

CONVERGENT PRESENTATIONS AND POLYGRAPHIC RESOLUTIONS OF ASSOCIATIVE ALGEBRAS

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Abstract – Several constructive homological methods based on noncommutative Gröbner bases are known to compute free resolutions of associative algebras. In particular, these methods relate the Koszul property for an associative algebra to the existence of a quadratic Gröbner basis of its ideal of relations. In this article, using a higher-dimensional rewriting theory approach, we give several improvements of these methods. We define polygraphs for associative algebras as higher-dimensional linear rewriting systems that generalise the notion of noncommutative Gröbner bases, and allow more possibilities of termination orders than those associated to monomial orders. We introduce polygraphic resolutions of associative algebras, giving a categorical description of higher-dimensional syzygies for presentations of algebras. We show how to compute polygraphic resolutions starting from a convergent presentation, and how these resolutions can be linked with the Koszul property.

Keywords – Higher-dimensional associative algebras, confluence and termination, linear rewriting, polygraphs, free resolutions, Koszulness.

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INTRODUCTION

An overview of rewriting methods in algebra

Rewriting theory. Throughout the twentieth century, several rewriting-based techniques have been developed for computations in an algebraic setting. Rewriting consists in orienting the relations of a presentation, and computational properties are deduced from the overlaps that may appear in the application of the oriented relations. A *rewriting step* is an application of a oriented relation on a term, and a *rewriting sequence* is a finite or infinite composition of rewriting steps. *Termination* is satisfied if there exist no infinite rewriting sequences, *confluence* holds if all the finite rewriting sequences that start on a given term end on a common term, and *convergence* is the conjunction of termination and confluence. A convergent presentation gives a way to compute normal forms in the presented algebraic object, with applications, in particular, to the decision of the word problem.

Rewriting theory has been in particular developed for monoids and groups [16] and for equational theories [5], and several techniques have been introduced to prove that a given oriented presentation is convergent, or to compute a convergent presentation from an arbitrary one. In particular, Newman’s lemma states that, under the termination hypothesis, confluence is equivalent to *local confluence*, i.e. confluence limited to the rewriting steps acting on a common term [45]. Moreover, the critical branching theorem states that local confluence can in turn be deduced from the confluence of *critical branchings*, which are the minimal overlaps of two oriented relations [41, 46, 35]. Together, these two results allow to deduce confluence from a local analysis of branchings. Another important result is Knuth-Bendix’s completion algorithm whose aim is to compute a convergent presentation from a terminating one, by progressive adjunction of new oriented relations when an obstruction to confluence is found.

In the 1980’s, Squier initiated a new applications of rewriting theory, in homological algebra. At that time, Jantzen asked if every finitely generated monoid with a decidable word problem always admit a finite convergent presentation [37, 38]. The answer was expected to be negative, but the proof was complicated by the observation that, for a given monoid, the result depends on the chosen finite set of generators, as shown by Kapur and Narendran on the example of the braid monoid B_3^+ [39]. To solve the problem, Squier introduced a homological finiteness condition that a monoid has to satisfy to admit a finite convergent presentation: if a monoid M admits a finite convergent presentation, then \mathbb{Z} admits a partial

resolution of length 3 by free, finitely generated left- $\mathbb{Z}M$ -modules. Using this condition, Squier could prove that a given finitely presented monoid, with a decidable word problem, admits no finite convergent presentation [52]. This seminal work had numerous subsequent developments. In particular, Squier later refined his homological condition into a homotopical condition [53], and his free resolution was extended in all dimensions by Kobayashi [42].

Rewriting methods for algebras. The rewriting point of view is also the main principle applied in numerous works in linear algebra, although with a different vocabulary, and in a somewhat restricted setting. The corresponding concepts have been introduced in particular to compute normal forms for different types of algebras presented by generators and relations, with applications to the decision of the ideal membership problem, and to the construction of bases, such as Poincaré-Birkhoff-Witt bases. For example, Shirshov introduced in [50] an algorithm to compute a linear basis of a Lie algebra presented through generators and relations. He used the notion of composition of elements in a free Lie algebra to describe the critical branchings. He gave an algorithm to compute bases in free algebras having the confluence property, and he proved the composition lemma, which is the analogue of Newman's lemma for Lie algebras. As an application, he deduced a constructive proof of the Poincaré-Birkhoff-Witt theorem.

Also, rewriting methods to compute with ideals of commutative polynomial rings were introduced by Buchberger with Gröbner basis theory [20, 22, 21]. Gröbner bases are sets of relations that satisfy the confluence property, plus a restricted form of termination, corresponding to a compatibility with a *monomial order*, i.e. a wellfounded total order on the monomials. Buchberger described critical branchings with the notion of S-polynomial, and gave an algorithm for the computation of Gröbner bases, using a linear counterpart to Newman's lemma, and which is the analogue of Knuth-Bendix's completion procedure for commutative algebras. Note that, in the same period, ideas in the spirit of Gröbner bases appear in several others works: by Hironaka and Grauert with standard bases for power series rings [34, 27], or for applications of Newman's lemma for universal algebras by Cohn [24]. The domain took foundation in several works on algorithmic methods in elimination theory by Macaulay with H-bases [43], by Janet with involutive bases [36], or Gunther with notions similar to Gröbner bases [33].

Bokut and Bergman have independently extended Gröbner bases to associative algebras, obtaining the analogue of Newman's lemma for free associative algebras, called respectively the composition lemma and the diamond lemma [15, 14]. Buchberger's algorithm has then been extended to this setting [44, 57]. Subsequently, rewriting methods were developed for a wide range of algebraic structures, such as Weyl algebras [49] or operads [25]. For a comprehensive treatment on noncommutative Gröbner bases, we refer the reader to [44, 56, 18].

Applications in homological algebra. At the end of the eighties, through Anick's and Green's works, noncommutative Gröbner bases have found new applications, in the shape of constructive methods to compute free resolutions of associative algebras [1, 2, 3, 28]. Their constructions provide small explicit resolutions to compute homological invariants of algebras presented by generators and a Gröbner basis: homology groups, Hilbert series, and Poincaré series. Anick's resolution consists in a complex generated by Anick's chains, i.e. certain iterated overlaps of the leading terms of the relations as in the later Kobayashi's resolution for monoids, and its differential is obtained by deforming the differential of a complex for an associated monomial algebra. This construction has many applications, such as an algorithm for the computation of Hilbert series [56]. The chains and the differential of the resolution are constructed recursively, making possible its implementation [7], but the differential is complicated to make explicit in

general. Sköldbberg introduced in [51] a homotopical method based on discrete Morse theory to derive Anick’s resolution from the bar resolution.

Anick’s resolution also gives a relation between the Koszul property of an algebra and the existence of a quadratic noncommutative Gröbner basis for its ideal of relations. Recall that a connected graded algebra A is *Koszul* if the Tor groups $\mathrm{Tor}_{k,(i)}^A(\mathbb{K}, \mathbb{K})$ vanish for $i \neq k$, where k is the homological degree, and i corresponds to the grading of the algebra. Koszul algebras were introduced by Priddy, and he proved that quadratic algebras having a Poincaré-Birkhoff-Witt basis are Koszul [48]. Thus, the Koszulness of a quadratic algebra A can be obtained by showing the existence of a quadratic Gröbner basis, either because Green proved in [28] that it implies the existence of a Poincaré-Birkhoff-Witt A , or because, in that case, Anick’s resolution is concentrated in the right degrees [2, 29]. Note that Backelin gave a characterisation of the Koszul property for quadratic algebras in terms of lattices [6, 8], and this condition was later interpreted by Berger’s X -confluence [11]. The quadratic Gröbner basis method to prove Koszulness has been extended to the case of operads by Dotsenko and Khoroshkin in [25].

Koszulness was later generalised by Berger in [12] to the case of N -homogeneous algebras, asking that $\mathrm{Tor}_{k,(i)}^A(\mathbb{K}, \mathbb{K})$ vanish for $i \neq \ell_N(k)$, where $\ell_N : \mathbb{N} \rightarrow \mathbb{N}$ is defined by $\ell_N(k) = \lfloor Nk \rfloor$ if $k = 2\lfloor k/N \rfloor$, and $\lfloor Nk \rfloor + 1$ if $k = 2\lfloor k/N \rfloor + 1$. For intuition, in a graded algebra A with N -homogeneous relations, the groups $\mathrm{Tor}_{k,(i)}^A(\mathbb{K}, \mathbb{K})$ always vanish for $i < \ell_N(k)$, so that the Koszul property corresponds to the limit case [13]. However, in the N -homogeneous case, a Gröbner basis concentrated in weight N is not enough to imply Koszulness: as shown by [12], one has to check an extra condition which, when the algebra is monomial, concerns the overlaps of the monomials of the relations.

Convergent presentations of associative algebras

Known constructions of free resolutions using confluence, such as Anick’s resolution, are not explicit and constructed inductively with respect to a monomial ordering. One of the objectives of this paper is to give such a construction in a higher-dimensional rewriting framework, in order to make explicit the contracting homotopy in terms of rewriting properties of the presentation, and using a non-monomial orientation of relations. Higher-dimensional rewriting theory is the theory of presentations by generators and relations of higher-dimensional categories [54, 55, 23]. The notion of *polygraph* is the main concept of the theory, extending to higher dimensions the notion of presentation by generators and relations for categories, and giving an unified paradigm of rewriting [30, 32]. In this work, we introduce a variation of the notion of polygraph for higher-dimensional associative algebras.

Higher-dimensional associative algebras and polygraphs. The first section of the paper introduces the main categorical notions used throughout this work. We define in 1.3.2 the category $\infty\mathrm{Alg}$ of ∞ -algebras and *morphisms of ∞ -algebras* as the category of ∞ -categories and ∞ -functors internal to the category Alg of associative algebras. Theorem 1.3.3 gives several interpretations of the structure of ∞ -algebra. In particular, the category $\infty\mathrm{Alg}$ is isomorphic to the category of internal ∞ -groupoids in the category Alg . Theorem 1.3.3 also makes explicit isomorphisms between these categories and full subcategories of the category of globular algebras and the category of globular bimodules. The latter interpretation is used to construct free ∞ -algebras in 2.1.3.

In Section 2, we adapt the set-theoretical notion of polygraph to presentations of higher-dimensional algebras. As in the set-theoretical case, the category $n\mathrm{Pol}(\mathrm{Alg})$ of *n-polygraphs for associative algebras*, called *n-polygraphs* for short, is constructed by induction on n in 2.2.1. The category $0\mathrm{Pol}(\mathrm{Alg})$ is the

category of sets. Then 1-polygraphs are pairs $\langle X_0 | X_1 \rangle$ made of a set X_0 and a *cellular extension* X_1 of the free algebra $\mathcal{A}(X_0)$ over X_0 , that is a set X_1 with two maps

$$\mathcal{A}(X_0) \xleftarrow[t]{s} X_1.$$

The elements of X_k are called the *k-cells* of X , and for a 1-cell α , the 0-cells $s(\alpha)$ and $t(\alpha)$ are its *source* and *target*. For instance the 1-polygraph

$$X = \langle x, y, z \mid xyz \xrightarrow{\gamma} x^3 + y^3 + z^3 \rangle \quad (1)$$

studied in Example 3.2.4 is generated by three 0-cells x , y , and z , and a 1-cell γ that reduces the monomial xyz into $x^3 + y^3 + z^3$. For $n \geq 2$, assume that the category $(n-1)\text{Pol}(\text{Alg})$ of $(n-1)$ -polygraphs is defined, together with the free $(n-1)$ -algebra functor $(n-1)\text{Pol}(\text{Alg}) \rightarrow (n-1)\text{Alg}$. The category $n\text{Pol}(\text{Alg})$ of n -polygraphs is defined as the pullback

$$\begin{array}{ccc} n\text{Pol}(\text{Alg}) & \longrightarrow & (n-1)\text{Alg}^+ \\ \downarrow \lrcorner & & \downarrow \\ (n-1)\text{Pol}(\text{Alg}) & \longrightarrow & (n-1)\text{Alg} \end{array}$$

where $(n-1)\text{Alg}^+$ is the category of $(n-1)$ -algebras A_{n-1} with a cellular extension, i.e. a set X with two maps

$$A_{n-1} \xleftarrow[t]{s} X$$

such that $s(\alpha)$ and $t(\alpha)$ have the same source and the same target. An n -polygraph is thus a sequence $\langle X_0 | \cdots | X_n \rangle$, made of a set X_0 and, for every $0 \leq k < n$, a cellular extension X_{k+1} of the free k -algebra over $\langle X_0 | \cdots | X_k \rangle$. If X is an n -polygraph, we denote by $\mathcal{A}(X)$ the free n -algebra over X . The *algebra presented* by a 1-polygraph X is the quotient algebra \bar{X} of the free algebra $\mathcal{A}(X_0)$ by the congruence generated by X_1 . A *coherent presentation* of an algebra A is a 2-polygraph $\langle X_0 | X_1 | X_2 \rangle$ whose underlying 1-polygraph $\langle X_0 | X_1 \rangle$ is a presentation of A , and such that the cellular extension X_2 is *acyclic*, meaning that every 1-sphere in the free 1-algebra $\mathcal{A}(X_1)$ is trivial with respect to the congruence generated by X_2 .

Convergent presentations. In Section 3, we study the rewriting properties of 1-polygraphs, whose 1-cells are not necessarily oriented with respect to a monomial ordering. Our approach is thus less restrictive than those known for associative algebras, which rely on a monomial order. For instance, there is no monomial order compatible with the rule γ of the 1-polygraph (1), see Example 3.2.4, whereas this orientation makes proving its confluence trivial.

A 1-polygraph X is *left-monomial* if, for every 1-cell α of X , the 0-cell $s(\alpha)$ is a monomial of $\mathcal{A}(X)$ that does not belong to the support of $t(\alpha)$, as defined in 2.3.2. We define in 3.1.1 a *rewriting step* of a left-monomial 1-polygraph X , as a 1-cell $\lambda f + 1_a$ of the free 1-algebra $\mathcal{A}(X)$ such that $\lambda \neq 0$ and $s(f)$ is not in the support of a . A composition of rewriting steps in $\mathcal{A}(X)$ is called a *positive* 1-cell of $\mathcal{A}(X)$. A 0-cell a of $\mathcal{A}(X)$ is *reduced* if there is no rewriting step of source a . The reduced 0-cells of $\mathcal{A}(X)$ form a linear subspace $\text{Red}(X)$ of the free algebra $\mathcal{A}(X_0)$, admitting as a basis the set $\text{Red}_m(X)$ of reduced monomials of $\mathcal{A}(X_0)$. We define in 3.2.2 the *termination* of X as the wellfoundedness of a binary relation

Indeed, nonoverlapping branchings are always confluent in the set-theoretic case, but we show that it is not the case for linear rewriting systems, for which confluence of non-overlapping branchings can depend on critical confluence. Finally, Section 4 gives a linearised version of Squier’s theorem, initially stated for presentations of monoids [53, 32].

Theorem 4.3.2. *Let X be a convergent left-monomial 1-polygraph. A cellular extension Y of $\mathcal{A}(X)$ that contains one 2-cell of the form (2) for every critical branching (f, g) of X , with h and k positive 1-cells of $\mathcal{A}(X)$, is acyclic.*

This result is then extended in Section 6 into a polygraphic resolution for the algebra \bar{X} presented by X , involving critical branchings in every dimension.

Polygraphic resolutions of associative algebras

A *polygraphic resolution* of an algebra A is an ∞ -polygraph X whose underlying polygraph is a presentation of A , and such that, for every $k \geq 1$, the cellular extension X_{k+1} of the free k -algebra $\mathcal{A}(A_k)$ is acyclic. In Sections 5 and 6 we construct such a resolution starting with a convergent presentation of the algebra A . Finally, in Section 7, we deduce a resolution of A by free A -bimodules from a polygraphic resolution of A .

Contractions of polygraphs. A method to construct a polygraphic resolution is to consider a contraction inducing a notion of normal form in every dimension, together with a homotopically coherent reduction of every cell to its normal form. This notion was introduced in [31] for presentations of categories, where it was called a normalisation strategy, and provides a constructive characterisation of the acyclicity of an $(\infty, 1)$ -polygraph. In 5.2, we introduce contractions for polygraphic resolutions of algebras. We prove that a polygraphic resolution of an algebra A is equivalent to the data of an ∞ -polygraph whose underlying 1-polygraph is a presentation of A and equipped with a contraction. Explicitely, a *unital section* of an ∞ -polygraph X is a section of the canonical projection $\pi : \mathcal{A}(X) \rightarrow \bar{X}$ mapping 1 to A . Given such a unital section ι , an ι -*contraction* of X is a homotopy $\sigma : \text{Id}_{\mathcal{A}(X)} \Rightarrow \iota\pi$ that satisfies $\sigma_a = 1_a$ on the images of ι and of σ , and an ι -contraction is called *right* if it satisfies $\sigma_{ab} = s_0(a)\sigma_b \star_0 \sigma_{\widehat{as_0(b)}}$. The main result of Section 5 relates the property for an ∞ -polygraph to be a polygraphic resolution to the existence of a right ι -contraction.

Theorem 5.2.6. *Let X be an ∞ -polygraph with a fixed unital section ι . Then X is a polygraphic resolution of \bar{X} if, and only if, it admits a right ι -contraction.*

The standard polygraphic resolution. Given an augmented algebra A and a linear basis \mathcal{B} of its positive part, Theorem 6.1.2 makes explicit a polygraphic resolution $\text{Std}(\mathcal{B})$, which is a cubical analogue of the standard resolution of the algebra A . We devote 6.1 to the proof of the acyclicity of the resolution by exhibiting a right contraction for the resolution $\text{Std}(\mathcal{B})$. We can apply this construction when X is a convergent presentation of A : Corollary 6.1.4 makes explicit a polygraphic resolution $\text{Std}(\text{Red}_m(X))$ for A , whose generating cells are finite families of nontrivial reduced monomials of the free algebra $\mathcal{A}(X)$. This resolution extends the coherent presentation given by Squier’s theorem.

Collapsing, and Squier’s polygraphic resolution. The polygraphic resolution $\text{Std}(\text{Red}_m(X))$ is too large in general to be used in practice. Following a construction given by Brown in [19], in 5.3 we define polygraphic *collapsing schemes*, which give, by a process similar to algebraic Morse theory for

chain complexes [51], a method to contract a polygraphic resolution of an algebra into a smaller one, see Theorem 5.3.4. We then use this result to contract the standard resolution $\text{Std}(\text{Red}_m(X))$ into a smaller one, denoted by $\text{Sq}(X)$ and called *Squier's polygraphic resolution*, containing only the critical n -branchings of X , which are the overlaps of n generating relations. Omitting the technical details, we obtain:

Theorem 6.2.4. *If X is a reduced convergent left-monomial presentation of an augmented algebra A , then $\text{Sq}(X)$ is a polygraphic resolution of A .*

By this result, any reduced convergent left-monomial presentation of an augmented algebra A extends to a polygraphic resolution of A whose generating cells correspond to the iterated overlaps of leading terms of relations: in this spirit, Squier's resolution is a categorical analogue of Anick's resolution of an augmented algebra presented by a Gröbner basis. In 6.3, we compute Squier's polygraphic resolutions for the symmetric algebra, the quantum deformation of the symmetric algebra, and the exterior algebra.

Free resolutions from polygraphic resolutions. In Section 7, we show that a polygraphic resolution of an algebra A induces free resolutions in categories of modules over A . Given an ∞ -polygraph X whose underlying 1-polygraph is a presentation of A , we construct in 7.1 a complex of A -bimodules, denoted by $A^e[X]$, whose boundary maps are induced by the source and target maps of the polygraph. We prove that if the polygraph X is acyclic, then the induced complex is acyclic, giving the following result.

Theorem 7.1.3. *If X is a polygraphic resolution of an algebra A , then the complex $A^e[X]$ is a free resolution of the A -bimodule A . Moreover, if X is of finite type, then so is $A^e[X]$.*

This bimodule resolution can be used to compute Hochschild homology, as in [12, Section 5]. Using these constructions, we deduce finiteness homological properties of an associative algebra A given by a convergent presentation. In 7.1.6, we introduce the property of *finite n -derivation type* for an associative algebra, that corresponds to admitting a polygraphic resolution with finitely many generating k -cells for $k < n$. We relate this property to a homological finiteness condition, *type FP_n* , and we prove that associative algebras admitting a finite convergent presentation are of finite n -derivation type for any natural number n , Proposition 7.1.7.

Confluence and Koszulness. In Section 7.2, we apply our constructions to study Koszulness of associative algebras. Given a map $\omega : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$, we call a polygraph X ω -concentrated if, for every natural number $n \geq 1$, the n -cells of X are concentrated in degree $\omega(n+1)$. Similarly, a free resolution F_* of bimodules is ω -concentrated if, for every natural number n , the bimodule F_n is generated in degree $\omega(n)$. As a consequence of Theorem 7.1.3, we deduce that if X is an ω -concentrated polygraphic resolution of a graded algebra A , then the free A -bimodule $A^e[X_n]$ is generated by its component of degree $\omega(n+1)$, Proposition 7.2.2. In particular, if $\omega = \ell_N$ for some $N \geq 2$, this proves that the algebra A is Koszul. Consequently, an algebra admitting a quadratic convergent presentation is Koszul. Finally, we discuss several examples applying our rewriting methods to prove or disprove Koszulness. For instance, the algebra presented by the 1-polygraph X given by (1) is trivially Koszul, whereas a proof of this property using Anick's resolution involves the computation of a resolution of infinite length.

Acknowledgements, organisation and conventions

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Organisation of the article. Section 1 presents some categorical background, and constructions on the structure of higher-dimensional associative algebra. Section 2 deals with the notion of polygraph for associative algebras, to define presentations, coherent presentations and resolutions for these algebras. Section 3 is devoted to the study of termination and confluence properties of polygraphs, making comparisons with Poincaré-Birkhoff-Witt bases and Gröbner bases. Section 4 deals with coherent presentations of associative algebras and the construction of such presentations using convergence. This is the first step of the construction achieved in Sections 5 and 6 of a polygraphic resolution of an associative algebra starting with a convergent presentation. Section 5 presents homotopical operations on polygraphic resolutions and a method to contract polygraphic resolutions into smaller ones. In Section 6, these operations are applied to construct Squier’s polygraphic resolution of an associative algebra starting with a reduced convergent presentation. Finally, Section 7 is devoted to the application of polygraphic resolutions to the construction of resolutions of associative algebras by free bimodules, leading to finiteness conditions and several necessary or sufficient conditions for an associative algebra to be Koszul.

Conventions. We fix a field \mathbb{K} for the whole article, and denote by \mathbf{Vect} the category of vector spaces over \mathbb{K} . The category of unital and associative algebras over \mathbb{K} is denoted by \mathbf{Alg} . In this article, we will say algebra for unital associative algebra if no confusion may arise.

1. HIGHER-DIMENSIONAL ASSOCIATIVE ALGEBRAS

This section introduces the higher-dimensional objects used throughout the paper: higher-dimensional vector spaces and higher-dimensional associative algebras, defined as (globular, strict) higher-dimensional categories internal to vector spaces and to associative algebras, respectively. Our notion of higher-dimensional vector space extends the 2-vector spaces defined by Baez and Crans in [9], but with a shift by 1 in the dimension: our n -vector spaces are n -categories in \mathbf{Vect} , instead of $(n - 1)$ -categories. The main result of the section, Theorem 1.3.3, explores the structure of ∞ -algebras, giving equivalences with other, simpler structures: this is used in the next section to build free ∞ -algebras.

1.1. Internal higher-dimensional categories

Let \mathcal{C} be a fixed category. The definitions of globular objects of \mathcal{C} and ∞ -categories of \mathcal{C} can be given in a more abstract setting, but we assume here that \mathcal{C} is concrete over the category of sets, and that the corresponding forgetful functor admits a left adjoint.

1.1.1. Indexed objects and morphisms. Fix $I \subseteq \mathbb{N}$. An I -indexed object of \mathcal{C} is a sequence $X = (X_n)_{n \in I}$ of objects of \mathcal{C} . In what follows, we just say *indexed object* for $I = \mathbb{N}$, and, if $n \in \mathbb{N}$, we say *n -indexed*

1. Higher-dimensional associative algebras

when $I = \{0, \dots, n\}$. If X and Y are I -indexed objects of \mathcal{C} , and p is an integer, an *indexed morphism of degree p from X to Y* is a sequence

$$f = (X_i \rightarrow Y_{i+p})_i$$

of morphisms of \mathcal{C} , where the index i ranges over the elements of I such that $i + p$ belongs to I . In particular, if $I = \mathbb{N}$ and $p = -1$, then i ranges over $\mathbb{N} \setminus \{0\}$. If X is an I -indexed object of \mathcal{C} , and $n \in I$, we abusively denote by X_n the I -indexed object of \mathcal{C} that is constantly equal to X_n . The I -indexed objects and indexed morphisms of degree 0 of \mathcal{C} form a category, denoted by $\text{Ind}(\mathcal{C})$. This category has limits and colimits, computed pointwise with respect to the ones of \mathcal{C} .

1.1.2. Internal globular objects. A *globular object of \mathcal{C}* is an indexed object $X = (X_n)_{n \geq 0}$ of \mathcal{C} equipped with indexed morphisms

$$X \xrightarrow{s} X, \quad X \xrightarrow{t} X \quad \text{and} \quad X \xrightarrow{i} X,$$

of respective degrees -1 , -1 and 1 , called the *source map*, the *target map* and the *identity map of X* , that satisfy the following equalities, collectively referred to as the *globular relations of X* :

$$ss = st, \quad ts = tt \quad \text{and} \quad si = ti = \text{Id}_X.$$

With the identity map and the last two relations removed, one gets a *semiglobular object of \mathcal{C}* . Remark that, usually, “globular objects” correspond to our semiglobular objects, while “reflexive globular objects” stand for our “globular objects”; we prefer the present terminology for its coherence with (semi)simplicial and (semi)cubical objects.

Given two globular objects X and Y of \mathcal{C} , a *globular morphism from X to Y* is an indexed morphism $F : X \rightarrow Y$ of degree 0 that commutes with the source, target and identity maps:

$$\begin{array}{ccc} X \xrightarrow{s} X & X \xrightarrow{t} X & X \xrightarrow{i} X \\ F \downarrow = \downarrow F & F \downarrow = \downarrow F & F \downarrow = \downarrow F \\ Y \xrightarrow{s} Y & Y \xrightarrow{t} Y & Y \xrightarrow{i} Y \end{array}$$

We denote by $\text{Glob}(\mathcal{C})$ the category of globular objects and globular morphisms of \mathcal{C} .

For $n \geq 0$, an *n -globular object of \mathcal{C}* is defined in the same way as a globular object of \mathcal{C} , but starting with a $\{0, \dots, n\}$ -indexed object of \mathcal{C} . We denote by $n\text{Glob}(\mathcal{C})$ the category they form with the corresponding n -globular morphisms of \mathcal{C} .

1.1.3. Sources, targets and spheres. Let X be globular object of \mathcal{C} , and fix $n \geq 0$. An element x of X_n is called an *n -cell of X* , and, if $n \geq 1$, the $(n - 1)$ -cells $s(x)$ and $t(x)$ are called the *source of x* and the *target of x* . We write $z : x \rightarrow y$ if z is an n -cell of X of source x and target y , and we use the more specific notation $z : x \Rightarrow y$ (resp. $z : x \Rrightarrow y$) when $n = 2$ (resp. $n = 3$). Since the globular relations imply that i is injective, and when no confusion occurs, we just write $i(x)$ or 1_x , or even just x , instead of any iterate image $i^p(x)$ of x through i .

Two n -cells x and y of X are called *parallel* if either $n = 0$, or $s(x) = s(y)$ and $t(x) = t(y)$. An *n -sphere of X* is a pair $\gamma = (x, y)$ of parallel n -cells of X , in which case we call x the *source of γ* and y the *target of γ* .

1.1.4. Composable cells. Let X be a globular object of \mathcal{C} , and fix $k \geq 0$. Define the k -source map of X and the k -target map of X as the indexed morphisms

$$X \xrightarrow{s_k} X_k \quad \text{and} \quad X \xrightarrow{t_k} X_k$$

of degree 0, given, on an n -cell x of X , by

$$s_k(x) = \begin{cases} s^{n-k}(x) & \text{if } n \geq k \\ i^{k-n}(x) & \text{if } n \leq k, \end{cases} \quad \text{and} \quad t_k(x) = \begin{cases} t^{n-k}(x) & \text{if } n \geq k \\ i^{k-n}(x) & \text{if } n \leq k. \end{cases}$$

The globular relations generalise, for $j < k$, to

$$s_j s_k = s_j t_k = s_j \quad \text{and} \quad t_j s_k = t_j t_k = t_j.$$

Recall that, since \mathcal{C} has pullbacks, then so does $\text{Ind}(\mathcal{C})$. For $k \geq 0$, denote by $X \star_k X$ the pullback

$$\begin{array}{ccc} X \star_k X & \longrightarrow & X \\ \downarrow \lrcorner & & \downarrow s_k \\ X & \xrightarrow{t_k} & X_k \end{array}$$

in $\text{Ind}(\mathcal{C})$. Explicitly, the n -cells of $X \star_k X$ are the pairs (x, y) of n -cells of X such that $t_k(x) = s_k(y)$ holds. An n -cell (x, y) of $X \star_k X$ is called a k -composable pair of n -cells of X . Note that, by definition of s_k and t_k , if $n \leq k$, then the n -cells of $X \star_k X$ are all pairs (x, x) for x an n -cell of X .

1.1.5. Internal ∞ -categories. An ∞ -category of \mathcal{C} is a globular object X of \mathcal{C} equipped, for every $k \geq 0$, with an indexed morphism

$$X \star_k X \xrightarrow{c_k} X$$

of degree 0, called the k -composition of X , whose value at (x, y) is denoted by $x \star_k y$, and such that the following relations are satisfied for all $0 \leq k < n$:

(i) (compatibility with the source and target maps) for every n -cell (x, y) of $V \star_k V$,

$$s(x \star_k y) = \begin{cases} s(x) & \text{if } k = n - 1 \\ s(x) \star_k s(y) & \text{otherwise} \end{cases} \quad \text{and} \quad t(x \star_k y) = \begin{cases} t(y) & \text{if } k = n - 1 \\ t(x) \star_k t(y) & \text{otherwise,} \end{cases}$$

(ii) (compatibility with the identity map) for every n -cell (x, y) of $V \star_k V$,

$$1_{x \star_k y} = 1_x \star_k 1_y,$$

(iii) (associativity) for all n -cells x, y and z of V such that (x, y) and (y, z) are n -cells of $V \star_k V$,

$$(x \star_k y) \star_k z = x \star_k (y \star_k z),$$

(iv) (neutrality) for every n -cell x of V ,

$$s_k(x) \star_k x = x = x \star_k t_k(x),$$

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- (v) (*exchange*) for every $j < k$, and all n -cells (x, x') and (y, y') of $V \star_k V$ such that (x, y) and (x', y') are n -cells of $V \star_j V$,

$$(x \star_k x') \star_j (y \star_k y') = (x \star_j y) \star_k (x' \star_j y').$$

Note that the compatibility of the compositions with the source and target maps ensures that the associativity axiom makes sense: if (x, y) and (y, z) are n -cells of $V \star_k V$, then so do $(x \star_k y, z)$ and $(x, y \star_k z)$. The compatibility of compositions with identities implies that we can still write x for 1_x with no ambiguity.

Given ∞ -categories X and Y of \mathcal{C} , an ∞ -functor from X to Y is a globular morphism $F : X \rightarrow Y$ that commutes with all k -compositions:

$$\begin{array}{ccc} X \star_k X & \xrightarrow{c_k} & X \\ F \times F \downarrow & = & \downarrow F \\ Y \star_k Y & \xrightarrow{c_k} & Y \end{array}$$

We denote by $\infty\text{Cat}(\mathcal{C})$ the category of ∞ -categories and ∞ -functors of \mathcal{C} . For $n \geq 0$, an n -category of \mathcal{C} is defined like an ∞ -category of \mathcal{C} , but starting with an n -globular object of \mathcal{C} , equipped with compositions c_0, \dots, c_{n-1} . We denote by $n\text{Cat}(\mathcal{C})$ the category of n -categories of \mathcal{C} and the corresponding n -functors.

1.1.6. Internal ∞ -groupoids. In an ∞ -category X of \mathcal{C} , for $n \geq 1$, an n -cell x is called *invertible* if there exists an n -cell x^- in X , of source $t(x)$ and target $s(x)$, such that the relations

$$x \star_{n-1} x^- = s(x) \quad \text{and} \quad x^- \star_{n-1} x = t(x)$$

are satisfied. An ∞ -groupoid of \mathcal{C} is an ∞ -category of \mathcal{C} in which all k -cells are invertible, for every $k \geq 1$. Similarly, for $n \geq 0$, an n -groupoid of \mathcal{C} is an n -category in which all k -cells are invertible, for every $k \geq 1$. We denote by $\infty\text{Gpd}(\mathcal{C})$ the category of ∞ -groupoids of \mathcal{C} and of ∞ -functors between them, and by $n\text{Gpd}(\mathcal{C})$ its full subcategory whose objects are n -groupoids.

1.2. Higher-dimensional vector spaces

1.2.1. Globular vector spaces. The objects and morphisms of the category $\text{Glob}(\text{Vect})$ are called *globular vector spaces* and *globular linear maps*. Explicitly, a globular vector space is a diagram

$$V_0 \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} V_1 \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} \cdots \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} V_n \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} V_{n+1} \begin{array}{c} \xleftarrow{s} \\ \xrightarrow{i} \\ \xleftarrow{t} \end{array} \cdots$$

of vector spaces and linear maps that satisfy the globular relations. If V is a globular vector space, and if a is an n -cell of V , for $n \geq 1$, then the *boundary of a* is the $(n-1)$ -cell of V denoted by $\partial(a)$ and defined by

$$\partial(a) = s(a) - t(a).$$

1.2.2. Higher-dimensional vector spaces. The categories $\infty\text{Cat}(\text{Vect})$ and $n\text{Cat}(\text{Vect})$ are denoted by ∞Vect and $n\text{Vect}$, and their objects and morphisms are called ∞ -vector spaces, linear ∞ -functors, n -vector spaces and linear n -functors, respectively. In particular, we have $0\text{Vect} = \text{Vect}$, and 1Vect is the category of 2-vector spaces of [9].

If V is a globular vector space, the pullback $V \star_k V$ is equipped with the vector space structure given by, for all n -cells (a, a') and (b, b') of $V \star_k V$ and all scalars λ and μ ,

$$\lambda(a, a') + \mu(b, b') = (\lambda a + \mu b, \lambda a' + \mu b').$$

Hence, if V is an ∞ -vector space, the linearity of its composition c_k is equivalent, in the same context, to

$$(\lambda a + \mu b) \star_k (\lambda a' + \mu b') = \lambda(a \star_k a') + \mu(b \star_k b'). \quad (3)$$

The following result states that globular vector spaces, ∞ -vector spaces and ∞ -groupoids of Vect are the same notions. This also holds for n -globular vector spaces, n -vector spaces and n -groupoids of Vect , which is essentially [40, Proposition 2.5]. The result is also similar to the equivalence between the categories of chain complexes in an abelian category \mathcal{A} and internal ∞ -categories in \mathcal{A} [17, Theorem 3.2].

1.2.3. Proposition. *The forgetful functors $\infty\text{Gpd}(\text{Vect}) \rightarrow \infty\text{Vect} \rightarrow \text{Glob}(\text{Vect})$ are isomorphisms. In particular :*

- (i) *A globular vector space V can be uniquely extended into an ∞ -vector space, by putting, for all natural numbers $0 \leq k < n$ and k -composable pair (a, b) of n -cells of V ,*

$$a \star_k b = a - t_k(a) + b. \quad (4)$$

- (ii) *If V is an ∞ -vector space, then, for every $n \geq 1$, every n -cell a of V is invertible, with inverse*

$$a^- = s(a) - a + t(a). \quad (5)$$

Proof. Let V be an ∞ -vector space, and (a, b) be a pair of k -composable n -cells of V . Let us prove (4), i.e. that composing a and b along their common k -boundary $t_k(a)$ is essentially the same as summing a and b , up to the lower-dimensional term $t_k(a)$. We first write

$$a \star_k b = (a - s_k(b) + s_k(b)) \star_k (t_k(a) - t_k(a) + b).$$

Then, using $t_k(a) = s_k(b)$, the linearity of the k -composition and the neutrality axioms, we obtain

$$a \star_k b = (a \star_k t_k(a)) - (s_k(b) \star_k t_k(a)) + (s_k(b) \star_k b) = a - t_k(a) + b.$$

As a consequence, given a globular vector space V , there exists at most one ∞ -vector space with V as underlying globular vector space, whose compositions are given by (4).

Conversely, let V be a globular vector space and define the k -composition c_k by (4). Let us check that the axioms of an ∞ -vector space are fulfilled. First, c_k is linear, which is obtained by replacing k -compositions by their definitions in both sides of the relations (3) and by using the linearity of t . Next, c_k is compatible with the source map:

$$s(a \star_k b) = s(a) - s(1_{s_k(b)}) + s(b) = s(a) - s_k(b) + s(b) = \begin{cases} s(a) & \text{if } k = n - 1, \\ s(a) \star_k s(b) & \text{otherwise.} \end{cases}$$

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We proceed symmetrically to get the compatibility of c_k with the target map, and its compatibility with the identity map comes from the linearity of the latter. For associativity, we use the compatibility of c_k with the source and target maps to get, by induction on $n \geq k$, that $t_k(a \star_k b) = t_k(b)$ and $s_k(b \star_k c) = s_k(b)$, and, then, we replace k -compositions in both sides of the associativity axiom to get the result. The neutrality relations are immediate consequences of the globular relations $s_i = t_i = \text{Id}_V$. Finally, the exchange relations are obtained by using $t_j t_k = t_j$ when $j < k$ to prove that both sides are equal to

$$a - t_k(a) + a' - t_j(a) + b - t_k(b) + b'.$$

We conclude that $\text{Glob}(\text{Vect})$ and ∞Vect are isomorphic by observing that any globular linear map F is automatically a linear ∞ -functor:

$$F(a \star_k b) = F(a - t_k(a) + b) = F(a) - t_k(F(a)) + F(b) = F(a) \star_k F(b).$$

Finally, let V be an ∞ -vector space and a be an n -cell of V , with $n \geq 1$. With a^- defined by (5), we check $s(a^-) = t(a)$ and $t(a^-) = s(a)$ and, then, we apply (4) to obtain $a \star_{n-1} a^- = s(a)$ and $a^- \star_{n-1} a = t(a)$. We conclude that $\text{Glob}(\text{Vect})$ and ∞Vect are isomorphic. \square

1.2.4. The graded case. Replacing Vect by the category GrVect of (non-negatively) graded vector spaces over \mathbb{K} and graded linear maps of degree 0, one obtains the category $\text{Glob}(\text{GrVect})$ of *globular graded vector spaces*: these are globular vector spaces V such that each vector space V_n is graded, i.e. each V_n admits a decomposition $V_n = \bigoplus_{i \in \mathbb{N}} V_n^{(i)}$, and the components of the source, target and identity maps are graded linear maps of degree 0. In such a graded globular vector space V , the n -cells of $V_n^{(i)}$ are said to be *homogeneous of degree i*. Similarly, the categories ∞GrVect and $\infty\text{Gpd}(\text{GrVect})$ are obtained by replacing of Vect by GrVect , and Proposition 1.2.3 extends in a straightforward way.

1.3. Higher-dimensional associative algebras

1.3.1. Globular associative algebras and bimodules. The objects and morphisms of the category $\text{Glob}(\text{Alg})$ are called *globular algebras* and *morphisms of globular algebras*. If A is an algebra, and $\text{Bimod}(A)$ is the category of A -bimodules, a *globular A -bimodule* is an object of $\text{Glob}(\text{Bimod}(A))$. In view of Theorem 1.3.3, the category $\text{Glob}(\text{Bimod})$ of *globular bimodules* is the one whose objects are pairs (A, M) formed by an algebra A and a globular A -bimodule M , and whose morphisms from (A, M) to (B, N) are pairs (F, G) made of a morphism $F : A \rightarrow B$ of algebras and a morphism $G : M \rightarrow N$ of bimodules (in the sense that $G(a m a') = F(a) G(m) F(a')$ holds for all a and a' in A and m in M).

1.3.2. Higher-dimensional associative algebras. The categories $\infty\text{Cat}(\text{Alg})$ and $n\text{Cat}(\text{Alg})$ are denoted by ∞Alg and $n\text{Alg}$, and their objects and morphisms are called *∞ -algebras*, *morphisms of ∞ -algebras*, *n -algebras* and *morphisms of n -algebras*, respectively. By construction of pullbacks in $\text{Glob}(\text{Alg})$, for a globular algebra A , the product of n -cells (a, a') and (b, b') of $A \star_k A$ is given by

$$(a, a')(b, b') = (ab, a'b').$$

This implies that, in an ∞ -algebra A , requiring that the composition $c_k : A \star_k A \rightarrow A$ is a morphism of algebras is equivalent to imposing the relation

$$ab \star_k a'b' = (a \star_k a')(b \star_k b') \tag{6}$$

for all k -composable pairs (a, a') and (b, b') of n -cells of A .

The following result states that the structure of ∞ -algebra boils down to the one of a globular bimodule that satisfies an extra condition, corresponding to the fact that the composition c_0 of an ∞ -algebra satisfies (6). This also holds for n -algebras and n -globular bimodules with the same condition.

1.3.3. Theorem. *The following categories are isomorphic:*

- (i) *The category ∞Alg of ∞ -algebras.*
- (ii) *The category $\infty\text{Gpd}(\text{Alg})$ of internal ∞ -groupoids in Alg .*
- (iii) *The full subcategory of $\text{Glob}(\text{Alg})$ whose objects are the globular algebras A that satisfy, for all n -cells a and b of A , the relations*

$$ab = as_0(b) + t_0(a)b - t_0(a)s_0(b) = s_0(a)b + at_0(b) - s_0(a)t_0(b). \quad (7)$$

- (iv) *The full subcategory of $\text{Glob}(\text{Bimod})$ whose objects are the pairs (A, M) such that M_0 is equal to A , with its canonical A -bimodule structure, and that satisfy, for all n -cells a and b of M , the relation*

$$as_0(b) + t_0(a)b - t_0(a)s_0(b) = s_0(a)b + at_0(b) - s_0(a)t_0(b). \quad (8)$$

Proof. First, one checks that (5) defines an inverse for every n -cell of an ∞ -algebra, for $n \geq 1$, so that ∞Alg is isomorphic to $\infty\text{Gpd}(\text{Alg})$.

Now, let A be an ∞ -algebra, and let us prove that (7) holds. For 0-cells a and b of A , (7) reads $ab = ab = ab$. If a and b are n -cells of A , for $n \geq 1$, Proposition 1.2.3 and (6) imply

$$ab = (a \star_0 t_0(a))(s_0(b) \star_0 b) = as_0(b) \star_0 t_0(a)b = as_0(b) - t_0(a)s_0(b) + t_0(a)b,$$

and symmetrically for the other part of (7). Moreover, forgetting the compatibility with the compositions of a morphism of ∞ -algebras gives a morphism of globular algebras.

Conversely, let A be a globular algebra that satisfies (7). Let us check that the unique possible composition \star_k on A , given by (4), satisfies (6). We start with $k = 0$, and pairs (a, a') and (b, b') in $A_n \star_0 A_n$. Writing c for $t_0(a)$ and d for $t_0(b)$, we obtain

$$(a \star_0 a')(b \star_0 b') = ab + ab' + a'b + a'b' - ad - a'd - cb - cb' + cd.$$

We use (7) on ab' and $a'b$ to get

$$ab' = ad + cb' - cd \quad \text{and} \quad a'b = cb + a'd - cd,$$

so that we conclude

$$(a \star_0 a')(b \star_0 b') = ab + a'b' - cd = ab \star_0 a'b'.$$

Now, let us fix $k \geq 1$, and pairs (a, a') and (b, b') in $A_n \star_k A_n$. We write c for $t_0(a)$ and d for $s_0(b)$, and we note that $c = t_0(a) = t_0 t_k(a) = t_0 s_k(a') = t_0(a')$ and, similarly, that $d = s_0(b')$. Then, we use the fact that \star_0 satisfies (6) and the exchange relation between \star_0 and \star_k to get

$$ab \star_k a'b' = (ad \star_0 cb) \star_k (a'd \star_0 cb') = (ad \star_k a'd) \star_0 (cb \star_k cb').$$

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By definition of \star_k , we obtain

$$ad \star_k a'd = ad - t_k(a)d + a'd = (a \star_k a')d$$

and

$$cb \star_k cb' = cb - ct_k(b) + cb' = c(b \star_k b').$$

Hence we get, using (6) for \star_0 again,

$$ab \star_k a'b' = (a \star_k a')d \star_0 c(b \star_k b') = (a \star_k a')(b \star_k b').$$

Moreover, if F is a morphism between two globular algebras that satisfy (6), then it automatically commutes with the compositions defined by (4), as in the case of globular vector spaces. This concludes the proof that ∞Alg is isomorphic to the category of **(iii)**.

Let A be a globular algebra that satisfies (7). By hypothesis, A_0 is an algebra and each A_n , for $n \geq 1$, is equipped with a structure of A_0 -bimodule by using its algebra product with iterated identities of elements of A_0 . Moreover, the source, target and identity maps commute with these A_0 -bimodule structures because they are morphisms of algebras, and (8) is satisfied by hypothesis. Finally, any morphism of globular algebras induces a morphism between the underlying globular bimodules.

Conversely, let A be an algebra, M be a globular A -bimodule with $M_0 = A$ and that satisfies (8). On each M_n , for $n \geq 1$, we define the product ab by

$$ab = as_0(b) + t_0(a)b - t_0(a)s_0(b),$$

and we check that this equips M with a structure of a globular algebra that satisfies the relation (7). Finally, a morphism of globular bimodules that satisfies (8) commutes with the product defined by (7). Thus ∞Alg is indeed isomorphic to the category described in **(iv)**. \square

1.3.4. The graded case. As in the case of vector spaces, we obtain the category $\text{Glob}(\text{GrAlg})$ of *globular graded algebras* by replacing Alg with the category GrAlg of graded algebras: in such an object A , the product of each A_n is graded so that, if f and g are n -cells of A , with f homogeneous of degree i and g homogeneous of degree j , then their product fg is a homogeneous n -cell of A of degree $i + j$. A globular graded algebra A is called *connected* if it satisfies $A_n^{(0)} = \mathbb{K}$ for every $n \geq 0$. The notions of graded ∞ -algebra and globular graded bimodule are obtained similarly, and Theorem 1.3.3 extends to this context.

2. POLYGRAPHS FOR ASSOCIATIVE ALGEBRAS

In [54, 55, 23], Street and Burroni have introduced categorical objects known as computads or polygraphs, to describe generating families and presentations of higher-dimensional categories. Here, we adapt these set-theoretic objects to provide bases and presentations of higher-dimensional algebras. As for the original polygraphs, which can be used to define homotopical resolutions of monoids, categories and n -categories, the polygraphs we define here give a notion of polygraphic resolution of an associative algebra.

2.1. Extended higher-dimensional associative algebras

2.1.1. Cellular extensions. Fix a natural number n , and let A be an n -algebra. A *cellular extension* of A is a set X equipped with maps

$$A_n \begin{array}{c} \xleftarrow{s} \\ \xleftarrow{t} \end{array} X$$

such that, for every x in X , the pair $(s(x), t(x))$ is an n -sphere of A .

Let X be a cellular extension of A . Denote by \equiv_X the congruence relation on the parallel n -cells of A generated by $s(x) \sim t(x)$ for every x in X (that is, \equiv_X is the smallest equivalence relation on the parallel n -cells of A , compatible with all the compositions of A and relating $s(x)$ and $t(x)$ for every x in X). Call X *acyclic* if, for every n -sphere (a, b) of A , we have $a \equiv_X b$. Every n -algebra A has two canonical cellular extensions: the empty one, and the one denoted by $\text{Sph}(A)$ that contains all the n -spheres of A and is, as a consequence, acyclic.

2.1.2. Extended higher-dimensional algebras. For $n \geq 0$, the category $n\text{Alg}^+$ of *extended n -algebras* is defined as the pullback

$$\begin{array}{ccc} n\text{Alg}^+ & \longrightarrow & (n+1)\text{Gph} \\ \downarrow \lrcorner & & \downarrow \\ n\text{Alg} & \longrightarrow & n\text{Gph} \end{array}$$

of the forgetful functors $n\text{Alg} \rightarrow n\text{Gph}$ (forgetting the algebra structures) and $(n+1)\text{Gph} \rightarrow n\text{Gph}$ (forgetting the $(n+1)$ -cells), where $n\text{Gph}$ is the category of set-theoretic n -graphs (or n -semiglobular objects in Set). In an explicit way, an object of $n\text{Alg}^+$ is a pair (A, X) formed by an n -algebra A and a cellular extension X of A , while a morphism from (A, X) to (B, Y) is a pair (F, φ) made of a morphism of n -algebras $F : A \rightarrow B$, and a map $\varphi : X \rightarrow Y$ that commutes with the source and target maps.

2.1.3. Free higher-dimensional algebras. Fix $n \geq 1$. The forgetful functor from $n\text{Alg}$ to $(n-1)\text{Alg}^+$, discarding the compositions of n -cells, admits a left adjoint, that maps an extended $(n-1)$ -algebra (A, X) to the n -algebra denoted by $A[X]$, called the *free n -algebra over (A, X)* , and defined as follows. First, we consider the A_0 -bimodule

$$M = (A_0 \otimes \mathbb{K}X \otimes A_0) \oplus A_{n-1}$$

obtained by the direct sum of the free A_0 -bimodule with basis X and of a copy of A_{n-1} , equipped with its canonical A_0 -bimodule structure. Thus M contains linear combinations of elements axb , for a and b in A_0 and x in X , and of an $(n-1)$ -cell c of A . We define the source, target and identity maps between M and A_{n-1} by

$$s(axb) = as(x)b, \quad s(c) = c, \quad t(axb) = at(x)b, \quad t(c) = c \quad \text{and} \quad i(c) = c,$$

for all x in X , a and b in A_0 , and c in A_{n-1} . Then we define the A_0 -bimodule $A[X]_n$ as the quotient of M by the A_0 -bimodule ideal generated by all the elements

$$(as_0(b) + t_0(a)b - t_0(a)s_0(b)) - (s_0(a)b + at_0(b) - s_0(a)t_0(b)),$$

where a and b range over $A_0 \otimes \mathbb{K}X \otimes A_0$. We check that the source and target maps are compatible with the quotient, so that, by Theorem 1.3.3, the A_0 -bimodule $A[X]_n$ extends A into a uniquely defined n -algebra $A[X]$. Note that, by construction of $A[X]$, the cellular extension X is acyclic if, and only if, for every $(n-1)$ -sphere (a, b) of A , there exists an n -cell $f : a \rightarrow b$ in the free n -algebra $A[X]$.

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2.1.4. Quotient higher-dimensional algebras. For $n \geq 0$ and (A, X) an extended n -algebra, the *quotient of A by X* is the n -algebra denoted by A/X , and obtained by quotient of the n -cells of A by the congruence \equiv_X . Thus, two n -cells a and b of A are identified in A/X if, and only if, there exists an $(n+1)$ -cell $f : a \rightarrow b$ in the free $(n+1)$ -algebra $A[X]$. As a consequence, the cellular extension X is acyclic if, and only if, the canonical projection $A \twoheadrightarrow A/X$ identifies all the parallel n -cells of A .

2.2. Polygraphs for associative algebras

2.2.1. Polygraphs of finite dimension. The category $n\text{Pol}(\text{Alg})$ of n -polygraphs for associative algebras, simply called n -polygraphs in this paper, and the *free n -algebra functor* from $n\text{Pol}(\text{Alg})$ to $n\text{Alg}$ are defined by induction on $n \geq 0$ as follows. Note that this is a special case of the general construction of polygraphs, or computads, for finitary monads on globular sets (here, the monad of ∞ -algebras) given in [10, Definition 2.1].

For $n = 0$, we define $0\text{Pol}(\text{Alg})$ as the category of sets. If X is a set, the *free 0-algebra functor* maps X to the free algebra $\mathcal{A}(X)$ over X . Assume that $n \geq 1$ and that the category $(n-1)\text{Pol}(\text{Alg})$ of $(n-1)$ -polygraphs is defined, together with the free $(n-1)$ -algebra functor $(n-1)\text{Pol}(\text{Alg}) \rightarrow (n-1)\text{Alg}$. The category $n\text{Pol}(\text{Alg})$ of n -polygraphs is defined as the pullback

$$\begin{array}{ccc} n\text{Pol}(\text{Alg}) & \longrightarrow & (n-1)\text{Alg}^+ \\ \downarrow \lrcorner & & \downarrow \\ (n-1)\text{Pol}(\text{Alg}) & \longrightarrow & (n-1)\text{Alg} \end{array} \quad (9)$$

of the free $(n-1)$ -algebra functor and the functor forgetting the cellular extension. The free n -algebra functor is obtained as the composite

$$n\text{Pol}(\text{Alg}) \longrightarrow (n-1)\text{Alg}^+ \longrightarrow n\text{Alg}$$

of the functor $n\text{Pol}(\text{Alg}) \longrightarrow (n-1)\text{Alg}^+$ of (9), followed by the functor mapping (A, X) to $A[X]$.

If X is an n -polygraph, we denote by $\mathcal{A}(X)$ the free n -algebra over X . Expanding the definition, an n -polygraph is a sequence (X_0, \dots, X_n) , written $\langle X_0 \mid \dots \mid X_n \rangle$, made of a set X_0 and, for every $0 \leq k < n$, a cellular extension X_{k+1} of the free k -algebra over $\langle X_0 \mid \dots \mid X_k \rangle$. The free n -algebra over X is given by

$$\mathcal{A}(X) = \mathcal{A}(X_0)[X_1] \cdots [X_n].$$

2.2.2. Polygraphs of infinite dimension. The category $\infty\text{Pol}(\text{Alg})$ of ∞ -polygraphs and the corresponding *free ∞ -algebra functor* are obtained as the limit of the functors $(n+1)\text{Pol}(\text{Alg}) \rightarrow n\text{Pol}(\text{Alg})$ of (9). Thus, by construction, an ∞ -polygraph X is a sequence $\langle X_0 \mid \dots \mid X_n \mid \dots \rangle$ such that $\langle X_0 \mid \dots \mid X_n \rangle$ is an n -polygraph for every $n \geq 0$, and n -polygraphs are the ∞ -polygraphs X such that $X_p = \emptyset$ for $p > n$.

Let X be an ∞ -polygraph. The elements of X_n are called the n -cells of X . We commit the abuse to also denote by X_n the underlying n -polygraph of X . We say that an ∞ -polygraph is of *finite type* if it has finitely many n -cells for every $n \geq 0$.

2.2.3. Higher-dimensional monomials. Let X be an ∞ -polygraph. A *monomial* of $\mathcal{A}(X)$ is an element of the free monoid X_0^* over X_0 , that is, a (possibly empty) product $x_1 \cdots x_p$ of elements of X_0 . The monomials of $\mathcal{A}(X)$ form a linear basis of the free algebra $\mathcal{A}(X_0)$, which means that every 0-cell a of $\mathcal{A}(X)$ can be uniquely written as a (possibly empty) linear combination

$$a = \sum_{i=1}^p \lambda_i u_i$$

of pairwise distinct monomials u_1, \dots, u_p of $\mathcal{A}(X)$, with $\lambda_1, \dots, \lambda_p$ nonzero scalars. This expression is called *the canonical decomposition* of a , and we define the *support* of a as the set $\text{supp}(a) = \{u_1, \dots, u_p\}$.

For $n \geq 1$, an *n-monomial* of $\mathcal{A}(X)$ is an n -cell of $\mathcal{A}(X)$ with shape $u\alpha v$, where α is an n -cell of X , and u and v are monomials of $\mathcal{A}(X)$. By construction of the free n -algebra over $(\mathcal{A}(X_{n-1}), X_n)$, and by freeness of $\mathcal{A}(X_{n-1})$, every n -cell a of $\mathcal{A}(X)$ can be written as a linear combination

$$a = \sum_{i=1}^p \lambda_i a_i + 1_c \quad (10)$$

of pairwise distinct n -monomials a_1, \dots, a_p and of an identity n -cell 1_c of $\mathcal{A}(X)$, and this decomposition is unique up to the relations

$$a s_0(b) + t_0(a) b - t_0(a) s_0(b) = s_0(a) b + a t_0(b) - s_0(a) t_0(b),$$

where a and b range over the n -monomials of $\mathcal{A}(X)$.

If a is an n -cell of $\mathcal{A}(X)$, the *size* of a is the minimum number of n -monomials of $\mathcal{A}(X)$ required to write a as in (10), and we denote by $\text{cell}(a)$ the subset of X_n that consists of the n -cells of X that appear in the corresponding n -monomials.

2.2.4. Notation. Assume that A is an ∞ -algebra. We use the following notational convention for the various types of cells that A contains, depending on the context. In a generic case, we write a, b, c , etc. for the n -cells of A . If $A = \mathcal{A}(X)$, for an ∞ -polygraph X , we write x, y, z the generating n -cells of $\mathcal{A}(X)$, i.e. the n -cells of X . If we focus on the lower dimensions, especially when we study presentations of algebras, we use specific notation to reflect the dimension of the cells: we write a, b, c , etc. for the 0-cells, f, g, h , etc. for the 1-cells, and F, G, H , etc. for the 2-cells of A . If $A = \mathcal{A}(X)$, for an ∞ -polygraph X , we write x, y, z the 0-cells of X , we use u, v, w for the monomials of $\mathcal{A}(X)$, and α, β, γ , etc. for the 1-cells of X ; the 2-cells of X are usually also written F, G, H , etc.

2.2.5. Lemma. *Let X be an ∞ -polygraph, and fix $n \geq 1$. Then, every nonidentity n -cell a of $\mathcal{A}(X)$ admits a decomposition*

$$a = a_1 \star_{n-1} \cdots \star_{n-1} a_p,$$

where a_1, \dots, a_p are n -cells of size 1 in $\mathcal{A}(X)$.

Proof. Write $a = \lambda_1 b_1 + \cdots + \lambda_p b_p + 1_c$ as in (10). If $p = 1$, then a is of size 1. Otherwise, put

$$d_i = \sum_{j=1}^i \lambda_j t(b_j) \quad \text{and} \quad e_i = \sum_{j=i}^p \lambda_j s(b_j)$$

2. Polygraphs for associative algebras

for each i in $\{1, \dots, p\}$, and $d_0 = e_{p+1} = 0$. Define, for each i in $\{1, \dots, p\}$, the n -cell of size 1

$$a_i = \lambda_i b_i + 1_c + 1_{d_{i-1}} + 1_{e_{i+1}}.$$

First, check $s(a_i) = c + d_{i-1} + e_i$ and $t(a_i) = c + d_i + e_{i+1}$, so that $a_1 \star_{n-1} \cdots \star_{n-1} a_p$ is a well-defined n -cell of $\mathcal{A}(X)$. Then, (4) gives

$$a_1 \star_{n-1} \cdots \star_{n-1} a_p = \sum_{i=1}^p \lambda_i b_i + \sum_{i=1}^p (1_c + 1_{d_{i-1}} + 1_{e_{i+1}}) - \sum_{i=1}^{p-1} (\lambda_i 1_{t(b_i)} + 1_c + 1_{d_{i-1}} + 1_{e_{i+1}}).$$

We conclude thanks to

$$d_{p-1} = \sum_{i=1}^{p-1} \lambda_i t(b_i) \quad \text{and} \quad e_{p+1} = 0. \quad \square$$

2.2.6. Polygraphs for graded associative algebras. The category $n\text{GrAlg}^+$ of *graded extended n -algebras* is defined similarly to $n\text{Alg}^+$: its objects are pairs (A, X) , where A is a graded n -algebra, and X is a graded cellular extension of A , meaning that $X = \coprod_{i \in \mathbb{N}} X^{(i)}$ and that the source and target of each x in $X^{(i)}$ are homogeneous of degree i . In that case, the free $(n+1)$ -algebra $A[X]$ and the quotient n -algebra A/X , defined as in the nongraded case, are also graded.

A *graded ∞ -polygraph* is an ∞ -polygraph X such that each set X_n is graded, for $n \geq 0$. This notion restricts to n -polygraphs, and, if $N \geq 2$, a 1-polygraph X is called *N -homogeneous* if X_0 is concentrated in degree 1 and X_1 is concentrated in degree N . We say *quadratic* and *cubic* instead of 2-homogeneous and 3-homogeneous, respectively. Given a fixed map $\omega : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$, we say that a graded ∞ -polygraph X is *ω -concentrated* if each graded set X_n , for $n \geq 0$, is concentrated in degree $\omega(n+1)$. In that case, because the source and target maps are graded, for $n \geq 1$, the source and target of every n -cell of X are homogeneous $(n-1)$ -cells of $\mathcal{A}(X)$ of degree $\omega(n+1)$.

2.3. Presentations, coherent presentations and polygraphic resolutions

2.3.1. Polygraphic presentations. Let X be a 1-polygraph. The *algebra presented by X* is the quotient algebra

$$\bar{X} = \mathcal{A}(X_0)/X_1.$$

When the context is clear, we denote by \bar{a} the image of a 0-cell a of $\mathcal{A}(X)$ through the canonical projection.

Let A be an algebra. We say that A is *presented by X* , or that X is a *presentation of A* , if A is isomorphic to \bar{X} . For example, if X_0 is a generating set of A , and if X_1 is the cellular extension of the free algebra over X_0 that contains a 1-cell $\alpha : a \rightarrow b$ for all 0-cells a and b of $\mathcal{A}(X_0)$ that are equal in A , then the 1-polygraph $\langle X_0 | X_1 \rangle$ is a presentation of A . We say that A is *N -homogeneous* if it admits a presentation by an N -homogeneous graded 1-polygraph.

2.3.2. Left-monomial 1-polygraphs. Let X be a 1-polygraph. We say that X is *left-monomial* if, for every 1-cell α of X , the source of α is a monomial of $\mathcal{A}(X)$ that does not belong to $\text{supp}(t(\alpha))$. In that case, the source of any 1-monomial of $\mathcal{A}(X)$ is a monomial of $\mathcal{A}(X)$. Note that, from X , one obtains

2.3. Presentations, coherent presentations and polygraphic resolutions

a left-monomial 1-polygraph that presents the same algebra as X as follows. For every 1-cell α of X , consider the boundary $\partial(\alpha) = s(\alpha) - t(\alpha)$ of α : if it is 0, discard α , otherwise, replace α with

$$u \xrightarrow{\alpha'} u - \frac{1}{\lambda} \partial(\alpha),$$

where u is any chosen monomial in $\text{supp}(\partial(\alpha))$ and λ is the coefficient of u in $\partial(\alpha)$.

2.3.3. Presentations and ideals. Let X be a 1-polygraph. We denote by $I(X)$ the ideal of the free algebra $\mathcal{A}(X_0)$ generated by the boundaries of the 1-cells of X . Since the algebra $\mathcal{A}(X_0)$ is free, the ideal $I(X)$ is made of all the linear combinations

$$\sum_{i=1}^p \lambda_i u_i \partial(\alpha_i) v_i,$$

where $u_1 \alpha_1 v_1, \dots, u_p \alpha_p v_p$ are pairwise distinct 1-monomials of $\mathcal{A}(X)$, and $\lambda_1, \dots, \lambda_p$ are nonzero scalars, so that the algebra \bar{X} presented by X is isomorphic to the quotient of $\mathcal{A}(X_0)$ by $I(X)$.

2.3.4. Lemma. *Let X be a 1-polygraph. For all 0-cells a and b of $\mathcal{A}(X)$, the following are equivalent:*

- (i) *The 0-cell $a - b$ belongs to the ideal $I(X)$.*
- (ii) *There exists a 1-cell $f : a \rightarrow b$ in $\mathcal{A}(X)$.*

As a consequence, $I(X)$ exactly contains the 0-cells a of $\mathcal{A}(X)$ such that $\bar{a} = 0$ holds in \bar{X} .

Proof. Let us assume that $a - b$ is in $I(X)$, that is, $a - b = \sum_{1 \leq i \leq p} \lambda_i u_i \partial(\alpha_i) v_i$. Then the following 1-cell f of $\mathcal{A}(X)$ has source a and target b :

$$f = \sum_{i=1}^p \lambda_i u_i \alpha_i v_i + \left(a - \sum_{i=1}^p \lambda_i u_i s(\alpha_i) v_i \right).$$

Conversely, let $f : a \rightarrow b$ be a 1-cell of $\mathcal{A}(X)$. Using Lemma 2.2.5, we decompose f into 1-cells of size 1:

$$f = f_1 \star_0 \cdots \star_0 f_p \quad \text{with} \quad f_i = \lambda_i u_i \alpha_i v_i + h_i.$$

Since $t(f_i) = s(f_{i+1})$, we have $a - b = \partial(f_1) + \cdots + \partial(f_p)$. Moreover, $\partial(f_i) = \lambda_i u_i \partial(\alpha_i) v_i$ implies that each $\partial(f_i)$ belongs to $I(X)$, and thus so does $a - b$.

Finally, if one applies the equivalence to the case $b = 0$, since $\bar{0} = 0$ holds in \bar{X} , we get that a is in $I(X)$ if, and only if, we have $\bar{a} = 0$ in \bar{X} . \square

2.3.5. Coherent presentations and polygraphic resolutions. Let A be an algebra. A *coherent presentation* of A is a 2-polygraph X such that $\langle X_0 | X_1 \rangle$ is a presentation of A , and such that X_2 is acyclic. A *polygraphic resolution* of A is an ∞ -polygraph X such that the 1-polygraph $\langle X_0 | X_1 \rangle$ is a presentation of A , and, for every $n \geq 1$, the cellular extension X_{n+1} of $\mathcal{A}(X_n)$ is acyclic. For example, if $X = \langle X_0 | X_1 \rangle$ is a presentation of A , then the 2-polygraph $\langle X_0 | X_1 | \text{Sph}(\mathcal{A}(X_1)) \rangle$ is a coherent presentation of A , and, if $X = \langle X_0 | X_1 | X_2 \rangle$ is a coherent presentation of A , then the ∞ -polygraph

$$\langle X_0 | X_1 | X_2 | \text{Sph}(\mathcal{A}(X_2)) | \text{Sph}(\mathcal{A}(\text{Sph}(\mathcal{A}(X_2)))) | \cdots \rangle$$

is a polygraphic resolution of A .

3. CONVERGENT PRESENTATIONS OF ASSOCIATIVE ALGEBRAS

This section develops a rewriting theory for associative algebras, based on the polygraphs introduced in Section 2. For that, the usual rewriting notions are introduced: rewriting steps, normal forms, termination, confluence and convergent presentations. As usual in rewriting theories, Theorem 3.4.2 asserts that a convergent presentation of an associative algebra yields a basis of that algebra, together with a mechanism to compute its product. In the end of the section, Propositions 3.5.2 and 3.5.4 detail how convergent presentations of associative algebras generalise other rewriting-like objects, namely Gröbner bases and Poincaré-Birkhoff-Witt bases.

3.1. Rewriting steps and normal forms

3.1.1. Rewriting steps and positive 1-cells. Assume that X is a left-monomial 1-polygraph. A *rewriting step of X* is a 1-cell $\lambda f + 1_\alpha$ of size 1 of the free 1-algebra $\mathcal{A}(X)$ that satisfies the condition

$$\text{supp}(\lambda s(f) + \alpha) = \{s(f)\} \amalg \text{supp}(\alpha),$$

that is, such that $\lambda \neq 0$ and $s(f) \notin \text{supp}(\alpha)$. A 1-cell of $\mathcal{A}(X)$ is called *positive* if it is a (possibly empty) 0-composite $f_1 \star_0 \cdots \star_0 f_p$ of rewriting steps of $\mathcal{A}(X)$.

3.1.2. Remark. The definition of rewriting step is more difficult here than in the set-theoretic case. In the set-theoretic case, a 1-polygraph X generates two different objects: a free 1-category X^* and a free 1-groupoid X^\top . In that case, we define a rewriting step as a size-one 1-cell of X^* , and their compositions generate all the 1-cells of X^* . But, in the case of associative algebras, there is no difference between the free 1-category and the free 1-groupoid, as stated by Theorem 1.3.3. Thus, we adopt a different point of view to define rewriting steps and positive 1-cells. Here, we isolate, among the 1-cells $\mathcal{A}(X)$, a set of positive 1-cells that will play the same role as X^* with respect to X^\top . First, the set of positive 1-cells is big enough for every 1-cell of $\mathcal{A}(X)$ to factor into a composite of positive 1-cells and opposites of positive 1-cells, as given by Lemmas 2.2.5 and 3.1.3. Second, the set of positive 1-cells is small enough for preventing a nontrivial 1-cell and its inverse to be positive at the same time, leaving termination possible.

3.1.3. Lemma. *Let X be a left-monomial 1-polygraph. Every 1-cell f of size 1 of $\mathcal{A}(X)$ can be decomposed into $f = g \star_0 h^-$, where each of g and h is either an identity or a rewriting step of X .*

Proof. Write $f = \lambda f' + 1_b$, where $f' : u \rightarrow \alpha$ is a 1-monomial of $\mathcal{A}(X)$. Let μ be the coefficient of u in b , possibly zero, so that $b = \mu u + c$ with c such that $\text{supp}(c)$ does not contain u . Put

$$g = (\lambda + \mu)f' + 1_c \quad \text{and} \quad h = \lambda 1_\alpha + \mu f' + 1_c.$$

The linearity of the 0-composition of $\mathcal{A}(X)$ gives $f = g \star_0 h^-$. Moreover, by hypothesis, u does not belong to any of $\text{supp}(\alpha)$ or $\text{supp}(c)$. As a consequence, each of the 1-cells g and h is either an identity (if $\lambda + \mu = 0$ for g , if $\mu = 0$ for h) or a rewriting step. \square

3.1.4. Reduced cells and normal forms. A 0-cell a of $\mathcal{A}(X)$ is called *reduced* if $\mathcal{A}(X)$ contains no rewriting step of source a . As a direct consequence of the definition, we have that a 0-cell is reduced if, and only if, it is a (possibly empty) linear combination of reduced monomials of $\mathcal{A}(X)$. The reduced 0-cells of $\mathcal{A}(X)$ form a linear subspace of the free algebra $\mathcal{A}(X_0)$ that is denoted by $\text{Red}(X)$. Because X is left-monomial, the set of reduced monomials of $\mathcal{A}(X)$, denoted by $\text{Red}_m(X)$, forms a basis of $\text{Red}(X)$.

If a is a 0-cell of $\mathcal{A}(X)$, a *normal form* of a is a reduced 0-cell b of $\mathcal{A}(X)$ such that there exists a positive 1-cell of source a and target b in $\mathcal{A}(X)$.

3.2. Termination

3.2.1. Binary relations on free algebras. Assume that X is a set and that \vdash is a binary relation on the free monoid X^* . We say that \vdash is *stable by product* if $u \vdash u'$ implies $vuw \vdash vu'w$ for all u, u', v and w in X^* . If Y is a left-monomial cellular extension of $\mathcal{A}(X)$, we say that \vdash is *compatible with Y* if $u \vdash v$ holds for every 1-cell $y : u \rightarrow a$ of Y and every monomial v in $\text{supp}(a)$.

The relation \vdash is extended to the 0-cells of the free algebra $\mathcal{A}(X)$ by putting $a \vdash b$ if the following two conditions hold:

- (i) $\text{supp}(a) \setminus \text{supp}(b) \neq \emptyset$,
- (ii) for every v in $\text{supp}(b) \setminus \text{supp}(a)$, there exists u in $\text{supp}(a) \setminus \text{supp}(b)$, such that $u \vdash v$.

As a consequence, if u is a monomial and a is a 0-cell of $\mathcal{A}(X)$, then $u \vdash a$ holds if, and only if, $u \vdash v$ holds for every v in $\text{supp}(a)$. For this reason, we use the same notation for the relation on X^* and for its extension to the 0-cells of $\mathcal{A}(X)$.

The relation \vdash on the 0-cells of $\mathcal{A}(X)$ corresponds to the restriction to finite subsets of X^* of the so-called multiset relation generated by \vdash . We refer to [5, Section 2.5] for the general definition and the main properties of multiset relations, and, in particular, the fact that \vdash is wellfounded on the 0-cells if, and only if, it is wellfounded on the monomials, the proof being based on König's lemma.

3.2.2. The rewrite order and termination. Let X be a left-monomial 1-polygraph. Define \succ_X as the smallest transitive binary relation on X_0^* that is stable by product and compatible with X_1 . We say that X *terminates* if the relation \succ_X is wellfounded. In that case, the reflexive closure \succcurlyeq_X of the relation \succ_X is a wellfounded order, called the *rewrite order* of X .

Assume that the 1-polygraph X terminates. Then the minimal 0-cells for the rewrite order of X are the reduced ones. Moreover, for every nonidentity positive 1-cell a of $\mathcal{A}(X)$, we have $s(a) \succcurlyeq_X t(a)$. This implies that the 1-algebra $\mathcal{A}(X)$ contains no infinite sequence of 0-composable rewriting steps

$$a_0 \xrightarrow{f_1} a_1 \xrightarrow{f_2} \dots \xrightarrow{f_{n-1}} a_{n-1} \xrightarrow{f_n} a_n \xrightarrow{f_{n+1}} \dots$$

As a consequence, every 0-cell of $\mathcal{A}(X)$ admits at least one normal form. If X terminates, induction on the wellfounded order \succcurlyeq_X is called *noetherian induction*.

3.2.3. Monomial orders. Let X be a set. A wellfounded total order on X^* , whose strict part is stable by product, is called a *monomial order* on $\mathcal{A}(X)$. A classical example of a monomial order is given, for any wellfounded total order relation $>$ on X , by the *deglex order* generated by $>$, defined by

3. Convergent presentations of associative algebras

- (i) $u \succ_{\text{deglex}} v$ for all monomials u and v of $\mathcal{A}(X)$ such that u has greater length than v , and
- (ii) $uxv \succ_{\text{deglex}} uyw$ for all $x \succ y$ of X , and monomials u, v and w of $\mathcal{A}(X)$ such that v and w have the same length.

Now, assume that X is a left-monomial 1-polygraph. If there exists a monomial order \succ on $\mathcal{A}(X_0)$ that is compatible with X_1 , then X terminates: the order \succ is wellfounded, and $a \succ_X b$ implies $a \succ b$ for all 0-cells a and b . However, the converse implication does not hold, as illustrated by the following example.

3.2.4. Example. The following left-monomial 1-polygraph terminates:

$$X = \langle x, y, z \mid xyz \xrightarrow{\gamma} x^3 + y^3 + z^3 \rangle$$

Indeed, for every monomial u of $\mathcal{A}(X)$, denote by $A(u)$ the number of factors xyz that occur in u , by $B(u)$ the number of y that u contains, and put $\Phi(u) = 3A(u) + B(u)$. It is sufficient to check that $\Phi(uxyzv)$ is strictly greater than each of $\Phi(ux^3v)$, $\Phi(uy^3v)$ and $\Phi(uz^3v)$, for all monomials u and v of $\mathcal{A}(X)$:

$$\Phi(uxyzv) = \Phi(u) + \Phi(v) + 4, \quad \Phi(ux^3v) = \begin{cases} \Phi(u) + \Phi(v) + 3 & \text{if } v = yzv' \\ \Phi(u) + \Phi(v) & \text{otherwise,} \end{cases}$$

$$\Phi(uy^3v) = \Phi(u) + \Phi(v) + 3, \quad \Phi(uz^3v) = \begin{cases} \Phi(u) + \Phi(v) + 3 & \text{if } u = u'xy \\ \Phi(u) + \Phi(v) & \text{otherwise.} \end{cases}$$

However, no monomial order on $\mathcal{A}(X_0)$ is compatible with X_1 , because, for such an order \succ , one of the monomials x^3, y^3, z^3 is always greater than xyz . Indeed, since \succ is total, one of x, y or z is greater than the other two. If it is x , the case of z being symmetric, $x \succ y$ implies $x^2 \succ yx$ and $x \succ z$ implies $yx \succ yz$, so that $x^2 \succ yz$, hence $x^3 \succ xyz$. Now, if $y \succ x$ and $y \succ z$, we get $y^2 \succ xy$, thus $y^2z \succ xyz$ and $y^3 \succ y^2z$.

3.2.5. Lemma. *If X is a terminating left-monomial 1-polygraph, then, as a vector space, $\mathcal{A}(X_0)$ admits the decomposition*

$$\mathcal{A}(X_0) = \text{Red}(X) + I(X).$$

Proof. Since X terminates, every 0-cell a of $\mathcal{A}(X)$ admits at least a normal form b , i.e. a reduced 0-cell b such that there exists a positive 1-cell $f : a \rightarrow b$ in $\mathcal{A}(X)$. We conclude by writing $a = b + (a - b)$, and by observing that b belongs to $\text{Red}(X)$, by hypothesis, and that $a - b$ is in $I(X)$, by Lemma 2.3.4. \square

3.3. Branchings and confluence

3.3.1. Branchings. Assume that X is a left-monomial 1-polygraph. A *branching* of X is a pair (f, g) of positive 1-cells of $\mathcal{A}(X)$ with the same source, this 0-cell being called the *source* of (f, g) . We do not distinguish the branchings (f, g) and (g, f) . A branching (f, g) of X is called *local* if both f and g are rewriting steps of $\mathcal{A}(X)$. For a branching (f, g) of X of source a , define the branching

$$\lambda u(f, g)v + b = (\lambda uf v + b, \lambda ug v + b),$$

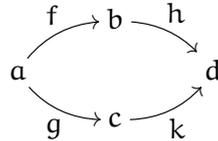
of X of source $\lambda uav + b$, for all scalar λ , monomials u and v and 0-cell b of $\mathcal{A}(X)$. Note that, if (f, g) is local and $\lambda \neq 0$, then $\lambda u(f, g) + b$ is also local.

3.3.2. Classification of local branchings. Assume that X is a left-monomial 1-polygraph. Given a local branching $(\lambda u_1 \alpha u_2 + a, \mu v_1 \beta v_2 + b)$ of X , we have two main possibilities, depending if $u_1 s(\alpha) u_2 = v_1 s(\beta) v_2$ or not. Moreover, in the equality case, there are three different situations, depending on the respective positions of $s(\alpha)$ and $s(\beta)$ in this common monomial. This analysis leads to a partition of the local branchings of X into the following four families.

- (i) *Aspherical* branchings: $\lambda(f, f) + b$, for all 1-monomial $f : u \rightarrow a$ of $\mathcal{A}(X)$, nonzero scalar λ , and 0-cell b of $\mathcal{A}(X)$, with $u \notin \text{supp}(b)$.
- (ii) *Additive* branchings: $(\lambda f + \mu v + c, \lambda u + \mu g + c)$, for all 1-monomials $f : u \rightarrow a$ and $g : v \rightarrow b$ of $\mathcal{A}(X)$, nonzero scalars λ and μ , and 0-cell c of $\mathcal{A}(X)$, with $u \neq v$ and $u, v \notin \text{supp}(c)$.
- (iii) *Peiffer* branchings: $\lambda(fv, ug) + c$, for all 1-monomials $f : u \rightarrow a$ and $g : v \rightarrow b$ of $\mathcal{A}(X)$, nonzero scalar λ , and 0-cell c of $\mathcal{A}(X)$, with $uv \notin \text{supp}(c)$.
- (iv) *Overlapping* branchings: $\lambda(f, g) + c$, for all 1-monomials $f : u \rightarrow a$ and $g : u \rightarrow b$ of $\mathcal{A}(X)$ such that (f, g) is neither aspherical nor Peiffer, every nonzero scalar λ , and every 0-cell c of $\mathcal{A}(X)$, with $u \notin \text{supp}(c)$.

The *critical branchings* of X are the overlapping branchings of X such that $\lambda = 1$ and $c = 0$, and that cannot be factored $(f, g) = u(f', g')v$ in a nontrivial way. Note that an overlapping branching has a unique decomposition $\lambda u(f, g)v + c$, with (f, g) critical.

3.3.3. Confluence. Assume that X is a left-monomial 1-polygraph. A branching (f, g) of X is called *confluent* if there exist positive 1-cells h and k of $\mathcal{A}(X)$ as in



If a is a 0-cell of $\mathcal{A}(X)$, we say that X is *confluent at a* (resp. *locally confluent at a* , resp. *critically confluent*) if every branching (resp. local branching, resp. critical branching) of X of source a is confluent. We say that X is *confluent* (resp. *locally confluent*, resp. *critically confluent*) if it is so at every 0-cell of $\mathcal{A}(X)$. Observe that confluence implies that every 0-cell of $\mathcal{A}(X)$ admits at most one normal form.

3.3.4. Proposition. *Let X be a terminating left-monomial 1-polygraph. The following assertions are equivalent:*

- (i) X is confluent.
- (ii) Every 0-cell of $I(X)$ admits 0 as a normal form.
- (iii) As a vector space, $\mathcal{A}(X_0)$ admits the direct decomposition $\mathcal{A}(X_0) = \text{Red}(X) \oplus I(X)$.

Proof. (i) \Rightarrow (ii). By Lemma 2.3.4, if a is in $I(X)$, then there exists a 1-cell $f : a \rightarrow 0$ in $\mathcal{A}(X)$. Since X is confluent, this implies that a and 0 have the same normal form, if any. And, since 0 is reduced, this implies that 0 is a normal form of a .

3. Convergent presentations of associative algebras

(ii) \Rightarrow (iii). By Lemma 3.2.5, it is sufficient to prove that $\text{Red}(X) \cap I(X)$ is reduced to 0. On the one hand, if a is in $\text{Red}(X)$, then a is reduced and, thus, admits itself as only normal form. On the other hand, if a is in $I(X)$, then a admits 0 as a normal form by hypothesis.

(iii) \Rightarrow (i). Consider a branching (f, g) of X , with $f : a \rightarrow b$ and $g : a \rightarrow c$. Since X terminates, each of b and c admits at least one normal form, say b' and c' respectively. Hence, there exist positive 1-cells $h : b \rightarrow b'$ and $k : c \rightarrow c'$ in $\mathcal{A}(X)$. Note that the difference $b' - c'$ is also reduced. Moreover, the 1-cell $(f \star_0 h)^- \star_0 (g \star_0 k)$ has b' as source and c' as target. This implies, by Lemma 2.3.4, that $b' - c'$ also belongs to $I(X)$. The hypothesis gives $b' - c' = 0$, so that (f, g) is confluent. \square

3.4. Convergence

3.4.1. Convergence. Let X be a left-monomial 1-polygraph. We say that X is *convergent* if it is both terminating and confluent. In that case, every 0-cell a of $\mathcal{A}(X)$ has a unique normal form, denoted by \widehat{a} , such that $\overline{a} = \overline{b}$ holds in \overline{X} if, and only if, $\widehat{a} = \widehat{b}$ holds in $\mathcal{A}(X)$. Hence, if X is a convergent presentation of an algebra A , the assignment of each element a of A to the normal form of any representative of a in $\mathcal{A}(X)$, written \widehat{a} by extension, defines a section $A \rightarrow \mathcal{A}(X)$ of the canonical projection, where A is seen as a 1-algebra with identity 1-cells only. The section is linear (the normal form of $\lambda a + \mu b$ is $\lambda \widehat{a} + \mu \widehat{b}$), it preserves the unit (termination implies $\widehat{1} = 1$), but $\widehat{ab} \neq \widehat{a}\widehat{b}$ in general.

3.4.2. Theorem. *Let A be an algebra and X be a convergent presentation of A . Then the set $\text{Red}_m(X)$ of reduced monomials of $\mathcal{A}(X)$ is a linear basis of A . As a consequence, the vector space $\text{Red}(X)$, equipped with the product defined by $a \cdot b = \widehat{ab}$, is an algebra that is isomorphic to A .*

Proof. If X is convergent, Proposition 3.3.4 induces that the following sequence of vector spaces is exact:

$$0 \longrightarrow I(X) \twoheadrightarrow \mathcal{A}(X_0) \twoheadrightarrow \text{Red}(X) \longrightarrow 0$$

Thus, since the algebra $\mathcal{A}(X_0)/I(X)$ is isomorphic to \overline{X} , convergence implies that $\text{Red}_m(X)$ is a linear basis of \overline{X} . We deduce that $\text{Red}(X)$ and \overline{X} are isomorphic as vector spaces. There remains to transport the product of \overline{X} to $\text{Red}(X)$ to get the result. \square

3.4.3. Example. Let A be the algebra presented by the 1-polygraph $X = \langle x, y \mid \alpha : xy \rightarrow x^2 \rangle$, which terminates, because $xy > x^2$ holds for the deglex order generated by $y > x$. This presentation is also confluent, because it has no critical branching (see Corollary 4.2.2). Hence, the set

$$\text{Red}_m(X) = \{y^i x^j \mid i, j \in \mathbb{N}\}$$

is a linear basis of the algebra A . Moreover, the product defined by

$$y^i x^j \cdot y^k x^l = \begin{cases} y^i x^{j+k+l} & \text{if } j \geq k \\ y^{i-j+k} x^{2j+l} & \text{if } j \leq k \end{cases}$$

turns $\text{Red}(X)$ into an algebra that is isomorphic to A .

3.5. Comparison with Gröbner bases and Poincaré-Birkhoff-Witt bases

Now, consider the presentation $Y = \langle x, y \mid \beta : x^2 \rightarrow xy \rangle$ of A . Termination of Y follows from the deglex order generated by $x > y$, but Y is not confluent, since it has a nonconfluent critical branching:

$$\begin{array}{c}
 \beta x \rightarrow xyx \\
 \swarrow \\
 x^3 \\
 \searrow \\
 x\beta \rightarrow x^2y \xrightarrow{\beta y} xy^2
 \end{array}$$

Thus the 0-cell $xyx - xy^2$ is both in $\text{Red}(Y)$ and $I(Y)$, proving that the sum $\text{Red}(Y) + I(Y)$ is not direct. As a consequence, $\text{Red}_m(Y)$ is not a linear basis of A .

3.4.4. Reduced convergent presentations. Let X be a left-monomial 1-polygraph. We say that X is *left-reduced* if, for every 1-cell α of X , the only rewriting step of X of source $s(\alpha)$ is α itself. We say that X is *right-reduced* if, for every 1-cell α of X , the 0-cell $t(\alpha)$ is reduced. We say that X is *reduced* if it is both left-reduced and right-reduced.

Assume that X is convergent. Then one obtains a reduced convergent left-monomial 1-polygraph Y , such that $\bar{X} \simeq \bar{Y}$, through the following operations. First, replace every 1-cell $a \rightarrow b$ with $a \rightarrow \hat{a}$. Then, if there exist parallel 1-cells, discard all of them but one. Finally, eliminate all the remaining 1-cells of X whose source contains the source of another 1-cell.

3.4.5. Completion of presentations. The completion procedure, developed by Buchberger for commutative algebras [20] and by Knuth and Bendix for term rewriting systems [41], adapts to terminating left-monomial 1-polygraphs as follows, to transform them into convergent ones.

Fix a left-monomial 1-polygraph X , and a well-founded strict order that is stable by product and compatible with X_1 . For each nonconfluent critical branching (f, g) of X , consider $a = c - d$, where c and d are arbitrary normal forms of $t(f)$ and $t(g)$, respectively. If $\text{supp}(a)$ contains a maximal element u , add the 1-cell $u \rightarrow b$ to X , where b is defined by $a = \lambda u + b$ and $u \notin \text{supp}(b)$; otherwise, the procedure fails. After the exploration of all the critical branchings of X , the procedure, if it has not failed, yields a terminating left-monomial 1-polygraph Y such that $\bar{X} \simeq \bar{Y}$. If Y is not confluent, restart with Y . The procedure either stops when it reaches a convergent left-monomial 1-polygraph, or runs forever.

3.5. Comparison with Gröbner bases and Poincaré-Birkhoff-Witt bases

3.5.1. Gröbner bases. Let X be a set and \preceq be a monomial order on the free algebra $\mathcal{A}(X)$. If a is a nonzero element of $\mathcal{A}(X)$, the *leading monomial* of a is the maximum element $\text{lm}(a)$ of $\text{supp}(a)$ for \preceq (or 0 if $\text{supp}(a)$ is empty), the *leading coefficient* of a is the coefficient $\text{lc}(a)$ of $\text{lm}(a)$ in a , and the *leading term* of a is the element $\text{lt}(a) = \text{lc}(a) \text{lm}(a)$ of $\mathcal{A}(X)$. Observe that, for a and b in $\mathcal{A}(X)$, we have $a \prec b$ if, and only if, either $\text{lm}(a) \prec \text{lm}(b)$ or $(\text{lt}(a) = \text{lt}(b) \text{ and } a - \text{lt}(a) \prec b - \text{lt}(b))$.

Let I be an ideal of $\mathcal{A}(X)$. A *Gröbner basis* for (I, \preceq) is a subset \mathcal{G} of I such that the ideals of $\mathcal{A}(X)$ generated by $\text{lm}(I)$ and by $\text{lm}(\mathcal{G})$ coincide.

3.5.2. Proposition. *If X is a convergent left-monomial 1-polygraph, and \preceq is a monomial order on $\mathcal{A}(X_0)$ that is compatible with X_1 , then the set $\partial(X_1)$ of boundaries of 1-cells of X is a Gröbner basis for $(I(X), \preceq)$.*

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Conversely, let X be a set, let \preceq be a monomial order on $\mathcal{A}(X)$, let I be an ideal of $\mathcal{A}(X)$ and \mathcal{G} be a subset of I . Define $\text{Lead}(\mathcal{G})$ as the 1-polygraph with 0-cells X and one 1-cell

$$\text{lm}(\alpha) \xrightarrow{\alpha_\alpha} \text{lm}(\alpha) - \frac{1}{\text{lc}(\alpha)} \alpha$$

for each α in \mathcal{G} . If \mathcal{G} is a Gröbner basis for (I, \preceq) , then $\text{Lead}(\mathcal{G})$ is a convergent left-monomial presentation of $\mathcal{A}(X)/I$, such that $I(\text{Lead}(\mathcal{G})) = I$, and \preceq is compatible with $\text{Lead}(\mathcal{G})_1$.

Proof. If X is convergent, then $\partial(\alpha)$ is in $I(X)$ for every 1-cell α of X . Since \preceq is compatible with X_1 , we have $\text{lm}(\partial(\alpha)) = s(\alpha)$ for every 1-cell α of X . Now, if α is in $I(X)$, it is a linear combination

$$\alpha = \sum_i \lambda_i u_i \partial(\alpha_i) v_i$$

of 1-cells $u_i \partial(\alpha_i) v_i$, where α_i is a 1-cell of X , and u_i and v_i are monomials of $\mathcal{A}(X)$. This implies that

$$\text{lm}(\alpha) = u_i s(\alpha_i) v_i = u_i \text{lm}(\partial(\alpha_i)) v_i$$

hold for some i . Thus $\partial(X_1)$ is a Gröbner basis for $(I(X), \preceq)$.

Conversely, assume that \mathcal{G} is a Gröbner basis for (I, \preceq) . By definition, \preceq is compatible with $\text{Lead}(\mathcal{G})_1$, hence $\text{Lead}(\mathcal{G})$ terminates, and $I(\text{Lead}(\mathcal{G})) = I$ holds, so that the algebra presented by $\text{Lead}(\mathcal{G})$ is indeed isomorphic to $\mathcal{A}(X)/I$. Moreover, the reduced monomials of $\mathcal{A}(\text{Lead}(\mathcal{G}))$ are the monomials of $\mathcal{A}(X)$ that cannot be decomposed as $u \text{lm}(\alpha) v$ with α in \mathcal{G} , and u and v monomials of $\mathcal{A}(X)$. Thus, if a reduced 0-cell α of $\mathcal{A}(\text{Lead}(\mathcal{G}))$ is in I , its leading monomial must be 0, because \mathcal{G} is a Gröbner basis of (I, \preceq) . As a consequence of Proposition 3.3.4, we get that $\text{Lead}(\mathcal{G})$ is confluent. \square

3.5.3. Poincaré-Birkhoff-Witt bases. Let A be an N -homogeneous algebra, for $N \geq 2$, let X be a generating set of A , concentrated in degree 1, and let \preceq be a monomial order on $\mathcal{A}(X)$. A *Poincaré-Birkhoff-Witt (PBW) basis* for (A, X, \preceq) is a subset \mathcal{B} of X^* such that:

- (i) \mathcal{B} is a linear basis of A , with $[u]_{\mathcal{B}}$ denoting the decomposition of an element u of X^* in the basis \mathcal{B} ,
- (ii) for all u and v in \mathcal{B} , we have $uv \succcurlyeq [uv]_{\mathcal{B}}$,
- (iii) an element u of X^* belongs to \mathcal{B} if, and only if, for every decomposition $u = vu'w$ of u in X^* such that u' has degree N , then u' is in \mathcal{B} .

3.5.4. Proposition. *If X is a convergent left-monomial N -homogeneous presentation of an algebra A , and \preceq is a monomial order on $\mathcal{A}(X_0)$ that is compatible with X_1 , then the set $\text{Red}_m(X)$ of reduced monomials of $\mathcal{A}(X)$ is a PBW basis for (A, X_0, \preceq) .*

Conversely, let A be an N -homogeneous algebra, let X be a generating set of A that is concentrated in degree 1, let \preceq a monomial order on $\mathcal{A}(X)$, and \mathcal{B} be a PBW basis of (A, X, \preceq) . Define $\tilde{\mathcal{B}}$ as the 1-polygraph with 0-cells X and with one 1-cell

$$uv \xrightarrow{\alpha_{u,v}} [uv]_{\mathcal{B}}$$

for all u and v in \mathcal{B} such that uv has degree N and $uv \neq [uv]_{\mathcal{B}}$. Then $\tilde{\mathcal{B}}$ is a convergent left-monomial N -homogeneous presentation of A , such that $\text{Red}_m(\tilde{\mathcal{B}}) = \mathcal{B}$, and \preceq is compatible with $\tilde{\mathcal{B}}_1$.

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into 1-cells of size 1. If $\alpha_i \prec \alpha$ holds for every $0 < i < p$, then there exist positive 1-cells g and h in $\mathcal{A}(X)$ and a 2-cell F in $\mathcal{A}(X)[Y]$ as in

$$\begin{array}{ccc} & \alpha_p & \\ f \nearrow & & \searrow h \\ \alpha_0 & & \alpha' \\ & \Downarrow F & \\ & g & \end{array}$$

Proof. Proceed by induction on p . If $p = 0$, then f is an identity, so taking $g = h = 1_{\alpha_0}$ and $F = 1_f$ proves the result. Otherwise, we construct

$$\begin{array}{ccccccc} & & & \alpha_p & & & \\ & & & \Downarrow F & & & \\ f_2 \star_0 \cdots \star_0 f_p & \nearrow & & & \searrow h_2 & & \\ & \alpha_1 & \xrightarrow{g_2} & b_2 & \xrightarrow{k_2} & & \\ f_1 \nearrow & & \searrow h_1 & & & & \\ \alpha_0 & \xrightarrow{g_1} & b_1 & \xrightarrow{k_1} & \alpha' & & \\ & & & \Downarrow G & & & \end{array}$$

Apply Lemma 3.1.3 to the 1-cell f_1 of size 1 to get the positive 1-cells g_1 and h_1 such that $f_1 = g_1 \star_0 h_1^-$. We have $\alpha_i \prec \alpha$ for every $1 < i < p$, so the induction hypothesis applies to $f_2 \star_0 \cdots \star_0 f_p$, providing the positive 1-cells g_2 and h_2 , and the 2-cell F . Then, consider the branching (h_1, g_2) , whose source α_1 satisfies $\alpha_1 \prec \alpha$: by hypothesis, the branching (h_1, g_2) is Y -confluent, giving the positive 1-cells k_1 and k_2 , and the 2-cell G . \square

4.1.3. Proposition (Coherent Newman's lemma). *Let X be a terminating left-monomial 1-polygraph, and Y be a cellular extension of $\mathcal{A}(X)$. If X is locally Y -confluent then it is Y -confluent.*

Proof. Prove that X is Y -confluent at every 0-cell α of $\mathcal{A}(X)$ by noetherian induction on α . If α is reduced, then $(1_\alpha, 1_\alpha)$ is the only branching of source α , and it is Y -confluent, taking $f' = g' = 1_\alpha$ and $F = 1_{1_\alpha}$. Now, let α be a nonreduced 0-cell of $\mathcal{A}(X)$ such that X is Y -confluent at every 0-cell $\beta \prec \alpha$, and let (f, g) be a branching of X of source α . If one of f or g is an identity, say f , then (f, g) is Y -confluent, taking $f' = g$ and $g' = 1_{t(g)}$, with $F = 1_g$. Otherwise, prove that (f, g) is Y -confluent thanks to the construction

$$\begin{array}{ccccccc} & & & & f_2 & & \\ & & & & \nearrow & & \searrow f'_2 \\ & & & b_1 & & b & \\ f_1 \nearrow & & & \searrow f'_1 & & \Downarrow G & \\ \alpha & & & \alpha'_1 & \xrightarrow{h} & b' & \\ & \Downarrow F & & & \Downarrow H & & \\ & g_1 & \nearrow & g'_1 & & & \\ & c_1 & \xrightarrow{g_2} & c & \xrightarrow{g'_2} & \alpha' & \\ & & & & & & \searrow k \end{array}$$

Since f and g are not identities, they admit decompositions $f = f_1 \star_0 f_2$ and $g = g_1 \star_0 g_2$ where f_1 and g_1 are rewriting steps, and f_2 and g_2 are positive 1-cells. By hypothesis, the local branching (f_1, g_1) is Y -confluent, yielding the positive 1-cells f'_1 and g'_1 and the 2-cell F . Since both $\alpha \succ b_1$ and $\alpha \succ c_1$ hold, the induction hypothesis applies to the branching (f_2, f'_1) to give f'_2 , h and G , and, then, to the branching $(g'_1 \star_0 h, g_2)$ to give h , g'_2 and H . \square

Taking $Y = \text{Sph}(\mathcal{A}(X))$ in Proposition 4.1.3, we deduce Newman's lemma [45]:

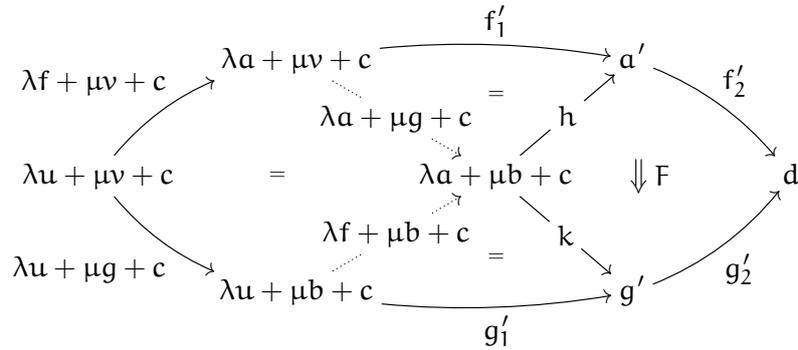
4.1.4. Corollary. *For terminating left-monoidal 1-polygraphs, confluence and local confluence are equivalent properties.*

4.2. The coherent critical branchings theorem

4.2.1. Theorem (The coherent critical branchings theorem). *Assume that X is a terminating left-monoidal 1-polygraph, and that Y is a cellular extension of $\mathcal{A}(X)$. If X is critically Y -confluent, then X is locally Y -confluent.*

Proof. We proceed by noetherian induction on the sources of the local branchings to prove that X is locally Y -confluent at every 0-cell of $\mathcal{A}(X)$. We note that a reduced 0-cell cannot be the source of a local branching, so X is locally Y -confluent at reduced 0-cells. Now, fix a nonreduced 0-cell α of $\mathcal{A}(X)$, and assume that X is locally Y -confluent at every $b \prec \alpha$. With a termination-based argument similar to that of Proposition 4.1.3, we deduce that X is Y -confluent at every $b \prec \alpha$. Then we proceed by case analysis on the type of the local branchings, noting that an aspherical branching $\lambda(f, f) + b$ is always Y -confluent.

For an additive branching, we construct



By linearity of the 0-composition, we have

$$(\lambda f + \mu v + c) \star_0 (\lambda a + \mu g + c) = \lambda f + \mu g + c = (\lambda u + \mu g + c) \star_0 (\lambda f + \mu b + c).$$

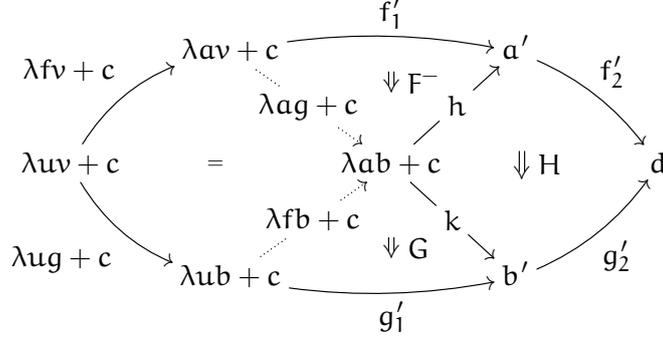
Note that the dotted 1-cells $\lambda a + \mu g + c$ and $\lambda f + \mu b + c$ are not positive in general, since u can be in $\text{supp}(b)$ or v in $\text{supp}(a)$. However, those 1-cells are of size 1, and Lemma 3.1.3 applies to both of them, to give positive 1-cells f'_1 , g'_1 , h and k that satisfy

$$f'_1 = (\lambda a + \mu g + c) \star_0 h \quad \text{and} \quad g'_1 = (\lambda f + \mu b + c) \star_0 k.$$

Now, $u \succ \alpha$, $v \succ b$, $\lambda \neq 0$ and $\mu \neq 0$ imply $\lambda u + \mu v + c \succ \lambda a + \mu b + c$. Thus, the branching (h, k) is Y -confluent by hypothesis, yielding the positive 1-cells f'_2 and g'_2 and the 2-cell F .

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Next, in the case of a Peiffer branching, we construct



Use the linearity of the 0-composition to obtain

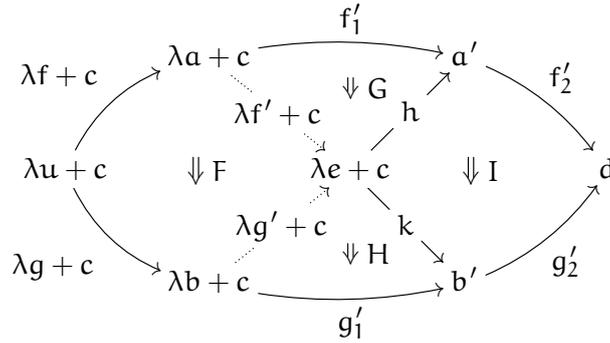
$$(\lambda f v + c) \star_0 (\lambda a g + c) = \lambda f g + c = (\lambda u g + c) \star_0 (\lambda f b + c).$$

Again, the dotted 1-cells $\lambda f b + c$ and $\lambda a g + c$ are not positive in general: this is the case, for example, if either $\text{supp}(ub) \cap \text{supp}(c)$ or $\text{supp}(av) \cap \text{supp}(c)$ is not empty. Let $a = \sum_{i=1}^p \mu_i u_i$ be the canonical decomposition of a . By linearity of the 0-composition, the 1-cell $\lambda a g + c$ admits the following decomposition in 1-cells of size 1:

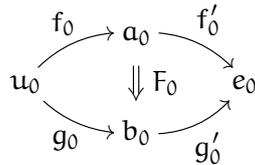
$$\lambda a g + c = g_1 \star_0 \cdots \star_0 g_p, \quad \text{with} \quad g_j = \sum_{1 \leq i < j} \lambda \mu_i u_i b + \lambda \mu_j u_j g + \sum_{j < i \leq p} \lambda \mu_i u_i v + c.$$

We have $u \succ u_i$ for every i , and $v \succ b$, giving $\lambda u v + v \succ s(g_j)$ for every j . Hence $\lambda a g + c$ is eligible to Lemma 4.1.2, yielding f'_1 , h and F . The cells g'_1 , k and G are obtained similarly from $\lambda f b + c$. Finally, $\lambda u v + c \succ \lambda a b + c$ implies, by induction hypothesis, that (h, k) is Y -confluent, giving f'_2 , g'_2 and H .

Finally, for an overlapping branching $(\lambda f + c, \lambda g + c)$, we construct



Consider the unique decomposition $(f, g) = v(f_0, g_0)w$, with (f_0, g_0) critical. Since (f_0, g_0) is Y -confluent by hypothesis, one obtains



4.2. The coherent critical branchings theorem

Define the positive 1-cells $f' = vf'_0w$ and $g' = vg'_0w$, and the 2-cell $F = vF_0w$. As previously, the dotted 1-cells are not positive in general, if $\text{supp}(c)$ intersects $\text{supp}(a)$ or $\text{supp}(b)$ for example. However, the 1-cell f' is positive, so that it is a 0-composite $f' = l_1 \star_0 \cdots \star_0 l_p$ of rewriting steps. As a consequence, we have the chain of inequalities

$$u \succ a = s(l_1) \succ (\cdots) \succ s(l_p) \succ e.$$

Since we have $\lambda \neq 0$ and $u \notin \text{supp}(c)$ by hypothesis, the inequality $\lambda u + c \succ \lambda s(l_i) + c$ holds for every i , so that the following decomposition of the 1-cell $\lambda f' + c$ satisfies the hypotheses of Lemma 4.1.2:

$$\lambda f' + c = (\lambda l_1 + c) \star_1 \cdots \star_1 (\lambda l_p + c).$$

This gives f'_1 , h and G . Proceed similarly with the 1-cell $\lambda g' + c$ to obtain g'_1 , k and H . Finally, apply the induction hypothesis on (h, k) , since $\lambda u + c \succ \lambda e + c$, to get f'_2 , g'_2 and I . \square

Taking $Y = \text{Sph}(\mathcal{A}(X))$ in Theorem 4.2.1, we deduce the critical branching theorem [46, 41, 35]:

4.2.2. Corollary. *For terminating left-monomial 1-polygraphs, local confluence and critical confluence are equivalent properties. In particular, a terminating left-monomial 1-polygraph with no critical branching is convergent.*

Proposition 3.5.2 and Corollaries 4.1.4 and 4.2.2 imply

4.2.3. Corollary (Buchberger's criterion). *Let X be a set, \preceq be a monomial order on the free algebra $\mathcal{A}(X)$, and I be an ideal of $\mathcal{A}(X)$. A subset \mathcal{G} of I is a Gröbner basis for (I, \preceq) if, and only if, the 1-polygraph $\text{Lead}(\mathcal{G})$ of Proposition 3.5.2 is critically confluent.*

4.2.4. Remark. The critical branching theorem for 1-polygraphs differs from its set-theoretic counterpart [32, 3.1.5]. Indeed, in the set-theoretic case, the termination hypothesis is not required, and nonoverlapping branchings are always confluent, independently of critical confluence. The following two counterexamples show that the linear case is different.

On the one hand, some local branchings can be nonconfluent without termination, even if critical confluence holds. Indeed, the 1-polygraph

$$\langle x, y, z, t \mid xy \xrightarrow{\alpha} xz, zt \xrightarrow{\beta} 2yt \rangle$$

has no critical branching, but it has a nonconfluent additive branching:

$$\begin{array}{c}
 \begin{array}{ccc}
 & & \begin{array}{c} 2x\beta \\ \rightarrow \\ 4xyt \end{array} \\
 \begin{array}{c} \alpha t + xzt \\ \rightarrow \\ 2xzt \end{array} & & \begin{array}{c} 4\alpha t \\ \rightarrow \\ 4xzt \end{array} \\
 \begin{array}{c} xyt + xzt \\ \rightarrow \\ 3xyt \end{array} & = & \begin{array}{c} xzt + x\beta \\ \rightarrow \\ xzt + 2xyt \end{array} \\
 & & \begin{array}{c} \alpha t + 2xyt \\ \rightarrow \\ 3xzt \end{array} \\
 \begin{array}{c} xyt + x\beta \\ \rightarrow \\ 3xyt \end{array} & & \begin{array}{c} 3x\beta \\ \rightarrow \\ 6xyt \end{array} \\
 & & \begin{array}{c} 6\alpha t \\ \rightarrow \\ \dots \end{array}
 \end{array}
 \end{array}$$

Next, let $f : a \rightarrow b$ be any 1-cell of $\mathcal{A}(X)$. By Lemmas 2.2.5 and 3.1.3, the 1-cell f factorises into

$$f = g_1 \star_0 h_1^- \star_0 \cdots \star_0 g_p \star_0 h_p^-,$$

where g_1, \dots, g_p and h_1, \dots, h_p are positive 1-cells of $\mathcal{A}(X)$. Then define η_f as the following composite 2-cell of $\mathcal{A}(X)[Y]$, with source $f \star_0 \eta_b$ and target η_a :

$$\begin{array}{ccccccc} a & \xrightarrow{g_1} & b_1 & \xrightarrow{h_1^-} & a_2 & \longrightarrow \cdots \longrightarrow & a_p & \xrightarrow{g_p} & b_p & \xrightarrow{h_p^-} & b \\ \eta_a \downarrow & \swarrow \eta_{g_1} & \downarrow \eta_{b_1} & \swarrow \eta_{h_1^-} & \downarrow \eta_{a_2} & & \downarrow \eta_{a_p} & \swarrow \eta_{g_p} & \downarrow \eta_{b_p} & \swarrow \eta_{h_p^-} & \downarrow \eta_b \\ \hat{a} & \xlongequal{\quad} & \hat{a} & \xlongequal{\quad} & \hat{a} & \xlongequal{\quad} \cdots \xlongequal{\quad} & \hat{a} & \xlongequal{\quad} & \hat{a} & \xlongequal{\quad} & \hat{a} \end{array}$$

Finally, for all parallel 1-cells $f, g : a \rightarrow b$ of $\mathcal{A}(X)$, the composite 2-cell

$$\begin{array}{ccc} & f \rightarrow b & \\ & \downarrow \eta_f & \eta_b = \\ a & \xrightarrow{\eta_a} & \hat{a} - \eta_b^- \rightarrow b \\ & \downarrow \eta_g & \eta_b = \\ & g \rightarrow b & \end{array}$$

of $\mathcal{A}(X)[Y]$ has source f and target g , thus concluding the proof that Y is acyclic. □

Composing Theorem 4.2.1 and Propositions 4.1.3 and 4.3.1 gives the analogue of Squier's theorem [53, Theorem 5.2] for convergent left-monomial 1-polygraphs:

4.3.2. Theorem (Squier's theorem). *Let X be a convergent left-monomial 1-polygraph. A cellular extension Y of $\mathcal{A}(X)$ that contains a 2-cell*

$$\begin{array}{ccc} & f \rightarrow b & \\ & \downarrow \eta_f & \eta_b = \\ a & \xrightarrow{\eta_a} & \hat{a} - \eta_b^- \rightarrow b \\ & \downarrow \eta_g & \eta_b = \\ & g \rightarrow b & \end{array}$$

for every critical branching (f, g) of X , with h and k positive 1-cells of $\mathcal{A}(X)$, is acyclic.

4.3.3. Example. From [47, 4.3], we consider the quadratic algebra A presented by

$$\langle x, y, z \mid x^2 + yz = 0, x^2 + \lambda zy = 0 \rangle,$$

where λ is a fixed scalar different from 0 and 1. Put $\mu = \lambda^{-1}$. The algebra A admits the presentation

$$X = \langle x, y, z \mid yz \xrightarrow{\alpha} -x^2, zy \xrightarrow{\beta} -\mu x^2 \rangle.$$

The deglex order generated by $z > y > x$ satisfies $yz > x^2$ and $zy > x^2$, proving that X terminates. However, X is not confluent. Indeed, it has two critical branchings:

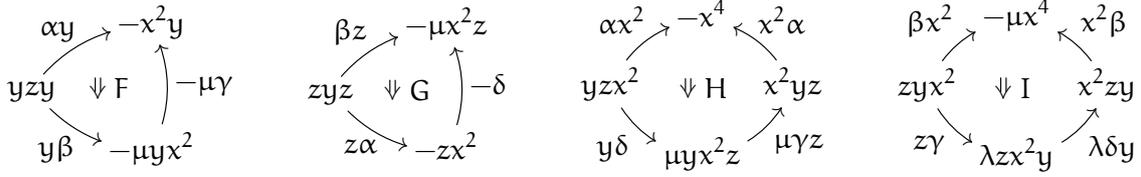
$$\begin{array}{ccc} & \alpha y \rightarrow -x^2 y & \\ & \downarrow & \\ yzy & & \\ & y\beta \rightarrow -\mu y x^2 & \end{array} \quad \text{and} \quad \begin{array}{ccc} & \beta z \rightarrow -\mu x^2 z & \\ & \downarrow & \\ zyz & & \\ & z\alpha \rightarrow -z x^2 & \end{array}$$

4. Coherent presentations of associative algebras

and neither of them is confluent, because the monomials x^2y , yx^2 , x^2z and zx^2 are reduced. The adjunction of the 1-cells

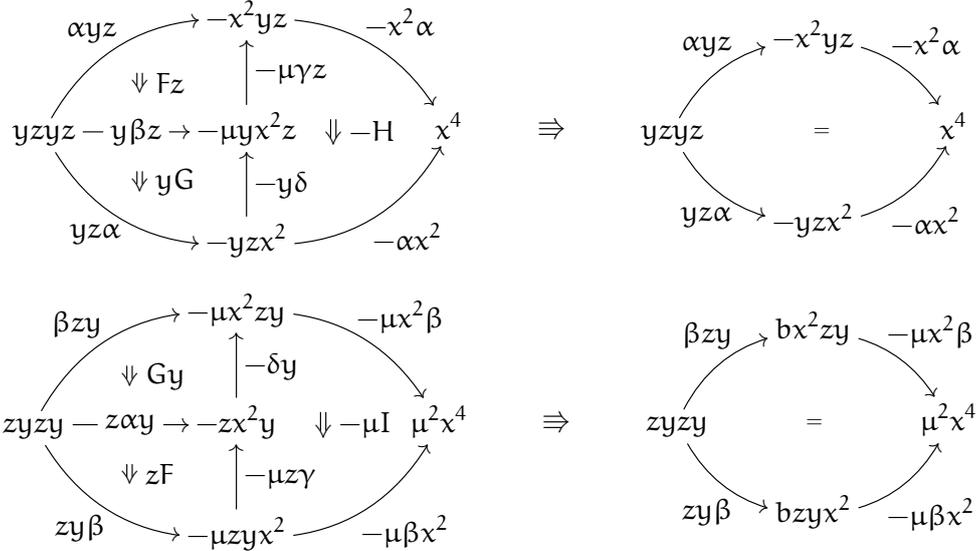
$$yx^2 \xrightarrow{\gamma} \lambda x^2 y \quad \text{and} \quad zx^2 \xrightarrow{\delta} \mu x^2 z$$

gives a left-monomial 1-polygraph Y that also presents A , since γ and δ induce relations that already hold in \bar{X} , and that also terminates, because of $yx^2 > x^2y$ and $zx^2 > x^2z$. Moreover, each one of the four critical branchings of Y is confluent:



Theorem 4.3.2 implies that the 2-polygraph $\langle Y | F, G, H, I \rangle$ is a coherent presentation of A .

This coherent presentation can be reduced to a smaller one by collapsing mechanism, that is formalised in Theorem 5.3.4, but hinted at on this example. First, some 2-cells may be removed without breaking acyclicity, because their boundary can also be filled by a composite of other 2-cells. Here, the “critical 3-branchings”, where three rewriting steps overlap, reveal two relations between 2-cells:



Since the boundaries of H and I can also be filled using F and G only, the 2-polygraph $\langle Y | F, G \rangle$ is also a coherent presentation of A . Next, the 1-cells γ and δ are redundant, because the corresponding relations can be derived from α and β , as testified by the 2-cells F and G : removing γ with F , and δ with G , proves that $\mathcal{A}(X)$ admits an empty acyclic cellular extension, so $\langle X | \emptyset \rangle$ is a coherent presentation of A .

4.3.4. Example (The standard coherent presentation). Assume that $A = \mathbb{K} \oplus A_+$ is an augmented algebra, and fix a linear basis \mathcal{B} of A_+ . For u and v in \mathcal{B} , write $u \otimes v$ for the product of u and v in the free algebra over \mathcal{B} , and uv for their product in A . Consider the 1-polygraph $\text{Std}(\mathcal{B})_1$ whose 0-cells are the elements of \mathcal{B} , and with a 1-cell

$$u \otimes v \xrightarrow{u|v} uv,$$

5. Construction and reduction of polygraphic resolutions

5. CONSTRUCTION AND REDUCTION OF POLYGRAPHIC RESOLUTIONS

In this section, we introduce concepts and techniques that are used in Section 6 to construct Squier's polygraphic resolution. First, we define contractions as sort of homotopy, and prove in Theorem 5.2.6 that they characterise ∞ -polygraphs that are polygraphic resolutions. Then, following [19], we define collapsing schemes of polygraphs, and Theorem 5.3.4 shows that they give a method to contract polygraphic resolutions into smaller ones. This process, similar to algebraic Morse theory for chain complexes [51], was already used to obtain minimal coherent presentations of Artin monoids in [26].

5.1. Linear homotopies

5.1.1. Homotopies. Let V and W be ∞ -vector spaces (resp. ∞ -algebras), and $F, G : V \rightarrow W$ be linear ∞ -functors (resp. morphisms of ∞ -algebras). A *homotopy from F to G* is an indexed linear map (resp. indexed morphism of algebras)

$$V \xrightarrow{\eta} W$$

of degree 1 that satisfies, writing η_a for $\eta(a)$,

(i) for every 0-cell a of V ,

$$s(\eta_a) = F(a) \quad \text{and} \quad t(\eta_a) = G(a), \quad (11)$$

(ii) for every $n \geq 1$ and every n -cell a of V ,

$$s(\eta_a) = F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(a)}, \quad (12)$$

$$t(\eta_a) = \eta_{s_{n-1}(a)} \star_{n-1} \cdots \star_1 \eta_{s_0(a)} \star_0 G(a), \quad (13)$$

with parentheses omitted according to the convention that \star_i binds more tightly than \star_j if $i < j$,

(iii) for every $n \geq 0$ and every n -cell a of V ,

$$\eta_{1_a} = 1_{\eta_a}. \quad (14)$$

For the definition of a homotopy to be licit, we have to check that the compositions involved in the right-hand side members of (12) and (13) are legal. Indeed, assume that η satisfies the axioms up to dimension $n - 1$, and fix an n -cell a of V . Then, for every $0 \leq k < n$,

$$\begin{aligned} t_k(F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(a)}) &= t_k(F(a)) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(a)} \\ &= F(t_k(a)) \star_0 \eta_{t_0 t_k(a)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1} t_k(a)} = s_k(\eta_{t_k(a)}). \end{aligned}$$

This proves that the right-hand side of (12) is well defined, and we proceed for (13).

5.1.2. Remarks. (i) The definition of homotopy requires precise shapes for the source and target of each cell η_a , and a compatibility with identities. These conditions could prevent such an η to exist, if the required relations could not be satisfied: if the formulas (12) and (13) were not linear in a (and compatible with the product in the case of ∞ -algebras), if the resulting cells $s(\eta_a)$ and $t(\eta_a)$ were not parallel, or if η_{1_a} was not parallel to 1_{η_a} .

But mapping a to $s(\eta_a)$ or to $t(\eta_a)$ is a linear map (resp. a morphism of algebras), because both are composites of linear maps (resp. morphisms of algebras). Moreover, $s(\eta_a)$ and $t(\eta_a)$ are parallel, because the globular relations and the compatibility of the source and target maps with the compositions imply

$$ss(\eta_x) = s(F(x)) \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-2} \eta_{t_{n-2}(x)} = s(\eta_{s(x)}) = st(\eta_x)$$

$$\text{and } ts(\eta_x) = t(\eta_{t(x)}) = \eta_{s_{n-2}(x)} \star_{n-2} \cdots \star_1 \eta_{s_0(x)} \star_0 t(G(x)) = tt(\eta_x).$$

Finally, replacing a by 1_a in (12) and (13) yields $s(\eta_{1_a}) = t(\eta_{1_a}) = \eta_a$.

(ii) Expanding (12) and (13), a homotopy η from F to G maps a 1-cell $a : b \rightarrow b'$ of V to a 2-cell

$$\begin{array}{ccccc} & F(a) & \xrightarrow{\quad} & F(b') & \xrightarrow{\quad} & \eta_{b'} \\ & \searrow & & \downarrow \eta_a & \searrow & \\ F(b) & & & & & G(b') \\ & \swarrow & & \downarrow \eta_b & \swarrow & \\ & G(b) & \xrightarrow{\quad} & G(a) & & \end{array}$$

of W , and a 2-cell $a : b \Rightarrow b' : c \rightarrow c'$ of V to a 3-cell

$$\begin{array}{ccc} \begin{array}{ccccc} F(b) & \xrightarrow{\quad} & F(c') & \xrightarrow{\quad} & \eta_{c'} \\ & \searrow & \downarrow \eta_{b'} & \searrow & \\ F(c) & \xrightarrow{\quad} & G(c) & \xrightarrow{\quad} & G(c') \\ & \swarrow & \downarrow \eta_b & \swarrow & \\ & G(b') & \xrightarrow{\quad} & G(a) & \end{array} & \xrightarrow{\eta_a} & \begin{array}{ccccc} F(b) & \xrightarrow{\quad} & F(c') & \xrightarrow{\quad} & \eta_{c'} \\ & \searrow & \downarrow \eta_b & \searrow & \\ F(c) & \xrightarrow{\quad} & G(c) & \xrightarrow{\quad} & G(c') \\ & \swarrow & \downarrow \eta_c & \swarrow & \\ & G(b') & \xrightarrow{\quad} & G(a) & \end{array} \end{array}$$

of W .

(iii) By definition, a homotopy is linear. Together with the relation $a \star_k b = a - t_k(a) + b$, this implies that a homotopy also satisfies the same compatibility condition with respect to compositions as the set-theoretic homotopies of [4, B.8].

5.1.3. Lemma. *Let V and W be ∞ -vector spaces (resp. ∞ -algebras), $F, G : V \rightarrow W$ be linear ∞ -functors (resp. morphisms of ∞ -algebras) and η be a homotopy from F to G . Then, for all $0 \leq k < n$, and every n -cell a of V ,*

$$F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_k \eta_{t_k(a)} = F(a) - t_k(F(a)) + \eta_{t_k(a)}$$

$$\text{and } \eta_{s_k(a)} \star_k \cdots \star_1 \eta_{s_0(a)} \star_0 G(a) = G(a) - s_k(F(a)) + \eta_{s_k(a)}.$$

As a consequence, for every $n \geq 1$ and every n -cell a of V ,

$$s(\eta_a) = F(a) - t(F(a)) + \eta_{t(F(a))} \quad \text{and} \quad t(\eta_a) = G(a) - s(G(a)) + \eta_{s(G(a))}.$$

5. Construction and reduction of polygraphic resolutions

Proof. We proceed by induction on k . By (12) and using the relation $b \star_k c = b - 1_{t_k(b)} + c$, we obtain

$$\begin{aligned} s_k(\eta_a) &= F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(a)} \\ &\quad - t_k(F(a) \star_0 \eta_{t_0(a)} \star_1 \cdots \star_{k-1} \eta_{t_{k-1}(a)}) \\ &\quad + \eta_{t_k(a)}. \end{aligned}$$

We apply the induction hypothesis to the first and second term, and use the linearity of t_k , to obtain

$$s_k(\eta_a) = (F(a) - t_{k-1}(F(a)) + \eta_{t_{k-1}(a)}) - (t_k(F(a)) - t_k t_{k-1}(F(a)) + t_k(\eta_{t_{k-1}(a)})) + \eta_{t_k(a)}.$$

Since $t_{k-1}(F(a))$ and $\eta_{t_{k-1}(a)}$ are of dimensions $k-1$ and k , respectively, we have $t_k t_{k-1}(F(a)) = t_{k-1}(F(a))$ and $t_k(\eta_{t_{k-1}(a)}) = \eta_{t_{k-1}(a)}$. So, cancelling out the terms in the former result, we obtain the first relation. The second one is proved in a symmetric way, and the final relation comes from the case $k = n-1$, with (12) and (13). \square

5.1.4. Lemma. *Assume that X is an ∞ -polygraph, and that A is an ∞ -algebra.*

- (i) *Let $F, G : \mathcal{A}(X) \rightarrow A$ be linear ∞ -functors. A homotopy η from F to G is uniquely and entirely determined by its values on the n -monomials of $\mathcal{A}(X)$, for $n \geq 0$, provided the relation*

$$\eta_{as_0(b)} + \eta_{t_0(a)b} - \eta_{t_0(a)s_0(b)} = \eta_{s_0(a)b} + \eta_{at_0(b)} - \eta_{s_0(a)t_0(b)} \quad (15)$$

is satisfied for all n -monomials a and b of $\mathcal{A}(X)$.

- (ii) *Let $F, G : \mathcal{A}(X) \rightarrow A$ be morphisms of ∞ -algebras. A homotopy η from F to G is uniquely and entirely determined by its values on the cells of X .*

Proof. (i) By induction on $n \geq 0$. For $n = 0$, assume that $\eta_u : F(u) \rightarrow G(u)$ is a fixed 1-cell of A for every monomial u of $\mathcal{A}(X)$. Extend η to every 0-cell a of $\mathcal{A}(X)$ by linearity: the resulting 1-cell η_a has source $F(a)$ and target $G(a)$, as required by (11) by linearity of F and G .

Now, fix $n \geq 1$, and assume that an $(n+1)$ -cell η_a has been chosen in A for every n -monomial a of $\mathcal{A}(X)$, with source and target given by (12) and (13), and in such a way that (15) holds. By construction, the n -cells of $\mathcal{A}(X)$ are the linear combinations of the n -monomials of $\mathcal{A}(X)$ and of identities of $(n-1)$ -cells of $\mathcal{A}(X)$, up to the relation

$$as_0(b) + t_0(a)b - t_0(a)s_0(b) = s_0(a)b + at_0(b) - s_0(a)t_0(b),$$

where a and b range over the n -monomials of $\mathcal{A}(X)$. Note that the values of η on identities of $(n-1)$ -cells are constrained by (14). Thus, one can extend η to any n -cell a of $\mathcal{A}(X)$, provided a decomposition of a into a linear combination of n -monomials and of an identity is chosen, and (15) implies that the result is independent of this choice. Finally, the source and target of the obtained $(n+1)$ -cell η_a satisfy (12) and (13) by linearity of F, G, η and all the compositions $\star_0, \dots, \star_{n-1}$.

(ii) By definition, a homotopy η between morphisms F and G of ∞ -algebras is a homotopy between the underlying linear ∞ -functors that commutes with identities and products. Thus, according to (i), the values of η on the 0-cells of $\mathcal{A}(X)$ can be uniquely reconstructed from its values on the 0-cells of X , and the corresponding 1-cells have the required source and target because F and G are morphisms of ∞ -algebras.

Now, if $n \geq 1$, we observe that the fact that η commutes with products ensures that (15) is automatically satisfied, so that η can be uniquely reconstructed from its values on the n -cells of $\mathcal{A}(X)$. Moreover, (12) and (13) are satisfied because F, G, η and the compositions $\star_0, \dots, \star_{n-1}$ are morphisms of ∞ -algebras. \square

5.2. Contractions of polygraphs for associative algebras

5.2.1. Unital sections and contractions. Let X be an ∞ -polygraph. A *unital section* of X is a linear ∞ -functor $\iota : \overline{X} \rightarrow \mathcal{A}(X)$ which is a section of the canonical projection $\pi : \mathcal{A}(X) \rightarrow \overline{X}$ and that satisfies $\iota(1) = 1$. Here the quotient algebra \overline{X} is seen as an ∞ -algebra whose n -cells are identities for every $n \geq 1$. Note that a unital section of π is not required to be a morphism of ∞ -algebras.

Let ι be a unital section of X . If a is an n -cell of $\mathcal{A}(X)$, write \widehat{a} for $\iota\pi(a)$ when no confusion occurs (note that \widehat{a} is an identity if $n \geq 1$). An ι -*contraction* of X is a homotopy σ from $\text{Id}_{\mathcal{A}(X)}$ to the composite linear ∞ -functor $\iota\pi$ that satisfies

$$\sigma_a = 1_a$$

for every n -cell a of $\mathcal{A}(X)$ that belongs to the image of ι or of σ . From the definition of $\iota\pi$, we note that, for every $n \geq 0$ and every n -cell a of $\mathcal{A}(X)$,

$$t(\sigma_a) = \begin{cases} \widehat{a} & \text{if } n = 0, \\ \sigma_{s(a)} & \text{if } n \geq 1. \end{cases}$$

An ι -contraction σ of X is called *right* if, for every $n \geq 0$ and all n -cells f and g of $\mathcal{A}(X)$ of respective 0-sources a and b , it satisfies the relation

$$\sigma_{fg} = a\sigma_g \star_0 \sigma_{f\widehat{b}}. \quad (16)$$

Note that the composition of the right-hand side of (16) is well defined, because the definition of an ι -contraction imposes

$$t_0(a\sigma_g) = a\widehat{b} = s_0(\sigma_{f\widehat{b}}).$$

5.2.2. Remarks. (i) The requirement $\sigma_a = 1_a$ whenever a is in the image of ι or σ could prevent ι -contractions to exist, if σ_a was not parallel to 1_a . On the contrary, let us prove that we always have $s(\sigma_a) = t(\sigma_a) = a$ by induction on the dimension.

Assume that a is a 0-cell of $\mathcal{A}(X)$. By definition of an ι -contraction, we have $s(\sigma_a) = a$ and $t(\sigma_a) = \widehat{a}$. Thus, if a is in the image of ι , then $\widehat{a} = a$, so that σ_a is parallel to 1_a .

Then, fix $n \geq 1$ and assume that $s(\sigma_a) = t(\sigma_a) = a$ hold for every k -cell a of $\mathcal{A}(X)$, with $k < n$, that lies in the image of ι or of σ . Fix an n -cell a of $\mathcal{A}(X)$. If a is in the image of ι , then a is an identity 1_b , with b in the image of ι by functoriality: because σ is a homotopy, it satisfies $\sigma_a = 1_{\sigma_b}$, so that, by induction hypothesis, $\sigma_a = 1_{1_b} = 1_a$. Now, assume that $a = \sigma_b$, for some $(n-1)$ -cell b of $\mathcal{A}(X)$. Lemma 5.1.3 gives

$$s(\sigma_a) = \sigma_b - 1_{t(\sigma_b)} + \sigma_{t(\sigma_b)}.$$

We distinguish two cases: if $n = 1$, then $t(\sigma_b) = \widehat{b}$, and, if $n > 1$, then $t(\sigma_b) = \sigma_{s(b)}$. Either way, $t(\sigma_b)$ is an $(n-1)$ -cell of $\mathcal{A}(X)$ that lies in the image of ι or of σ , so that the induction hypothesis implies $\sigma_{t(\sigma_b)} = 1_{t(\sigma_b)}$, and thus $s(\sigma_a) = \sigma_b = a$. Moreover, $t(\sigma_a) = \sigma_{s(a)}$, and Lemma 5.1.3 implies

$$t(\sigma_a) = \sigma_{s(\sigma_b)} = \sigma_b - \sigma_{1_{t(b)}} + \sigma_{\sigma_{t(b)}}.$$

By definition of σ , we have $\sigma_{1_{t(b)}} = 1_{\sigma_{t(b)}}$, and, by induction hypothesis, $\sigma_{\sigma_{t(b)}} = 1_{\sigma_{t(b)}}$, giving $t(\sigma_a) = \sigma_b = a$.

5. Construction and reduction of polygraphic resolutions

(ii) Expanding the definition, an ι -contraction σ of X maps a 0-cell a to a 1-cell $\sigma_a : a \rightarrow \hat{a}$, a 1-cell $f : a \rightarrow b$ to a 2-cell

$$\begin{array}{ccc} & f & \\ & \nearrow & \\ a & & b \\ & \searrow & \\ & \sigma_a & \\ & \searrow & \\ & & \hat{a} \end{array} \quad \begin{array}{c} \sigma_b \\ \downarrow \sigma_f \\ \sigma_a \end{array}$$

and a 2-cell $F : f \Rightarrow g : a \rightarrow b$ to a 3-cell

$$\begin{array}{ccc} & f & \\ & \nearrow & \\ a & & b \\ & \searrow & \\ & \sigma_a & \\ & \searrow & \\ & & \hat{a} \end{array} \quad \begin{array}{c} \sigma_b \\ \downarrow \sigma_f \\ \sigma_a \end{array} \quad \begin{array}{c} \Downarrow F \\ \Downarrow \sigma_g \\ \Downarrow \sigma_f \end{array} \quad \begin{array}{ccc} & f & \\ & \nearrow & \\ a & & b \\ & \searrow & \\ & \sigma_a & \\ & \searrow & \\ & & \hat{a} \end{array} \quad \begin{array}{c} \sigma_b \\ \downarrow \sigma_f \\ \sigma_a \end{array}$$

(iii) If σ is a right ι -contraction, then (16) reads, on 0-cells a and b ,

$$\begin{array}{ccc} & a\sigma_b & \\ & \nearrow & \\ ab & & \hat{a}\hat{b} \\ & \searrow & \\ & \sigma_{ab} & \\ & \searrow & \\ & & \hat{a}\hat{b} \end{array} \quad \begin{array}{c} \sigma_{\hat{a}\hat{b}} \\ = \\ \sigma_{ab} \end{array}$$

and, on 1-cells $f : a \rightarrow b$ and $g : c \rightarrow d$,

$$\begin{array}{ccc} & fg & \\ & \nearrow & \\ ac & & bd \\ & \searrow & \\ & \sigma_{ac} & \\ & \searrow & \\ & & \hat{a}\hat{c} \end{array} \quad \begin{array}{c} \sigma_{bd} \\ \downarrow \sigma_{fg} \\ \sigma_{ac} \end{array} = \begin{array}{ccc} & fg & \\ & \nearrow & \\ ac & & bd \\ & \searrow & \\ & \sigma_{ac} & \\ & \searrow & \\ & & \hat{a}\hat{c} \end{array} \quad \begin{array}{c} \sigma_{bd} \\ \downarrow \sigma_{fg} \\ \sigma_{ac} \end{array} \quad \begin{array}{ccc} & fg & \\ & \nearrow & \\ ac & & bd \\ & \searrow & \\ & \sigma_{ac} & \\ & \searrow & \\ & & \hat{a}\hat{c} \end{array} \quad \begin{array}{c} \sigma_{bd} \\ \downarrow \sigma_{fg} \\ \sigma_{ac} \end{array} \quad \begin{array}{ccc} & fg & \\ & \nearrow & \\ ac & & bd \\ & \searrow & \\ & \sigma_{ac} & \\ & \searrow & \\ & & \hat{a}\hat{c} \end{array} \quad \begin{array}{c} \sigma_{bd} \\ \downarrow \sigma_{fg} \\ \sigma_{ac} \end{array}$$

5.2.3. Lemma. Let X be an ∞ -polygraph, ι be a unital section of X , and σ be a right ι -contraction of X . Then, for every $n \geq 0$ and all n -cells f and g of $\mathcal{A}(X)$, putting $\hat{b} = s_0(g)$, we have

$$\sigma_{fg} = f\sigma_g - f\hat{b} + \sigma_{f\hat{b}}.$$

Proof. On the one hand, (16) implies

$$\sigma_{fg} = a\sigma_g \star_0 \sigma_{f\hat{b}} = a\sigma_g - a\hat{b} + \sigma_{f\hat{b}},$$

and, on the other hand, the fact that \star_0 is a morphism of algebras yields

$$f\sigma_g = a\sigma_g \star_0 f\hat{b} = a\sigma_g - a\hat{b} + f\hat{b}.$$

Combining these two expressions gives the result. \square

5.2. Contractions of polygraphs for associative algebras

5.2.4. Reduced and essential monomials. Fix a unital section ι of X . A monomial u of $\mathcal{A}(X)$ is ι -reduced if $u = \widehat{u}$ holds. A non- ι -reduced monomial u of $\mathcal{A}(X)$ is ι -essential if $u = xv$, with x a 0-cell of X and v an ι -reduced monomial of $\mathcal{A}(X)$.

If σ is an ι -contraction of X , for $n \geq 1$, an n -cell α of $\mathcal{A}(X)$ is σ -reduced if it is an identity or in the image of σ . If σ is a right ι -contraction of X , and $n \geq 1$, a non- σ -reduced n -monomial α of $\mathcal{A}(X)$ is σ -essential if there exist an n -cell α of X and an ι -reduced monomial of $\mathcal{A}(X)$ such that $\alpha = \alpha v$.

5.2.5. Lemma. *Let X be an ∞ -polygraph, and ι be a unital section of X . A right ι -contraction σ of X is uniquely and entirely determined by its values on the ι -essential monomials and, for every $n \geq 1$, on the σ -essential n -monomials of $\mathcal{A}(X)$.*

Proof. By Lemma 5.1.4 (i), we know that the homotopy underlying σ is uniquely and entirely determined by its values on the n -monomials of $\mathcal{A}(X)$, provided (15) is satisfied. There remains to check that the values of σ on the ι -essential monomials and on the σ -essential n -monomials completely determines its values on the other monomials and n -monomials, and that (15) is automatically satisfied.

If u is a non- ι -essential monomial, then either $u = 1$, or $u = xv$ with x a 0-cell of X and v a non- ι -reduced monomial. In the former case, $\sigma_1 = 1$ is forced because 1 is ι -reduced. In the latter case, (16) imposes

$$\sigma_{xv} = x\sigma_v \star_0 \sigma_{x\widehat{v}}.$$

Then proceed by induction on the length of v to define σ_v from the values of σ on ι -reduced monomials.

Now, for every n -monomial $u\alpha v$ of $\mathcal{A}(X)$, with α an n -cell of X of 0-source a , and u and v monomials of $\mathcal{A}(X)$,

$$\sigma_{u\alpha v} = u\alpha\sigma_v \star_0 u\sigma_{\alpha\widehat{v}} \star_0 \sigma_{u\widehat{a\widehat{v}}}$$

is imposed by (16). Then, if αv is σ -reduced, then $\alpha v = \sigma_b$ for some $(n-1)$ -cell b of $\mathcal{A}(X)$, which implies $\sigma_{\alpha v} = 1_{\sigma_b}$.

To check (15), fix n -monomials a and b of $\mathcal{A}(X)$, and put $c = s_0(a)$, $c' = t_0(a)$, $d = s_0(b)$ and $d' = t_0(b)$. On the one hand,

$$\begin{aligned} \sigma_{ad} + \sigma_{c'b} - \sigma_{c'd} &= c\sigma_d \star_0 \sigma_{a\widehat{d}} + c'\sigma_b \star_0 \sigma_{c'\widehat{d}} - c'\sigma_d \star_0 \sigma_{c'\widehat{d}} \\ &= \sigma_{a\widehat{d}} + c'\sigma_b + c\sigma_d - c'\sigma_d - c\widehat{d}, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \sigma_{cb} + \sigma_{ad'} - \sigma_{cd'} &= c\sigma_b \star_0 \sigma_{c\widehat{d}} + c\sigma_d' \star_0 \sigma_{a\widehat{d}} - c\sigma_d' \star_0 \sigma_{c\widehat{d}} \\ &= \sigma_{a\widehat{d}} + c\sigma_b - c\widehat{d}. \end{aligned}$$

So, we are left with proving

$$c'\sigma_b + c\sigma_d = c\sigma_b + c'\sigma_d \tag{17}$$

under the hypothesis that there exists an n -cell α in $\mathcal{A}(X)$ with 0-source c and 0-target c' . Since the 0-composition of $\mathcal{A}(X)$ is a morphism of ∞ -algebras, we have

$$a\widehat{d} \star_0 c'\sigma_d = a\sigma_d = c\sigma_d \star_0 a\widehat{d}.$$

5. Construction and reduction of polygraphic resolutions

Replacing the 0-composition by its linear expression in the leftmost and the rightmost terms yields

$$ad - c'd + c'\sigma_d = c\sigma_d - c\widehat{d} + a\widehat{d}.$$

Similarly, using $s_0(\sigma_b) = d$ and $t_0(\sigma_b) = \widehat{d}$, and decomposing $a\sigma_b$, we obtain

$$ad - c'd + c'\sigma_b = c\sigma_b - c\widehat{d} + a\widehat{d}.$$

The difference between the two equations so obtained gives (17). \square

5.2.6. Theorem. *Let X be an ∞ -polygraph with a fixed unital section ι . Then X is a polygraphic resolution of \overline{X} if, and only if, it admits a right ι -contraction.*

Proof. Assume that X is a polygraphic resolution of \overline{X} , and define a right ι -contraction σ of X thanks to Lemma 5.2.5. If xu is an essential monomial of $\mathcal{A}(X)$, then xu and \widehat{xu} have the same image in \overline{X} , so that, by definition of \overline{X} , there exists a 1-cell $\sigma_{xu} : xu \rightarrow \widehat{xu}$ in $\mathcal{A}(X)$. Assume that σ is defined on the n -cells of $\mathcal{A}(X)$, for $n \geq 1$, and let αu be a σ -essential n -monomial of $\mathcal{A}(X)$. The n -cells defining $s(\sigma_{\alpha u})$ and $t(\sigma_{\alpha u})$ are parallel, so, by hypothesis, there exists an $(n+1)$ -cell $\sigma_{\alpha u}$ with these source and target in $\mathcal{A}(X)$.

Conversely, let σ be an ι -contraction of X , and a and b be parallel n -cells of $\mathcal{A}(X)$, for $n \geq 1$. We have $t(\sigma_a) = \sigma_{s(a)} = \sigma_{s(b)} = t(\sigma_b)$ by hypothesis, so that the $(n+1)$ -cell $\sigma_a \star_n \sigma_b^-$ is well defined, with source $s(\sigma_a)$ and target $s(\sigma_b)$. The fact that $t_k(a) = t_k(b)$ holds for every $0 \leq k < n$ implies that

$$(\sigma_a \star_n \sigma_b)^- \star_{n-1} \sigma_{t_{n-1}(a)}^- \star_{n-2} \cdots \star_0 \sigma_{t_0(a)}^-$$

is a well-defined n -cell of $\mathcal{A}(X)$, with source a and target b , thus proving that X_{n+1} is acyclic. \square

5.3. Collapsing polygraphic resolutions

5.3.1. Collapsing schemes. Let X be an ∞ -polygraph, and Y be an indexed subset of X . A *collapsing scheme of X onto Y* is an injective indexed partial map $\varphi : X \rightarrow X$ of degree 1 that satisfies, writing φ_x for $\varphi(x)$,

(i) as an indexed set, X admits the partition

$$X = \text{im}(\varphi) \amalg Y \amalg \text{dom}(\varphi), \quad (18)$$

(ii) for every x in $\text{dom}(\varphi)$, the boundary of φ_x satisfies

$$\partial(\varphi_x) = \lambda x + \alpha, \quad (19)$$

where λ is a nonzero scalar, and α is an n -cell of $\mathcal{A}(X)$ such that x does not belong to the set $\text{cell}(\alpha)$ of n -cells of X that appear in α ,

(iii) putting $x \triangleright_{\varphi} y$, for all x in $\text{dom}(\varphi)$ and y in $\text{cell}(\partial(\varphi_x))$, defines a wellfounded order on the cells of X .

Let φ be a fixed collapsing scheme of X onto Y . If x is an n -cell of $\text{dom}(\varphi)$, using the same notation as in (19), define the n -cell \tilde{x} and the $(n+1)$ -cell $\tilde{\varphi}_x$ of $\mathcal{A}(X)$ by

$$\tilde{x} = -\frac{1}{\lambda} \alpha \quad \text{and} \quad \tilde{\varphi}_x = \frac{1}{\lambda} (\varphi_x - t(\varphi_x)) + \tilde{x},$$

so that $s(\tilde{\varphi}_x) = x$ and $t(\tilde{\varphi}_x) = \tilde{x}$ are satisfied, and $x \triangleright_{\varphi} y$ holds for every $y \in \text{cell}(\tilde{x})$.

5.3.2. Lemma. *Let X be an ∞ -polygraph, Y be an indexed subset of X , and φ be a collapsing scheme of X onto Y . There is a structure of ∞ -polygraph on Y , with source \bar{s} and target \bar{t} , and a morphism of ∞ -algebras $\pi : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$, such that, for every $n \geq 1$ and every n -cell x of Y ,*

$$\bar{s}(x) = \pi(s(x)) \quad \text{and} \quad \bar{t}(x) = \pi(t(x)), \quad (20)$$

and, for every $n \geq 0$ and every n -cell x of X ,

$$\pi(x) = \begin{cases} x & \text{if } x \in Y, \\ \pi(\tilde{x}) & \text{if } x \in \text{dom}(\varphi), \\ 1_{\bar{s}(x)} & \text{if } x \in \text{im}(\varphi). \end{cases} \quad (21)$$

Proof. By induction on $n \geq 0$. For $n = 0$, we only have to check that (21) defines a map from the 0-cells of X to the 0-cells of $\mathcal{A}(Y)$: this holds by induction on the wellfounded order \triangleright_{φ} , using the fact that $x \triangleright_{\varphi} y$ is satisfied for every x in $\text{dom}(\varphi)$ and every y in $\text{cell}(\tilde{x})$.

Now, fix $n \geq 1$, assume that Y_{n-1} is an $(n-1)$ -polygraph, with \bar{s} and \bar{t} as source and target maps, and suppose that (21) defines a morphism of $(n-1)$ -algebras $\pi : \mathcal{A}(X_{n-1}) \rightarrow \mathcal{A}(Y_{n-1})$.

First, we define \bar{s} and \bar{t} on the n -cells of Y by (20). Let us check that the globular relations are satisfied. If $n > 1$ and x is an n -cell of Y , the definition of \bar{s} and the fact that π commutes with the source map in dimension $n-1$ give

$$\bar{s}(\bar{s}(x)) = \bar{s}(\pi(s(x))) = \pi(s(s(x))).$$

Similarly, we obtain $\bar{s}(\bar{t}(x)) = \pi(s(t(x)))$, so that $\bar{s}\bar{s} = \bar{s}\bar{t}$ is deduced from $ss = st$, and, with the same reasoning, $\bar{t}\bar{s} = \bar{t}\bar{t}$ also holds.

Next, since \triangleright_{φ} is wellfounded and $x \triangleright_{\varphi} y$ holds for every x in $\text{dom}(\varphi)$ and every y in $\text{cell}(\tilde{x})$, we deduce that (21) defines π as a map from the set of n -cells of X to the set of n -cells of $\mathcal{A}(Y)$. Now, let us check that π commutes with the source and target maps, to prove that π induces a morphism of n -algebras from $\mathcal{A}(X_n)$ to $\mathcal{A}(Y_n)$. Assume that x is an n -cell of X , and that $\bar{s}(\pi(y)) = \pi(s(y))$ and $\bar{t}(\pi(y)) = \pi(t(y))$ hold for every n -cell y of X such that $x \triangleright_{\varphi} y$. We distinguish three cases.

If x is in Y , then the definitions of π , \bar{s} and \bar{t} give

$$\bar{s}(\pi(x)) = \bar{s}(x) = \pi(s(x)) \quad \text{and} \quad \bar{t}(\pi(x)) = \bar{t}(x) = \pi(t(x)).$$

If x is in $\text{im}(\varphi)$ (and, thus, $n \geq 1$), then the definition of $\pi(x)$, the compatibility of \bar{s} with identities, and the definition of \bar{s} produce

$$\bar{s}(\pi(x)) = \bar{s}(1_{\bar{s}(x)}) = \bar{s}(x) = \pi(s(x)).$$

5. Construction and reduction of polygraphic resolutions

For the same reasons, we deduce $\bar{t}(\pi(x)) = \pi(s(x))$, leaving $\pi(s(x)) = \pi(t(x))$ to prove. Let y be the $(n-1)$ -cell of X such that $\varphi_y = x$. The linearity of π on the $(n-1)$ -cells, and the definitions of \tilde{y} and $\pi(y)$ imply

$$\pi(s(x)) - \pi(t(x)) = \pi(\partial(x)) = \pi(y) - \pi(\tilde{y}) = 0.$$

If x is in $\text{dom}(\varphi)$,

$$\bar{s}(\pi(x)) = \bar{s}(\pi(\tilde{x})) = \pi(s(\tilde{x})) \quad \text{and} \quad \bar{t}(\pi(x)) = \bar{t}(\pi(\tilde{x})) = \pi(t(\tilde{x}))$$

come from the definition of $\pi(x)$ and the induction hypothesis, since every $y \in \text{cell}(\tilde{x})$ satisfies $x \triangleright_{\varphi} y$. Moreover, the globular relations, and $s(\tilde{\varphi}_x) = x$ and $t(\tilde{\varphi}_x) = \tilde{x}$, imply $s(x) = s(\tilde{x})$ and $t(x) = t(\tilde{x})$. \square

5.3.3. Lemma. *Let X be an ∞ -polygraph, Y be an indexed subset of X , and φ be a collapsing scheme of X onto Y . There exist a unique morphism of ∞ -algebras $\iota : \mathcal{A}(Y) \rightarrow \mathcal{A}(X)$, and a unique homotopy η from $\text{Id}_{\mathcal{A}(X)}$ to the composite morphism of ∞ -algebras $\iota\pi$, that satisfy, for every n -cell x of Y ,*

$$\eta_{s_{n-1}(x)} \star_{n-1} \cdots \star_1 \eta_{s_0(x)} \star_0 \iota(x) = x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)} \quad (22)$$

and, for every n -cell x of X ,

$$\eta_x = \begin{cases} 1_{x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)}} & \text{if } x \in Y \text{ or } x \in \text{im}(\varphi), \\ \tilde{\varphi}_x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)} \star_n \eta_{\tilde{x}} & \text{if } x \in \text{dom}(\varphi). \end{cases} \quad (23)$$

Moreover, ι is a section of the projection $\pi : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$, and $\pi(\eta_a) = 1_{\pi(a)}$ holds for every n -cell a of $\mathcal{A}(X)$.

Proof. Proceed by induction on the dimension.

If x is a 0-cell of Y , put $\iota(x) = x$, as required by (22), and extend ι into a morphism of algebras, that must satisfy $\pi\iota = \text{Id}_{\mathcal{A}(Y)}$ because π is the identity on Y .

Then, assume that x is a 0-cell of X and define a 1-cell $\eta_x : x \rightarrow \iota\pi(x)$ of $\mathcal{A}(X)$, by induction on the wellfounded order \triangleright_{φ} . Fix a 0-cell y of X , and assume that η_y is defined by (23) for every 0-cell y of X such that $x \triangleright_{\varphi} y$, in such a way that $\pi(\eta_y) = 1_{\pi(y)}$ holds. First, extend η to every 0-cell a of $\mathcal{A}(X)$, such that $x \triangleright_{\varphi} y$ holds for every y in $\text{cell}(a)$, using Lemma 5.1.4 (ii). We distinguish two cases.

If x is in Y , then $\pi(x) = x$ holds by definition of π , hence $\iota\pi(x) = x$ follows by definition of ι . Put $\eta_x = 1_x$ as required by (23), so that $\pi(\eta_x) = 1_{\pi(x)}$ is satisfied.

If x is in $\text{dom}(\varphi)$, then $\pi(x) = \pi(\tilde{x})$ holds by definition of π . Define η_x by

$$\begin{array}{ccc} x & \xrightarrow{\eta_x} & \pi(x) \\ & \searrow \tilde{\varphi}_x & \nearrow \eta_{\tilde{x}} \\ & \tilde{x} & \end{array} \quad \begin{array}{c} \\ \\ = \end{array}$$

as required by (23), with $\pi(\eta_x) = 1_{\pi(x)}$ because $\pi(\tilde{\varphi}_x) = \pi(\eta_{\tilde{x}}) = 1_{\pi(x)}$ hold by definition of π and by induction hypothesis on $\eta_{\tilde{x}}$.

Now, fix $n \geq 1$, and assume that ι and η are defined by (22) and (23), and that they satisfy $\pi\iota = \text{Id}_{\mathcal{A}(Y)}$ and $\pi(\eta_a) = 1_{\pi(a)}$, up to dimension $n-1$.

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Let x be an n -cell of Y . First, the compositions involved in (22) are legal, because the facts that η is a homotopy and that $\iota\pi$ commutes with each source map s_k , for $0 \leq k < n$, imply

$$\begin{aligned} t_k(\eta_{s_k(x)}) &= \eta_{s_{k-1}(x)} \star_{k-1} \cdots \star_1 \eta_{s_0(x)} \star_0 \iota\pi(s_k(x)) \\ &= s_k(\eta_{s_{k-1}(x)} \star_{k-1} \cdots \star_1 \eta_{s_0(x)} \star_0 \iota(x)), \end{aligned}$$

and similarly for the right-hand side of (22). Then, the two sides of (22) are parallel, because

$$\begin{aligned} s(\eta_{s_{n-1}(x)} \star_{n-1} \cdots \star_1 \eta_{s_0(x)} \star_0 \iota(x)) &= s(\eta_{s(x)}) \\ &= s(x \star_0 \eta_{t_0(x)} \star_1 \cdots \eta_{t_{n-1}(x)}) \end{aligned}$$

hold, a similar computation giving the same equality between the targets. Thus, $\iota(x)$ can be defined by (22), because every other cell involved in the left-hand side is invertible. Moreover, $\pi\iota(x) = x$ holds by induction hypothesis on η .

Next, consider an n -cell x of X , and define η_x by induction on the wellfounded order \triangleright_φ . Assume that η_y is defined for every n -cell y of X such that $x \triangleright_\varphi y$, and extend η , thanks to Lemma 5.1.4 (ii), to every n -cell a of $\mathcal{A}(X)$ such that $x \triangleright_\varphi y$ holds for every $y \in \text{cell}(a)$. Define η_x as in (23), and proceed by case analysis to check that it satisfies (12), (13), and $\pi(\eta_x) = 1_{\pi(x)}$.

If x is in Y , then

$$\pi(x) = x \quad \text{and} \quad \eta_x = 1_{x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)}}$$

hold. So (12) is satisfied, and (13) is equivalent to (22). Moreover, $\pi(\eta_x) = 1_{\pi(x)}$ holds by induction hypothesis on η .

If x is in $\text{dom}(\varphi)$, then

$$\pi(x) = \pi(\tilde{x}) \quad \text{and} \quad \eta_x = \tilde{\varphi}_x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)} \star_n \eta_{\tilde{x}}$$

are satisfied. Note that compositions involved in the definition of η_x are valid, because x and \tilde{x} are parallel. Now, observe that

$$s(\eta_x) = s(\tilde{\varphi}_x) \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)} \quad \text{and} \quad t(\eta_x) = t(\eta_{\tilde{x}})$$

hold. By definition, $s(\tilde{\varphi}_x) = x$, which implies (12). Then (13) follows from the induction hypothesis applied to \tilde{x} , which satisfies $x \triangleright_\varphi y$ for every $y \in \text{cell}(\tilde{x})$, and from the fact that $\pi(x) = \pi(\tilde{x})$ holds. Finally, the definition of π and the induction hypothesis on η imply $\pi(\eta_x) = 1_{\pi(x)}$.

If x is in $\text{im}(\varphi)$, we have

$$\pi(x) = 1_{\tilde{s}(x)} \quad \text{and} \quad \eta_x = 1_{x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)}}$$

So (12) is satisfied, and

$$\eta_{s_{n-1}(x)} \star_{n-1} \cdots \star_1 \eta_{s_0(x)} \star_0 \iota\pi(x) = \eta_{s(x)}$$

holds. Thus, (13) is equivalent to

$$\eta_{s(x)} = x \star_0 \eta_{t_0(x)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(x)}.$$

6. Squier's polygraphic resolution of associative algebras

Recall that the right-hand side of the last equality is equal to $x - t(x) + \eta_{t(x)}$, and let y be the $(n-1)$ -cell of $\text{dom}(\varphi)$ such that $\varphi_y = x$. By induction hypothesis on η , and using $a \star_k b = a - t_k(a) + b$, we obtain

$$\eta_y = \tilde{\varphi}_y \star_0 \eta_{t_0(y)} \star_1 \cdots \star_{n-1} \eta_{t_{n-1}(y)} \star_n \eta_{\tilde{y}} = \tilde{\varphi}_y - \tilde{y} + \eta_{\tilde{y}}.$$

Now, $\tilde{\varphi}_y = \frac{1}{\lambda}(x - t(x)) + \tilde{y}$, satisfied by definition, implies

$$\eta_y = \frac{1}{\lambda}(x - t(x)) + \eta_{\tilde{y}}.$$

So, by definition of y and by linearity of η , we obtain

$$\eta_{s(x)} - \eta_{t(x)} = \lambda(\eta_y - \eta_{\tilde{y}}) = x - t(x),$$

which, in turn, implies (13). Finally, $\pi(\eta_x) = 1_{\pi(x)}$ is satisfied by induction hypothesis on η . \square

5.3.4. Theorem. *Let A be an algebra and X be a polygraphic resolution of A . If Y is an indexed subset of X , and φ is a collapsing scheme of X onto Y , then Y , equipped with the structure of ∞ -polygraph of Lemma 5.3.2, is a polygraphic resolution of A .*

Proof. First, prove that the algebras \bar{X} and \bar{Y} , respectively presented by X and Y , are isomorphic. If a and b are 0-cells of $\mathcal{A}(X)$ (resp. of $\mathcal{A}(Y)$) that are identified in \bar{X} (resp. in \bar{Y}), there exists a 1-cell $f : a \rightarrow b$ in $\mathcal{A}(X)$ (resp. in $\mathcal{A}(Y)$); because π (resp. ι) is a morphism of ∞ -algebras, $\pi(f) : \pi(a) \rightarrow \pi(b)$ (resp. $\iota(f) : \iota(a) \rightarrow \iota(b)$) is a 1-cell of $\mathcal{A}(Y)$ (resp. of $\mathcal{A}(X)$), meaning that $\pi(a)$ and $\pi(b)$ are identified in \bar{Y} (resp. that $\iota(a)$ and $\iota(b)$ are identified in \bar{X}). As a consequence, the morphisms of ∞ -algebras $\pi : \mathcal{A}(X) \rightarrow \mathcal{A}(Y)$ and $\iota : \mathcal{A}(Y) \rightarrow \mathcal{A}(X)$ induce morphisms of algebras $\bar{\pi} : \bar{X} \rightarrow \bar{Y}$ and $\bar{\iota} : \bar{Y} \rightarrow \bar{X}$. Moreover, the equality $\pi\iota = \text{Id}_{\mathcal{A}(Y)}$ induces $\bar{\pi}\bar{\iota} = \text{Id}_{\bar{Y}}$. Finally, for every 0-cell a of $\mathcal{A}(X)$, the 1-cell $\eta_a : a \rightarrow \iota\pi(a)$ of $\mathcal{A}(X)$ proves that $\bar{\iota}\bar{\pi}(a) = a$.

Now, prove that, for every $n \geq 1$ and every n -sphere (a, b) of $\mathcal{A}(Y)$, there exists an $(n+1)$ -cell of source a and target b in $\mathcal{A}(Y)$. Apply the morphism of ∞ -algebras ι from Lemma 5.3.3 to both a and b , to obtain an n -sphere $(\iota(a), \iota(b))$ of $\mathcal{A}(X)$. Since X is a polygraphic resolution of A , there exists an $(n+1)$ -cell $f : \iota(a) \rightarrow \iota(b)$ in $\mathcal{A}(X)$. Apply the projection π , and use $\pi\iota = \text{Id}_{\mathcal{A}(Y)}$, to obtain an $(n+1)$ -cell $\pi(f) : a \rightarrow b$ in $\mathcal{A}(Y)$. \square

6. SQUIER'S POLYGRAPHIC RESOLUTION OF ASSOCIATIVE ALGEBRAS

In this section, given a convergent presentation of an algebra, the coherent presentation given by Squier's theorem, Theorem 4.3.2, is extended into a polygraphic resolution. The first step is to construct a polygraphic analogue (but with cubical cells instead of simplicial ones) of the standard resolution of an algebra, which is proved to be a resolution in Theorem 6.1.2 by building a contraction. Using Theorem 5.3.4, this explicit but very large resolution is then contracted into Squier's polygraphic resolution, with one n -cell for each critical n -branching of the initial convergent presentation. Several examples are given at the end of the section.

6.1. The standard polygraphic resolution of an associative algebra

6.1.1. Cubical faces. Let $A = \mathbb{K} \oplus A_+$ be an augmented algebra. For $n \geq 1$, define the vector space

$$A^{(n)} = \bigoplus_{\substack{1 \leq k \leq n \\ i_1 + \dots + i_k = n}} A_+^{\otimes i_1+1} \otimes \dots \otimes A_+^{\otimes i_k+1}.$$

Vertical bars are used to denote the innermost products of copies of A_+ , so that $A^{(n)}$ is made of the linear combinations of elements $a_0^1 | \dots | a_{i_1}^1 \otimes \dots \otimes a_0^k | \dots | a_{i_k}^k$ with n vertical bars. In particular, each $a_0 | \dots | a_n$ of $A_+^{\otimes n+1}$ belongs to $A^{(n)}$. If \mathcal{B} is a linear basis of A_+ , the elements $u_0^1 | \dots | u_{i_1}^1 \otimes \dots \otimes u_0^k | \dots | u_{i_k}^k$, with each u_i^j in \mathcal{B} , form a basis of the vector space $A^{(n)}$.

For $1 \leq i \leq n$, let d_i^- (resp. d_i^+) be the linear map from $A^{(n)}$ to $A^{(n-1)}$ that replaces the i^{th} vertical bar, counting from the left, with a tensor (resp. the product of A_+). For example,

$$d_2^-(a \otimes b | c \otimes d | e) = a \otimes b | c \otimes d \otimes e \quad \text{and} \quad d_1^+(a \otimes b | c \otimes d | e) = a \otimes bc \otimes d | e.$$

It follows from the definition that these maps satisfy the cubical relations:

$$d_j^\beta d_i^\alpha = d_i^\alpha d_{j+1}^\beta \quad \text{if } i \leq j. \quad (24)$$

We define d_i^j as d_i^- if j is odd, and as d_i^+ if j is even, and we put

$$D_{i_1, \dots, i_k}^n = d_{i_1}^{n-i_1} \dots d_{i_k}^{n-i_k}$$

for all $1 \leq k \leq n$ and $1 \leq i_1 < \dots < i_k \leq n$. So, for example, $D_1^3 = d_1^+$, $D_2^3 = d_2^-$ and $D_{1,3}^3 = d_1^+ d_3^+$.

6.1.2. Theorem. Assume that $A = \mathbb{K} \oplus A_+$ is an augmented algebra, and that \mathcal{B} is a linear basis of A_+ . Setting $\text{Std}(\mathcal{B})_n = \mathcal{B}^{n+1}$ and, for every $u_0 | \dots | u_n$ in \mathcal{B}^{n+1} ,

$$s(u_0 | \dots | u_n) = \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k}^{n+1}(u_0 | \dots | u_n), \quad (25)$$

$$t(u_0 | \dots | u_n) = \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k}^n(u_0 | \dots | u_n), \quad (26)$$

defines a polygraphic resolution of A .

6.1.3. Remark. The formulas (25) and (26) become clearer when visualised on cubes. The general idea is that, in the boundary of an n -cell x of $\text{Std}(\mathcal{B})$, the terms corresponding to the cases $k = 1$ give the $2n$ faces $d_i^\alpha(x)$ of dimension $n - 1$ of x . The other terms, involving more than one face map d_i^α , are faces of lower dimensions, corresponding to the common boundaries of the $(n - 1)$ -faces. For example, if $u | v$ is a 1-cell, then $s(u | v) = d_1^-(u | v) = u \otimes v$ and $t(u | v) = d_1^+(u | v) = uv$. If $u | v | w$ is a 2-cell, then

$$\begin{aligned} s(u | v | w) &= d_1^+(u | v | w) + d_2^-(u | v | w) - d_1^+ d_2^-(u | v | w) = u | v \otimes w + uv | w - uv \otimes w \\ \text{and } t(u | v | w) &= d_1^-(u | v | w) + d_2^+(u | v | w) - d_1^- d_2^+(u | v | w) = u \otimes v | w + u | vw - u \otimes vw. \end{aligned}$$

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Then, noticing that $t(u|v \otimes w) = s(uv|w) = uv \otimes w$ and $t(u \otimes v|w) = s(u|vw) = u \otimes vw$, we deduce, using $a \star_0 b = a - t_0(a) + b$,

$$s(u|v|w) = u|v \otimes w \star_0 uv|w \quad \text{and} \quad t(u|v|w) = u \otimes v|w \star_0 u|vw.$$

So, $u|v|w$ has, indeed, the shape of a 2-cube, with the following two figures relating the face maps and their results:

$$\begin{array}{ccc} \begin{array}{ccc} d_2^- & \rightarrow & d_1^+ d_2^- \\ & \searrow & \downarrow u|v|w \\ d_1^- d_2^- & & d_1^+ d_2^+ \\ & \swarrow & \downarrow \\ d_1^- & \rightarrow & d_1^- d_2^+ \\ & \searrow & \downarrow \\ & & d_2^+ \end{array} & = & \begin{array}{ccc} u|v \otimes w & \rightarrow & uv \otimes w \\ & \searrow & \downarrow u|v|w \\ u \otimes v \otimes w & & uvw \\ & \swarrow & \downarrow \\ u \otimes v|w & \rightarrow & u \otimes vw \\ & \searrow & \downarrow \\ & & u|vw \end{array} \end{array}$$

A consequence of these observations is that the underlying 2-polygraph of $\text{Std}(\mathcal{B})$ is the standard coherent presentation $\text{Std}(\mathcal{B})_2$ of A , as given in Example 4.3.4.

Finally, similar computations prove that the source of a 3-cell $u|v|w|x$ of $\text{Std}(\mathcal{B})$, as given by (25), is equal to the composite

$$\begin{array}{ccc} & & uv|w \otimes x \\ & \nearrow & \searrow \\ u|v \otimes w \otimes x & & uvw \otimes x \\ \downarrow [u|v|w \otimes x] & & \downarrow [u|vw \otimes x] \\ u \otimes v \otimes w \otimes x & \rightarrow & u \otimes vw \otimes x \\ \downarrow [u \otimes v|w \otimes x] & & \downarrow [u|vw|x] \\ u \otimes v \otimes w|x & & u \otimes vw|x \\ \downarrow [u \otimes v|w|x] & & \downarrow [u \otimes vw|x] \\ u \otimes v \otimes wx & \rightarrow & u \otimes vwx \\ \downarrow [u \otimes v|wx] & & \downarrow [u|vwx] \\ & & u \otimes v|wx \end{array}$$

where we have framed the cells that appear explicitly in (25):

$$\begin{aligned} d_1^-(u|v|w|x) &= u \otimes v|w|x, & d_2^+(u|v|w|x) &= u|vw|x, & d_3^-(u|v|w|x) &= u|v|w \otimes x, \\ d_1^- d_2^+(u|v|w|x) &= u \otimes vw|x, & d_1^- d_3^-(u|v|w|x) &= u \otimes v|w \otimes x, & d_2^+ d_3^-(u|v|w|x) &= u|vw \otimes x, \\ d_1^- d_2^+ d_3^-(u|v|w|x) &= u \otimes vw \otimes x. \end{aligned}$$

Before giving the proof of Theorem 6.1.2, let us state a consequence for augmented algebras given with a convergent presentation.

6.1.4. Corollary. *Assume that A is an augmented algebra, and that X is a convergent left-monomial presentation of A . Then the algebra A admits a polygraphic resolution $\text{Std}(\text{Red}_m(X))$, whose n -cells are the $(n+1)$ -uples $u_0 | \cdots | u_n$ of nontrivial reduced monomials of $\mathcal{A}(X)$, with source and target*

$$s(u_0 | \cdots | u_n) = \sum_{\substack{0 \leq i \leq n+1 \\ n+1-i \text{ even}}} d_i(u_0 | \cdots | u_n) + s'(u_0 | \cdots | u_n)$$

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$$\text{and } t(u_0 | \cdots | u_n) = \sum_{\substack{0 \leq i \leq n+1 \\ n+1-i \text{ odd}}} d_i(u_0 | \cdots | u_n) + t'(u_0 | \cdots | u_n),$$

where $s'(u_0 | \cdots | u_n)$ and $t'(u_0 | \cdots | u_n)$ are identities, and where the maps d_i are defined by

$$d_i(u_0 | \cdots | u_n) = \begin{cases} u_0 \otimes u_1 | \cdots | u_n & \text{if } i = 0, \\ u_0 | \cdots | \widehat{u_{i-1} u_i} | \cdots | u_n & \text{if } 1 \leq i \leq n, \\ u_0 | \cdots | u_{n-1} \otimes u_n & \text{if } i = n + 1. \end{cases}$$

Proof. The algebra A being augmented, for reduced monomials u and v in $\mathcal{A}(X)$, the normal form \widehat{uv} belongs to the vector space spanned by nontrivial reduced monomials of $\mathcal{A}(X)$. By Theorem 3.4.2, the reduced monomials of $\mathcal{A}(X)$ form a linear basis $\text{Red}_m(X)$ of A , and the product in A of two reduced monomials u and v is given by $u \cdot v = \widehat{uv}$. Apply Theorem 6.1.2 to $\text{Red}_m(X)$ to obtain $\text{Std}(\text{Red}_m(X))$. Then, put $d_0 = d_0^-$, $d_i = d_i^+$ for $1 \leq i \leq n$, and $d_{n+1} = d_n^-$, and define $s'(u_0 | \cdots | u_n)$ and $t'(u_0 | \cdots | u_n)$ as the rest of the terms that occur in the source and target maps of $\text{Std}(\text{Red}_m(X))$. Finally, observe that $s'(u_0 | \cdots | u_n)$ and $t'(u_0 | \cdots | u_n)$ are identities because they are linear combinations of identities: each $d_i^-(u_0 | \cdots | u_n)$, for $1 \leq i < n$, and composites of two or more face maps. \square

Fix an augmented algebra $A = \mathbb{K} \oplus A_+$ and a linear basis \mathcal{B} of A_+ for the rest of the section, and define $\text{Std}(\mathcal{B})$ as in Theorem 6.1.2.

6.1.5. Lemma. *The source and target map given in Theorem 6.1.2 are linear maps that satisfy the globular relations, so that they equip $\text{Std}(\mathcal{B})$ with a structure of ∞ -polygraph.*

Proof. The source and target maps of $\text{Std}(\mathcal{B})$ are linear by definition. Now, let us check that they satisfy the globular relations. Let x be a fixed n -cell of $\text{Std}(\mathcal{B})$. If $k > 1$, then $D_{i_1, \dots, i_k}^{n+1}(x)$ has dimension strictly lower than $n - 2$, so that s_{n-2} is the identity on such a cell. So (25) give

$$s_{n-2}s_{n-1}(x) = \sum_{1 \leq i \leq n} s_{n-2}(d_i^{n+1-i}(x)) + \sum_{\substack{1 < k \leq n \\ 1 \leq i_1 < \cdots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k}^{n+1}(x).$$

Fix i in $\{1, \dots, n\}$. By definition,

$$s_{n-2}(d_i^{n+1-i}(x)) = \sum_{\substack{1 \leq k < n \\ 1 \leq j_1 < \cdots < j_k < n}} (-1)^{k+1} D_{j_1, \dots, j_k}^n d_i^{n+1-i}(x).$$

Now, fix $1 \leq k < n$ and $1 \leq j_1 < \cdots < j_k < n$. Let l be the smallest element of $\{1, \dots, k\}$ such that $i \leq j_l$, or $k + 1$ if $i > j_k$. Then, the cubical relations (24) imply

$$D_{j_1, \dots, j_k}^n d_i^{n+1-i} = D_{j_1, \dots, j_l}^n d_i^{n+1-i} D_{j_{l+1}, \dots, j_{k+1}}^{n+1}.$$

Then, for each $1 \leq p \leq k + 1$, we define i_p as j_p if $p < l$, as i if $p = l$, and as $j_p + 1$ if $p > l$, to obtain

$$D_{j_1, \dots, j_k}^n d_i^{n+1-i} = D_{i_1, \dots, i_{l-1}}^n D_{i_l, \dots, i_{k+1}}^{n+1}.$$

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As a consequence

$$s_{n-2}(d_i^{n+1-i}(x)) = \sum_{\substack{1 \leq l \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^k D_{i_1, \dots, i_{l-1}}^n D_{i_l, \dots, i_{k+1}}^{n+1}(x).$$

The terms corresponding to $l = 1$ cancel out with the remaining part of $s_{n-2}s_{n-1}(x)$, leaving, if we sum over $1 \leq l < k \leq n$ instead of $1 < l \leq k \leq n$,

$$s_{n-2}s_{n-1}(x) = \sum_{\substack{1 \leq l < k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^k D_{i_1, \dots, i_l}^n D_{i_{l+1}, \dots, i_k}^{n+1}(x).$$

Similar computations, starting with (26), lead to the same expression for $s_{n-2}t_{n-1}(x)$, so that $ss = st$. A similar argument proves that both $ts(x)$ and $tt(x)$ are equal to

$$\sum_{\substack{1 \leq l < k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^k D_{i_1, \dots, i_l}^{n+1} D_{i_{l+1}, \dots, i_k}^n(x). \quad \square$$

6.1.6. Proposition. *Let ι be the unital section of $\text{Std}(\mathcal{B})$ given by the inclusion of \mathcal{B} into $\mathcal{A}(\text{Std}(\mathcal{B}))$. The assignment $\sigma_{u_0 | \dots | u_n \otimes u_{n+1}} = u_0 | \dots | u_{n+1}$ defines a right ι -contraction of $\text{Std}(\mathcal{B})$.*

The proof of Proposition 6.1.6 uses the following lemma.

6.1.7. Lemma. *With the same hypotheses as in Proposition 6.1.6, fix $n \geq 1$, and assume that the assignment $\sigma_{u_0 | \dots | u_k \otimes u_{k+1}} = u_0 | \dots | u_{k+1}$ defines a right ι -contraction of $\text{Std}(\mathcal{B})$ up to dimension $n - 1$. Then, the relations*

$$\sigma_{D d_{n+1}^-(x)} - \sigma_{D d_n^+ d_{n+1}^-(x)} = D(x) - D d_n^+(x) \quad (27)$$

$$\text{and } \sigma_{D d_{n+1}^-(x)} - \sigma_{D d_n^- d_{n+1}^-(x)} = D(x) - D d_n^-(x) - D d_{n+1}^+(x) + D d_n^- d_{n+1}^+(x) \quad (28)$$

are satisfied for every $D = d_{i_1}^{\alpha_1} \dots d_{i_k}^{\alpha_k}$, with $1 \leq i_1 < \dots < i_k < n$ and $\alpha_1, \dots, \alpha_k$ in $\{-, +\}$, and every $(n+1)$ -cell $x = u_0 | \dots | u_{n+1}$ of $\text{Std}(\mathcal{B})$.

Proof. We distinguish two cases. First, assume that every α_i is a $+$. Then, on the one hand, $D d_{n+1}^-(x)$ has shape $v_0 | \dots | v_p | u_n \otimes u_{n+1}$, so that

$$\sigma_{D d_{n+1}^-(x)} = \sigma_{v_0 | \dots | v_p | u_n \otimes u_{n+1}} = v_0 | \dots | v_p | u_n | u_{n+1} = D(x)$$

$$\text{and } \sigma_{D d_n^+ d_{n+1}^-(x)} = \sigma_{v_0 | \dots | v_p u_n \otimes u_{n+1}} = v_0 | \dots | v_p u_n | u_{n+1} = D d_n^+(x).$$

And, on the other hand, since σ is a right ι -contraction up to dimension $n-1$, and $\iota\pi(u_n \otimes u_{n+1}) = u_n u_{n+1}$, we have

$$\begin{aligned} \sigma_{D d_n^- d_{n+1}^-(x)} &= \sigma_{v_0 | \dots | v_p \otimes u_n \otimes u_{n+1}} \\ &= v_0 | \dots | v_p \otimes u_n | u_{n+1} - v_0 | \dots | v_p \otimes u_n u_{n+1} + v_0 | \dots | v_p | u_n u_{n+1} \\ &= D d_n^-(x) - D d_n^- d_{n+1}^+(x) + D d_{n+1}^+(x). \end{aligned}$$

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Otherwise, write $D = D_1 d_i^- D_2$, where the indices occurring in D_1 are strictly smaller than i , the ones that occur in D_2 are strictly greater than i , and all exponents that occur in D_2 are $+s$. Put

$$\begin{aligned} y &= D_1(u_0 | \cdots | u_i), & v_0 | \cdots | v_p &= D_2(u_{i+1} | \cdots | u_{n-1}), \\ z &= y \otimes v_0 \cdots v_p u_n u_{n+1} - \sigma_{y \otimes v_0 \cdots v_p} u_n u_{n+1}. \end{aligned}$$

The fact that σ is a right ι -contraction implies, on the one hand,

$$\begin{aligned} \sigma_{D d_{n+1}^-}(x) &= \sigma_{y \otimes v_0 | \cdots | v_p | u_n \otimes u_{n+1}} = y \otimes v_0 | \cdots | v_p | u_n | u_{n+1} - z = D(x) - z \\ \text{and } \sigma_{D d_n^+ d_{n+1}^-}(x) &= \sigma_{y \otimes v_0 | \cdots | v_p u_n \otimes u_{n+1}} = y \otimes v_0 | \cdots | v_p u_n | u_{n+1} - z = D d_n^+(x) - z, \end{aligned}$$

and, on the other hand,

$$\begin{aligned} \sigma_{D d_n^- d_{n+1}^-}(x) &= \sigma_{y \otimes v_0 | \cdots | v_p \otimes u_n \otimes u_{n+1}} \\ &= y \otimes v_0 | \cdots | v_p \otimes u_n | u_{n+1} - y \otimes v_0 | \cdots | v_p \otimes u_n u_{n+1} \\ &\quad + y \otimes v_0 | \cdots | v_p | u_n u_{n+1} - z \\ &= D d_n^-(x) - D d_n^- d_{n+1}^+(x) + D d_{n+1}^+(x) - z. \end{aligned} \quad \square$$

Proof of Proposition 6.1.6. We proceed by induction on the dimension. By definition, $\iota\tau(u_0 \otimes \cdots \otimes u_n) = u_0 \cdots u_n$, so that the ι -reduced 0-cells of $\mathcal{A}(\text{Std}(\mathcal{B}))$ are the 0-cells of $\text{Std}(\mathcal{B})$; as a consequence, the ι -essential monomials of $\mathcal{A}(\text{Std}(\mathcal{B}))$ are the monomials $u \otimes v$, for u and v in \mathcal{B} . Define σ on 0-cells as the unique right ι -contraction that satisfies $\sigma_{u \otimes v} = u|v$, which is legitimate since the equalities $s(\sigma_{u \otimes v}) = u \otimes v$ and $t(\sigma_{u \otimes v}) = \widehat{u \otimes v} = uv$ hold.

Now, fix $n \geq 1$, and assume that the assignment $\sigma_{u_0 | \cdots | u_k \otimes u_{k+1}} = u_0 | \cdots | u_{k+1}$ defines a right ι -contraction of $\text{Std}(\mathcal{B})$ up to dimension $n-1$. Fix an n -cell $u_0 | \cdots | u_n$ of $\text{Std}(\mathcal{B})$ and an element u_{n+1} of \mathcal{B} . Note that the n -monomial $u_0 | \cdots | u_n \otimes u_{n+1}$ is not in the image of σ , so that it is σ -essential, and put $\sigma_{u_0 | \cdots | u_n \otimes u_{n+1}} = u_1 | \cdots | u_{n+1}$. To ensure that this defines a unique right ι -contraction on the n -cells of $\text{Std}(\mathcal{B})$, we must check

$$s(u_0 | \cdots | u_{n+1}) = u_0 | \cdots | u_n \otimes u_{n+1} - t(u_0 | \cdots | u_n) \otimes u_{n+1} + \sigma_{t(u_0 | \cdots | u_n) \otimes u_{n+1}} \quad (29)$$

$$\text{and } t(u_0 | \cdots | u_{n+1}) = \sigma_{s(u_0 | \cdots | u_n) \otimes u_{n+1}}. \quad (30)$$

Put $x = u_0 | \cdots | u_{n+1}$. Let us compute each one of the three terms of the right-hand side of (29). First, we note that

$$u_0 | \cdots | u_n \otimes u_{n+1} = d_{n+1}^-(x) = D_{n+1}^{n+2}(x).$$

Next, we use (26), and $D_{i_1, \dots, i_k}^n d_{n+1}^- = D_{i_1, \dots, i_k, n}^n$ to obtain

$$\begin{aligned} t(u_0 | \cdots | u_n) \otimes u_{n+1} &= \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \cdots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k}^n (u_0 | \cdots | u_n) \otimes u_{n+1} \\ &= \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \cdots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k}^n d_{n+1}^-(x). \end{aligned}$$

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$$= \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k, n+1}^n(\mathbf{x}).$$

Finally, for the last term, we use (26), split the sum in two parts depending if $i_k = n$ or not (changing the bounds of the indices appropriately), apply (27) to each D_{i_1, \dots, i_k} , and merge the sums again, to obtain

$$\begin{aligned} \sigma_{t(u_0 | \dots | u_n) \otimes u_{n+1}} &= \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^{k+1} \sigma_{D_{i_1, \dots, i_k}^n d_{n+1}^-(\mathbf{x})} \\ &= \sum_{\substack{1 \leq k < n \\ 1 \leq i_1 < \dots < i_k < n}} (-1)^{k+1} \left(\sigma_{D_{i_1, \dots, i_k}^n d_{n+1}^-(\mathbf{x})} - \sigma_{D_{i_1, \dots, i_k}^n d_n^+ d_{n+1}^-(\mathbf{x})} \right) \\ &= \sum_{\substack{1 \leq k < n \\ 1 \leq i_1 < \dots < i_k < n}} (-1)^{k+1} \left(D_{i_1, \dots, i_k}^n(\mathbf{x}) - D_{i_1, \dots, i_k}^n d_n^+(\mathbf{x}) \right) \\ &= \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^{k+1} D_{i_1, \dots, i_k}^n(\mathbf{x}). \end{aligned}$$

We note that $D_{i_1, \dots, i_k}^n = D_{i_1, \dots, i_k}^{n+2}$, and sum the three results to obtain that the right-hand side of (29) is indeed equal to $s(u_0 | \dots | u_{n+1})$ as given by (25).

Now, let us compute the right-hand side of (30). Here, we use (25), split the sum in two parts depending if $i_k = n$ or not, apply (28), and merge the sums again, to obtain

$$\begin{aligned} \sigma_{s(u_0 | \dots | u_n) \otimes u_{n+1}} &= \sum_{\substack{1 \leq k \leq n \\ 1 \leq i_1 < \dots < i_k \leq n}} (-1)^{k+1} \sigma_{D_{i_1, \dots, i_k}^{n+1} d_{n+1}^-(\mathbf{x})} \\ &= \sum_{\substack{1 \leq k < n \\ 1 \leq i_1 < \dots < i_k < n}} (-1)^{k+1} \left(\sigma_{D_{i_1, \dots, i_k}^{n+1} d_{n+1}^-(\mathbf{x})} - \sigma_{D_{i_1, \dots, i_k}^{n+1} d_n^- d_{n+1}^-(\mathbf{x})} \right) \\ &= \sum_{\substack{1 \leq k < n \\ 1 \leq i_1 < \dots < i_k < n}} (-1)^{k+1} \left(\begin{array}{c} D_{i_1, \dots, i_k}^{n+1}(\mathbf{x}) - D_{i_1, \dots, i_k}^{n+1} d_n^-(\mathbf{x}) \\ -D_{i_1, \dots, i_k}^{n+1} d_{n+1}^+(\mathbf{x}) + D_{i_1, \dots, i_k}^{n+1} d_n^- d_{n+1}^+(\mathbf{x}) \end{array} \right) \\ &= \sum_{\substack{1 \leq k \leq n+1 \\ 1 \leq i_1 < \dots < i_k \leq n+1}} (-1)^{k+1} D_{i_1, \dots, i_k}^{n+1}(\mathbf{x}). \end{aligned}$$

So (30) holds, which concludes the proof. \square

Proof of Theorem 6.1.2. Lemma 6.1.5 states that the definition of $\text{Std}(\mathcal{B})$ makes sense. Then, as noted in Remark 6.1.3, the 1-polygraph underlying $\text{Std}(\mathcal{B})$ is a presentation of the algebra A . Finally, Proposition 6.1.6 exhibits a right contraction of $\text{Std}(\mathcal{B})$, so that Lemma 5.2.5 and Theorem 5.2.6 conclude the proof of Theorem 6.1.2. \square

6.2. Squier's polygraphic resolution

6.2.1. The rightmost rewriting step. Assume that X is a reduced convergent left-monomial 1-polygraph, and let u be a monomial of $\mathcal{A}(X)$. Because X is reduced, there is at most one rewriting step of source u in $\mathcal{A}(X)$ for each submonomial of u . As a consequence, the set of rewriting steps of $\mathcal{A}(X)$ of source u is finite. Moreover, this set, extended with the identity 1_u , is totally ordered by $f \sqsubseteq g$ if $f = 1_u$, or if, writing $f = v\alpha v'$ and $g = w\beta w'$, the length of v is smaller than the length of w . The maximal element for this order is denoted by ρ_u . By definition, we have $\rho_u = 1_u$ if, and only if, u is reduced. Furthermore, if u and v are monomials of $\mathcal{A}(X)$, then the inequalities $\rho_u v \sqsubseteq \rho_{uv_1} v_2 \sqsubseteq \rho_{uv}$ hold for every factorisation $v = v_1 v_2$.

6.2.2. Critical, subcritical and supercritical cells. Fix an augmented algebra A , and a reduced convergent left-monomial presentation X of A . Let $u_0 | \cdots | u_n$ be an n -cell of $\text{Std}(\text{Red}_m(X))$. For i in $\{0, \dots, n\}$, define $\delta_i(u_0 | \cdots | u_n)$ as the pair (v_i, w_i) of (possibly trivial) monomials of $\mathcal{A}(X)$ such that $u_i = v_i w_i$ and

- (i) if $i = 0$, then v_0 is of length 1,
- (ii) if $i \geq 1$, then v_i is the shortest left-factor of u_i that satisfies $\rho_{u_0 \cdots u_{i-1} v_i} w_i = \rho_{u_0 \cdots u_i}$.

Define i -critical, i -subcritical and i -supercritical n -cells of $\text{Std}(\text{Red}_m(X))$ by induction on $0 \leq i \leq n$ as follows. Fix $0 \leq i \leq n$, and assume that $x = u_0 | \cdots | u_n$ is j -critical for every $0 \leq j < n$. Call x

- (i) *i-subcritical* if $\delta_i(x) = (1, u_i)$,
- (ii) *i-critical* if $\delta_i(x) = (u_i, 1)$,
- (iii) *i-supercritical* otherwise.

An n -cell of $\text{Std}(\text{Red}_m(X))$ is *critical* if it is i -critical for every $0 \leq i \leq n$, and *subcritical* (resp. *supercritical*) if it is i -subcritical (resp. i -supercritical) for some $0 \leq i \leq n$.

6.2.3. Remark. A nontrivial monomial u_0 of $\mathcal{A}(X)$ is either critical (if it is of length 1) or supercritical (otherwise), but never subcritical.

Let $u_0 | u_1$ be a 1-cell of $\text{Std}(\text{Red}_m(X))$ such that u_0 is critical. By definition of ρ , we have the inequality $\rho_{u_0} u_1 \sqsubseteq \rho_{u_0 u_1}$, with $\rho_{u_0} u_1 = 1_{u_0 u_1}$ by hypothesis on u_0 . Then $u_0 | u_1$ is subcritical if, and only if, the inequality is an equality, i.e. $u_0 u_1$ is reduced. Otherwise, u_1 being reduced, $\rho_{u_0 u_1} = \alpha w$ for a 1-cell α of X and a right-factor w of u_1 : then $u_0 | u_1$ is critical if $w = 1$, and supercritical otherwise.

Finally, assume that $x = u_0 | u_1 | u_2$ is a 2-cell of $\text{Std}(\text{Red}_m(X))$, with $u_0 | u_1$ critical. Denote by b the branching $(\rho_{u_0 u_1} u_2, \rho_{u_0 u_1 u_2})$ of X . Then, by definition, x is subcritical if, and only if, $\rho_{u_0 u_1} u_2 = \rho_{u_0 u_1 u_2}$, i.e. b is aspherical. Otherwise, u_2 being reduced, the branching b is an overlap of shape $(\alpha u_2, w\beta w')$. Then x is critical in the case $w' = 1$, i.e. when the branching b is critical, and supercritical otherwise.

6.2.4. Theorem (Squier's polygraphic resolution). Let A be an augmented algebra and X be a reduced convergent left-monomial presentation of A . Define $\text{Sq}(X)$ to be the indexed subset of $\text{Std}(\text{Red}_m(X))$ consisting of its critical cells, and φ as the indexed partial map of degree 1 from $\text{Sq}(X)$ to itself, given by

$$\varphi(u_0 | \cdots | u_n) = u_0 | \cdots | u_{i-1} | v_i | w_i | u_{i+1} | \cdots | u_n$$

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for every i -supercritical n -cell $u_0 | \cdots | u_n$ with $\delta_i(u_0 | \cdots | u_n) = (v_i, w_i)$.

Then φ is a collapsing scheme of $\text{Std}(\text{Red}_m(X))$ onto $\text{Sq}(X)$. As a consequence, $\text{Sq}(X)$, equipped with the structure of ∞ -polygraph induced by φ , is a polygraphic resolution of A .

Proof. First, φ induces a bijection between the i -supercritical n -cells and the $(i+1)$ -subcritical $(n+1)$ -cells of $\text{Std}(\text{Red}_m(X))$. Indeed, if $u_0 | \cdots | u_{n+1}$ is an $(i+1)$ -subcritical $(n+1)$ -cell, put $\psi(u_0 | \cdots | u_{n+1}) = u_0 | \cdots | u_i u_{i+1} | \cdots | u_n$. This assignment makes sense because, if $\delta_{i+1}(u_0 | \cdots | u_{n+1}) = (1, u_{i+1})$, then $u_i u_{i+1}$ is reduced, hence $u_0 | \cdots | u_i u_{i+1} | \cdots | u_n$ is an i -supercritical n -cell. Moreover, ψ is inverse to φ .

Second, the partition $\text{Std}(\text{Red}_m(X)) = \text{im}(\varphi) \amalg \text{Sq}(X) \amalg \text{dom}(\varphi)$ is obtained by observing that $\text{Sq}(X)$, $\text{dom}(\varphi)$ and $\text{im}(\varphi)$ respectively consist of the critical, supercritical and subcritical cells of $\text{Std}(\text{Red}_m(X))$.

Third, let $u_0 | \cdots | u_n$ be an i -subcritical n -cell of $\text{Std}(\text{Red}_m(X))$. Note that the definition of φ implies $u_0 | \cdots | u_n = \varphi(u_0 | \cdots | u_i u_{i+1} | \cdots | u_n)$. By Corollary 6.1.4, the boundary of $u_0 | \cdots | u_n$ satisfies

$$\partial(u_0 | \cdots | u_n) = \sum_{0 \leq j \leq n+1} (-1)^{n+1-j} d_j(u_0 | \cdots | u_n) + \partial'(u_0 | \cdots | u_n),$$

where $\partial'(u_0 | \cdots | u_n)$ is an identity. Observe that $u_0 | \cdots | u_i u_{i+1} | \cdots | u_n = d_{i+1}(u_0 | \cdots | u_n)$, and that this $(n-1)$ -cell is distinct from each other $(n-1)$ -cell of $\text{Std}(\text{Red}_m(X))$ that appears in $\partial(u_0 | \cdots | u_n)$. Hence, $\partial(u_0 | \cdots | u_n)$ has the required form.

Finally, check that the relation \triangleright_φ induced by φ is wellfounded, by proving that it is included into a wellfounded order \triangleright . Put $u_0 | \cdots | u_n \triangleright v_0 | \cdots | v_n$ if either

- (i) $v_0 \cdots v_n$ is a proper submonomial of $u_0 \cdots u_n$, or
- (ii) $u_0 \cdots u_n \succ_X \alpha$, with α an n -cell of $\mathcal{A}(\text{Std}(\text{Red}_m(X)))$ such that $v_0 | \cdots | v_n$ belongs to $\text{supp}(\alpha)$, or
- (iii) $u_0 \cdots u_n = v_0 \cdots v_n$, and there exists i in $\{0, \dots, n\}$ such that $u_0 = v_0, \dots, u_{i-1} = v_{i-1}$, and $l(u_i) > l(v_i)$.

The order relation \triangleright so defined is wellfounded as a lexicographic product of wellfounded orders. Now, let $u_0 | \cdots | u_n$ be an i -subcritical n -cell of $\text{Std}(\text{Red}_m(X))$, and prove $u_0 | \cdots | u_i u_{i+1} | \cdots | u_n \triangleright y$ for every other $(n-1)$ -cell y in $\text{supp}(\partial(u_0 | \cdots | u_n))$, that is, in the support of a $d_j(u_0 | \cdots | u_n)$, for a $j \neq i+1$. Consider the possible values of j .

If $j = 0$ or $j = n+1$, the only possible $(n-1)$ -cells for y are $u_1 | \cdots | u_n$ or $u_0 | \cdots | u_{n-1}$. These two $(n-1)$ -cells are proper submonomials of $u_0 \cdots u_n$, so that Condition (i) applies.

If $1 \leq j \leq n$, then y is in the support of $u_1 | \cdots | \widehat{u_j u_{j+1}} | \cdots | u_n$, i.e. $y = u_0 | \cdots | u_{j-1} v | u_{j+2} | \cdots | u_n$, for $v \in \text{supp}(\widehat{u_j u_{j+1}})$. If $u_j u_{j+1}$ is not reduced, then $u_0 \cdots u_{j-1} v u_{j+2} \cdots u_n$ is not a proper submonomial of $u_0 \cdots u_n$, because, otherwise, X would not terminate, so that Condition (ii) applies. On the contrary, if $u_j u_{j+1}$ is reduced, then none of Conditions (i) and (ii) is satisfied, but Condition (iii) applies, because $u_0 | \cdots | u_{i-1}$ critical implies that $u_k u_{k+1}$ is nonreduced for each $k < i$, so that $j > i+1$ holds. \square

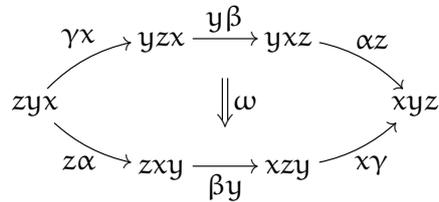
6.3. Examples: the symmetric algebra and variations

6.3.1. Example. Consider the symmetric algebra on three generators x, y and z , presented by

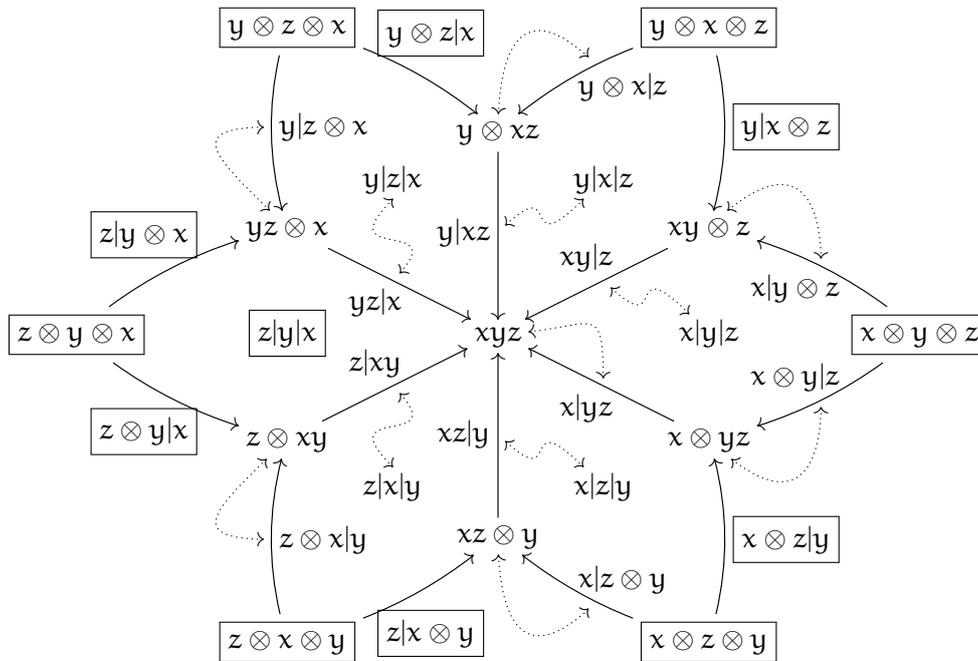
$$\text{Sym}(x, y, z) = \langle x, y, z \mid yx \xrightarrow{\alpha} xy, zx \xrightarrow{\beta} xz, zy \xrightarrow{\gamma} yz \rangle.$$

6.3. Examples: the symmetric algebra and variations

The 1-polygraph $\text{Sym}(x, y, z)$ is convergent, with one critical branching of source zyx , giving rise, by Theorem 4.3.2, to an acyclic extension with exactly one 2-cell:



Note that $\text{Sym}(x, y, z)$ is also reduced, so that Theorem 6.2.4 applies to it: we obtain a polygraphic resolution $\text{Sq}(\text{Sym}(x, y, z))$ with a unique 2-cell and no n -cell for $n \geq 3$, and the boundary of the 2-cell is obtained by collapsing the standard polygraphic resolution, through the collapsing scheme φ of Theorem 6.2.4. The cells required to understand the computation are drawn on the following figure:



The identifications induced by φ are given by the dotted arrows, and the cells that belong to $\text{Sq}(\text{Sym}(x, y, z))$ are boxed. After the collapse, by identification of $y|x$, $z|x$ and $z|y$ to α , β and γ , respectively, and by omission of the \otimes sign, we recover the former 2-cell ω as $z|y|x$.

6.3.2. Example. Consider the symmetric algebra on a set X of generators: given a fixed linear order on X , it admits a convergent presentation $\text{Sym}(X)$ whose relations are the $\alpha_{xy} : yx \rightarrow xy$, for all $x < y$ in X . The polygraphic resolution obtained by Theorem 6.2.4 has exactly one n -cell $\omega_{x_0 \dots x_n}$ for all $x_0 < \dots < x_n$ of X , that corresponds to the critical n -cell $x_n | \dots | x_0$ of the standard resolution. One can compute that Squier's resolution $\text{Sq}(\text{Sym}(X))$ has one 3-cell ω_{xyz} for each triple $x < y < z$ of elements of X , similar to the permutohedron 3-cell ω of Example 6.3.1, and one 4-cell ω_{xyzt} for each quadruple $x < y < z < t$,

6.3. Examples: the symmetric algebra and variations

Observe that $d_0(x_n | \cdots | x_0)$ and $d_{n+1}(x_n | \cdots | x_0)$ belong to $\mathcal{A}(\text{Sq}(\text{Sym}(X)))$, because $x_{n-1} | \cdots | x_0$ and $x_n | \cdots | x_1$ are critical, whereas, for $1 \leq i \leq n$, the $(n-1)$ -cell $d_i(x_n | \cdots | x_0)$ is supercritical. Now, fix $1 \leq i \leq n$. By definition of π , the $(n-1)$ -cell $x_n | \cdots | x_i x_{i+1} | \cdots | x_0$ is mapped to the $(n-1)$ -cell c such that

$$x_n | \cdots | x_i x_{i+1} | \cdots | x_0 = \partial(\varphi(x_n | \cdots | x_i x_{i+1} | \cdots | x_0)) + c.$$

Note that $\varphi(x_n | \cdots | x_i x_{i+1} | \cdots | x_0) = x_n | \cdots | x_i | x_{i+1} | \cdots | x_0$. As before, the boundary of the latter n -cell is an alternated sum of $d_j(x_n | \cdots | x_i | x_{i+1} | \cdots | x_0)$, for $0 \leq j \leq n+1$, plus an identity. A careful examination of the different possibilities shows that, for $j > i+1$ or $j < i-1$, the $(n-1)$ -cell $d_j(x_n | \cdots | x_i | x_{i+1} | \cdots | x_0)$ is subcritical: so, π maps it to an identity. Repeating the process for each of $j = i+1$ and $j = i-1$ gives, by induction respectively upwards from $i+1$ to $n+1$ and downwards from $i-1$ to 0 , that

$$\pi(d_{i+1}(x_n | \cdots | x_i | x_{i+1} | \cdots | x_0)) = x_i \otimes x_n | \cdots | \widehat{x_i} | \cdots | x_0 + a$$

$$\text{and } \pi(d_{i-1}(x_n | \cdots | x_i | x_{i+1} | \cdots | x_0)) = x_n | \cdots | \widehat{x_{i+1}} | \cdots | x_0 \otimes x_{i+1} + b,$$

where a and b are some identities, and $\widehat{x_i}$ means that x_i is omitted (and not a normal form). So, we have

$$\pi(d_i(x_n | \cdots | x_i | x_{i+1} | \cdots | x_0)) = x_i \otimes x_n | \cdots | \widehat{x_i} | \cdots | x_0 + x_n | \cdots | \widehat{x_{i+1}} | \cdots | x_0 \otimes x_{i+1} + a$$

for some identity a . Wrapping up the results, we obtain, for some identity a :

$$\partial(\omega_{x_0 \cdots x_n}) = \sum_{0 \leq i \leq n+1} (-1)^{i+1} (x_i \omega_{x_0 \cdots \widehat{x_i} \cdots x_n} - \omega_{x_0 \cdots \widehat{x_i} \cdots x_n} x_i) + a. \quad (31)$$

6.3.3. Example. Consider the quantum deformation of the symmetric algebra, whose generators commute up to some constant parameters, i.e. $yx = q_{xy}xy$, where q_{xy} is a scalar. This algebra, on a set X of generators equipped with a linear order, is presented by the convergent 1-polygraph

$$\langle X \mid (yx \xrightarrow{\alpha_{xy}} q_{xy}xy)_{x < y} \rangle.$$

Squier's resolution, given by Theorem 6.2.4, is similar to the one of the symmetric algebra of Example 6.3.2. For example, it has one 2-cell ω_{xyz} for each triple $x < y < z$ of generators:

$$\begin{array}{ccccc} & & \alpha_{yzx} & & \\ & & \nearrow & & \\ zy & & q_{yz}yZX & \xrightarrow{q_{yz}y\alpha_{xz}} & q_{xz}q_{yz}yXZ & \xrightarrow{q_{xz}q_{yz}\alpha_{xy}z} & q_{xy}q_{xz}q_{yz}xyZ \\ & & \searrow & & \Downarrow \omega_{xyz} & & \\ & & z\alpha_{xy} & & q_{xy}q_{xz}XZY & \xrightarrow{q_{xy}q_{xz}\alpha_{zy}} & q_{xy}q_{xz}X\alpha_{yz} \end{array}$$

6.3.4. Example. Consider the exterior algebra on a set X , equipped with an arbitrary linear order. As an associative algebra, it is presented by the 1-polygraph

$$\langle X \mid (yx \xrightarrow{\alpha_{xy}} -xy)_{x < y}, (xx \xrightarrow{\delta_x} 0)_x \rangle.$$

7.1. Free bimodules resolutions from polygraphic resolutions

7.1.1. Free bimodules. Let X be an ∞ -polygraph, and A be the algebra presented by the underlying 1-polygraph of X . Denote by $A^e = A^o \otimes A$ the enveloping algebra of A , and, for $n \geq 0$, by $A^e[X_n]$ the free A -bimodule over X_n . An element $a \otimes \alpha \otimes b$ of $A^e[X_n]$ is written $a[\alpha]b$. The inclusion $\alpha \mapsto [\alpha]$ of X_n into $A^e[X_n]$ is extended to all the n -cells of $\mathcal{A}(X)$ as follows. In dimension 0, define $[u]$ for every monomial u of $\mathcal{A}(X)$, by induction on the length of u , by

$$[1] = 0 \quad \text{and} \quad [uv] = [u]\bar{v} + \bar{u}[v],$$

and, then, extend the bracket to all 0-cells by linearity. In dimension $n \geq 1$, define the bracket on identities and on n -monomials by

$$[1_a] = 0 \quad \text{and} \quad [u\alpha v] = \bar{u}[\alpha]\bar{v}.$$

This definition, extended by linearity to linear combinations of identities and n -monomials is compatible with (7) in Theorem 1.3.3, yielding a well-defined linear map on the n -cells of $\mathcal{A}(X)$. Then (4) and (5) in Proposition 1.2.3 imply, for any possible n -cells a and b in $\mathcal{A}(X)$:

$$[a \star_k b] = [a] + [b] \quad \text{and} \quad [a^-] = -[a].$$

7.1.2. The chain complex of a polygraph. Assume that X is an ∞ -polygraph, and let A be the algebra presented by X . Let $A^e[X]$ be the chain complex of A -bimodules

$$0 \longleftarrow A \xleftarrow{\mu} A^e \xleftarrow{\delta_0} A^e[X_0] \xleftarrow{\delta_1} \dots \longleftarrow A^e[X_k] \xleftarrow{\delta_{k+1}} A^e[X_{k+1}] \longleftarrow \dots$$

with $\mu(a \otimes b) = ab$, and, for every $k \geq 0$,

$$\delta_k[\alpha] = \begin{cases} 1 \otimes \alpha - \alpha \otimes 1 & \text{if } k = 0, \\ [s_{k-1}(\alpha)] - [t_{k-1}(\alpha)] & \text{if } k \geq 1. \end{cases}$$

Let us check that $A^e[X]$ is indeed a chain complex. First, prove by induction on the length that $\delta_0([u]) = 1 \otimes u - u \otimes 1$ holds for every monomial u of $\mathcal{A}(X)$, yielding $\mu\delta_0 = 0$. Then, for every $(n+1)$ -cell α of X , with $n \geq 1$, the globular relations imply

$$\delta_n\delta_{n+1}[\alpha] = [s_{n-1}s_n(\alpha)] + [t_{n-1}s_n(\alpha)] - [s_{n-1}t_n(\alpha)] - [t_{n-1}t_n(\alpha)] = 0.$$

7.1.3. Theorem. *If X is a polygraphic resolution of an algebra A , then the complex $A^e[X]$ is a free resolution of the A -bimodule A . Moreover, if X is of finite type, then so is $A^e[X]$.*

Proof. Fix a unital section ι of X , writing \hat{a} for $\iota(a)$, and a right ι -contraction σ of X , thanks to Proposition 5.2.6. Using σ , let us construct a contracting homotopy h of $A^e[X]$. Define $h_{-1} : A \rightarrow A^e$ and $h_0 : A^e \rightarrow A^e[X_0]$ by

$$h_{-1}(a) = a \otimes 1 \quad \text{and} \quad h_0(a \otimes b) = a[\hat{b}],$$

for all a and b in A , and, for $n \geq 1$, define a morphism of A -modules $h_n : A^e[X_{n-1}] \rightarrow A^e[X_n]$ by

$$h_n([\alpha]a) = [\sigma_{\alpha\hat{a}}],$$

7. Free resolutions of associative algebras

for every a in A , and every $(n-1)$ -cell α of X .

Now, check that $h_{n+1}([a]b) = [\sigma_{\widehat{ab}}]$ holds for every n -cell a of $\mathcal{A}(X)$, and every b in A . If $a = u$ is a monomial of $\mathcal{A}(X)$, we prove, by induction on the length of u , that $h_1([u]a) = [\sigma_{\widehat{ua}}]$ holds. If $u = 1$, then both members of the equality are 0. If u and v are monomials of $\mathcal{A}(X)$, then, on the one hand, by definition of the bracket,

$$h_1([uv]a) = \bar{u}[\sigma_{\widehat{va}}] + [\sigma_{\widehat{uv}a}],$$

and, on the other hand, using the fact that σ is a right ι -contraction,

$$[\sigma_{\widehat{uv}a}] = \bar{u}[\sigma_{\widehat{va}}] + [\sigma_{\widehat{uv}a}].$$

For $n \geq 1$, the n -cells of $\mathcal{A}(X)$ are linear combinations of the n -monomials of $\mathcal{A}(X)$ and of an identity n -cell of $\mathcal{A}(X)$. Suppose that a is an n -monomial $u\alpha v$, where u and v are monomials of $\mathcal{A}(X)$ and α is an n -cell of X . On the one hand, we have

$$h_{n+1}([u\alpha v]a) = h_{n+1}(\bar{u}[\alpha]\bar{v}a) = \bar{u}[\sigma_{\widehat{\alpha v}a}].$$

On the other hand, (16) gives, with $b = s_0(\alpha)$,

$$\sigma_{u\alpha v\widehat{a}} = \mathbf{u}b\sigma_{v\widehat{a}} \star_0 \mathbf{u}\sigma_{\widehat{\alpha v}a} \star_0 \sigma_{\widehat{ubv}a},$$

from which we obtain, using the fact that $\sigma_{v\widehat{a}}$ and $\sigma_{\widehat{ubv}a}$ are identities,

$$[\sigma_{u\alpha v\widehat{a}}] = \bar{\mathbf{u}}\bar{b}[\sigma_{v\widehat{a}}] + \bar{u}[\sigma_{\widehat{\alpha v}a}] + [\sigma_{\widehat{ubv}a}] = \bar{u}[\sigma_{\widehat{\alpha v}a}].$$

Finally, prove that h is a contracting homotopy for the complex $A^e[X]$. We have $h_{-1}\mu(a \otimes b) = ab \otimes 1$ and $\delta_0 h_0(a \otimes b) = a \otimes b - ab \otimes 1$, thus $h_{-1}\mu + \delta_0 h_0 = \text{Id}_{A^e}$. Then, for every $[x]a$ in $A^e[X_0]$, we have $h_0 \delta_0([x]a) = [\widehat{x\widehat{a}}] - \bar{x}[\widehat{a}]$ and

$$\delta_1 h_1([x]a) = \delta_1([\sigma_{\widehat{x\widehat{a}}}] = [\widehat{x\widehat{a}}] - [\widehat{x\widehat{a}}] = \bar{x}[\widehat{a}] + [x]a - [\widehat{x\widehat{a}}],$$

which give $h_0 \delta_0 + \delta_1 h_1 = \text{Id}_{A^e[X_0]}$. Now, for $n \geq 1$ and $[\alpha]a$ be in $A^e[X_n]$, we have

$$\begin{aligned} \delta_n h_{n+1}([\alpha]a) &= \delta_n([\sigma_{\widehat{\alpha\widehat{a}}}] = [\alpha\widehat{a}] + [\sigma_{t_{n-1}(\alpha)\widehat{a}}] - [\sigma_{s_{n-1}(\alpha)\widehat{a}}], \\ &= [\alpha]a + h_{n-1}([t_{n-1}(\alpha)]a) - h_{n-1}([s_{n-1}(\alpha)]a). \end{aligned}$$

Thus $h_{n-1}\delta_{n-1} + \delta_n h_n = \text{Id}_{A^e[X_n]}$. □

7.1.4. Example. Consider the polygraphic resolution $\text{Sq}(\text{Sym}(X))$ of the symmetric algebra on a set X , obtained in Example 6.3.2. The associated free resolution of the symmetric algebra is generated by one element $[\omega_{x_0 \dots x_n}]$ for every tuple $x_n > \dots > x_0$ in X . From (31), we deduce

$$\delta_n[\omega_{x_0 \dots x_n}] = \sum_{0 \leq i \leq n+1} (-1)^{i+1} (x_i[\omega_{x_0 \dots \widehat{x}_i \dots x_n}] - [\omega_{x_0 \dots \widehat{x}_i \dots x_n}]x_i).$$

Thus, we recover (up to the sign of the differential) the usual Koszul bimodule complex of the symmetric algebra over X . In the case of a quantum deformation of the symmetric algebra of Example 6.3.3, we get the quantum version of the Koszul complex, obtained for instance by Wambst in [58].

7.1.5. Remark. The construction of the resolution $A^e[X]$ of A -bimodules can be adapted to obtain a resolution of \mathbb{K} by right A -modules. For $n \geq 0$, denote by $A^o[X_n]$ the free right A -module generated by X_n . The mapping inclusion $\alpha \mapsto [\alpha]$ of X_n into $A^o[X_n]$ is extended to all the n -cells of $\mathcal{A}(X)$ by setting

$$[1] = 0, \quad [xu] = [x]\bar{u} + [u], \quad [1_u] = 0, \quad [x\alpha] = 0, \quad [\alpha u] = [\alpha]\bar{u},$$

for every 0-cell x of X , every n -cell α of X , with $n \geq 1$, and every monomial u of $\mathcal{A}(X)$. The differentials are then defined in the same way as the ones of $A^e[X]$, except for $\delta_0[x] = x$ for every 0-cell x of X . A symmetric construction would give a resolution of \mathbb{K} by left A -modules, but the construction of the contracting homotopy requires the notion of a *left* contraction.

7.1.6. Algebras of finite derivation type. For $n \geq 1$, an algebra is of *finite n -derivation type*, FDT_n for short, if it admits a polygraphic resolution with finitely many k -cells for every $k < n$, and it is of *finite ∞ -derivation type*, FDT_∞ for short, if it admits a polygraphic resolution of finite type. In particular, an algebra is FDT_1 if it is finitely generated, FDT_2 if it is finitely presented, and FDT_3 if it admits a finite coherent presentation. The property FDT_3 corresponds to the finite derivation type condition originally defined by Squier for monoids in [53]. The property FDT_n , for $n \geq 3$ for higher-dimensional categories were introduced in [31, 2.3.6]. By definition, FDT_∞ implies FDT_n , and FDT_n implies FDT_p for all $n > p$.

For $n \geq 1$, an algebra A is of *homological type bi-FP $_n$* (over \mathbb{K}) if there is an exact sequence of A -bimodules

$$0 \longleftarrow A \longleftarrow F_0 \longleftarrow F_1 \longleftarrow \cdots \longleftarrow F_{n-1} \longleftarrow F_n,$$

where each F_i is a finitely generated, free A -bimodule, and it is of *homological type bi-FP $_\infty$* if it is $bi-FP_n$, for all $n > 0$. Theorems 7.1.3 and 6.2.4 give the following implications.

7.1.7. Proposition. (i) FDT_n implies $bi-FP_n$, for every $n \geq 1$, and FDT_∞ implies $bi-FP_\infty$.

(ii) An algebra that admits a finite convergent presentation is FDT_∞ .

7.2. Convergence and Koszulness

7.2.1. Koszul algebras. Let us define the map $\ell_N : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$, for a fixed $N \geq 2$, by

$$\ell_N(2p) = Np \quad \text{and} \quad \ell_N(2p + 1) = Np + 1.$$

So, if X is an ℓ_N -concentrated graded ∞ -polygraph, then its 0-cells are in degree $\ell_N(1) = 1$, its 1-cells in degree $\ell_N(2) = N$, its 2-cells in degree $\ell_N(3) = N + 1$, and so on; in particular, the 1-polygraph underlying X is N -homogeneous. An N -homogeneous algebra A is said to be *Koszul* if there exists a resolution

$$0 \longleftarrow A \longleftarrow M_0 \longleftarrow M_1 \longleftarrow M_2 \longleftarrow \cdots$$

by projective graded A -bimodules such that each M_n is generated by $M_n^{(\ell_N(n))}$. Note that [13, Proposition 4.4] implies that one obtains the same notion of Koszulness by replacing bimodules by left or right modules. As a consequence of the definition, if A is Koszul, the vector spaces $\text{Tor}_{n,(i)}^A(\mathbb{K}, \mathbb{K})$ vanish for $i \neq \ell_N(n + 1)$, where the first grading in the Tor refers to the homological degree and the second one to the internal grading of the algebra A . This property of the Tor groups is an equivalent definition of Koszul algebras, as proved by Berger [12, Theorem 2.11]. From Theorem 7.1.3, we deduce the following sufficient condition for an algebra to be Koszul.

7. Free resolutions of associative algebras

7.2.2. Proposition. *Assume that X is an ω -concentrated polygraphic resolution of a graded algebra A , for some map $\omega : \mathbb{N} \setminus \{0\} \rightarrow \mathbb{N} \setminus \{0\}$. Then each graded A -bimodule $A^e[X_n]$ is generated by its component of degree $\omega(n+1)$. As a consequence, if $\omega = \ell_{\mathbb{N}}$ for some $N \geq 2$, then A is Koszul. In particular, if an algebra admits a quadratic convergent presentation, then it is Koszul.*

Proof. The A -bimodule $A^e[X_n]$ is generated by the n -cells of X which, by hypothesis, are homogeneous of degree $\omega(n+1)$. Thus, when $\omega = \ell_{\mathbb{N}}$, the resolution $A^e[X]$ satisfies the required properties to prove that A is Koszul. If a graded algebra A admits a quadratic convergent presentation X , then the critical n -cells of $\text{Std}(\text{Red}_m(X))$ are all of the form $x_0 | \cdots | x_n$, where each x_i belongs to X . As a consequence, the generators of $A^e[X_n]$ lie in degree $n+1 = \ell_2(n+1)$. \square

7.2.3. Example. The Koszulness of the (quantum) symmetric algebra is a consequence of Example 7.1.4.

7.2.4. Proposition. *If a graded algebra A has an N -homogeneous presentation that admits an empty acyclic cellular extension, then A is Koszul. In particular, if A admits an N -homogeneous terminating presentation with no critical branching, then A is Koszul.*

7.2.5. Example. Consider the cubical algebra A presented by the convergent left-monomial 1-polygraph X of Example 3.2.4. Since X has no critical branching, A is Koszul, and admits the resolution

$$0 \longleftarrow A \longleftarrow A^e \longleftarrow A^e[x, y, z] \longleftarrow A^e[\gamma] \longleftarrow 0$$

Thus $\text{Tor}_{0,(0)}^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}$, $\text{Tor}_{1,(1)}^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}^3$, $\text{Tor}_{2,(3)}^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}$, while $\text{Tor}_{n,(i)}^A(\mathbb{K}, \mathbb{K})$ vanishes for other values of n and i .

Let us compare with the resolution obtained when starting with another presentation. Using the deglex order induced by $x < y < z$, the leading monomial of $z^3 + y^3 + x^3 - xyz$ is z^3 . The corresponding terminating presentation Y of A has x, y and z as 0-cells, and $\alpha : z^3 \rightarrow xyz - x^3 - y^3$ as only 1-cell. This presentation is not confluent, because neither of its two critical branchings is:

$$\begin{array}{ccc} \alpha z \rightarrow xyz^2 - x^3z - y^3z & & \alpha z^2 \rightarrow xyz^3 - x^3z^2 \\ z^4 \swarrow & & \searrow \alpha z^2 \rightarrow xyz^3 - x^3z^2 \\ z\alpha \rightarrow zxyz - zx^3 - zy^3 & & \xrightarrow{\begin{array}{c} xy\alpha - x^3z^2 \\ -y^3z^2 \end{array}} \rightarrow \begin{array}{c} xyxyz - xy^4 - yxz^3 \\ -x^3z^2 - y^3z^2 \end{array} \\ z^5 \swarrow & & \searrow z^2\alpha \rightarrow z^2xyz - z^2x^3 - z^2y^3 \end{array}$$

The adjunction of the 1-cell $\beta : zy^3 \rightarrow zxyz - zx^3 + y^3z + x^3z - xyz^2$ to Y yields convergent left-monomial presentation Z of A , with three critical branchings. In that case, Squier's resolution 6.2.4 obtained from Z is way larger than $\text{Sq}(X)$, with cells in every dimension. This induces a resolution $A^e[Z]$ of A by A -bimodules of infinite length, with a non-trivial differential, making homological computations harder than with $A^e[X]$.

7.2.6. Example. In Example 4.3.3, we have seen that the quadratic 1-polygraph

$$X = \langle x, y, z \mid yz \xrightarrow{\alpha} -x^2, zy \xrightarrow{\beta} -\lambda^{-1}x^2 \rangle.$$

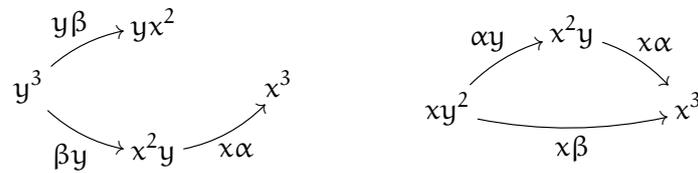
can be extended into a coherent presentation without any 2-cell. By Proposition 7.2.4, it follows that the presented algebra is Koszul. Moreover we obtain that $\text{Tor}_{0,(0)}^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}$, $\text{Tor}_{1,(1)}^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}^3$, $\text{Tor}_{2,(2)}^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}^2$, and $\text{Tor}_{n,(i)}^A(\mathbb{K}, \mathbb{K})$ vanishes for other values of n and i .

7.2.7. Proposition. Let X be a polygraphic resolution of an algebra A , whose underlying n -polygraph is $\ell_{\mathbb{N}}$ -concentrated, for $\mathbb{N} \geq 2$. If $X_n^{(i)}$ has strictly more elements than $X_{n+1}^{(i)}$ for any $i > \ell_{\mathbb{N}}(n + 1)$, then A is not Koszul.

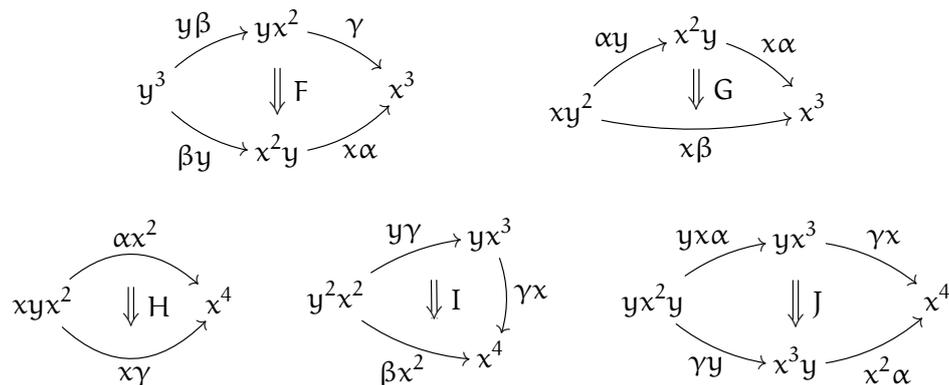
7.2.8. Example. Consider the quadratic algebra A presented by the terminating 1-polygraph

$$X = \langle x, y \mid xy \xrightarrow{\alpha} x^2, y^2 \xrightarrow{\beta} x^2 \rangle,$$

which has two critical branchings, only one of them being confluent:



Adding the 1-cell $\gamma : yx^2 \rightarrow x^3$ to X gives a convergent polygraph Y , so that, by Theorem 4.3.2, the following 2-cells extend the 1-polygraph Y into a coherent presentation of A :



The standard polygraphic resolution has seven critical 3-cells, inducing the same amount of 3-cells in $Sq(Y)$, with the following 0-sources: xyx^2y , xy^2x^2 , xy^3 , yx^2y , y^2x^2y , y^3x^2 and y^4 . Thus, $Sq(Y)$ has three 2-cells and two 3-cells in degree 4, preventing A to be Koszul.

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