2 \Gamma_{k-2}^- \dots \Gamma_1^- \sigma_{x'} = \sigma_{\partial_{j-1}^- A},\end{aligned}$$

$$\begin{aligned}\partial_j^+ \sigma_A &= \partial_j^+ \Gamma_1^- A \circ_1 \partial_j^+ \varepsilon_2 \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_1^- \partial_{j-1}^+ A \circ_1 \varepsilon_2 \Gamma_{k-2}^- \dots \Gamma_{j-1}^- \varepsilon_{j-1} \partial_{j-1}^+ \Gamma_{j-2}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_1^- \Gamma_{k-2}^- \dots \Gamma_1^- \sigma_{t_0(f_{j-1})} \circ_1 \varepsilon_2 \varepsilon_{k-2} \dots \varepsilon_{j-1} \partial_{j-1}^+ \Gamma_{j-2}^- \dots \Gamma_1^- \sigma_{x'} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{j-1})} \circ_1 \varepsilon_2 \varepsilon_{k-2} \dots \varepsilon_1 \partial_1^+ \sigma_{x'} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{j-1})} \circ_1 \varepsilon_1 \dots \varepsilon_1 \widehat{x} \\ &= \Gamma_{k-1}^- \dots \Gamma_1^- \sigma_{t_0(f_{j-1})} = \Gamma_{k-1}^- \dots \Gamma_2^- \sigma_{\sigma_{t_0(f_{j-1})}} \\ &= \sigma_{\Gamma_{k-2}^- \dots \Gamma_1^- \sigma_{t_0(f_{j-1})}} = \sigma_{\partial_{j-1}^+ A}.\end{aligned}$$

□

We can now state the main theorem of this section.

5.2.8. Theorem. *Every convergent 1-polygraph X extends to an acyclic ω-groupoid $\mathrm{C}_\omega(X)^\top$.*

Proof. In §5.2.4, we have defined a family of 1-cells σ_x in $X_1^\top$ with boundary $x^\partial = (x, \widehat{x})$, for every 0-cell x in X_0 such that $x \neq \widehat{x}$. We have also defined a family of $(k+1)$-cells σ_f in $\mathrm{C}_{k+1}(X)^\top$, for every $0 < k < n$, with boundary f^∂, for every k-cell f in $\mathrm{C}_k(X)$ which is not of the form σ_g for some g in $\mathrm{C}_{k-1}(X)^\top$. Then σ is a contraction on $\mathrm{C}_\omega(X)$ by Lemma 5.1.2 and the claim follows from Theorem 5.1.3. □

5.3. A refined acyclic ω-groupoid from convergence

Finally, we refine the construction leading to Theorem so that it generates an acyclic ω-groupoid from a ARS without introducing any generating cells of dimension higher than 2. We begin with a technical lemma, which is an immediate consequence of [20, Prop. 2.1].

5.3.1. Lemma. *In every ω-groupoid, each k-square with thin faces can be filled by a thin cell.*

Proof. Let S be a k-square. Applying the folding maps, as defined in §3.2.3, yields a k-square $T = \Phi_k(S)$, which satisfies $\partial_1^- \Psi_k T = \partial_1^+ \Psi_k T$ by [1, Prop. 3.6] and has a unique thin filler B by [20, Prop. 2.1(iii)]. Applying the unfolding maps yields a k-cell $A = \overline{\Phi}_k(S, B)$ which is a filler of S by Lemma 3.2.4. □

5.3.2. Theorem. *Every convergent 1-polygraph X extends to an acyclic ω-groupoid $\mathrm{C}_\omega^{tr}(X)^\top$ which is generated by the $(\omega, 0)$-polygraph defined by*

$$\mathrm{C}_0^{tr}(X) := X_0, \qquad \mathrm{C}_1^{tr}(X) := X_1, \qquad \mathrm{C}_2^{tr}(X) := \{A_2(\eta_x, f) \mid f \in X_1,\ \partial_1^+(f) = x,\ \eta_x \neq f\},$$

where the boundary of $A_2(\eta_x, f)$ is given by (4.3.3), and which has no k-generators for $k > 2$.

Proof. Let X be a convergent 1-polygraph equipped with the normal form section and with normalisation strategy σ. We consider the acyclic ω-groupoid $\mathrm{C}_\omega(X)^\top$ from Theorem 5.3. Let (f_1, f_2) be a local branching with source x such that $f_1, f_2 \neq \eta_x$, let x' be the target of η_x. The 2-generator $A_2(f_1, f_2)$ has the same faces as the 2-cell

$$\left(\Gamma_1^+ \eta_x \circ_2 A_2(\eta_x, f_2)\right) \circ_1 \left(T_1 A_2(\eta_x, f_1) \circ_2 \Gamma_1^- \sigma_{x'}\right) = \begin{array}{ccccc} x & = & x & \xrightarrow{f_2} & y_2 \\ \| & \Gamma_1^+\eta_x & \downarrow & A_2(\eta_x, f_2) & \downarrow \sigma_{y_2} \\ x & \longrightarrow & x' & \longrightarrow & \widehat{x} \\ f_1 \downarrow & T_1 A_2(\eta_x, f_1) & \downarrow & \Gamma_1^- \sigma_{x'} & \| \\ y_1 & \xrightarrow[\sigma_{y_1}]{} & \widehat{x} & = & \widehat{x} \end{array} \tag{5.3.3}$$

We can thus replace $A_2(f_1, f_2)$ by this 2-cell that depends only on the generators $A_2(\eta_x, f_1)$, $A_2(\eta_x, f_2)$.

Let (f_1, f_2, f_3) be a local 3-branching with source x and let x' be the target of η_x. Suppose $f_1 = \eta_x$ and $f_2, f_3 \neq \eta_x$. The 3-generator $A_3(\eta_x, f_2, f_3)$ has faces $A_2(\eta_x, f_2)$, $A_2(\eta_x, f_3)$, $A_2(f_2, f_3)$ and three thin cells. We replace $A_2(f_2, f_3)$ by (5.3.3), so that $A_3(\eta_x, f_2, f_3)$ has the same faces as the 3-cell

$$\left(\Gamma_1^- \Gamma_1^+ \eta_x \circ_3 \Gamma_1^- A_2(\eta_x, f_3)\right) \circ_2 \left(\Gamma_2^- T_1 A_2(\eta_x, f_2) \circ_3 \Gamma_2^- \Gamma_1^- \sigma_{x'}\right). \tag{5.3.4}$$

The cases where $f_2 = \eta_x$ or $f_3 = \eta_x$ lead to similar thin cells.

Now suppose $f_1, f_2, f_3 \neq \eta_x$. If we replace $A_2(f_1, f_2)$, $A_2(f_1, f_3)$ and $A_2(f_2, f_3)$ by (5.3.3), then the 3-generator $A_3(f_1, f_2, f_3)$ has the same faces as the 3-cell

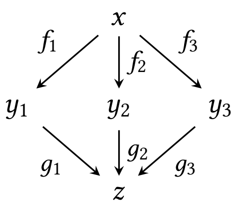
$$\Big(\big(\big(\Gamma_1^-\Gamma_1^+\eta_x \circ_1 R_1\Gamma_1^-\Gamma_1^-\eta_x\big) \circ_3 \Gamma_1^- A_2(\eta_x, f_3)\big) \circ_2 \big(T_2\Gamma_1^- T_1 A_2(\eta_x, f_2) \circ_3 \Gamma_2^-\Gamma_1^-\sigma_{x'}\big)\Big) \\ \circ_1 \Gamma_2^- \left(T_1 A_2(\eta_x, f_1) \circ_2 \Gamma_1^-\sigma_{x'}\right). \tag{5.3.5}$$

So again we replace $A_3(f_1, f_2, f_3)$ by this thin cell.

Lemma 5.3.1 implies that, if we replace the faces of any 4-generator in $\mathrm{C}_\omega(X)^\top$ by the thin 3-cells described in formulas (5.3.4) and (5.3.5), then the 4-generator itself can be replaced by a thin cell. The same argument applies inductively in all higher dimensions.

This allows constructing a trucacted $(\omega, 0)$-polygraph $\mathrm{C}_\omega^{tr}(X)$ from the acyclic ω-groupoid $\mathrm{C}_\omega(X)^\top$, retaining only the 0-generators, the 1-generators and the 2-generators of the form $A_2(\eta_x, f)$, where $f \in X_1$ and $f \neq \eta_x$. By construction, it freely generates an acyclic ω-groupoid $\mathrm{C}_\omega^{tr}(X)^\top$. In particular, it has no k-generators and no non-thin k-cells for any $k \geqslant 3$. □

5.3.6. Example. To illustrate the difference between Theorem 5.3 and Theorem 5.3.2, we consider the 1-polygraph X defined by the diagram

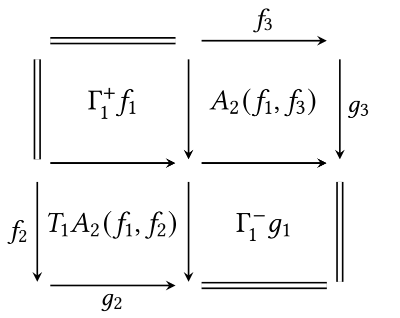


It is convergent, and z is the normal form of every 0-cell. We define the normalisation strategy σ by $\sigma_x = f_1 \circ_1 g_1$, $\sigma_{y_i} = g_i$ for every $1 \leqslant i \leqslant 3$, and $\sigma_z = 1_z$, and set $\eta_x = f_1$ and $f_1 < f_2 < f_3$.

The ARS X has the critical 2-branchings (f_1, f_2), (f_1, f_3), (f_2, f_3) and the critical 3-branching (f_1, f_2, f_3). The $(\omega, 0)$-polygraph $\mathrm{C}_\omega(X)$ extending X has the 2-generators $A_2(f_1, f_2)$, $A_2(f_1, f_3)$, $A_2(f_2, f_3)$ and the 3-generator $A_3(f_1, f_2, f_3)$. The ω-groupoid $\mathrm{C}_\omega(X)^\top$ freely generated this way is acyclic.

By contrast, the $(2, 0)$-polygraph $\mathrm{C}_2^{tr}(X)$ extending X has the 2-generators $A_2(f_1, f_2)$, $A_2(f_1, f_3)$, but no 3-generator. The 2-groupoid $\mathrm{C}_2^{tr}(X)^\top$ freely generated this alternative way also acyclic. The critical 2-branching (f_2, f_3), for instance, converges to z via the confluence (g_2, g_3), and it gives rise to the 1-square $S = (f_2, f_3, g_3, g_2)$, filled with the 2-cell

$(\Gamma_1^+ f_1 \circ_2 A_2(f_1, f_3)) \circ_1 (T_1 A_2(f_1, f_2) \circ_2 \Gamma_1^- g_1) =$



The critical 3-branching (f_1, f_2, f_3) converges to z via the confluence (g_1, g_2, g_3). This induces the 2-square S defined by

$$\partial_1^+ S = \Gamma_1^- g_1, \qquad \partial_2^+ S = \Gamma_1^- g_2, \qquad \partial_3^+ S = \Gamma_1^- g_3,$$
$$\partial_1^- S = (\Gamma_1^+ f_1 \circ_2 A_2(f_1, f_3)) \circ_1 (T_1 A_2(f_1, f_2) \circ_2 \Gamma_1^- g_1), \qquad \partial_2^- S = A_2(f_1, f_3), \qquad \partial_3^- S = A_2(f_1, f_2).$$

It can be filled by the thin 3-cell

$$(\Gamma_1^- \Gamma_1^+ f_1 \circ_3 \Gamma_1^- A_2(f_1, f_3)) \circ_2 (\Gamma_2^- T_1 A_2(f_1, f_2) \circ_3 \Gamma_2^- \Gamma_1^- g_1).$$

Then $\mathrm{C}_\omega^{tr}(X)^\top$ is indeed acyclic; the 2-generator $A_2(f_2, f_3)$ and the 3-generator $A_3(f_1, f_2, f_3)$ are no longer needed.

5.4. Concluding remarks

The only 3-confluence fillers in the proof of Theorem 5.3.2 are thin cells. The 2-confluence fillers employed are normalising, as explained in Remark 4.3.8, and the cube law holds *a fortiori*. Hence, the cube law always holds for any ARS, since rewriting rules have no application context and the critical branching lemma from classical rewriting is trivial.

By contrast, in algebraic rewriting systems (string, term, linear, etc.), the cube law is not inherent and must be proved separately – as, for instance, in the λ-calculus (see 4.2.3). Future work will apply the cubical constructions developed in this paper to such systems. Note also that, unlike for ARS, convergent algebraic extensions generally do not terminate after finitely many steps (see Theorem 5.3.2).

In globular higher-dimensional rewriting, the constructions of ω-groupoids and related structures from polygraphs are known as *polygraphic resolutions*, as mentioned in the introduction, and contractions may be regarded as contracting homotopies. This topological terminology is justified by the folk model structure on strict ω-categories and the fact that polygraphic resolutions are cofibrant approximations [25, 32, 33]. In the cubical case, much less is known; polygraphic resolutions as cofibrant approximations remain an avenue for future work. The proof of Theorem 5.3.2 has been inspired in particular by a categorical approach to Tietze transformations in globular polygraphs [16], which appears worth exploring via cubical categories as well.

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A. Appendices

A.1. Axioms of cubical categories

We give a comprehensive axiomatisation of cubical categories, which were outlined in Subsection 2.1.

A.1.1. Cubical categories. Cubical categories satisfy the following axioms, for all $i, j, k \in \mathbb{N}$ such that $1 \leqslant i, j \leqslant k$:

$$\partial^{\alpha}_{k,i}\varepsilon_{k,j} = \begin{cases} \varepsilon_{k-1,j-1}\partial^{\alpha}_{k-1,i} & \text{if } i < j, \\ id_{C_{k-1}} & \text{if } i = j, \\ \varepsilon_{k-1,j}\partial^{\alpha}_{k-1,i-1} & \text{if } i > j, \end{cases}$$

$$\varepsilon_{k+1,i}\varepsilon_{k,j+1} = \varepsilon_{k+1,j}\varepsilon_{k,i} \quad \text{if } i \leqslant j, \qquad \varepsilon_{k+1,i}\varepsilon_{k,j} = \varepsilon_{k+1,j}\varepsilon_{k,i+1} \quad \text{if } i > j,$$

$$(a \circ_{k,i} b) \circ_{k,j} (c \circ_{k,i} d) = (a \circ_{k,j} c) \circ_{k,i} (b \circ_{k,j} d),$$
$$a \circ_{k,i} (b \circ_{k,i} c) = (a \circ_{k,i} b) \circ_{k,i} c,$$

$$\varepsilon_{k+1,i}(a \circ_{k,j} b) = \begin{cases} \varepsilon_{k+1,i}a \circ_{k+1,j+1} \varepsilon_{k+1,i}b & \text{if } i \leqslant j, \\ \varepsilon_{k+1,i}a \circ_{k+1,j} \varepsilon_{k+1,i}b & \text{if } i > j, \end{cases}$$

$$a \circ_{k,i} \varepsilon_{k,i}\partial^{+}_{k,i}a = \varepsilon_{k,i}\partial^{-}_{k,i}a \circ_{k,i} a = a,$$

$$\partial^{\alpha}_{k,i}(a \circ_{k,j} b) = \begin{cases} \partial^{\alpha}_{k,i}a \circ_{k,j-1} \partial^{\alpha}_{k,i}b & \text{if } i < j, \\ \partial^{-}_{k,i}a & \text{if } i = j \text{ and } \alpha = -, \\ \partial^{+}_{k,i}b & \text{if } i = j \text{ and } \alpha = +, \\ \partial^{\alpha}_{k,i}a \circ_{k,j} \partial^{\alpha}_{k,i}b & \text{if } i > j, \end{cases}$$

A.1.2. Connections. Cubical categories with connections satisfy the following additional axioms:

$$\partial^{\alpha}_{k,i}\Gamma^{\beta}_{k,j} = \begin{cases} \Gamma^{\beta}_{k-1,j-1}\partial^{\alpha}_{k-1,i} & \text{if } i < j, \\ id_{C_{k-1}} & \text{if } i = j, j+1 \text{ and } \alpha = \beta, \\ \varepsilon_{k-1,j}\partial^{\alpha}_{k-1,j} & \text{if } i = j, j+1 \text{ and } \alpha = -\beta, \\ \Gamma^{\beta}_{k-1,j}\partial^{\alpha}_{k-1,i-1} & \text{if } i > j+1, \end{cases}$$

$$\Gamma^{\alpha}_{k+1,i}\varepsilon_{k,j} = \begin{cases} \varepsilon_{k+1,j+1}\Gamma^{\alpha}_{k,i} & \text{if } i < j, \\ \varepsilon_{k+1,i}\varepsilon_{k,i} & \text{if } i = j, \\ \varepsilon_{k+1,j}\Gamma^{\alpha}_{k,i-1} & \text{if } i > j, \end{cases} \qquad \Gamma^{\alpha}_{k+1,i}\Gamma^{\beta}_{k,j} = \begin{cases} \Gamma^{\beta}_{k+1,j+1}\Gamma^{\alpha}_{k,i} & \text{if } i < j, \\ \Gamma^{\alpha}_{k+1,j}\Gamma^{\alpha}_{k,j} & \text{if } i = j+1 \text{ and } \alpha = \beta, \\ \Gamma^{\beta}_{k+1,j}\Gamma^{\alpha}_{k,i-1} & \text{if } i > j+1. \end{cases}$$

$$\Gamma^+_{k,i} a \circ_{k,i} \Gamma^-_{k,i} a = \varepsilon_{k,i+1} a, \qquad \Gamma^+_{k,i} a \circ_{k,i+1} \Gamma^-_{k,i} a = \varepsilon_{k,i} a,$$

$$\Gamma^\alpha_{k+1,i}(a \circ_{k,j} b) = \begin{cases} \Gamma^\alpha_{k+1,i} a \circ_{k,j+1} \Gamma^\alpha_{k+1,i} b & \text{if } i < j, \\ (\Gamma^-_{k+1,i} a \circ_{k,i} \varepsilon_{k+1,i+1} b) \circ_{k,i+1} (\varepsilon_{k+1,i} b \circ_{k,i} \Gamma^-_{k+1,i} b) & \text{if } i = j \text{ and } \alpha = -, \\ (\Gamma^+_{k+1,i} a \circ_{k,i} \varepsilon_{k+1,i} a) \circ_{k,i+1} (\varepsilon_{k+1,i+1} a \circ_{k,i} \Gamma^+_{k+1,i} b) & \text{if } i = j \text{ and } \alpha = +, \\ \Gamma^\alpha_{k+1,i} a \circ_{k,j} \Gamma^\alpha_{k+1,i} b & \text{if } i > j, \end{cases}$$

A.1.3. Functors. A functor $F : \mathcal{C} \to \mathcal{D}$ of cubical ω-categories is a family of maps $(F_k : \mathcal{C}_k \to \mathcal{D}_k)_{0 \leqslant k}$ satisfying

$$F_k(a \circ_{k,i} b) = F_k a \circ_{k,i} F_k b, \qquad F_{k-1}\partial^\alpha_{k,i} = \partial^\alpha_{k,i} F_k, \qquad F_k \varepsilon_{k,i} = \varepsilon_{k,i} F_{k-1}, \qquad F_k \Gamma^\alpha_{k,j} = \Gamma^\alpha_{k,j} F_{k-1},$$

for all $i, j, k \in \mathbb{N}$ such that $1 \leqslant i \leqslant k$ and $1 \leqslant j < k$, and all $\circ_{k,i}$-composable $a, b \in \mathcal{C}_k$.

A.1.4. Inverses. The inversion maps R_i and T_i defined in §2.2.1 are compatible with

i) the face maps

$$\partial^\alpha_i R_j f = \begin{cases} R_{j-1}\partial^\alpha_i f & \text{if } i < j, \\ \partial^{-\alpha}_i f & \text{if } i = j, \\ R_j \partial^\alpha_i f & \text{if } i > j, \end{cases} \qquad \partial^\alpha_i T_j f = \begin{cases} T_{j-1}\partial^\alpha_i f & \text{if } i < j, \\ \partial^\alpha_{i+1} f & \text{if } i = j, \\ \partial^\alpha_{i-1} f & \text{if } i = j+1, \\ T_j \partial^\alpha_i f & \text{if } i > j+1, \end{cases}$$

ii) the compositions

$$R_i(f \circ_j g) = \begin{cases} R_i g \circ_i R_i f & \text{if } i = j, \\ R_i f \circ_j R_i g & \text{if } i \neq j, \end{cases} \qquad T_i(f \circ_j g) = \begin{cases} T_i f \circ_{i+1} T_i g & \text{if } j = i, \\ T_i f \circ_i T_i g & \text{if } j = i+1, \\ T_i f \circ_j T_i g & \text{if } j \neq i, i+1, \end{cases}$$

iii) the degeneracies

$$R_i \varepsilon_j f = \begin{cases} \varepsilon_j R_i f & \text{if } i < j, \\ \varepsilon_i f & \text{if } i = j, \\ \varepsilon_j R_{i-1} f & \text{if } i > j, \end{cases} \qquad T_i \varepsilon_j f = \begin{cases} \varepsilon_j T_{i-1} f & \text{if } j < i, \\ \varepsilon_{i+1} f & \text{if } j = i, \\ \varepsilon_i f & \text{if } j = i+1, \\ \varepsilon_j T_i f & \text{if } j > i+1, \end{cases}$$

iv) the connections

$$R_i \Gamma^\alpha_j f = \begin{cases} \Gamma^\alpha_j R_i f & \text{if } i < j, \\ \varepsilon_{i+1} R_i f \circ_{i+1} \Gamma^+_i f & \text{if } i = j, \alpha = -, \\ \Gamma^-_i f \circ_i \varepsilon_{i+1} R_i f & \text{if } i = j, \alpha = +, \\ \varepsilon_{i-1} R_{i-1} f \circ_i \Gamma^+_{i-1} f & \text{if } i = j+1, \alpha = -, \\ \Gamma^-_{i-1} f \circ_i \varepsilon_{i-1} R_{i-1} f & \text{if } i = j+1, \alpha = +, \\ \Gamma^\alpha_j R_{i-1} f & \text{if } i > j+1, \end{cases} \qquad T_i \Gamma^\alpha_j f = \begin{cases} \Gamma^\alpha_j T_i f & \text{if } i < j, \\ \Gamma^\alpha_i f & \text{if } i = j, \\ \Gamma^\alpha_j T_{i-1} f & \text{if } i > j, \end{cases}$$

$$T_{i+1} \Gamma^\alpha_i T_i f = T_i \Gamma^\alpha_{i+1} f, \qquad T_i \Gamma^\alpha_{i+1} T_i f = T_{i+1} \Gamma^\alpha_i f,$$

v) other inversion maps

$$R_iR_jf = \begin{cases} f & \text{if } i = j, \\ R_jR_if & \text{if } i \neq j, \end{cases} \qquad T_iT_jf = \begin{cases} f & \text{if } i = j, \\ T_jT_if & \text{if } |i-j| \geqslant 2, \end{cases}$$

$$T_iR_jf = \begin{cases} R_{i+1}T_if & \text{if } j = i, \\ R_iT_if & \text{if } j = i+1, \\ R_jT_if & \text{if } j \neq i, i+1. \end{cases} \qquad T_iT_{i+1}T_if = T_{i+1}T_iT_{i+1}f,$$

A.2. Cubical polygraphs and free cubical categories

In this appendix, we detail the construction of the cubical polygraphs used in Section 5. Cubical polygraphs form systems of generators for cubical categories, defined inductively on the dimension. Our presentation follows the method developed by Métayer in the globular setting [33]. We first introduce the notion of *cubical extension*, a set of $(n+1)$-generators adjoined to a cubical n-category. Lemma A.2.2 makes the construction of the free cubical $(n+1)$-category generated by a cubical n-category and equipped with a cubical extension explicit. This construction is then used to define cubical polygraphs recursively by adjoining cubical extensions to freely generated cubical categories.

A.2.1. Cubical extensions. For $n \in \mathbb{N}$, a *precubical n-set* is a family $C = (C_k)_{0\leqslant k\leqslant n}$ of k-cells with face maps $\partial^\alpha_{k,i} : C_k \to C_{k-1}$, for $1 \leqslant i \leqslant k \leqslant n$, satisfying the cubical relations (2.1.2). A functor $F : C \to \mathcal{D}$ of precubical sets is a family of maps $(F_k : C_k \to \mathcal{D}_k)_{k\in\mathbb{N}}$ that preserve face maps, that is $F_{k-1}\partial^\alpha_{k,i} = \partial^\alpha_{k,i}F_k$, for every $1 \leqslant i \leqslant n$. We denote by PreCub_n the category of precubical n-sets and their functors. We denote by $\mathsf{Cub}_n{}^\Gamma$ the category of cubical n-categories and their functors as defined in §2.1.1.

The *category of cubical extensions of cubical n-categories* is defined by the following pullback in CAT

$$\begin{array}{ccc} (\mathsf{Cub}_n{}^\Gamma)^+ & \dashrightarrow & \mathsf{PreCub}_{n+1} \\ \vdots & & \downarrow \\ \mathsf{Cub}_n{}^\Gamma & \xrightarrow{U_n} & \mathsf{PreCub}_n \end{array}$$

where the bottom arrow is the forgetful functor and the right arrow the truncation functor.

Explicitly, a cubical extension of a cubical n-category C consists of a set X_{n+1} of $(n+1)$-*generators* and a set of face maps $\partial^\alpha_{n+1,i} : X_{n+1} \to C_n$, for $1 \leqslant i \leqslant n+1$, that satisfy the cubical relations (2.1.2). A morphism of cubical extensions $F : (C, X) \to (\mathcal{D}, Y)$ consists of a functor between the cubical n-categories $G : C \to \mathcal{D}$ and a map $H : X \to Y$ such that $\partial^\alpha_{n+1,i}H = G_n\partial^\alpha_{n+1,i}$ for all $1 \leqslant i \leqslant n+1$.

Consider the forgetful functor

$$W_n : \mathsf{Cub}_{n+1}{}^\Gamma \to (\mathsf{Cub}_n{}^\Gamma)^+$$

sending a cubical $(n+1)$-category C to the pair $(C_{\leqslant n}, C_{n+1})$, where $C_{\leqslant n}$ is the n-category made of k-cells of C, for $k \leqslant n$, and C_{n+1} is the set of $(n+1)$-cells viewed as a cubical extension. It has a left adjoint L_n, which maps a cubical n-category C, equipped with a cubical extension X_{n+1}, to the freely generated cubical $(n+1)$-category $C[X_{n+1}]$. For Gray categories and polygraphs, a proof of the existence of this adjoint functor has been given by Lucas [28], although no explicit construction is given there. We provide a fully syntactic construction of the free functor L_n using a type system analogous to that of Métayer in the globular case [33, Section 4.1]. Our syntax differs from the globular one in several respects: we introduce constants for degeneracy and connection maps rather than identity maps, and we quotient by the cubical axioms instead of the globular ones. Another difference concerns the type of $(n+1)$-cells: in the globular case one uses n-globes; here the corresponding types are n-squares.

A.2.2. Lemma. *The forgetful functor* $W_n : \mathsf{Cub}_{n+1}{}^{\Gamma} \to (\mathsf{Cub}_n{}^{\Gamma})^+$ *has a left adjoint* L_n.

Proof. Consider (C, X_{n+1}) in $(\mathsf{Cub}_n{}^{\Gamma})^+$, with face maps of X_{n+1} denoted $\partial^{\alpha}_{n+1,i}$ for all $1 \leqslant i \leqslant n+1$. We define a formal syntax $\mathcal{E}$ formed by

- **i)** a constant symbol $\mathbf{c}_x$, for each $x \in X_{n+1}$,
- **ii)** a constant symbol $\mathbf{e}_{i,c}$, for each $c \in C_n$ and $1 \leqslant i \leqslant n+1$,
- **iii)** a constant symbol $\mathbf{g}^{\alpha}_{i,c}$, for each $c \in C_n$ and $1 \leqslant i \leqslant n$,
- **iv)** a binary function symbol $\circ_i$, for each $1 \leqslant i \leqslant n+1$.

Then $\mathcal{E}$ is the smallest set of that contains all constants and is closed under the operation $A \circ_i B$, for all $A, B \in \mathcal{E}$ and $1 \leqslant i \leqslant n+1$. A *type* is any n-square in C_n. For every $A \in \mathcal{E}$ and every type S, we recursively defined the judgement $A : S$ – *A has type S*: following rules:

- **i)** $\mathbf{c}_x : \partial x$, for every $x \in X_{n+1}$,
- **ii)** $\mathbf{e}_{i,c} : S$, for every n-cell c in C, where

$$S^{\alpha}_{j} = \begin{cases} \varepsilon_{n,i}\partial^{\alpha}_{n,j-1}c & \text{if } i < j, \\ c & \text{if } i = j, \\ \varepsilon_{n,i-1}\partial^{\alpha}_{n,j}c & \text{if } i > j, \end{cases}$$

- **iii)** $\mathbf{g}^{\alpha}_{i,c} : S$, for every n-cell c in C, where

$$S^{\beta}_{j} = \begin{cases} \Gamma^{\alpha}_{n,i}\partial^{\beta}_{n,j-1}c & \text{if } i < j-1, \\ c & \text{if } j = i, i+1 \text{ and } \alpha = \beta, \\ \varepsilon_{n,i}\partial^{\alpha}_{n,i}c & \text{if } j = i, i+1 \text{ and } \alpha = -\beta, \\ \Gamma^{\alpha}_{n,i-1}\partial^{\beta}_{n,j}c & \text{if } i > j, \end{cases}$$

- **iv)** $(A \circ_i B) : U$, for expressions $A : S$ and $B : T$, where

$$U^{\alpha}_{j} = \begin{cases} S^{\alpha}_{j} \circ_{n,i} T^{\alpha}_{j} & \text{if } i < j, \\ S^{-}_{i} & \text{if } i = j \text{ and } \alpha = -, \\ T^{+}_{i} & \text{if } i = j \text{ and } \alpha = +, \\ S^{\alpha}_{j} \circ_{n,i-1} T^{\alpha}_{j} & \text{if } i > j. \end{cases}$$

An expression A is *typable* if $A : S$ for some type S. A simple structural induction shows that typable expressions are uniquely type. Let $\mathcal{E}_T \subseteq \mathcal{E}$ denote the set of typable expressions. By uniqueness of types, there exist unique maps $d_i^\alpha : \mathcal{E}_T \to C_n$, for $1 \leqslant i \leqslant n+1$, such that $d_i^\alpha(\mathbf{c}_x) = \partial_{n+1,i}^\alpha(x)$ and $A : (d_i^\alpha(A))_{i,\alpha}$ for all $x \in C_n$ and $A \in \mathcal{E}_T$.

We write $\rhd_i$ for the relation of being $\circ_i$-composable on C_n. We extend this relation to $\mathcal{E}_T$ by setting $A \rhd_i B$ if $d_i^-(A) = d_i^+(B)$. Let $\sim$ be the smallest equivalence on $\mathcal{E}_T$ generated by the following conditions, for all $1 \leqslant i, j \leqslant n$, $A, B, C, D \in \mathcal{E}_T$ and $c, d \in C_n$:

i) $A \circ_i (B \circ_i C) \sim (A \circ_i B) \circ_i C$, if $A \rhd_i B \rhd_i C$,

ii) if $i < j$, $A \rhd_i B$, $C \rhd_i D$, $A \rhd_j C$ and $B \rhd_j D$, then

$$(A \circ_i B) \circ_j (C \circ_i D) \sim (A \circ_j C) \circ_i (B \circ_j D),$$

iii) $\mathbf{e}_{i,c} \circ_i A \sim A$, if $d_i^-(A) = c$,

iv) $A \circ_i \mathbf{e}_{i,c} \sim A$, if $d_i^+(A) = c$,

v) if $c \rhd_i d$, then

$$\mathbf{e}_{i,c \circ_j d} \sim \begin{cases} \mathbf{e}_{i,c} \circ_{j+1} \mathbf{e}_{i,d} & \text{if } i \leqslant j, \\ \mathbf{e}_{i,c} \circ_j \mathbf{e}_{i,d} & \text{if } i > j, \end{cases}$$

vi) $\mathbf{e}_{i,\varepsilon_{n,j}c} \sim \mathbf{e}_{j+1,\varepsilon_{n,i}c}$, if $i \leqslant j$,

vii) if $c \rhd_i d$, then

$$\mathbf{g}^\alpha_{i,c \circ_{n,j} d} \sim \begin{cases} \mathbf{g}^\alpha_{i,c} \circ_{j+1} \mathbf{g}^\alpha_{i,d} & \text{if } i < j, \\ (\mathbf{g}^-_{i,c} \circ_i \mathbf{e}_{i+1,d}) \circ_{i+1} (\mathbf{e}_{i,d} \circ_i \mathbf{g}^-_{i,d}) & \text{if } i = j \text{ and } \alpha = -, \\ (\mathbf{g}^+_{i,c} \circ_i \mathbf{e}_{i,c}) \circ_{i+1} (\mathbf{e}_{i+1,c} \circ_i \mathbf{g}^+_{i,d}) & \text{if } i = j \text{ and } \alpha = +, \\ \mathbf{g}^\alpha_{i,c} \circ_j \mathbf{g}^\alpha_{i,d} & \text{if } i > j, \end{cases}$$

viii) $\mathbf{g}^+_{i,c} \circ_i \mathbf{g}^-_{i,c} \sim \mathbf{e}_{i+1,c}$ and $\mathbf{g}^+_{i,c} \circ_{i+1} \mathbf{g}^-_{i,c} \sim \mathbf{e}_{i,c}$,

ix)

$$\mathbf{g}^\alpha_{i,\varepsilon_{n,j}c} \sim \begin{cases} \mathbf{e}_{j+1,\Gamma^\alpha_{n,i}c} & \text{if } i < j, \\ \mathbf{g}^\alpha_{i,\varepsilon_{n,i}c} \sim \mathbf{e}_{i,\varepsilon_{n,i}c} & \text{if } i = j, \\ \mathbf{g}^\alpha_{i,\varepsilon_{n,j}c} \sim \mathbf{e}_{j,\Gamma^\alpha_{n,i-1}c} & \text{if } i > j, \end{cases}$$

x) $\mathbf{g}^\alpha_{i,\Gamma^\beta_{n,j}c} \sim \mathbf{g}^\beta_{j+1,\Gamma^\alpha_{n,i}c}$ if $i < j$ and $\mathbf{g}^\alpha_{i,\Gamma^\alpha_{n,i}c} \sim \mathbf{g}^\alpha_{i+1,\Gamma^\alpha_{n,i}c}$.

Let $\cong$ be the congruence generated by $\sim$ on $\mathcal{E}_T$. We define $X^*_{n+1} \coloneqq \mathcal{E}_T/\cong$, and write $[A]$ for the equivalence class of an expression A. we define the operations

$$\partial^\alpha_{n+1,i}([A]) \coloneqq d^\alpha_i(A) \quad \text{and} \quad [A_1] \circ_{n+1,i} [A_2] \coloneqq [A_1 \circ_i A_2],$$

on X^*_{n+1} whenever $A_1 \rhd_i A_2$. We further define maps $\varepsilon_{n+1,i}, \Gamma^\alpha_{n+1,i} : C_n \to X^*_{n+1}$, for every $c \in C_n$, by

$$\varepsilon_{n+1,i}(c) \coloneqq [\mathbf{e}_{i,c}] \quad \text{and} \quad \Gamma^\alpha_{n+1,i}(c) \coloneqq [\mathbf{g}^\alpha_{i,c}],$$

Finally, we define $L_n(C, X_{n+1})$ to be the cubical $(n+1)$-category with underlying n-category C_n, set of $(n+1)$-cells X^*_{n+1}, and structure induced by the operations just introduced.

It is routine to check that this construction produces a cubical $(n+1)$-category, and that it extends to make $L_n : (\mathsf{Cub}_n{}^\Gamma)^+ \to \mathsf{Cub}_{n+1}{}^\Gamma$ functorial.

Next, we check the adjunction $L_n \dashv W_n$. Let (C, X_{n+1}) be in $(\mathsf{Cub}_n{}^\Gamma)^+$, let $\mathcal{D}$ be in $\mathsf{Cub}_{n+1}{}^\Gamma$ and let

$$f \coloneqq (g : C \to \mathcal{D}_{\leqslant n}, h : X_{n+1} \to \mathcal{D}_{n+1})$$

be a morphism $f : (C, X_{n+1}) \to W_n(\mathcal{D})$ in $(\mathsf{Cub}_n{}^\Gamma)^+$.

We recursively define a map $f' : \mathcal{E}_T \to \mathcal{D}$, for all $x \in X_{n+1}$, $c \in C_n$ and $A, B \in \mathcal{E}_T$, $1 \leqslant i \leqslant n+1$, as

$$f'(\mathbf{c}_x) = h(x), \qquad f'(\mathbf{e}_{i,c}) = \varepsilon_{n+1,i}(g(c)), \qquad f'(\mathbf{g}^\alpha_{i,c})\Gamma^\alpha_{n+1,i}(g(c)), \qquad f'(A \circ_i B) = f'(A) \circ_{n+1,i} f'(B).$$

It is compatible with $\cong$ in the sense that $f'(A) = f'(B)$ whenever $A \cong B$, hence it induces a well-defined map $f^* : L_n(C, X_{n+1}) \to \mathcal{D}$. It is straightforward to check that f^* is a cubical $(n+1)$-functor. Hence we obtain a map of type

$$(\mathsf{Cub}_n{}^\Gamma)^+((C, X_{n+1}), W_n(\mathcal{D})) \to \mathsf{Cub}_{n+1}{}^\Gamma(L_n(C, X_{n+1}), \mathcal{D}).$$

It is also easy to check that this map is natural in (C, X_{n+1}) and $\mathcal{D}$, and that it is invertible, the inverse sending a cubical $(n+1)$-functor f to the pair (g, h) where g is the n-truncation of f and h is the map between the sets of $(n+1)$-cells. This yields a natural isomorphism between the above hom-sets, which establishes $L_n \dashv W_n$. □

The construction of the left adjoint for cubical (n, p)-categories proceeds as above, after adjoining inverse as constants to the syntax and the associated invertibility axioms to the congruence $\sim$.

A.2.3. Lemma. *The forgetful functor $W_{(n,p)} : \mathsf{Cat}_{(n+1,p)} \to \mathsf{Cat}_{(n,p)}{}^+$ has a left adjoint $L_{(n,p)}$.*

A.2.4. Cubical polygraphs. We can now construct cubical polygraphs along the lines of their globular siblings [2]. We recursively define the categories CubPol_n of *cubical n-polygraphs* and the functors $F_n : \mathsf{CubPol}_n \to \mathsf{Cub}_n{}^\Gamma$, which send a cubical n-polygraph to the cubical n-category $F_n(X) = X^*$ freely generates by it:

i) The category CubPol_0 is Set and the functor F_0 the identity.

ii) Given CubPol_n and F_n, the category CubPol_{n+1} is defined by the pullback

$$\begin{array}{ccc} \mathsf{CubPol}_{n+1} & \overset{J_n}{\dashrightarrow} & (\mathsf{Cub}_n{}^{\Gamma})^{+} \\ \vdots & & \downarrow \\ \mathsf{CubPol}_n & \xrightarrow{F_n} & \mathsf{Cub}_n{}^{\Gamma} \end{array}$$

in CAT, and the functor F_{n+1} is defined as the composition

$$\mathsf{CubPol}_{n+1} \overset{J_n}{\to} (\mathsf{Cub}_n{}^{\Gamma})^{+} \overset{L_n}{\to} \mathsf{Cub}_{n+1}{}^{\Gamma}.$$

Explicitly, a cubical n-polygraph is a family $(X_0, \ldots, X_n)$, where each X_{k+1} is a cubical extension of $X^*_{\leqslant k}$ for every $k < n$. The category CubPol_ω of *cubical ω-polygraphs* is the projective limit of the following diagram in CAT

$$\mathsf{CubPol}_0 \overset{V_0}{\longleftarrow} \mathsf{CubPol}_1 \longleftarrow \cdots \longleftarrow \mathsf{CubPol}_n \overset{V_n}{\longleftarrow} \mathsf{CubPol}_{n+1} \longleftarrow \ldots$$

where, for every $n \geqslant 1$, the functor V_n is the truncation functor forgetting the $(n+1)$-dimensional cubical extension.

Finally, adding inverses both to the definition of cubical polygraphs and to the construction of the free cubical category in Lemma A.2.3 leads to the notion of *cubical (n,p)-polygraphs* for all $p \leqslant n$. Each cubical (n,p)-polygraph X freely generates a cubical (n,p)-category, denoted $X^{\top}$.