Noncommutative linear rewriting: applications and generalizations

Philippe Malbos

Abstract – These notes were intended for a lecture given at the Kobe summer school in July 2015. We present the notion of polygraphic presentation for algebras and their properties of confluence and termination, as introduced in [GHM17]. These presentations, called linear polygraphs, are rewriting systems that generalize noncommutative Gröbner bases, as studied by Bokut, [Bok76], Bergman, [Ber78] and Mora, [Mor94], in the sense that the orientation of the rewriting rules does not depend of a monomial order. We give a description of an algorithm given by Anick, [Ani86], for computing a free resolution of right modules over an algebra presented by a noncommutative Gröbner basis. Finally, we briefly sketch a method to compute polygraphic resolutions for an algebra presented by a confluent and terminating linear polygraph introduced in [GHM17].

M.S.C. 2010 – 18C10, 18D05, 18G10, 16S37, 68Q42.

1. Introduction

The aim of these lecture notes is to provide a summary of the theory of linear rewriting and the application of this theory to the construction of free resolutions for associative algebras. In Section 2 we present linear polygraphs as an algebraic setting for linear rewriting without a monomial order and we review fundamental notion of linear polygraphs. In Section 3 we recall several historical constructions on linear rewriting systems for associative algebras, and we show how the confluence properties is studied in these different approaches. We relate the notion of convergent linear polygraph with the notion of noncommutative Gröbner basis. In Section 4, we describe an algorithmic way to compute free resolutions for algebras using a method introduced by Anick. Section 5 deals with extension of linear polygraphs, seen as higher dimensional linear rewriting systems, into polygraphic resolutions for algebras. We show how to construct such a resolution starting from a convergent presentation. In the last section, we show how to relate Koszulness for algebras with the property of confluence.

Rewriting and linear rewriting

Rewriting in computer science. The notion of rewriting system comes from combinatorial algebra. It was introduced by Thue when he considered systems of transformation rules for rewriting combinatorial objects such as strings, trees or graphs. Its main motivation was to solve the word problem for finitely presented semigroups by using an orientation of relations, [Thu14]. Afterwards, the word problem have been considered in many contexts in algebra and in computer science. Far beyond the precursor works on this decidability problem on strings, rewriting theory has been mainly developed in theoretical computer science for equational reasoning in various situations: theory of programming languages for analysis, verification and optimization, automated deduction, automated theorem proving... Rewriting theory is also present in many others computational formalisms such as Petri nets or logical systems. Depending on the context of application, rewriting theory has numerous variants corresponding to different syntaxes of the formulas being transformed: string, term, graph, circuit, term modulo, tree, λ-term, higher-order term, higher-dimensional term...

Rewriting in algebra. Rewriting appears also on various forms in algebra for universal algebras (term rewriting in Lawvere theories), [BN98, Klo92, Ter03, MM16], monoids (string rewriting in monoids), [BO93, GGM15, HM17], monoidal categories, [GM12a], linear structures, such as algebras of various type: commutative, [Buc65, Buc70, Buc06], associative, [Bok76, Ber78], Lie, [Shi62], as well as on topological objects, such as Reidemeister moves, knots or braids, [Bur01].

These notes focus on various aspects of rewriting in associative algebras. Rewriting theory gives algorithmic methods to study associative algebras presented by generators and defining relations. The relations are oriented as *rewriting rules* providing linear bases of normal forms with respect the defining relations. In particular, rewriting methods can be used to provide procedures for decision problems, such as the word problem, ideal membership, or to compute quadratic bases, e.g., Poincaré-Birkhoff-Witt bases, Hilbert series, syzygies of presentations, homology groups and Poincaré series.

We have to be careful when we rewrite over a field. Rewriting rules that relate elements in a ring or in an algebra need to be compatible with the linear structure in the following way. For a rewriting rule

$$f \to g$$

relating two elements of an algebra on a ground field $\mathbb{K}$, then for any scalar λ in $\mathbb{K}$ we would like a rewriting

$$\lambda f \to \lambda g$$

and for any other element h of the algebra we would like a rewriting

$$f + h \to g + h.$$

Taken together, these two reductions lead to losing termination of rewriting. Indeed, it that case from the rule $f \to g$, we deduce the reductions $-f \to -g$ and $-f + (f + g) \to -g + (f + g)$. Finally, we deduce the following reduction

$$g \to f.$$

As a consequence, the system will never terminate. Further to this remark, it is necessary to adapt the notion of rewriting system to linear situations. In the example presented above the reduction $-f + (f + g) \to -g + (f + g)$ appears as the source of the nontermination problem. In these notes, we will see two possibilities to fix this problem.

- By choosing an orientation of the rules induced by a *monomial order*, which is well-founded by definition, see 2.4.1. This is the most common used method, in particular in the noncommutative Gröbner basis theory.
- By using the structure of *linear* 2*-polygraph* introduced in [GHM17] and with an appropriated notion of reduction, explained in Subsection 2.2.

Noncommutative Gröbner bases: applications and generalizations

Gröbner basis theory. Gröbner basis theory for ideals in commutative polynomial rings was introduced by Buchberger in [Buc65]. A subset G of an ideal I in the polynomial ring $\mathbb{K}[x]$ of commutative polynomials is a *Gröbner basis* of I with respect to a given monomial order $\prec$, if the leading term ideal of I is generated by the set of leading monomials of G, that is

$$\langle \operatorname{lt}_\prec(I) \rangle = \langle \operatorname{lt}_\prec(G) \rangle.$$

Buchberger introduced the notion of S*-polynomial* to describe the obstructions to local confluence and gave an algorithm for computation of Gröbner bases, [Buc65, Buc06], see also [Buc87] for an historical account. Any ideal I of a commutative polynomial ring $\mathbb{K}[x]$ has a finite Gröbner basis. Indeed, the Buchberger algorithm on a finite family of generators of an ideal I always terminates and returns a Gröbner basis of the ideal I.

Shirshov introduced in [Shi62] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of *composition* of elements in a free Lie algebra, that corresponds to the notion of S-polynomial in the work of Buchberger. He gave an algorithm to compute bases in free algebras having the computational properties of the Gröbner bases. He proved that irreducible elements for such a basis forms a linear basis of the Lie algebra. This result is called now the *Composition Lemma* for Lie algebras.

Subsequently, the Gröbner basis theory has been developed for other types of algebras, such as associative algebras by Bokut in [Bok76] and by Bergman in [Ber78]. They prove Newman's Lemma for rewriting systems in free associative algebras compatible with a monomial order stating that local confluence and confluence are equivalent properties. This result was called *Composition Lemma* by Bokut and *Diamond Lemma* for ring theory by Bergman, see also [Mor94, Ufn95]. In general, the Buchberger algorithm does not terminate for ideals in a noncommutative polynomial ring $\mathbb{K}\langle x \rangle$. Indeed, its termination would give a decision procedure of the undecidable word problem. Even if the ideal is finitely generated it may not have a finite Gröbner basis. However, when $\mathbb{K}$ is a field an infinite Gröbner basis can be computed, [Mor94, Ufn98]. We survey the constructions and the results of Bokut and Bergman in Section 3.

Note that ideas in the spirit of the Gröbner basis approach appear in several others works. Let us mention works by Hironaka in [Hir64] and Grauert in [Gra72] that compute bases of ideals in rings of power series having analogous properties to Gröbner bases but without a constructive method for computing such bases. In [Coh65], Cohn gave a method to decide the word problem by a normal form algorithm based on a confluence property. Finally, Janet [Jan20], Thomas [Tho37] and Pommaret [Pom78] developed the notion of involutive bases that are particular cases of Gröbner bases in the context of partial differential algebra. Much more recently, Gröbner basis theory was developed in various noncommutative contexts such as Weyl algebras, see [SST00], or operads [DK10].

Computing normal forms. The main purpose of noncommutative Gröbner basis theory for associative algebras is to compute linear bases. Consider an algebra **A** presented by a set of generators X and a set R of defining relations, that is **A** is the quotient of the free algebra $\mathbb{K}\langle X \rangle$ by the ideal generated by R. The set of monomials on X forms a linear basis of the free algebra $\mathbb{K}\langle X \rangle$. One application of the Gröbner basis theory is to compute a basis of the algebra **A** in the form of a reduced subset of monomials. The computation is based on a monomial order on the set of monomials on X and the confluence property of a rewriting system compatible with this order. The set of monomials in normal form with respect to a Gröbner basis forms a linear basis of the algebra **A**.

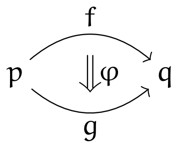
The Buchberger algorithm that computes Gröbner bases is the analogue of the Knuth-Bendix completion procedure in a linear setting. Several frameworks unify Buchberger and Knuth-Bendix algorithms, in particular a Gröbner basis corresponds to a confluent and terminating presentation of an algebra, see [Buc87]. This correspondence is well known in the case of associative and commutative algebras, as recalled in the papers by Bokut [Bok76], Bergman[Ber78], Mora [Mor94]. For a fuller treatment on noncommutative Gröbner bases for associative algebras, we refer the reader to the books [BD16, Chapter 2] and [Ufn95] and to [KR00, Chapter 2] and [BW93, Chapters 4-5] for commutative Gröbner bases.

Computation of free resolutions. In homological algebra, constructive methods based on noncommutative Gröbner bases were developed to compute projective resolutions for algebras. In particular, Anick and Green constructed small explicit free resolutions for algebras given by noncommutative Gröbner bases, [Ani85, Ani86, AG87, Gre99]. Their constructions provide resolutions to compute homological invariants (homology groups, Hilbert and Poincaré series) of algebras presented by generators and relations given by a Gröbner basis. The chains of these resolutions are given by iterated overlaps of the leading terms of the Gröbner basis and the differentials are constructed by Noetherian induction.

Linear polygraphs. All the constructions mentioned above rely on a monomial order, that is a well-founded order of the monomials compatible with the multiplication. The termination orders in linear polygraphs introduced in [GHM17] are less restrictive. A linear polygraph is a higher-dimensional linear rewriting system for presentation of an algebra that allows more possibilities of termination orders than those associated to Gröbner bases using monomial orders. A set-theoretical 2-polygraph describes a string rewriting system, see [GM18]. It is defined by a data $(\Sigma_0, \Sigma_1, \Sigma_2)$ made of a 1-*polygraph*, that is an oriented graph

$$\Sigma_0 \underset{t_0}{\overset{s_0}{\leftleftarrows}} \Sigma_1$$

where Σ_0 and Σ_1 denote respectively the sets of 0-cells and 1-cells and s_0, t_0 denote the source and target maps, with a *cellular extension* Σ_2 of the free category Σ_1^*, that is a set of globular 2-cells relating parallel 1-cells:



A 2-polygraph corresponds to a string rewriting system, where the rules are describe by a the globular 2-cells, see [GM18].

A *linear* 2-*polygraph* corresponds to the same notion for rewriting in a free algebra or a free algebroid. It is constructed in the same manner as a 2-polygraph, but the cellular extension is linear in the sense that it is constructed on 1-spheres in the free 1-algebroid over generating 1-cells. Explicitly, we define a linear 2-polygraph as a triple $(\Lambda_0, \Lambda_1, \Lambda_2)$ such that (Λ_0, Λ_1) is a 1-polygraph and Λ_2 is a cellular extension of the free algebroid Λ_1^ℓ generated by the 1-polygraph (Λ_0, Λ_1), that is given by two maps

$$\Lambda_1^\ell \underset{t_1}{\overset{s_1}{\leftleftarrows}} \Lambda_2$$

satisfying *globular relations* $s_0 s_1 = s_0 t_1$ and $t_0 s_1 = t_0 t_1$. All the categorical background will be introduced in Section 2. In the free 2-algebroid Λ_2^ℓ, any 2-cell being invertible, the notion of rewriting step induced by a linear polygraph needs to be defined with attention. In Section 2 and Section 3, we recall from [GHM17] properties of termination, confluence and local confluence for linear 2-polygraphs. We state the Newman Lemma for linear 2-polygraphs in Theorem 3.2.11 showing that a terminating left-monomial linear 2-polygraph is confluent if and only if it is locally confluent. We give a formulation of a critical branching lemma for linear 2-polygraphs in Theorem 3.3.7. The formulation of this result differs from the critical branching lemma for 2-polygraphs in the sense that the termination hypothesis is required, as we will explain with several examples in Section 3.3. Finally, we explain how to recover noncommutative Gröbner bases as a special case of convergent linear 2-polygraphs in Section 3.6.

Polygraphic resolutions of algebroids. We recall in Section 5.1 the notion of linear syzygies for linear polygraphs. When the linear 2-polygraph is convergent, we show that all the syzygies can be generated by confluence diagrams induced by the critical branchings, this is the Squier Theorem 5.1.6.

In Section 5.2, we recall from [GHM17] the notion of *polygraphic resolution* for an algebra giving a categorical description of higher-dimensional syzygies of its presentations. A polygraphic resolution for an algebra **A** is an acyclic polygraphic extension of a presentation of **A**. That is a linear ∞-polygraph, which satisfies an acyclicity condition. Theorem 5.2.6 from [GHM17] shows that any convergent linear 2-polygraph Λ extends to an acyclic linear ∞-polygraph, presenting the same algebra and whose n-cells, for $n \geqslant 3$, are indexed by the critical $(n-1)$-fold branchings. From this point of view, this resolution is similar to Anick's resolution associated with a Gröbner basis.

Finally, we show how a polygraphic resolution of an algebra **A** induces a free resolution in the category of right-modules (resp. left-modules, resp. bi-modules) over **A**.

Confluence and Koszulness. In the last section of these notes, we show how Anick's resolution leads to relate the Koszul property for an associative algebra to the existence of a quadratic Gröbner basis for its ideal of relations. We also show how to prove this property using convergent linear 2-polygraphs.

In Subsection 6.1, we recall the notion of Koszulness for quadratic algebras and N-homogeneous algebras. Koszulness for quadratic algebras was introduced by Priddy, [Pri70]. A connected graded algebra **A** is *Koszul* if the Tor groups $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K},\mathbb{K})$ vanish for $i \neq n$, where the grading n is the homological degree and the grading i corresponds to the internal grading of the algebra. This notion was generalized by Berger to the case of N-homogeneous algebras, [Ber01].

In [Ani86], Anick showed how its resolution can be used to prove Koszulness of a quadratic algebra. Indeed, if an algebra **A** admits a presentation whose relations are defined by a quadratic Gröbner basis, then Anick's resolution associated to this Gröbner basis is concentrated in the right bidegree, and thus the algebra **A** is Koszul, see Theorem 6.2.3. For the N-homogeneous case, a Gröbner basis concentrated in weight N is not enough to imply Koszulness: an extra condition has to be checked as shown by Berger in [Ber01].

Finally, we present a sufficient polygraphic condition of Koszulness of graded algebras given in [GHM17]. Using a graded version of Theorem 5.2.9, one shows that an N-homogeneous algebra having a ℓ_N-concentrated polygraphic resolution is Koszul, Theorem 6.2.7.

2. Linear rewriting

In this section we recall the categorical description of linear rewriting given in [GHM17] using the notion of linear polygraph. This notion extends to associative algebras the categorical notion of 2-polygraph used to describe presentations of monoids by generators and relations. This approach is based on presentations by generators and relations of higher-dimensional categories, independently introduced by Burroni and Street under the respective names of *polygraphs* in [Bur93] and *computads* in [Str76, Str87]. Higher-dimensional rewriting has unified several paradigms of rewriting. These notes concern only rewriting in algebras, for a deeper discussion on categorical description of string rewriting systems by 2-polygraphs, we refer the reader to [GM18]. Note that there is a shift by 1 in the dimension: in these lecture notes the linear 2-polygraphs are linear 1-polygraphs in [GM18].

2.1. Linear 2-polygraphs

2.1.1. Categories. Recall that a *(small) category* (or 1*-category*) is a data **C** made of a set $\mathbf{C}_0$, whose elements are called 0*-cells* (or *objects*) of **C**, for every 0-cells p and q a set $\mathbf{C}(p,q)$, whose elements are called 1*-cells* (or *arrows*) of **C** with *source* p and *target* q, for every 0-cell p a specified 1-cell 1_p in $\mathbf{C}(p,p)$, called the *identity of* p, and for every 0-cells p, q and r a *composition* map

$$\star_0^{p,q,r} : \mathbf{C}(p,q) \times \mathbf{C}(q,r) \to \mathbf{C}(p,r),$$

that is associative and such that the identities are local units for this composition.

A monoid M with product $\cdot$ and identity element 1_M corresponds to a category **M** with only one 0-cell, denoted by $*$, and the 1-cells of $\mathbf{M}(*,*)$ are the elements of the monoid M. The identity arrow 1_* of **M** corresponds to the identity element 1_M and the composition of $u \star_0 v$ of 1-cells in $\mathbf{M}(*,*)$ corresponds to the product $u \cdot v$ in the monoid M. The associativity and unitary properties of the composition, making **M** into a category, are induced by the corresponding properties of the product $\cdot$ of the monoid. In this way, any monoid can be thought of as a one-0-cell category and a category can be thought of as a "monoid with many 0-cells". In a similar way, the notion of algebroid describes the concept of associative algebra with many 0-cells.

2.1.2. Algebroids. A 1-*algebroid* over a ground field $\mathbb{K}$ is a category enriched over the monoidal category of vector spaces over $\mathbb{K}$ with its usual tensor product. Explicitly, a 1-algebroid **A** is specified by the following data:

i) a set $\mathbf{A}_0$ of 0-*cells*, that we will denote by p, q...

ii) for every 0-cells p and q, a vector space $\mathbf{A}(p,q)$, whose elements are the 1-*cells* of **A**, with *source* p and *target* q, that we will denote by f, g...

iii) for every 0-cells p, q and r, a linear map

$$\star_0 : \mathbf{A}(p,q) \otimes \mathbf{A}(q,r) \longrightarrow \mathbf{A}(p,r)$$

called the 0-*composition* of **A** and whose image on $f \otimes g$ is denoted by $f \star_0 g$ or fg. This composition is *associative*, that is the relation:

$$(f \star_0 g) \star_0 h = f \star_0 (g \star_0 h),$$

holds for any 0-composable 1-cells f, g and h, and *unitary*, that is, for any 0-cell p, there is a 1-cell 1_p such that for any 1-cell f in $\mathbf{A}(p,q)$, the following relation holds

$$1_p \star_0 f = f \star_0 1_q = f.$$

A 1-cell f with source p and target q will be graphically represented by

$$p \xrightarrow{\;f\;} q$$

2.1.3. Remarks. An 1-*algebra* is an 1-algebroid with a single one 0-cell, that can be identified to an (unital associative) algebras over $\mathbb{K}$. We will denote by **Alg** the category of algebras over $\mathbb{K}$. The notion of 1-algebroid was first introduced by Mitchell as *ring with several objects* called $\mathbb{K}$-*category* in [Mit72], terminology *linear category* appears also in the literature. A small $\mathbb{Z}$-category is called a *ringoid* and a one-0-cell ringoid is a ring.

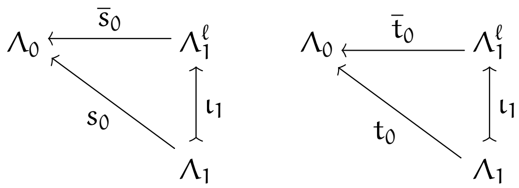
2.1.4. One-dimensional polygraphs. An algebroid can be defined by generators and relations. The generators are described by one-dimensional polygraphs. A 1-*polygraph* is a directed graph

$$\Lambda_0 \overset{s_0}{\underset{t_0}{\longleftleftarrows}} \Lambda_1$$

given by a set Λ_0 of 0-*cells*, a set Λ_1 of 1-*cells* together with two maps s_0 and t_0 sending a 1-cell x on its *source* $s_0(x)$ and its *target* $t_0(x)$. A 1-polygraph with only one 0-cell will be identified to a set.

We will denote by Λ_1^* the free 1-category generated by the 1-polygraph (Λ_0, Λ_1). Its set of 0-cells is Λ_0 and for any 0-cells p and q, the elements of the hom-set $\Lambda_1^*(p, q)$ are paths from p to q in the 1-polygraph (Λ_0, Λ_1). The composition is the concatenation of paths and the identity on a 0-cell p is the empty path with source and target p. If the 1-polygraph has only one 0-cell, Λ_1^* will be identified to the free monoid on the set Λ_1.

2.1.5. Free 1-algebroid. The free 1-algebroid on a 1-polygraph (Λ_0, Λ_1) is the 1-algebroid, denoted by Λ_1^ℓ, whose set of 0-cells is Λ_0, and for any 0-cells p and q, $\Lambda_1^\ell(p, q)$ is the free vector space on $\Lambda_1^*(p, q)$. In other words, the space $\Lambda_1^\ell(p, q)$ has for basis the set of paths from p to q in the 1-polygraph Λ. If Λ_0 is reduced to only one 0-cell, Λ_1^ℓ is the free algebra with basis Λ_1. The source and target maps s_0 and t_0 are extended into maps on Λ_1^ℓ, denoted by $\bar{s}_0$ and $\bar{t}_0$, in a natural way making the following two diagrams commutative:



where ι_1 denotes the inclusion of 1-cells of Λ_1 in the free algebroid Λ_1^ℓ.

2.1.6. Quivers and path algebras. The terminology *directed graph* is used in graph theory. The same notion is also called *quiver* in representation theory. A linear representation of a quiver (Λ_0, Λ_1) is a functor ρ from the free category Λ_1^* to the category **Vect** of vectors spaces. The path algebra of a quiver (Λ_0, Λ_1) is the *category algebra* of the free category Λ_1^*. That is, it is the $\mathbb{K}$-algebra whose underlying space is spanned by the set of 1-cells in Λ_1^* and the product on the basis elements is defined by $u \cdot v = u \star_0 v$ if u and v are 0-composable 1-cells in Λ_1^* and $u \cdot v = 0$ otherwise. When the set Λ_0 is finite, then $\sum_{p \in \Lambda_0} 1_p$ is the identity of the path algebra. Note that, we can obtain the path algebra of a quiver Λ from the free 1-algebroid Λ_1^ℓ by forgetting the 1-category structure.

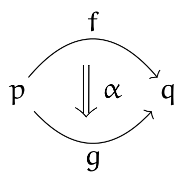
2.1.7. Linear 2-polygraph. A *cellular extension* of the 1-algebroid Λ_1^ℓ is a set Λ_2 equipped with two maps

$$\Lambda_1^\ell \overset{s_1}{\underset{t_1}{\Longleftarrow}} \Lambda_2$$

such that, for every α in Λ_2, the pair $(s_1(\alpha), t_1(\alpha))$ is a 1-*sphere* in Λ_1^ℓ, that is, the following *globular relations* hold

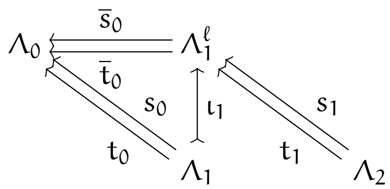
$$s_0 s_1(\alpha) = s_0 t_1(\alpha) \quad \text{and} \quad t_0 s_1(\alpha) = t_0 t_1(\alpha).$$

An element of the cellular extension Λ_2 will be graphically represented by a 2-cell with the following globular shape



that relates parallel 1-cells f and g in Λ_1^ℓ, also denoted by $f \overset{\alpha}{\Longrightarrow} g$ or by $\alpha : f \Rightarrow g$.

We define a *linear 2-polygraph* as a triple $(\Lambda_0, \Lambda_1, \Lambda_2)$, where (Λ_0, Λ_1) is a 1-polygraph and Λ_2 is a cellular extension of the free 1-algebroid Λ_1^ℓ:



The elements of Λ_2 are called the 2-*cells* of Λ, or the *rewriting rules* of Λ.

In the sequel, we will consider polygraphs with one 0-cell denoted $*$.

2.1.8. Presentations of algebras by generators and relations. Given a linear 2-polygraph Λ. The *algebra presented by* Λ is the quotient algebra of the free algebra Λ_1^ℓ by the cellular extension Λ_2. That is, it is the algebra $\mathbf{A}$ obtained by identifying in Λ_1^ℓ all the 1-cells $s_1(a)$ and $t_1(a)$, for every 2-cell a in Λ_2^ℓ. We denote by $\overline{f}$ the image of a 1-cell f of Λ_1^ℓ through the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$. We say that a linear 2-polygraph Λ is a *presentation* of an algebra $\mathbf{A}$ if the algebra presented by Λ is isomorphic to $\mathbf{A}$. Two linear 2-polygraphs are said to be *Tietze equivalent* if they present isomorphic algebras.

2.1.9. First toy example. Here our first toy example that we will use through this lecture:

$$\Lambda = \langle * \mid x, y, z \mid xyz \overset{\gamma}{\Longrightarrow} x^3 + y^3 + z^3 \rangle.$$

The free 1-algebroid generated by $\Lambda_1 = \{x, y, z\}$ is the free algebra $\mathbb{K}\langle x, y, z\rangle$. The algebra presented by the linear 2-polygraph Λ is the quotient of the free algebra $\mathbb{K}\langle x, y, z\rangle$ by the ideal generated by the 1-cell $xyz - x^3 - y^3 - z^3$.

2.1.10. Other toy examples. We will consider the two following Tietze equivalent linear 2-polygraphs:

$$\Lambda = \langle * \mid x, y \mid x^2 \overset{\beta}{\Longrightarrow} yx \rangle, \qquad \Lambda' = \langle * \mid x, y \mid yx \overset{\beta'}{\Longrightarrow} x^2 \rangle.$$

2.1.11. Two-dimensional algebras. We define a 2-*algebra* $\mathbf{A}$ as an internal 1-category in the category **Alg**. Explicitly, it is defined by a diagram

$$\mathbf{A}_1 \underset{i_2}{\overset{t_1,\ s_1}{\leftleftarrows\!\rightarrow}} \mathbf{A}_2 \overset{\star_1}{\longleftarrow} \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \tag{1}$$

where $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ is the algebra defined by the following pullback diagram in the category **Alg**:

$$\begin{array}{ccc} \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \longrightarrow & \mathbf{A}_2 \\ \big\downarrow & \lrcorner & \big\downarrow{\scriptstyle s_1} \\ \mathbf{A}_2 & \underset{t_1}{\longrightarrow} & \mathbf{A}_1 \end{array}$$

Elements of the algebra $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ are pairs (a, a') of 1-*composable* 2-cells a and a', that is satisfying $t_1(a) = s_1(a')$. The morphisms of algebras s_1, t_1 and $\star_1$ satisfy the axioms in such a way that Diagram (1) defines a 1-category. Explicitly, the following diagrams commute in the category **Alg**:

$$
\begin{array}{ccc}
\mathbf{A}_1 & \xrightarrow{i_2} & \mathbf{A}_2 \\
& {\scriptstyle id}\searrow & \downarrow{\scriptstyle s_1} \\
& & \mathbf{A}_1
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{A}_1 & \xrightarrow{i_2} & \mathbf{A}_2 \\
& {\scriptstyle id}\searrow & \downarrow{\scriptstyle t_1} \\
& & \mathbf{A}_1
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \xrightarrow{\star_1} & \mathbf{A}_2 \\
{\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle s_1} \\
\mathbf{A}_2 & \xrightarrow[s_1]{} & \mathbf{A}_1
\end{array}
\qquad
\begin{array}{ccc}
\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \xrightarrow{\star_1} & \mathbf{A}_2 \\
{\scriptstyle \pi_2}\downarrow & & \downarrow{\scriptstyle t_1} \\
\mathbf{A}_2 & \xrightarrow[t_1]{} & \mathbf{A}_1
\end{array}
$$

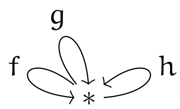
$$
\begin{array}{ccc}
\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \xrightarrow{\star_1 \times_{\mathbf{A}_1} id} & \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \\
{\scriptstyle id \times_{\mathbf{A}_1} \star_1}\downarrow & & \downarrow{\scriptstyle \star_1} \\
\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \xrightarrow[\star_1]{} & \mathbf{A}_2
\end{array}
\qquad
\begin{array}{ccccc}
\mathbf{A}_1 \times_{\mathbf{A}_1} \mathbf{A}_2 & \xrightarrow{i_2 \times_{\mathbf{A}_1} id} & \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 & \xleftarrow{id \times_{\mathbf{A}_1} i_2} & \mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_1 \\
& {\scriptstyle \pi_2}\searrow & \downarrow{\scriptstyle \star_1} & \swarrow{\scriptstyle \pi_1} & \\
& & \mathbf{A}_2 & &
\end{array}
$$

where π_1 and π_2 denote respectively first and second projection. Note that the linear structure and the product in the algebra $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$ are given by

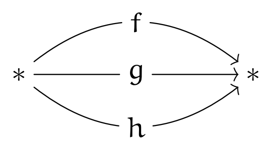
$$
\begin{aligned}
(a, a') + (b, b') &= (a + b, a' + b'), \\
\lambda(a, a') &= (\lambda a, \lambda a'), \\
(a, a')(b, b') &= (ab, a'b'),
\end{aligned}
$$

for all pair of 1-composable 2-cells (a, a') and (b, b') and scalar λ in $\mathbb{K}$.

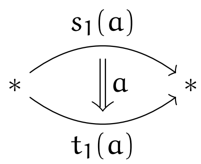
2.1.12. Notations. For a 1-cell f, the identity 2-cell $i_2(f)$ is denoted by 1_f, or f if there is no possible confusion. The 1-composite $\star_1(a, a')$ of 1-composable 2-cells a and a', will be denoted by $a \star_1 a'$. Elements of the algebra $\mathbf{A}_1$, called 1-*cells of* $\mathbf{A}$, are graphically pictured as follows



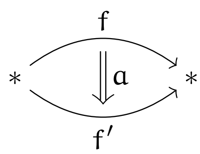
or



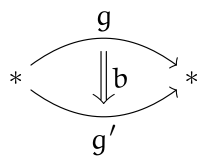
The elements of $\mathbf{A}_2$, called 2-*cells of* $\mathbf{A}$ are graphically represented by



Given 2-cells



and



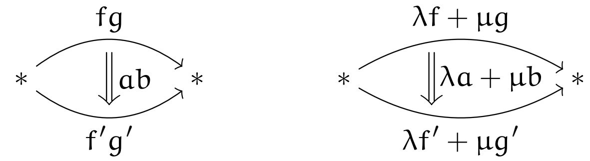
we denote by ab their product in the algebra $\mathbf{A}_2$. The source and target maps s_1 and t_1 being morphisms of algebras, we have

$$s_1(ab) = s_1(a)s_1(b), \qquad \text{and} \qquad t_1(ab) = t_1(a)t_1(b),$$

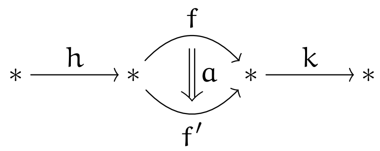
and for any scalars λ and μ in $\mathbb{K}$, we have

$$s_1(\lambda a + \mu b) = \lambda s_1(a) + \mu s_1(b), \qquad \text{and} \qquad t_1(\lambda a + \mu b) = \lambda t_1(a) + \mu t_1(b).$$

Hence

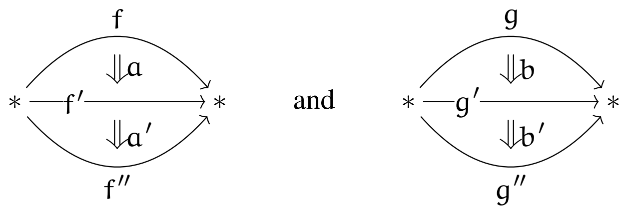


Given 1-cells h, f, f' and k in $\mathbf{A}_1$ and a 2-cell a in $\mathbf{A}_2$ such that



we will denote by $hak : hfk \Rightarrow hf'k$ the 0-composite $1_h \star_0 a \star_0 1_k$.

2.1.13. Properties of 1-composition. Given 1-composable 2-cells:



in $\mathbf{A}_2 \star_{\mathbf{A}_1} \mathbf{A}_2$, the 1-composition $\star_1$ being linear, $a \star_1 a' + b \star_1 b'$ is a 2-cell from $f + g$ to $f' + g'$ and we have

$$(a + b) \star_1 (a' + b') = a \star_1 a' + b \star_1 b'.$$

and, for any scalar λ in $\mathbb{K}$, $\lambda(a \star_1 a')$ is a 2-cell from λf to $\lambda f''$ and we have

$$(\lambda a) \star_1 (\lambda a') = \lambda(a \star_1 a').$$

Finally, the compatibility with the product induces the following relation:

$$(a \star_1 a')(b \star_1 b') = ab \star_1 a'b'. \tag{2}$$

Relation (2) corresponds to the *exchange law* in the 2-algebra $\mathbf{A}$ between the 1-composition and the product.

2.1.14. Remarkable identities in a 2-algebra. The following properties hold in a 2-algebra $\mathbf{A}$

i) for any 1-composable 2-cells a and a' in $\mathbf{A}$, we have

$$a \star_1 a' = a + a' - t_1(a), \tag{3}$$

ii) any 2-cell a in $\mathbf{A}$ is invertible for the $\star_1$-composition, and its inverse is given by

$$a^- = -a + s_1(a) + t_1(a). \tag{4}$$

iii) for any 2-cells a and b in $\mathbf{A}$, we have

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b) = s_1(a)b + at_1(b) - s_1(a)t_1(b). \tag{5}$$

Relation (3) is a consequence of the linearity of the 1-composition $\star_1$. Indeed, for any (a, a') in $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2$, we have

$$\begin{aligned} a \star_1 a' &= (a - s_1(a') + s_1(a')) \star_1 (t_1(a) - t_1(a) + a'), \\ &= a \star_1 t_1(a) - s_1(a') \star_1 t_1(a) + s_1(a') \star_1 a', \\ &= a - t_1(a) + a'. \end{aligned}$$

2.1.15. Exercice. Show identities (4) and (5).

2.1.16. The free 2-algebra on a linear 2-polygraph. The *free 2-algebra over a linear 2-polygraph* Λ is the 2-algebra, denoted by Λ_2^ℓ, defined as follows. In dimension 1, it is the free 1-algebra Λ_1^ℓ over Λ_1. For dimension 2, we consider the following diagram in the category of Λ_1^ℓ-bimodule

$$\Lambda_1^\ell \underset{i_2}{\overset{t_1}{\underset{s_1}{\leftleftarrows\!\rightarrow}}} \Lambda_2^{\mathcal{M}}$$

where $\Lambda_2^{\mathcal{M}}$ is the Λ_1^ℓ-bimodule $\left(\Lambda_1^\ell \otimes \mathbb{K}\Lambda_2 \otimes \Lambda_1^\ell\right) \oplus \Lambda_1^\ell$ and where the maps s_1, t_1 and i_2 are defined by:

$$s_1(f\alpha g) = fs_1(\alpha)g, \qquad t_1(f\alpha g) = ft_1(\alpha)g \qquad \text{and} \qquad s_1(h) = t_1(h) = i_2(h) = h,$$

for all 2-cell α in Λ_2, and 1-cells f, g, h in Λ_1^ℓ. The quotient of the Λ_1^ℓ-bimodule $\Lambda_2^{\mathcal{M}}$ by the equivalence relation generated by

$$as_1(b) + t_1(a)b - t_1(a)s_1(b) \sim s_1(a)b + at_1(b) - s_1(a)t_1(b),$$

for all a and b in $\Lambda_1^\ell \otimes \mathbb{K}\Lambda_2 \otimes \Lambda_1^\ell$, has a structure of algebra, denoted by Λ_2^ℓ, and whose product is given by

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b).$$

We prove that the source and target maps are compatible with this quotient, so giving a structure of 2-algebra:

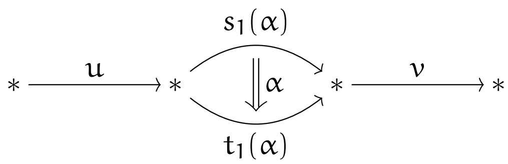
$$\Lambda_1^\ell \underset{i_2}{\overset{t_1}{\underset{s_1}{\leftleftarrows\!\rightarrow}}} \Lambda_2^\ell$$

2.1.17. Monomials. A *monomial* in the free 2-algebra Λ_2^ℓ is a 1-cell of the free monoid Λ_1^* over Λ_1. The set monomials of Λ_2^ℓ, also denoted by Λ_1^*, forms a linear basis of the free algebra Λ_1^ℓ. As a consequence, every nonzero 1-cell f of Λ_1^ℓ can be uniquely written as a linear combination of pairwise distinct monomials $u_1, \ldots, u_p$:

$$f = \lambda_1 u_1 + \ldots + \lambda_p u_p$$

with $\lambda_i \in \mathbb{K} \setminus \{0\}$, for all $i = 1, \ldots, p$. The set of monomials $\{u_1, \ldots, u_p\}$ will be called the *support* of f and denoted by $\mathrm{Supp}(f)$.

2.1.18. 2-monomials. A 2-*monomial* of a free 2-algebra Λ_2^ℓ is a 2-cell of Λ_2^ℓ with shape $u\alpha v$, where α is a 2-cell in Λ_2, and u and v are monomials in Λ_1^*:



By construction of the free 2-algebra Λ_2^ℓ, and by freeness of Λ_1^ℓ, every non-identity 2-cell a of Λ_2^ℓ can be written as a linear combination of pairwise distinct 2-monomials $a_1, \ldots, a_p$ and of an 1-cell h of Λ_1^ℓ:

$$a = \lambda_1 a_1 + \ldots + \lambda_p a_p + h. \tag{6}$$

2.1.19. Exercise. Prove that the decomposition in (6) is unique up to the following relations

$$a s_1(b) + t_1(a) b - t_1(a) s_1(b) = s_1(a) b + a t_1(b) - s_1(a) t_1(b), \tag{7}$$

for all 2-monomials a and b in Λ_2^ℓ.

2.1.20. Monomial linear 2-polygraphs. A linear 2-polygraph Λ is *left-monomial* if, for every 2-cell α of Λ_2, the source $s_1(\alpha)$ is a monomial in $\Lambda_1^* \setminus \mathrm{Supp}(t_1(\alpha))$. Note that a non-left monomial linear 2-polygraph would produce useless ambiguity only due to the linear structure.

A linear 2-polygraph Λ is *monomial* if it is left-monomial and for every 2-cell α of Λ_2, $t_1(\alpha) = 0$ holds. A *monomial algebra* is an algebra admitting a presentation by a monomial linear 2-polygraph.

2.1.21. Degrees and length. For monomials u and v in Λ_1^*, we denote by $\mathrm{Occ}_v(u)$ the number of different occurrences of the monomial v in the monomial u. For instance $\mathrm{Occ}_{x^2}(x^4) = 3$ and $\mathrm{Occ}_y(x^4) = 0$. For a subset M of monomials in Λ_1^*, we denote

$$\mathrm{Occ}_M(u) = \sum_{v \in M} \mathrm{Occ}_v(u).$$

The *length* of a monomial u in Λ_1^*, denoted by $\ell(u)$, is equal to $\mathrm{Occ}_{\Lambda_1}(u)$.

2.1.22. Exercise. Show that any linear 2-polygraph is Tietze equivalent to a left-monomial linear 2-polygraph.

2.1.23. Examples. The linear 2-polygraph Λ given in Example 2.1.9 is left-monomial. The linear 2-polygraph $\langle * \mid x, y \mid x^2 + y^2 \Rightarrow 2xy \rangle$ is not left-monomial, but it is Tietze equivalent to the following left-monomial 2-polygraph:

$$\Lambda' = \langle * \mid x, y \mid xy \overset{\alpha'}{\Longrightarrow} \frac{1}{2}(x^2 + y^2) \rangle.$$

The linear 2-polygraphs $\langle * \mid x \mid x^2 \Rightarrow 0 \rangle$ and $\langle * \mid x, y \mid xy \Rightarrow 0 \rangle$ are monomials.

2.2. Linear rewriting steps

2.2.1. Elementary 2-cells. Let Λ be a linear 2-polygraph. An *elementary* 2-cell of the free 2-algebra Λ_2^ℓ is a 2-cell of Λ_2^ℓ with shape

$$\lambda \; * \underset{t_1(a)}{\overset{s_1(a)}{\Downarrow a}} * \quad + \quad * \xrightarrow{\;g\;} *$$

where a is a 2-monomial, g is a 1-cell of Λ_1^ℓ and λ is a nonzero scalar in $\mathbb{K}$.

2.2.2. Example. With the polygraph Λ' of Example 2.1.23, the 2-cell

$$2x\alpha'y + y^3 : 2x^2y^2 \Rightarrow x^3y + xy^3 - y^3$$

is elementary and the 2-cell

$$x\alpha' + \alpha'y : x^2y + xy^2 \Rightarrow \frac{1}{2}(x^3 + xy^2 + x^2y + y^3)$$

is not elementary.

2.2.3. Exercise. Show that any 2-cell in a free 2-algebra Λ_2^ℓ can be decomposed into a 1-composition of elementary 2-cells of Λ_2^ℓ

2.2.4. Rewriting steps. Let Λ be a left-monomial linear 2-polygraph. A *rewriting step of* Λ is an elementary 2-cell

$$\lambda \; * \underset{f}{\overset{u}{\Downarrow a}} * \quad + \quad * \xrightarrow{\;g\;} *$$

of Λ_2^ℓ such that λ is a nonzero scalar and u is not in the support of g.

2.2.5. Examples. For the linear 2-polygraph given in Example 2.1.9, the 2-cell

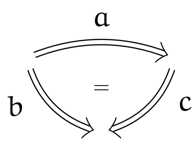
$$3x\gamma - 3xz^3 : 3x^2yz - 3xz^3 \Longrightarrow 3x^4 + 3xy^3$$

is a rewriting step. For a linear 2-polygraph having a rule $\alpha : u \Rightarrow f$, the 2-cell

$$-\alpha + (u + f) : -u + (u + f) \Longrightarrow -f + (u + f)$$

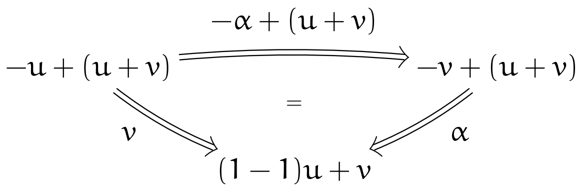
is not a rewriting step because the monomial u appears in the context $u + f$.

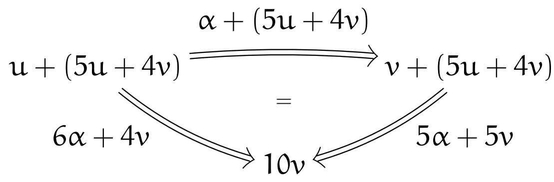
2.2.6. Exercise, [GHM17, Lemma 3.1.2]. Let Λ be a left-monomial linear 2-polygraph and let a be an elementary 2-cell of the 2-algebra Λ_2^ℓ. Show that a can be factorised in the 2-algebra Λ_2^ℓ into



where b and c are either identities of rewriting steps.

2.2.7. Example. Let Λ be a linear 2-polygraph and let $\alpha : u \Rightarrow v$ be a 2-cell of Λ_2. The 2-cell $-\alpha + (u + v)$ and $\alpha + (5u + 4v)$ are not rewriting steps of Λ. They can be decomposed respectively as follows:





2.2.8. Rewriting sequences. A 2-cell a of Λ_2^ℓ is *positive*, or a *rewriting sequence*, if it is an identity or a 1-composite

$$f_0 \overset{a_1}{\Longrightarrow} f_1 \Longrightarrow \cdots \Longrightarrow f_{k-1} \overset{a_k}{\Longrightarrow} f_k$$

of rewriting steps of Λ.

2.2.9. Reduced cells. A 1-cell f of Λ_1^ℓ is called *reduced*, or *irreducible*, *with respect to* Λ_2, if there is no rewriting step of Λ with source f. As a consequence, a 1-cell is reduced if and only if it is the zero 1-cell of Λ_1^ℓ, or a linear combination of reduced monomials in Λ_1^*. The reduced 1-cells of Λ_1^ℓ form a vector subspace of Λ_1^ℓ, denoted by Λ_1^{ir}. Since Λ is left-monomial, the set of reduced monomials of Λ_1^*, denoted by Λ_1^{irm}, forms a basis of the vector space Λ_1^{ir}.

We denote by $s_1(\Lambda)$ the set of *redex* of a reduced left-monomial linear 2-polygraph Λ defined by

$$s_1(\Lambda) = \{s_1(\alpha) \mid \alpha \text{ in } \Lambda_2\}.$$

In [Ani86], a redex is called an *obstruction*. The number of possible application of rules of Λ_2 to a monomial u is $\mathrm{Occ}_{s_1(\Lambda)}(u)$.

2.2.10. Reduced linear 2-polygraphs. We say that a linear 2-polygraph Λ is *left-reduced* if, for every 2-cell α in Λ_2, the 1-cell $s_1(\alpha)$ is reduced with respect to $\Lambda_2 \setminus \{\alpha\}$. We say that Λ is *right-reduced* if, for every 2-cell α of Λ, the 1-cell $t_1(\alpha)$ is reduced. The linear polygraph Λ is *reduced* if it is both left-reduced and right-reduced.

2.2.11. Exercise. Show that any left-monomial linear 2-polygraph is Tietze equivalent to a reduced left-monomial linear 2-polygraph.

2.2.12. Normal forms. If f is a 1-cell of Λ_1^ℓ, a *normal form for* f *with respect to* Λ_2 is a reduced 1-cell g of Λ_1^ℓ such that there exists a positive 2-cell $a : f \Rightarrow g$ in Λ_2^ℓ.

2.3. Termination of linear 2-polygraphs

We recall the notions of rewrite relation and termination for linear 2-polygraphs from [GHM17, 3.2]. Let us fix a left-monomial linear 2-polygraph Λ.

2.3.1. Termination. The *rewrite relation of* Λ is the smallest transitive binary relation on Λ_1^*, denoted by $\prec_\Lambda$, such that

- **i)** the relation $\prec_\Lambda$ is *compatible with* Λ_2, that is $w \prec_\Lambda u$ for every 2-cell $\alpha : u \Rightarrow f$ of Λ and every monomial w in $\mathrm{Supp}(f)$,
- **ii)** the relation $\prec_\Lambda$ is *compatible with products*, that is $u' \prec_\Lambda u$ implies $vu'w \prec_\Lambda vuw$ for every monomials u, u', v and w of Λ_1^*.

We say that the 2-polygraph Λ *terminates* if the rewrite relation $\prec_\Lambda$ is well-founded, that is, there is no infinite descending chains in Λ_1^*:

$$u_1 \succ_\Lambda u_2 \succ_\Lambda \ldots \succ_\Lambda u_n \succ_\Lambda u_{n+1} \succ_\Lambda \ldots$$

2.3.2. Example. Consider the linear 2-polygraph $\Lambda = \langle * \mid x, y \mid xy \overset{\alpha}{\Longrightarrow} x^2 + y^2 \rangle$. We have $xy \succ_\Lambda x^2$ and $xy \succ_\Lambda y^2$. Following compatibility with products we have

$$x^2y \succ_\Lambda xy^2 \succ_\Lambda x^2y.$$

Hence the relation $\prec_\Lambda$ is not well-founded and the polygraph Λ is not terminating. Note that, we have an infinite sequence of rewriting steps:

$$x^2y \overset{x\alpha}{\Longrightarrow} x^3 + xy^2 \overset{x^3 + \alpha y}{\Longrightarrow} x^3 + y^3 + x^2y \Longrightarrow \ldots$$

2.3.3. The rewrite relation on 1-cells. The rewrite relation $\prec_\Lambda$ is extended to the 1-cells of Λ_1^ℓ by setting, for any 1-cells f and g, $g \prec_\Lambda f$ if the following two conditions hold

- **i)** there exists a monomial in $\mathrm{Supp}(f)$ which is not in $\mathrm{Supp}(g)$,
- **ii)** for any monomial v in $\mathrm{Supp}(g) \setminus \mathrm{Supp}(f)$, there exists a monomial u in $\mathrm{Supp}(f) \setminus \mathrm{Supp}(g)$, such that $v \prec_\Lambda u$

2.3.4. Proposition. *The rewrite relation $\prec_\Lambda$ is well-founded on* 1*-cells if and only if it is well-founded on monomials.*

If Λ terminates, then for every rewriting step a of Λ, we have $t_1(a) \prec_\Lambda s_1(a)$. This implies that the 2-algebra Λ_2^ℓ contains no infinite sequence of pairwise 1-composable rewriting steps

$$f_0 \overset{a_1}{\Longrightarrow} f_1 \Longrightarrow \cdots \Longrightarrow f_{k-1} \overset{a_k}{\Longrightarrow} f_k \Longrightarrow \cdots$$

so that every 1-cell of Λ_1^ℓ admits at least one normal form with respect to Λ_2.

2.4. Monomial orders

2.4.1. Monomial orders. A total order $\prec$ on the set of monomials Λ_1^* is a *monomial order* if the following conditions are satisfied

i) $\prec$ is a *well-order*, that is, there is no infinite descending chains in Λ_1^*.

$$u_1 \succ u_2 \succ u_3 \succ \ldots \succ u_n \succ u_{n+1} \succ \ldots$$

ii) $\prec$ is *compatible with the multiplicative structure* on monomials, that is

$$u \prec u' \quad \text{implies} \quad vuw \prec vu'w,$$

for all monomials u, u', v and w in Λ_1^*.

2.4.2. Example. Given a total order relation $\prec$ on Λ_1, we define the *left degree-wise lexicographic order generated by* $\prec$, or *deglex order generated by* $\prec$, as the order $\prec_{\text{deglex}}$ on Λ_1^* that compare two monomials first by degree and then lexicographically. It is defined by

i) $y_1 \ldots y_p \prec_{\text{deglex}} x_1 \ldots x_q$, if $p < q$,

ii) $y_1 \ldots y_{j-1} y_j \ldots y_p \prec_{\text{deglex}} y_1 \ldots y_{j-1} x_j \ldots x_p$, if $y_j \prec x_j$.

2.4.3. Exercise. Show that the order $\prec_{\text{deglex}}$ is a monomial order.

2.4.4. Exercise. Explain why the pure lexicographic order is not a monomial order. Show that it is neither a well-order nor compatible with the product of monomials.

2.4.5. Polygraph compatible with a monomial order. A linear 2-polygraph Λ is say to be *compatible with a monomial order* $\prec$ if for every 2-cell $\alpha : u \Rightarrow f$ of Λ_2, then $w \prec u$ for any monomial w in the support of f. The monomial order $\prec$ is thus a well-founded rewrite relation for Λ. It follows that any linear 2-polygraph compatible with a monomial order is terminating. The converse is false in general as we will see in Exercise 2.4.7.

2.4.6. Example. Consider the linear 2-polygraph $\Lambda = \langle * \mid x, y \mid x^2 \overset{\alpha}{\Longrightarrow} xy - y^2 \rangle$. It is Tietze equivalent to the linear 2-polygraph of Example 2.3.2, but it is terminating. Indeed, having $xy \prec x^2$ and $y^2 \prec x^2$, the linear 2-polygraph Λ is compatible with the deglex order $\prec_{\text{deglex}}$ induced by $y \prec x$, hence it is terminating. An other way to prove that Λ is terminating, is to count the number of occurrence of x in monomials. For any u in Λ_1^*, let denote by $A(u)$ the number of occurrence of x in u. To prove that the linear 2-polygraph Λ terminates, it is sufficient to check that, for every rewriting step $a : s_1(a) \Rightarrow f$, we have $A(s_1(a)) > A(v)$, for any monomial v in $\text{Supp}(f)$.

2.4.7. Exercise, [GHM17, Ex. 3.2.4]. Show that the linear 2-polygraph Λ given in Example 2.1.9 is terminating. Show that Λ is not compatible with a monomial order.

2.4.8. Exercise, [Ber78, Ex. 5.2.1.]. Examine termination of the linear 2-polygraph $\langle * \mid x, y \mid \alpha \rangle$ in each of the following situations

$$x^2y \overset{\alpha}{\Longrightarrow} yx, \qquad yx \overset{\alpha}{\Longrightarrow} x^2y, \qquad x^2y^2 \overset{\alpha}{\Longrightarrow} yx, \qquad yx \overset{\alpha}{\Longrightarrow} x^2y^2.$$

2.4.9. Noetherian induction principle. Let us recall the principle of noetherian induction for terminating rewriting systems, see [Hue80] for more details. Let Λ be a left-monomial terminating linear 2-polygraph. The principle can be used to prove by induction a property formulated on the 1-cells of Λ_1^ℓ. Given a property $\mathcal{P}(f)$ of the 1-cells f of Λ_1^ℓ. In order to show that $\mathcal{P}(f)$ holds for any 1-cell f of Λ_1^ℓ, it suffices to show that

i) $\mathcal{P}(f)$ holds for f reduced with respect to Λ_2,

ii) $\mathcal{P}(f)$ holds under the assumption that $\mathcal{P}(g)$ is hold for every $g \prec f$.

2.4.10. Leading terms. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. For a nonzero 1-cell f of Λ_1^ℓ, the *leading monomial of* f *with respect to* $\prec$ is the monomial of f, denoted by $\mathrm{lm}(f)$, such that $w \prec \mathrm{lm}(f)$, for any monomial w in the support of f. The *leading coefficient of* f is the coefficient $\mathrm{lc}(f)$ of $\mathrm{lm}(f)$ in f, and the *leading term of* f is the 1-cell $\mathrm{lt}(f) = \mathrm{lc}(f)\,\mathrm{lm}(f)$ of Λ_1^ℓ. We also define $\mathrm{lt}(0) = \mathrm{lc}(0) = \mathrm{lm}(0) = 0$.

Note that for any 1-cells f and g in Λ_1^ℓ, we have $f \prec g$ if and only if either $\mathrm{lm}(f) \prec \mathrm{lm}(g)$ or ($\mathrm{lm}(f) = \mathrm{lm}(g)$ and $f - \mathrm{lt}(f) \prec g - \mathrm{lt}(g)$). The following property

$$\mathrm{lt}(fg) = \mathrm{lt}(f)\,\mathrm{lt}(g),$$

for any 1-cells f and g is also useful.

2.4.11. Leading polygraph. Given a monomial order $\prec$ on Λ_1^ℓ and a nonzero 1-cell g in Λ_1^ℓ, we define the 2-cell:

$$\alpha_{g,\prec} : \mathrm{lm}(g) \;\Longrightarrow\; \mathrm{lm}(g) - \frac{1}{\mathrm{lc}(g)} g.$$

For any set $\mathcal{G}$ of nonzero 1-cells in Λ_1^ℓ, the *leading 2-polygraph* associated to $\mathcal{G}$ with respect to $\prec$ is the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ whose set of 1-cells is Λ_1 and

$$\Lambda(\mathcal{G}, \prec)_2 = \{\alpha_{g,\prec} \mid g \in \mathcal{G}\}.$$

By definition, the leading polygraph $\Lambda(\mathcal{G} \prec)$ is compatible with the monomial order $\prec$.

A monomial w in Λ_1^* is $\mathcal{G}$*-reduced with respect to the monomial order* $\prec$ if it reduced with respect to $\Lambda(\mathcal{G}, \prec)_2$, that is, there is no factorization $w = u\,\mathrm{lm}(g)v$, with u and v monomials in Λ_1^* and g in $\mathcal{G}$. A set $\mathcal{G}$ of 1-cells is *reduced with respect to the monomial order* $\prec$ if for any 1-cell g in $\mathcal{G}$, any monomial in the support of g is $(\mathcal{G} \setminus \{g\})$-reduced.

3. Convergence in linear rewriting systems

3.1. Ideal of a linear 2-polygraph

3.1.1. The ideal of a linear 2-polygraph. Given a linear 2-polygraph Λ. We denote by $I(\Lambda)$ the two-sided ideal of the free algebra Λ_1^ℓ generated by the following set of 1-cells

$$\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}.$$

The ideal $I(\Lambda)$ is made of the linear combinations

$$\sum_{i=1}^{p} \lambda_i u_i (s_1(\alpha_i) - t_1(\alpha_i)) v_i,$$

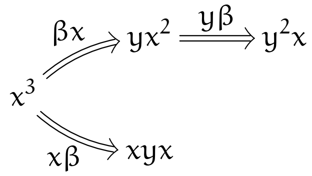
for pairwise distinct 2-monomials $u_1\alpha_1 v_1, \ldots, u_p \alpha_p v_p$ of Λ_1^ℓ, and nonzero scalars $\lambda_1, \ldots, \lambda_p$. Note that the algebra presented by Λ is isomorphic to the quotient of the free algebra Λ_1^ℓ by the ideal $I(\Lambda)$.

3.1.2. Exercise. Let Λ be a linear 2-polygraph. Given 1-cells f and g in Λ_1^ℓ, show that the 1-cell $f - g$ belongs to $I(\Lambda)$ if and only if there exists a 2-cell $a : f \Rightarrow g$ in Λ_2^ℓ.

3.1.3. Suppose that Λ is a terminating left-monomial linear 2-polygraph. Every 1-cell f of Λ_1^ℓ admits at least a normal form $\widetilde{f}$. That is, $\widetilde{f}$ is reduced and there exists a positive 2-cell $a : f \Rightarrow \widetilde{f}$ in Λ_2^ℓ. As a consequence, we have a decomposition $f = \widetilde{f} + (f - \widetilde{f})$, with $\widetilde{f}$ in Λ_1^{ir} and $f - \widetilde{f}$ in $I(\Lambda)$ by Exercice 3.1.2. It follows that the vector space Λ_1^ℓ admits the following decomposition

$$\Lambda_1^\ell = \Lambda_1^{ir} + I(\Lambda). \tag{8}$$

3.1.4. Example. Note that the decomposition (8) is not direct in general. Indeed, consider the linear 2-polygraph Λ from Example 2.1.10. It is terminating thanks to the deglex order generated by $x > y$. Consider the two following reduction sequences reducing the 1-cell x^3:



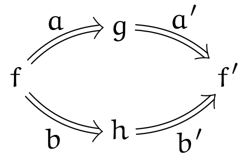
Thus the 1-cell

$$xyx - y^2x = (x^2 - yx)x - x(x^2 - yx) + y(x^2 - yx)$$

is both in Λ_1^{ir} and $I(\Lambda)$. It follows that the sum $\Lambda_1^{ir} + I(\Lambda)$ is not direct. We will see in the next section a sufficient condition on the linear 2-polygraph Λ to have a direct decomposition.

3.2. Confluence and convergence

3.2.1. Branchings and confluence. Let Λ be a left-monomial linear 2-polygraph. A *branching of* Λ is a non-ordered pair (a, b) of positive 2-cells of Λ_2^ℓ with a common *source* $s_1(a) = s_1(b)$. A branching (a, b) is *local* if both a and b are rewriting steps of Λ. A branching (a, b) of Λ is *confluent* if there exist positive 2-cells a' and b' of Λ as in the following diagram



We say that Λ is *confluent* (resp. *locally confluent*) if every branching (resp. local branching) of Λ is confluent. An immediate consequence of the confluence property is that every 1-cell of Λ_1^ℓ admits at most one normal form.

Under termination hypothesis, we have the following characterization of the confluence.

3.2.2. Proposition. *Let Λ be a terminating left-monomial linear 2-polygraph. The following conditions are equivalent.*

i) Λ *is confluent.*

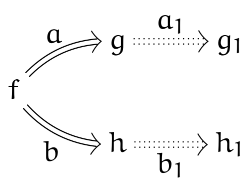
ii) *Every* 1*-cell of* $I(\Lambda)$ *admits* 0 *as a normal form with respect to* Λ_2*.*

iii) *The vector space* Λ_1^ℓ *admits the direct decomposition* $\Lambda_1^\ell = \Lambda_1^{ir} \oplus I(\Lambda)$.

Proof. **i)** $\Rightarrow$ **ii)**. Let f be a 1-cell in the ideal $I(\Lambda)$, then there exists a 2-cell $\alpha : f \Rightarrow 0$ in Λ_2^ℓ. The polygraph Λ being confluent, the 1-cells f and 0 have the same normal form. Finally, 0 being reduced, this implies that 0 is a normal form for f.

ii) $\Rightarrow$ **iii)**. Prove that $\Lambda_1^{ir} \cap I(\Lambda) = 0$. If f is in Λ_1^{ir}, then $\widehat{f} = f$ is reduced and, thus, admits itself as normal form. If f is in $I(\Lambda)$, then $\widehat{f} = 0$ by **ii)**. Hence $\Lambda_1^{ir} \cap I(\Lambda) = 0$.

iii) $\Rightarrow$ **i)**. Given a branching $(f \overset{a}{\Longrightarrow} g, f \overset{b}{\Longrightarrow} h)$. Since Λ terminates, the 1-cells g and h admit normal forms, say g_1 and h_1 respectively, and there exist positive 2-cells a_1 and b_1 in Λ_2^ℓ:



with g_1 and h_1 reduced. It follows that $g_1 - h_1$ is also reduced. Moreover, the 2-cell $(a \star_1 a_1)^- \star_1 (b \star b_1)$ has g_1 as source and h_1 as target. This implies that $g_1 - h_1$ is also in $I(\Lambda)$. As $\Lambda_1^{ir} \cap I(\Lambda) = 0$, we have $g_1 - h_1 = 0$, hence the branching (a, b) is confluent. □

3.2.3. Convergence. We say that a left-monomial linear 2-polygraph Λ is *convergent* if it terminates and it is confluent. In that case, every 1-cell f of Λ_1^ℓ has a unique normal form, denoted by $\widehat{f}$, such that $\overline{f} = \overline{g}$ holds in $\overline{\Lambda}$ if and only if $\widehat{f} = \widehat{g}$ holds in Λ_1^ℓ.

As a consequence, if Λ is a convergent presentation of an algebra **A**, the assignment of every 1-cell f of **A** to the normal form $\widehat{f}$, defines a section $\iota : \mathbf{A} \longrightarrow \Lambda_1^\ell$ of the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$. The section ι is a linear map, i.e., it satisfies $\widehat{\lambda f + \mu g} = \lambda\widehat{f} + \mu\widehat{g}$, and it preserves the identities because Λ terminates.

3.2.4. Exercise. Show that the section ι is not a morphism of algebras in general.

3.2.5. Suppose that Λ is a convergent linear 2-polygraph. By Proposition 3.2.2 the following sequence of vector spaces is exact:

$$0 \longrightarrow I(\Lambda) \longrightarrow \Lambda_1^\ell \longrightarrow \Lambda_1^{ir} \longrightarrow 0.$$

The vector space Λ_1^{ir} admits Λ_1^{irm} as a basis, hence Λ_1^{irm} forms a linear basis of the quotient algebra $\Lambda_1^\ell/I(\Lambda)$. The polygraph Λ being convergent, any 1-cell of Λ_1^ℓ has a unique normal form, hence the product defined by $f \cdot g = \widehat{fg}$ is associative. Indeed, for any 1-cells f, g and h, we have

$$(f \cdot g) \cdot h = \widehat{fg} \cdot h = \widehat{\widehat{fg}h} = \widehat{f\widehat{gh}} = f \cdot \widehat{gh} = f \cdot (g \cdot h).$$

It follows that this product equips Λ_1^{ir} with a structure of algebra in such a way that Λ_1^{ir} is isomorphic to the quotient algebra $\Lambda_1^\ell/I(\Lambda)$. We have thus proved the following result.

3.2.6. Theorem ([GHM17, Thm 3.4.2]). *Let* **A** *be an algebra and* Λ *be a convergent presentation of* **A**. *The set* Λ_1^{irm} *of reduced monomials is a linear basis of* **A**. *Moreover, the vector space* Λ_1^{ir} *equipped with the product defined by* $f \cdot g = \widehat{fg}$, *for any* 1*-cells* f *and* g *in* Λ_1^{ir}, *is an algebra isomorphic to* **A**.

3.2.7. Exercise. Compute a linear basis of the algebra presented by $\langle \ast \mid x, y \mid xy = x^2 \rangle$.

3.2.8. Exercise. Compute a linear basis for the symmetric algebra on k variables presented by

$$\langle x_1, \ldots, x_k \mid x_i x_j \overset{\tau_{ij}}{\Longrightarrow} x_j x_i \mid 1 \leqslant i < j \leqslant k \rangle$$

and for the skew-polynomial algebra on k variables presented by

$$\langle x_1, \ldots, x_k \mid x_i x_j \overset{\tau_{ij}}{\Longrightarrow} q_i^j x_j x_i \mid 1 \leqslant i < j \leqslant k \rangle,$$

where the q_i^j are scalars in $\mathbb{K}$.

3.2.9. Exercise: Poincaré-Birkhoff-Witt theorem, [Bok76, §1], [Ber78, Thm. 3.1]. Consider an ordered bases $x_1 \prec x_2 \prec \ldots \prec x_k$ of a Lie algebra $\mathfrak{g}$. Consider the following ideals of the free tensor algebra $T(\mathfrak{g})$ over $\mathfrak{g}$:

$$I = \langle x_j x_i - x_i x_j \mid 1 \leqslant i < j \leqslant k \rangle,$$
$$J = \langle x_j x_i - x_i x_j + [x_i, x_j] \mid 1 \leqslant i < j \leqslant k \rangle.$$

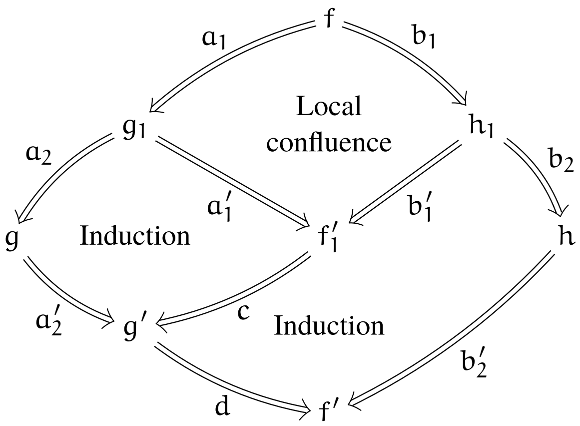
Show that the symmetric algebra $S(\mathfrak{g}) = T(\mathfrak{g})/I$ and the enveloping algebra $U(\mathfrak{g}) = T(\mathfrak{g})/J$ are isomorphic as vector spaces.

3.2.10. From local to global confluence. The Newman lemma, also called the diamond lemma, states that for terminating rewriting systems local confluence and confluence are equivalent properties. This result was proved by Newman in [New42] for abstract rewriting systems. A short and simple proof of this result was given by Huet in [Hue80] using the principle of noetherian induction. Let us recall the arguments of this proof for linear 2-polygraphs.

3.2.11. Theorem (Newman's Lemma). *Let Λ be a terminating left-monomial linear 2-polygraph. Then Λ is confluent if and only if it is locally confluent.*

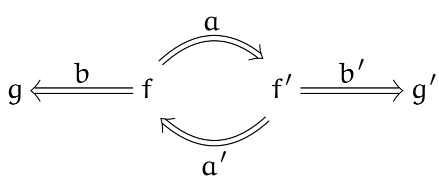
Proof. The proof works as for abstract rewriting systems. One implication is trivial. Suppose Λ locally confluent and prove that it is confluent at every 1-cell f of Λ_1^ℓ. We proceed by noetherian induction on f using the principle given in 2.4.9. If f is reduced, the only branching with source f is $(1_f, 1_f)$ which is confluent.

Suppose that f is a nonreduced 1-cell of Λ_1^ℓ and such that Λ is confluent at every 1-cell $g \prec f$. Consider a branching (a, b) of Λ with source f. If a or b is an identity, then (a, b) is confluent. Otherwise, we prove that the branching (a, b) is confluent by induction. Since a and b are not identities, they admit decompositions $a = a_1 \star_1 a_2$ and $b = b_1 \star_1 b_2$ where a_1 and b_1 are rewriting steps, and a_2 and b_2 are positive 2-cells. By local confluence, the local branching (a_1, b_1) is confluent. Hence there exist positive 2-cells a_1' and b_1' as indicated in the following diagram



We have $g_1 \prec_\Lambda f$ and $h_1 \prec_\Lambda f$. Then we apply the induction hypothesis on the branching (a_2, a_1') to get positive 2-cells a_2' and c, and, then, to the branching $(b_1' \star_1 c, b_2)$ to get positive 2-cells d and b_2', which complete the proof. □

3.2.12. Example, [Hue80]. The requirement of noetherianity is necessary to prove confluence from local confluence. Indeed, consider the 2-polygraph generated by the following four 2-cells

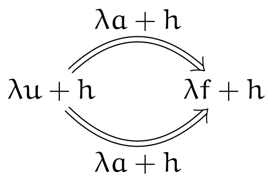


It is locally confluent but it is not confluent.

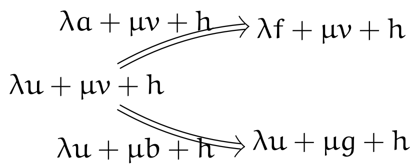
3.3. Critical branching lemma

3.3.1. Local branchings. A case analysis leads to a partition of the local branchings of a left-monomial linear 2-polygraph Λ into the following four families, see [GHM17, 3.3.2] for details.

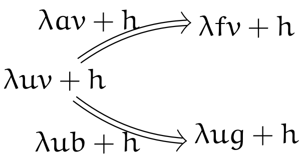
i) *Aspherical* branchings, for all 2-monomial $\alpha : u \Rightarrow f$ of Λ_2^ℓ, nonzero scalar λ, and 1-cell h of Λ_1^ℓ such that the monomial u is not in the support of h:



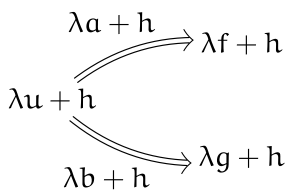
ii) *Additive* branchings, for all 2-monomials $\alpha : u \Rightarrow f$ and $\beta : v \Rightarrow g$ of Λ_2^ℓ, nonzero scalars λ and μ, and 1-cell h of Λ_1^ℓ such that the monomials u and v are not in the support of h:



iii) *Peiffer* branchings, for all 2-monomials $\alpha : u \Rightarrow f$ and $\beta : v \Rightarrow g$ of Λ_2^ℓ, nonzero scalar λ, and 1-cell h of Λ_1^ℓ such that the monomial uv is not in the support of h:



iv) *Overlapping* branchings, for all 2-monomials $\alpha : u \Rightarrow f$ and $\beta : u \Rightarrow g$ of Λ_2^ℓ such that the branching (α, β) is neither aspherical nor Peiffer, and all nonzero scalar λ and 1-cell h of Λ_1^ℓ such that the monomial u is not in the support of h:



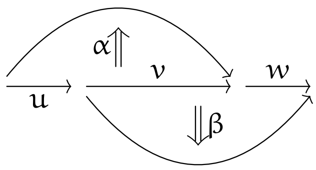
3.3.2. Critical branchings. A *critical branching of a left-monomial linear 2-polygraph* Λ is an overlapping branchings, as defined in 3.3.1, with $\lambda = 1$ and $h = 0$, and that is minimal for the relation on branchings defined by

$$(a, b) \sqsubseteq (waw', wbw') \quad \text{for any } w \text{ and } w' \text{ in } \Lambda_1^*.$$

By case analysis on the source of critical branchings, they must have one of the following two shapes

with α, β in Λ_2. When the linear 2-polygraph Λ is reduced, the first case cannot occur since, otherwise, the monomial $s_1(\alpha)$ would be reducible by β.

3.3.3. Exercise. Let Λ be a reduced linear 2-polygraph. Show that for any critical branching



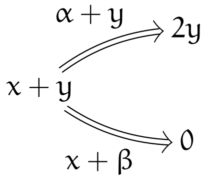
the monomial u, v and w are reduced and cannot be identities or null.

3.3.4. Critical branching lemma. By Newman's lemma 3.2.11, for terminating rewriting systems, local confluence and confluence are equivalent properties. It turns out that one can decide whether a rewriting system is convergent by checking local confluence. For string rewriting systems, that is 2-polygraphs, the critical branching lemma states that local confluence is equivalent to the confluence of all critical branching, see [GM18, 3.1.5] for details. For linear 2-polygraphs the critical branching lemma given in [GHM17] differs from the case of 2-polygraphs. Indeed, in the linear setting the termination hypothesis is required. Moreover, nonoverlapping branchings may be non confluent as illustrated by the following example in which an additive branching is nonconfluent.

3.3.5. Example. Some local branchings can be nonconfluent without termination, even if critical confluence holds. Indeed, consider the linear 2-polygraph

$$\langle \, * \mid x, y \mid x \overset{\alpha}{\Longrightarrow} y,\ y \overset{\beta}{\Longrightarrow} -x \,\rangle$$

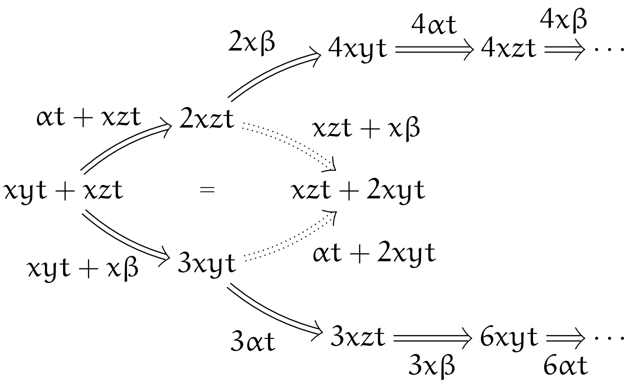
It has no critical branching, but it has a nonconfluent additive branching:



Here another example from [GHM17, Rem. 4.2.4] for instance the following linear 2-polygraph

$$\langle * \mid x, y, z, t \mid xy \overset{\alpha}{\Longrightarrow} xz,\ zt \overset{\beta}{\Longrightarrow} 2yt \rangle$$

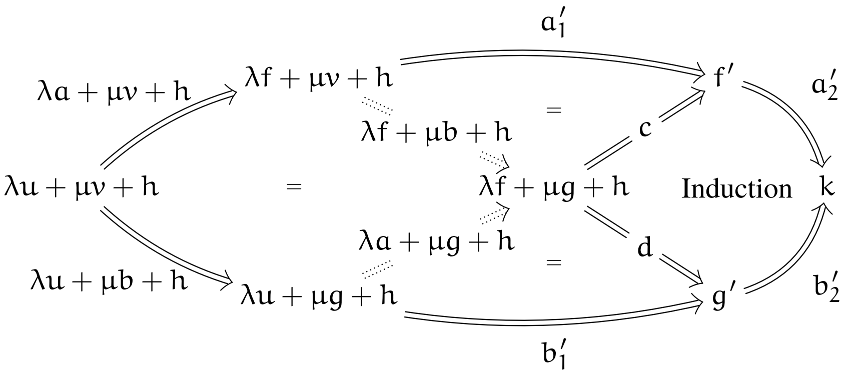
has no critical branching, but it has a nonconfluent additive branching:



3.3.6. If a linear 2-polygraph Λ is terminating and with any critical branching confluent, we can show that such an additive branching is confluent by noetherian induction on the sources of the branchings. Let consider an additive branching $(\lambda u + \mu v + h, \lambda u + \mu g + h)$ as in 3.3.1 and suppose that Λ is locally confluent at every $g \prec_\Lambda \lambda u + \mu v + h$. By linearity of the 1-composition, the following equation

$$(\lambda a + \mu v + h) \star_1 (\lambda f + \mu b + h) = (\lambda u + \mu b + h) \star_1 (\lambda a + \mu g + h)$$

holds in the free 2-algebra Λ_2^ℓ:



Note that the dotted 2-cells $\lambda a + \mu g + h$ and $\lambda f + \mu b + h$ may be not positive in general. Indeed, the monomial u can be in the support of g or the monomial v can be in the support of f, as illustrated in Example 3.3.5. However, those 2-cells are elementary, hence there exist, see Exercise 2.2.6, positive 2-cells a'_1, b'_1, c and d that satisfy

$$a'_1 = (\lambda f + \mu b + h) \star_1 c \qquad \text{and} \qquad b'_1 = (\lambda a + \mu g + h) \star_1 d.$$

We have $f \prec_\Lambda u$ and $g \prec_\Lambda v$, hence $\lambda f + \mu g + h \prec_\Lambda \lambda u + \mu v + h$. Thus, the branching (c, d) is confluent by induction hypothesis, yielding the positive 2-cells a_2' and b_2'.

Under terminating hypothesis, all local branching given in 3.3.1 are confluent if all critical branching are confluent, see [GHM17, 4.2] for a proof of this result.

3.3.7. Theorem (Critical branching lemma, [GHM17, Cor. 4.2.2]). *A terminating left-monomial linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent.*

As consequence of the critical branching lemma and of Newman's lemma 3.2.11, a terminating left-monomial linear 2-polygraph is confluent if all its critical branchings are confluent. In particular a terminating left-monomial 2-polygraph with no critical branching is convergent.

3.3.8. Example. The linear 2-polygraph given in Example 2.1.9 is terminating, see Exercise 2.4.7. Moreover, it does not have critical branching, hence it is convergent.

3.3.9. The Knuth-Bendix completion procedure. Let us recall the completion procedure introduced in [KB70] to the setting of linear 2-polygraphs. Let Λ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on Λ_1^*. A *Knuth-Bendix completion* of Λ is a linear 2-polygraph $\mathcal{KB}(\Lambda)$ obtained by the following procedure that examines the confluence of the set of critical branchings.

Input: Λ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on Λ_1^*.
$\mathcal{KB}(\Lambda) := \Lambda$
$\mathcal{C}b := \{$ critical branchings with respect to Λ_2 $\}$
while $\mathcal{C}b \neq \emptyset$ **do**
  Picks a branching in $\mathcal{C}b$:

$$u \overset{f}{\Rightarrow} v, \qquad u \underset{g}{\Rightarrow} w$$

  $\mathcal{C}b := \mathcal{C}b \setminus \{(f, g)\}$
  Reduce v to a normal form $\widehat{v}$ with respect to $\mathcal{KB}(\Lambda)_2$
  Reduce w to a normal form $\widehat{w}$ with respect to $\mathcal{KB}(\Lambda)_2$

$$u \overset{f}{\Rightarrow} v \Rightarrow \widehat{v}, \qquad u \underset{g}{\Rightarrow} w \Rightarrow \widehat{w}$$

  $g = \widehat{v} - \widehat{w}$
  if $g \neq 0$ **then**
    $\mathcal{KB}(\Lambda)_2 := \mathcal{KB}(\Lambda)_2 \cup \{ \alpha_{g,\prec} : \mathrm{lm}(g) \Rightarrow \mathrm{lm}(g) - \frac{1}{\mathrm{lc}(g)} g \}$
    $\mathcal{C}b := \mathcal{C}b \cup \{$ critical branching created by $\alpha_{g,\prec}$ $\}$
  end
end

If the procedure stops, it returns a finite convergent left-monomial linear 2-polygraph $\mathcal{KB}(\Lambda)$. Otherwise, it builds an increasing sequence of left-monomial linear 2-polygraphs, whose limit is also denoted by $\mathcal{KB}(\Lambda)$. Note that, if the starting linear 2-polygraph Λ is convergent, then the Knuth-Bendix completion of Λ is Λ itself. The linear 2-polygraph $\mathcal{KB}(\Lambda)$ obtained by this procedure depends on the order of examination of the critical branchings. Finally, since all the operations of adding new rules performed by the procedure are Tietze transformations, the linear 2-polygraph $\mathcal{KB}(\Lambda)$ is Tietze-equivalent to Λ.

3.3.10. Exercice, [GHM17, Rem. 4.2.4]. Prove that the following linear 2-polygraph has a nonconfluent Peiffer branching

$$\langle * \mid x, y, z \mid xy \overset{\alpha}{\Longrightarrow} 2x,\ yz \overset{\beta}{\Longrightarrow} z \rangle.$$

3.3.11. Weyl algebras. Let $\mathbb{K}$ be a field of characteristic zero. The *Weyl algebra* of dimension n over $\mathbb{K}$ is the algebra presented by the linear 2-polygraph whose 1-cells are

$$x_1, \ldots, x_n, \partial_1, \ldots, \partial_n$$

and with the following 2-cells:

$$x_ix_j \Rightarrow x_jx_i, \quad \partial_i\partial_j \Rightarrow \partial_j\partial_i, \quad \partial_ix_j \Rightarrow x_j\partial_i, \qquad \text{for any } 1 \leqslant i < j \leqslant n,$$

$$\partial_ix_i \Rightarrow x_i\partial_i + 1, \qquad \text{for any } 1 \leqslant i \leqslant n.$$

This polygraph is convergent with the following six families of confluent critical branchings:

$$x_ix_jx_k \Rightarrow x_jx_ix_k \Rightarrow x_jx_kx_i \Rightarrow x_kx_jx_i, \qquad x_ix_jx_k \Rightarrow x_ix_kx_j \Rightarrow x_kx_ix_j \Rightarrow x_kx_jx_i$$

$$\partial_i\partial_j\partial_k \Rightarrow \partial_j\partial_i\partial_k \Rightarrow \partial_j\partial_k\partial_i \Rightarrow \partial_k\partial_j\partial_i, \qquad \partial_i\partial_j\partial_k \Rightarrow \partial_i\partial_k\partial_j \Rightarrow \partial_k\partial_i\partial_j \Rightarrow \partial_k\partial_j\partial_i$$

$$\partial_ix_jx_k \Rightarrow x_j\partial_ix_k \Rightarrow x_jx_k\partial_i \Rightarrow x_kx_j\partial_i, \qquad \partial_ix_jx_k \Rightarrow \partial_ix_kx_j \Rightarrow x_k\partial_ix_j \Rightarrow x_kx_j\partial_i$$

$$\partial_i\partial_jx_k \Rightarrow \partial_j\partial_ix_k \Rightarrow \partial_jx_k\partial_i \Rightarrow x_k\partial_j\partial_i, \qquad \partial_i\partial_jx_k \Rightarrow \partial_ix_k\partial_j \Rightarrow x_k\partial_i\partial_j \Rightarrow x_k\partial_j\partial_i$$

$$\partial_ix_jx_k \Rightarrow x_i\partial_ix_j + x_j \Rightarrow x_ix_j\partial_i + x_j \Rightarrow x_jx_i\partial_i + x_j, \qquad \partial_ix_jx_k \Rightarrow \partial_ix_jx_i \Longrightarrow x_j\partial_ix_i \Rightarrow x_jx_i\partial_i + x_j$$

$$\partial_i\partial_jx_j \Rightarrow \partial_j\partial_ix_j \Longrightarrow \partial_jx_j\partial_i \Rightarrow x_j\partial_j\partial_i + \partial_i, \qquad \partial_i\partial_jx_j \Rightarrow \partial_ix_j\partial_j + \partial_i \Rightarrow x_j\partial_i\partial_j + \partial_i \Rightarrow x_j\partial_j\partial_i + \partial_i$$

where $1 \leqslant i < j \leqslant n$.

3.3.12. Exercice. In his seminal paper on the diamond lemma, Bergman point out that he was first led to the ideas of his paper with the following American Mathematical Monthly Advanced Problem 5082, [Ber78, 2.1.].

> *Let* R *be a ring in which, if either* $x + x = 0$ *or* $x + x + x = 0$*, it follows that* $x = 0$*. Suppose that* a, b, c *and* $a + b + c$ *are all idempotents in* R*. Does it follows that* $ab = 0$*?*

Solve this problem. [Hints. Consider the following linear 2-polygraph:

$$\Lambda = \langle * \mid a, b, c \mid a^2 \Rightarrow a,\ b^2 \Rightarrow b,\ c^2 \Rightarrow c,\ ba \Rightarrow -ab - bc - cb - ac - ca \rangle.$$

1/ List all critical branchings of Λ. 2/ Compute a convergent left-monomial linear 2-polygraph $\mathcal{KB}(\Lambda)$ by applying the Knuth-Bendix completion procedure to Λ. 3/ List all irreducible monomials with respect to $\mathcal{KB}(\Lambda)_2$. 4/ Conclude that $ab \neq 0$.]

3.4. Composition Lemma

3.4.1. Compositions in free Lie algebras. Shirshov introduced in [Shi62] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of *composition* of elements in a free Lie algebra, that corresponds to the notion of S*-polynomial* in the work of Buchberger, [Buc65]. This work remained unknown outside the USSR and the two theories were developed in parallel. The algorithm *completes* a given set of elements in a free algebra by adding all nontrival compositions. This algorithm corresponds to the completion algorithm given by Knuth-Bendix for term rewriting systems, [KB70], and by Buchberger for commutative polynomials, [Buc65]. The Shirshov completion constructs a set, that may be infinite, such that every composition of its elements is trivial. Such a subset is called a *Lie Gröbner-Shirshov basis*. The key result in [Shi62] states that the set of irreducible elements for a Gröbner-Shirshov basis $\mathcal{S}$ forms a linear basis of the Lie algebra with defining relations $\mathcal{S}$. This result is called now the *Composition-Diamond Lemma* for Lie algebras. For a recent account of the theory of Gröbner-Shirshov we refer the reader to [BC14].

In this subsection we summarize without proofs an analogue of Shirshov's composition-diamond lemma for associative algebras given by Bokut in [Bok76].

3.4.2. Compositions. Bokut introduced in [Bok76] the notion of composition of elements of a free associative algebra as follows. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. Given two 1-cells f and g in Λ_1^ℓ and a monomial w in Λ_1^*. There are two kinds of compositions:

i) if $w = \mathrm{lm}(f)v = u\,\mathrm{lm}(g)$ with $\ell(\mathrm{lm}(f)) + \ell(\mathrm{lm}(g)) > \ell(w)$, for some monomials u and v in Λ_1^*, then the 1-cell

$$(f, g)_w = \frac{1}{\mathrm{lc}(f)} fv - \frac{1}{\mathrm{lc}(g)} ug$$

is called the *intersection composition* of f and g with respect to w.

ii) if $w = \mathrm{lm}(f) = u\,\mathrm{lm}(g)v$, for some monomials u and v in Λ_1^*, then the 1-cell

$$(f, g)_w = \frac{1}{\mathrm{lc}(f)} f - \frac{1}{\mathrm{lc}(g)} ugv$$

is called the *inclusion composition* of f and g with respect to w.

A composition $(f,g)_w$ can also be called an S-*polynomial* of f and g with respect to w. A composition $(f,g)_w$ is either zero or satisfy $(f,g)_w \prec w$. Moreover the composition $(f,g)_w$ is in the ideal $\langle f,g\rangle$ generated by f and g. Note that a composition $(f,g)_w$ depends on the two polynomials f and g as well as the monomial w. Indeed, in some cases two polynomials f and g may overlap with different combinations creating several compositions.

3.4.3. Example. Consider the polynomial $f = x^2 - xy$. With respect to the deglex order generated by $x > y$, we have

$$(f,f)_{x^3} = x^3 - xyx - x^3 + x^2y = x^2y - xyx.$$

Compare with Example 3.1.4.

3.4.4. Gröbner-Shirshov bases. Let $\mathcal{G}$ be a set of nonzero 1-cells in Λ_1^ℓ. Given a monomial w in Λ_1^*, a 1-cell h is *trivial modulo* $(\mathcal{G}, w)$ if there exists a decomposition

$$h = \sum_{i\in I} \lambda_i u_i g_i v_i,$$

with λ_i in $\mathbb{K}\setminus\{0\}$, u_i, v_i in Λ_1^* and g_i in $\mathcal{G}$ such that $u_i\,\mathrm{lm}(g_i)v_i \prec w$.

A set $\mathcal{G}$ of nonzero 1-cells in Λ_1^ℓ is a *Gröbner-Shirshov basis* with respect to the monomial ordering $\prec$ if every composition $(f,g)_w$ of 1-cells in $\mathcal{G}$ is trivial modulo $(\mathcal{G}, w)$. A Gröbner-Shirshov basis $\mathcal{G}$ is *minimal* if there is no inclusion composition with elements of $\mathcal{G}$. A minimal Gröbner-Shirshov basis $\mathcal{G}$ is called *closed under composition* in [Bok76]. Finally, a Gröbner-Shirshov basis $\mathcal{G}$ is *reduced* if the set $\mathcal{G}$ is reduced with respect to the monomial order $\prec$.

3.4.5. Exercise. Let $\mathcal{G}$ be a minimal Gröbner-Shirshov basis in a free algebra Λ_1^ℓ. Suppose that there exists a decomposition

$$w = u_1\,\mathrm{lm}(g_1)v_1 = u_2\,\mathrm{lm}(g_2)v_2,$$

with $u_1, v_1, u_2, v_2 \in \Lambda_1^*$ and $g_1, g_2 \in \mathcal{G}$. Show that $u_1g_1v_1 - u_2g_2v_2$ is trivial modulo $(\mathcal{G}, w)$.

3.4.6. Theorem (The Composition Lemma, [Bok76, Prop. 1 & Cor. 1]). *Let Λ_1^ℓ be a free algebra and let $\prec$ be a monomial order on Λ_1^ℓ. Let $\mathcal{G}$ be a set of 1-cells in Λ_1^ℓ and let I be the ideal generated by $\mathcal{G}$. The following conditions are equivalent*

- **i)** *$\mathcal{G}$ is a Gröbner-Shirshov basis,*
- **ii)** *For any f in I, there exists a factorization $\mathrm{lm}(f) = u\,\mathrm{lm}(g)v$ for some u, v in Λ_1^* and g in $\mathcal{G}$,*
- **iii)** *The set of $\mathcal{G}$-reduced monomial forms a linear basis of the algebra given by the quotient of the free algebra Λ_1^ℓ by the ideal I.*

3.5. Reduction operators

3.5.1. Reduction operators. Another approach of rewriting in associative algebras were developed by Bergman in [Ber78]. With a functional description of linear rewriting reductions he obtained an equivalent result of the composition lemma 3.4.6. Given Λ_1^ℓ a free algebra over a set Λ_1, he defines a *reduction* *system* as a set S of pairs $\sigma = (w_\sigma, f_\sigma)$, where w_σ is a monomial of Λ_1^ℓ and f_σ is a 1-cell of Λ_1^ℓ. Given σ in S and two monomials u, v in Λ_1^*, he considers the $\mathbb{K}$-linear map $r_{u\sigma v} : \Lambda_1^\ell \longrightarrow \Lambda_1^\ell$ defined by

$$r_{u\sigma v}(w) = \begin{cases} uf_\sigma v & \text{if } w = uw_\sigma v, \\ w & \text{otherwise.} \end{cases}$$

The endomorphism $r_{u\sigma v}$ is called *reduction by* σ. Note that this notion of reduction corresponds to the notion of rewriting step given in 2.2.4.

A 1-cell f in Σ_1^ℓ is *irreducible under* S if every reduction by elements of S acts trivially on f, that is $uw_\sigma v$ is not in the support of f, for any σ in S and monomials u, v in Σ_1^*. As in the case of linear 2-polygraphs, we denote by Λ_1^{ir} the vector subspace of Λ_1^ℓ of all irreducible 1-cells of Λ_1^ℓ.

3.5.2. Reduction-unique. Bergman introduced the notion of confluence for reduction systems as follows. A finite sequence of reductions $r_1, \ldots, r_n$ is *final* on a 1-cell f, if the 1-cell $r_n \ldots r_1(f)$ is irreducible. A 1-cell f of Λ_1^ℓ is *reduction-finite* if for any infinite sequence $(r_n)_{n\geqslant 1}$ of reductions, r_i acts trivially on $r_{i-1} \ldots r_1(f)$ for a sufficiently large i. A 1-cell f is *reduction-unique* if it is reduction-finite and if its images under all final sequences of reduction are the same. This common image is denoted by $r_S(f)$. A reduction system S is *reduction-unique* if all 1-cells of Λ_1^ℓ are reduction-unique under S.

3.5.3. Exercise, [Ber78, Lemma 1.1.].

1) Show that the set of reduction-unique 1-cells of Λ_1^ℓ forms a subspace of Λ_1^ℓ denoted by Λ_1^{ru} and that $r_S : \Lambda_1^{\text{ru}} \to \Lambda_1^{\text{irr}}$ defines a linear map.

2) Given monomials w_f, w_g and w_h in the support of the 1-cells f, g and h respectively, such that the product $w_f w_g w_h$ is in Λ_1^{ru}. Show that for any finite composition of reductions r, then $fr(g)h$ is in Λ_1^{ru} and that $r_S(fr(g)h) = r_S(fgh)$ holds.

3.5.4. Ambiguities. A 5-tuple (σ, τ, u, v, w) with σ, τ in S and u, v, w monomials in Λ_1^*, such that $w_\sigma = uv$ and $w_\tau = vw$ (resp. $\sigma \neq \tau$, $w_\sigma = v$ and $w_\tau = uvw$) is an *overlap ambiguity* (resp. *inclusion ambiguity*) of S. Such an ambiguity is *resolvable* if there exist compositions of reductions r and r′ that satisfy the *confluence condition*:

$$r(f_\sigma w) = r'(uf_\tau) \quad \big(\text{resp.} \quad r(uf_\sigma w) = r'(f_\tau)\,\big).$$

3.5.5. Reduction system compatible with a monomial order. The diamond lemma obtained by Bergman concern reduction systems compatible with a monomial order. A reduction system S is *compatible* with a monomial order $\prec$, if for any $\sigma = (w_\sigma, f_\sigma)$ in S, we have $w \prec w_\sigma$ for any monomial w in the support of f_σ.

Given a reduction system compatible with a monomial order $\prec$. For a monomial w in Σ_1^*, we denote by $I_{\prec w}$ the subspace of Λ_1^ℓ defined by

$$I_{\prec w} = \text{Span}_{\mathbb{K}}\big(u(w_\sigma - f_\sigma)v \mid (w_\sigma, f_\sigma) \in S \text{ and } uw_\sigma v \prec w\big).$$

An overlap ambiguity (resp. inclusion ambiguity) (σ, τ, u, v, w) is *resolvable relative to* $\prec$ if

$$f_\sigma w - uf_\tau \in I_{\prec uvw}, \quad \big(\text{resp. } uf_\sigma w - f_\tau \in I_{\prec uvw}\,\big).$$

Let $\mathcal{G}$ be a subset of 1-cells of Λ_1^ℓ and let $\prec$ be a monomial order on Λ_1^ℓ. We denote by $S(\mathcal{G},\prec)$ the *reduction system generated by* $\mathcal{G}$ with respect to $\prec$ defined by

$$S(\mathcal{G},\prec) = \{\,(\mathrm{lm}(f), \mathrm{lm}(f) - \frac{1}{\mathrm{lc}(f)} f) \mid f \in \mathcal{G}\,\}.$$

3.5.6. Theorem (The Diamond Lemma, [Ber78, Thm. 1.2]). *Let* S *be a reduction system compatible with a monomial order* $\prec$*. The following conditions are equivalent.*

i) *All the ambiguities of* S *are resolvable.*

ii) *All the ambiguities of* S *are resolvable relative to* $\prec$*.*

iii) S *is reduction-unique.*

A fourth equivalent condition is given in [Ber78, Thm. 1.2] as follows. Consider the algebra **A** given as the quotient of the free algebra Λ_1^ℓ by the two-side ideal

$$I(S) = \{\, w_\sigma - f_\sigma \mid \sigma \in S\,\}.$$

If the reduction system S is compatible with a monomial order $\prec$, the confluence conditions **i)** - **iii)** above hold if and only if the set Λ_1^{irm} of irreducible monomial under S is a linear basis of the algebra **A**. In this case, the $\mathbb{K}$-algebra **A** is isomorphic to the $\mathbb{K}$-algebra Λ_1^{ir}, whose product is given by $f \cdot g = r_S(fg)$, for any 1-cells f and g in Λ_1^{ir}.

3.6. Noncommutative Gröbner bases

3.6.1. Noncommutative Gröbner bases. Let Λ_1^ℓ be a free algebra over a set Λ_1 and let $\prec$ be a monomial order on Λ_1^ℓ. A *(noncommutative) Gröbner basis* of an ideal I of Λ_1^ℓ with respect to the monomial order $\prec$ is a subset $\mathcal{G}$ of I such that the ideal generated by the leading monomials of the 1-cells of I coincides with the ideal generated by the leading monomials of the 1-cells of $\mathcal{G}$:

$$\langle\, \mathrm{lm}(I)\,\rangle = \langle\, \mathrm{lm}(\mathcal{G})\,\rangle.$$

Equivalently, for every 1-cell f in I, there exists g in $\mathcal{G}$ with $\mathrm{lm}(f) = u\,\mathrm{lm}(g)v$, where u and v are monomials of Λ_1^ℓ.

The two following results show that the notion of noncommutative Gröbner basis corresponds to the notion of left-monomial convergent linear 2-polygraph compatible with a monomial order.

3.6.2. Proposition. *Let* Λ *be a convergent left-monomial linear* 2*-polygraph, compatible with a monomial order* $\prec$ *on* Λ_1^ℓ*. The set of* 1*-cells* $\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}$ *is a Gröbner basis of the ideal* $I(\Lambda)$ *for the monomial order* $\prec$*.*

3.6.3. Exercise. Prove Proposition 3.6.2.

3.6.4. Proposition. *Let* I *be an ideal of a free* 1*-algebra* Λ_1^ℓ. *Let* $\mathcal{G}$ *be a Gröbner basis for* I *with respect to a monomial order* $\prec$. *Then the leading* 2*-polygraph* $\Lambda(\mathcal{G}, \prec)$ *is convergent and* $\mathrm{I}(\Lambda(\mathcal{G}, \prec)) = \mathrm{I}$ *holds.*

Proof. Suppose that $\mathcal{G}$ is a Gröbner basis of the ideal I with respect to $\prec$. By definition, the ideal $\mathrm{I}(\Lambda(\mathcal{G}, \prec))$ is equal to the ideal I generated by $\mathcal{G}$. Prove that the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is convergent. Its termination is a consequence of its compatibility with the monomial order $\prec$. The monomials in Λ_1^* reduced with respect to $\Lambda(\mathcal{G}, \prec)$ are the monomials that cannot be decomposed as $u\,\mathrm{lm}(g)v$ with g in $\mathcal{G}$ and u and v monomials in Λ_1^*. As a consequence, if a reduced 1-cell f of Λ_1^ℓ is contained in the ideal I, its leading monomial must be 0, because $\mathcal{G}$ is a Gröbner basis of I. By Proposition 3.2.2, we deduce that the linear 2-polygraph $\Lambda(\mathcal{G}, \prec)$ is confluent. □

The following theorem summarizes results obtained in this section. Note that some equivalences are tautological or reformulations.

3.6.5. Theorem. *Let* I *be an ideal of a free algebra* Λ_1^ℓ *over a set* Λ_1. *Let* $\prec$ *be a monomial order on* Λ_1^ℓ. *For a subset* $\mathcal{G}$ *of* I, *the following conditions are equivalent.*

i) *The set* $\mathcal{G}$ *is a Gröbner basis with respect to* $\prec$.

ii) *The leading polygraph* $\Lambda(\mathcal{G}, \prec)$ *is convergent.*

iii) *The leading polygraph* $\Lambda(\mathcal{G}, \prec)$ *is confluent.*

iv) *The leading polygraph* $\Lambda(\mathcal{G}, \prec)$ *is locally confluent.*

v) *All the critical branchings of the leading polygraph* $\Lambda(\mathcal{G}, \prec)$ *are confluent.*

vi) *The set* $\mathcal{G}$ *is a Gröbner-Shirshov basis with respect to* $\prec$.

vii) *All the ambiguities of the reduction system* $S(\mathcal{G}, \prec)$ *are resolvable.*

viii) *All the ambiguities of the reduction system* $S(\mathcal{G}, \prec)$ *are resolvable relative to* $\prec$.

ix) *The reduction system* $S(\mathcal{G}, \prec)$ *is reduction-unique.*

x) $\Lambda_1^\ell = \Lambda_1^{\mathrm{ir}} \oplus \mathrm{I}$.

xi) *Every* 1*-cell of* I *admits* 0 *as a normal form with respect to* $\Lambda(\mathcal{G}, \prec)_2$.

xii) *For any* f *in* I, *there exists a decomposition* $\mathrm{lm}(f) = u\,\mathrm{lm}(g)v$ *for some* u, v *in* Λ_1^* *and* g *in* $\mathcal{G}$.

xiii) *The set of* $\mathcal{G}$*-reduced monomials forms a linear basis of the algebra given by the quotient of* Λ_1^ℓ *by the ideal* I.

3.6.6. Exercise. Prove the equivalences of Theorem 3.6.5.

3.6.7. Example. Consider the linear 2-polygraph Λ given in Example 2.1.9. For the deglex order $\prec_{\text{deglex}}$ induced by the alphabetic order $x \prec y \prec z$, the leading monomial of $f = z^3 + y^3 + x^3 - xyz$ is z^3, so that

$$\Lambda(\{f\}, \prec_{\text{deglex}}) = \langle * \mid x, y, z \mid z^3 \overset{\alpha_f}{\Longrightarrow} xyz - x^3 - y^3 \rangle.$$

The left-monomial linear 2-polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$ is compatible with the monomial order $\prec_{\text{deglex}}$, hence it is terminating. It is not confluent, because neither of its two critical branchings is confluent:

$$z^4 \overset{\alpha_f z}{\Longrightarrow} xyz^2 - x^3z - y^3z \qquad z^4 \overset{z\alpha_f}{\Longrightarrow} zxyz - zx^3 - zy^3$$

$$z^5 \overset{\alpha_f z^2}{\Longrightarrow} xyz^3 - x^3z^2 - y^3z^2 \xRightarrow{xy\alpha_f - x^3z^2 - y^3z^2} xyxyz - xy^4 - xyx^3 - x^3z^2 - y^3z^2 \qquad z^5 \overset{z^2\alpha_f}{\Longrightarrow} z^2xyz - z^2x^3 - z^2y^3$$

In particular, $\{f\}$ does not form a Gröbner basis of the ideal $I(\Lambda)$ We add to the polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$ the following 2-cell

$$\beta : zy^3 \Rightarrow zxyz - zx^3 + y^3z + x^3z - xyz^2.$$

This new rule makes the two previous critical branchings confluent and create a new critical branching

$$z^3y^3 \overset{z^2\beta}{\Longrightarrow} z^3xyz - z^3x^3 + z^2y^3z + z^2x^3z - z^2xyz^2 \qquad z^3y^3 \overset{\alpha y^3}{\Longrightarrow} xyzy^3 - x^3y^3 - y^6$$

which is also confluent. Finally, the convergent linear 2-polygraph $\langle * \mid x, y, z \mid \alpha_f, \beta \rangle$ is Tietze equivalent to the initial linear 2-polygraph $\Lambda(\{f\}, \prec_{\text{deglex}})$. In particular, the set of 1-cells $\{f, s_1(\beta) - t_1(\beta)\}$ forms a Gröbner basis of the ideal $I(\Lambda)$ with respect to the order $\prec_{\text{deglex}}$.

3.6.8. Example. The algebra presented by the following linear 2-polygraph

$$\langle * \mid x, y, z \mid x^2 = 0,\ xy = zx \rangle$$

does not have a finite Gröbner bases on three generators x, y and z. Indeed, the first relation is oriented as $x^2 \Rightarrow 0$ and the orientation $xy \Rightarrow zx$ induce the addition of the 2-cells $xz^n x \Rightarrow 0$, for all integer $n \geqslant 1$. Another way is to orient the relation as $zx \Rightarrow xy$. But in this case, we need to add the 2-cells $xy^n x \Rightarrow 0$, for all integer $n \geqslant 1$.

3.6.9. Exercise. Show that we can compute a Gröbner bases for the algebra given in Example 3.6.8 with four generators. [Hint. Add a generator t and the relations $xy \Rightarrow t$ and $zx \Rightarrow t$.]

3.6.10. Exercise. Consider the ideal I generated by the linear 2-polygraph Λ of Example 3.1.4.

1) Show that $\{ xy^k x - xy^{k+1} \mid k \geqslant 0 \}$ is a Gröbner basis of the ideal I with respect to a monomial order with $x \succ y$.

2) Compute a Gröbner basis for the ideal I reduced to only one element.

4. Anick's resolution

In two seminal papers, Anick introduced a method to compute a free resolution for an algebra starting with a Gröbner basis of its ideal of relations. First he gave the construction for monomial algebras in [Ani85] then for associative augmented algebras in [Ani86]. Resolutions for path algebras using the same method were obtained by Anick and Green in [AG87]. For a deeper discussion on the theory of Gröbner bases for path algebras and how to apply this theory to the construction of free resolutions for path algebras, we refer the reader to [Gre99]. Let us mention that Anick's resolution has been achieved by other methods. In particular, Anick's resolution for a homogeneous algebra can be constructed by a deformation of the resolution computed on the associated monomial algebra, see [DK09, Section 2.4.] for details, see also the Backelin construction, [Bac78]. Anick's resolution can be also obtained using algebraic Morse theory with a Morse matching on the bar resolution, see [Skö06, Section 3.2.] for details. Morse theory allows to construct, starting from a chain complex, a new chain complex such that the homology of the two complexes coincides. This method was applied to the computation of minimal resolutions starting from Anick's resolution, [JW09].

Note also that others constructions of free resolutions using convergent rewriting systems were obtained by several authors, [Bro92, Kob90, Gro90, Kob05, GM12b]. Finally, let us mention that noncommutative Gröbner bases where developed by Dotsenko and Khoroshkin for shuffle operads in [DK10], giving operadic versions of Newman's lemma and Buchberger's algorithm. Anick's resolution for shuffle operads was constructed by Dotsenko and Khoroshkin in [DK09, DK13]. Using this construction, they prove that a shuffle operad with a quadratic Gröbner basis is Koszul, [DK13].

The nth chains in Anick's resolution are generated by the n-fold overlaps of the leading terms of the Gröbner basis and the differentials are constructed by Noetherian induction with respect to the monomial order. The chains defined by Anick are recall in Subsection 4.2. The construction of the resolution is given in Subsection 4.3. In a first part of this section, we briefly recall the definition of the homology of associative algebras.

4.1. Homology of an algebra

4.1.1. Functor Tor. Let us recall the definition of the bifunctor $\mathrm{Tor}^{\mathbf{R}}$, where $\mathbf{R}$ is a fixed ring. Let M be a left $\mathbf{R}$-module and N be a right $\mathbf{R}$-module. Given a projective resolution $\mathcal{P}$ of the right $\mathbf{R}$-module N:

$$\mathcal{P} \;:\quad \cdots \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_0} P_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

we associate the *deleted complex*:

$$\mathcal{P}_N \;:\quad \cdots \longrightarrow P_n \xrightarrow{d_{n-1}} P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \xrightarrow{d_0} P_0 \longrightarrow 0$$

obtained by suppressing the module N. Note that, we have not lost any information in the complex $\mathcal{P}_N$, as $N = \operatorname{coker}(d_0)$ by exactness of complex $\mathcal{P}$. Then, applying the functor $- \otimes_{\mathbf{R}} M$, we form a complex of $\mathbb{Z}$-modules denoted by $\mathcal{P}_N \otimes_{\mathbf{R}} M$:

$$\mathcal{P}_N \otimes_{\mathbf{R}} M \;:\quad \cdots \longrightarrow P_n \otimes_{\mathbf{R}} M \xrightarrow{\overline{d}_{n-1}} P_{n-1} \otimes_{\mathbf{R}} M \longrightarrow \cdots \longrightarrow P_1 \otimes_{\mathbf{R}} M \xrightarrow{\overline{d}_0} P_0 \otimes_{\mathbf{R}} M \longrightarrow 0$$

where $\overline{d}_{n-1}$ denotes the map $d_{n-1} \otimes \mathrm{Id}_M$.

We defined the $\mathbb{Z}$-module $\mathrm{Tor}^{\mathbf{R}}(M, N)$ as the homology of the complex $\mathcal{P}_N \otimes_{\mathbf{R}} M$:

$$\mathrm{Tor}^{\mathbf{R}}_n(N, M) = H_n(\mathcal{P}_N \otimes_{\mathbf{R}} M) = \operatorname{Ker} \overline{d}_{n-1} / \operatorname{Im} \overline{d}_n.$$

In this way, we define a bifunctor $\mathrm{Tor}^{\mathbf{R}}$ with values in the category of $\mathbb{Z}$-modules.

Following the definitions, the functor $\mathrm{Tor}^{\mathbf{R}}_0(N, -)$ is naturally equivalent to $N \otimes_{\mathbf{R}} -$ and the functor $\mathrm{Tor}^{\mathbf{R}}_n(-, M)$ is naturally equivalent to $- \otimes_{\mathbf{R}} M$. Indeed, we have $\mathrm{Tor}^{\mathbf{R}}_0(N, M) = \operatorname{coker}(\overline{d}_0)$. Furthermore, the functor $N \otimes_{\mathbf{R}} -$ is right exact, hence

$$\operatorname{coker}(\overline{d}_0) = P_0 \otimes_{\mathbf{R}} M / \operatorname{Im}(\overline{d}_0) = P_0 \otimes_{\mathbf{R}} M / \ker(\varepsilon \otimes \mathrm{Id}_M) = N \otimes_{\mathbf{R}} M.$$

This proves that

$$\mathrm{Tor}^{\mathbf{R}}_0(N, M) = N \otimes_{\mathbf{R}} M.$$

4.1.2. Contracting homotopy. Recall that a method to prove that a complex

$$\cdots \longrightarrow M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{d_0} M_0 \xrightarrow{\varepsilon} N \longrightarrow 0$$

is acyclic is to construct a *contracting homotopy*, that is a sequence of morphisms of abelian groups

$$(\cdots) \longleftarrow M_{n+1} \xleftarrow{i_{n+1}} M_n \xleftarrow{i_n} M_{n-1} \longleftarrow (\cdots) \longleftarrow M_1 \xleftarrow{i_1} M_0 \xleftarrow{i_0} N$$

such that

$$\varepsilon\iota_0 = \mathrm{Id}_N, \qquad d_0\iota_1 + \iota_0\varepsilon = \mathrm{Id}_{M_0}, \qquad d_n\iota_{n+1} + \iota_n d_{n-1} \;=\; \mathrm{Id}_{M_n},$$

for every $n \geqslant 1$.

4.1.3. Homology of an algebra. Let $\mathbf{A}$ be an associative algebra over a field $\mathbb{K}$. For $n \geqslant 0$, the n-*th homology space* of the algebra $\mathbf{A}$ with coefficient in a left $\mathbf{A}$-module M is defined by

$$H_n(\mathbf{A}, M) = \mathrm{Tor}^{\mathbf{A}}_n(\mathbb{K}, M).$$

In practice, to compute the n-th homology spaces $H_n(\mathbf{A}, \mathbb{K})$, for all $n \geqslant 0$, we construct a free resolution of $\mathbb{K}$, seen as a trivial right-$\mathbf{A}$-module:

$$\mathcal{F} \;: \cdots \longrightarrow F_n \xrightarrow{d_{n-1}} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

and we compute the homology of the complex $\mathcal{F}_{\mathbb{K}} \otimes_{\mathbf{A}} \mathbb{K}$.

4.1.4. Minimal complex. A complex of free right **A**-modules

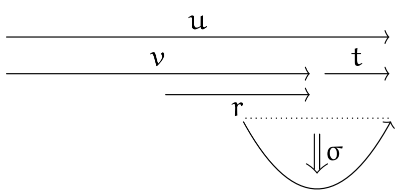
$$\cdots \longrightarrow F_{n+1} \xrightarrow{d_n} F_n \xrightarrow{d_{n-1}} F_{n-1} \longrightarrow \cdots$$

is *minimal* if all induced maps $\overline{d}_n = d_n \otimes \mathrm{Id}_{\mathbb{K}} : F_{n+1} \otimes_{\mathbf{A}} \mathbb{K} \longrightarrow F_n \otimes_{\mathbf{A}} \mathbb{K}$ are zero. A resolution is *minimal* if the associated complex is minimal. Note that a minimal free resolution is one in which each free module has the minimal number of generators as illustrated in the following example.

4.2. Anick's chains

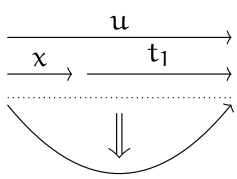
4.2.1. Anick's chains, [Ani86]. Let Λ be a reduced left-monomial linear 2-polygraph. The *Anick* n*-chains* of the linear 2-polygraph Λ and their *tails* are defined by induction as follows.

- The unique (-1)*-chain* is the empty monomial, denoted by 1, it is its own tail.
- The 0*-chains* are the 1-cells in Λ_1, and the *tail of* 0*-chain* x in Λ_1 is x itself.
- For $n \geqslant 1$, suppose that the $(n-1)$-chains and their tails constructed. An n*-chain* is a monomial u in Λ_1^* of the form
$$u = vt$$
where the monomials v and t satisfy the following conditions:
  i) v is $(n-1)$-chain,
  ii) t is a reduced monomial with respect to Λ, called the *tail* of u,
  iii) if r is the tail of v, then $\mathrm{Occ}_{s_1(\Lambda)}(rt) = 1$,
  iv) the unique reduction on rt is rightmost, that is, given by a 2-cell σ in Λ reducing the ending of the monomial rt:



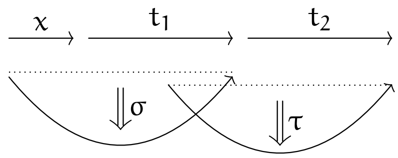
We will denote by $\Omega_n(\Lambda)$ the set of n-chains of the linear 2-polygraph Λ.

4.2.2. Anick's chains and overlapping. The linear 2-polygraph Λ being reduced, we have the following description of Anick's chains. A 1-chain u is necessarily in $s_1(\Lambda)$. Indeed, a 1-chain is a non reduced monomial u written $u = xt$, where x is a 1-cell in Λ_1 and t is a monomial reduced with respect to Λ:



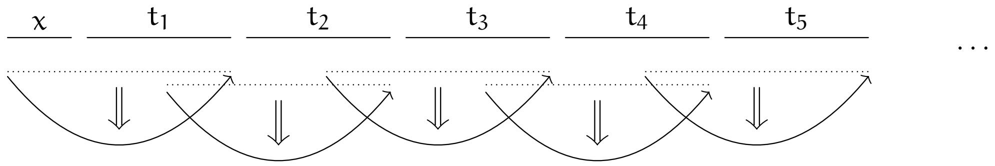
and such that there is only one 2-cell of Λ that can be applied on the monomial u.

A 2-chains u is the source of a critical branching. Indeed, $u = xt_1t_2$, where xt_1 is the source of a 2-cell σ in Λ_2 and there is a rightmost reduction τ reducing t_1t_2 and thus overlapping σ:



Moreover, u is not the source of a critical triple branching, as we have $\mathrm{Occ}_{s_1(\Lambda)}(u) = 2$. In this way, there is a 1-1 correspondence between $\Omega_2(\Lambda)$ and the set of critical branchings of the 2-polygraph Λ.

For $n \geqslant 3$, a n-chain u corresponds to a n-fold overlapping composed by $(n-1)$ chained critical branchings. Note that it may possible that $\mathrm{Occ}_{s_1(\Lambda)}(u) > n$, see Example 4.2.5.

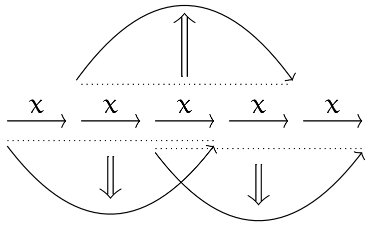


4.2.3. Proposition ([Ani86]). *Suppose* $n \geqslant 1$. *If* $u = x_{i_1} \dots x_{i_t}$ *is an* n*-chain, then there is a unique* $s \leqslant t$ *such that* $x_{i_1} \dots x_{i_s}$ *is an* $(n-1)$*-chain. Moreover,* $x_{i_{s+1}} \dots x_{i_t}$ *is reduced.*

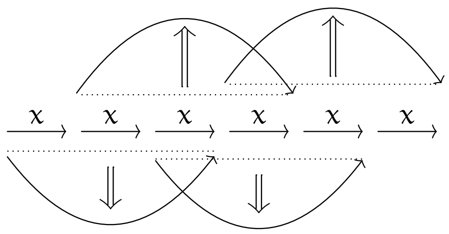
Indeed, suppose that there is two $(n-1)$-chains $x_{i_1} \dots x_{i_s}$ and $x_{i_1} \dots x_{i_{s'}}$ which factorise u. By uniqueness of the reduction on the tail, condition **iii)** in 4.2.1, necessarily we have $s = s'$.

4.2.4. Notation. If u is a n-chain with $(n-1)$-chain v and tail t, we will denote $u = v|t$. An n-chain will be denoted by $x|t_1|t_2| \dots |t_n$.

4.2.5. Example, [Ani86]. Let Λ be a reduced left-monomial linear 2-polygraph with $\Lambda_1 = \{x\}$ and $s_1(\Lambda) = \{x^3\}$. The 1-cell x is the unique 0-chain. The monomial $x^3 = x|x^2$ is the unique 1-chain, xx is not a 1-chain because $\mathrm{Occ}_{\Lambda_2}(x^2) = 0$. The monomial $x^4 = x^3|x$ is the unique 2-chain. Note that $x^5 = x^3x^2$ is not a 2-chain because $\mathrm{Occ}_{\Lambda_2}(x^2x^2) = 2$: on the monomial x^5 there are 3 possible reductions. Here x^5 links three obstructions, with the first one intersect with the last, hence it form a critical triple branching:



The monomial $x^6 = x^4|x^2$ is the unique 3-chain, note that $x^5 = x^4x$ is not a 3-chain because $\mathrm{Occ}_{\Lambda_2}(xx) = 0$. Note that there are 4-obstructions on the 3-chain x^6:



Thus we have

$$\Omega_0(\Lambda) = \Lambda_1, \quad \Omega_1(\Lambda) = s_1(\Lambda), \quad \Omega_2(\Lambda) = \{x^4\}, \quad \Omega_3(\Lambda) = \{x^6\}.$$

More generally, we show that for any integer $n \geqslant 0$, we have

$$\Omega_{2n-1}(\Lambda) = \{x^{3n}\}, \quad \Omega_{2n}(\Lambda) = \{x^{3n+1}\}.$$

4.2.6. Example, [Ani86]. Suppose that $\Lambda_1 = \{x, y\}$ and $s_1(\Lambda) = \{x^2yxy, xyxy^2\}$. Then we have

$$\Omega_0(\Lambda) = \{x, y\}, \quad \Omega_1(\Lambda) = \{x|xyxy, x|yxy^2\}, \quad \Omega_2(\Lambda) = \{x|xyxy|y, x|xyxy|xy^2\},$$

and $\Omega_n(\Lambda)$ is empty for $n \geqslant 3$.

4.2.7. Exercise, [Ani85]. Let Λ be a linear 2-polygraph such that $\Lambda_1 = \{x, y, z\}$. Determine Anick's chains in the following situations

1) $s_1(\Lambda) = \{xyzx, zxy\}$,

2) $s_1(\Lambda) = \{xyzx, xxy\}$. In this case, show that the number of n-chains equals the $(n+2)$nd Fibonacci number when $n \geqslant 1$.

4.3. Anick's resolution

Let Λ be a convergent reduced left-monomial linear 2-polygraph, compatible with a monomial order $\prec$ on Λ_1^ℓ. Let denote by $\mathbf{A}$ the algebra presented by Λ. We define a section $\iota : \mathbf{A} \longrightarrow \Lambda_1^\ell$ of the canonical projection $\pi : \Lambda_1^\ell \longrightarrow \mathbf{A}$, sending every 1-cell f of $\mathbf{A}$ to the unique normal form $\widehat{f}$ of any representative 1-cell of f in Λ_1^ℓ, as in 3.2.3. In the construction of the following resolution, the convergence hypothesis is used to guarantee the unicity of this normal form.

4.3.1. Anick's resolution. Let $\mathbf{A}[\Omega_n(\Lambda)] = \mathbb{K}[\Omega_n(\Lambda)] \otimes_{\mathbb{K}} \mathbf{A}$ be the free right $\mathbf{A}$-module over the set of n-chains $\Omega_n(\Lambda)$. We identify $\mathbf{A}[\Omega_0(\Lambda)]$ to $\mathbf{A}[\Lambda_1]$ and $\mathbf{A}[\Omega_{-1}(\Lambda)]$ to $\mathbf{A}$. Anick constructs in [Ani86] a free resolution of right $\mathbf{A}$-modules, that we will denote by $\mathcal{A}(\Lambda)$, and defined by

$$\cdots \longrightarrow \mathbf{A}[\Omega_n(\Lambda)] \xrightarrow{d_n} \mathbf{A}[\Omega_{n-1}(\Lambda)] \longrightarrow \cdots \longrightarrow \mathbf{A}[\Omega_1(\Lambda)] \xrightarrow{d_1} \mathbf{A}[\Lambda_1] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

where the differentials d_n are constructed by induction on n together with the contracting homotopy

$$\iota_n : \mathrm{Ker}\, d_{n-1} \longrightarrow \mathbf{A}[\Omega_n(\Lambda)].$$

The applications d_n are morphisms of right $\mathbf{A}$-module and the applications ι_n are linear maps.

4.3.2. The applications d_n and ι_n are constructed by noetherian induction with respect to the monomial order $\prec$. From the monomial order $\prec$ on Λ_1^ℓ, we define a partial order $\prec_{\Omega_n}$ on the set of elements $u \otimes t$ such that $u \in \Omega_n(\Lambda)$ and $t \in \Lambda_1^*$ by setting

$$u \otimes t \prec_{\Omega_n} u' \otimes t' \quad \text{if and only if} \quad \widehat{ut} \prec \widehat{u't'}.$$

This order is total on the set of n-chains. Indeed, by Proposition 4.2.3, if $ut = u't'$, then $u = u'$ and then $t = t'$.

Given a linear combination $h = \sum_{i=1}^{l} \lambda_i u_i \otimes t_i$ in $\mathbf{A}[\Omega_n(\Lambda)]$, the *leading term* of h with respect to $\prec_{\Omega_n}$ is the term $u_k \otimes t_k$ such that $u_i \otimes t_i \prec_{\Omega_n} u_k \otimes t_k$ for any $i \in \{1, \ldots, l\} \setminus \{k\}$.

4.3.3. For the first steps of the resolution

$$\mathbf{A}[\Lambda_1] \underset{\iota_0}{\overset{d_0}{\rightleftarrows}} \mathbf{A} \underset{\iota_{-1}}{\overset{\varepsilon}{\rightleftarrows}} \mathbb{K} \longrightarrow 0$$

we set $\iota_{-1} = \eta : \mathbb{K} \hookrightarrow \mathbf{A}$ the embedding of $\mathbb{K}$ in $\mathbf{A}$ and we define the *augmentation* $\varepsilon : \mathbf{A} \to \mathbb{K}$ by $\varepsilon(x) = 0$, for all $x \in \Lambda_1$. Hence $\mathbf{A} = \mathbb{K} \oplus \operatorname{Ker} \varepsilon$ and we have $\varepsilon\iota_{-1} = \mathrm{Id}_{\mathbb{K}}$. Then we set

$$d_0(x \otimes 1) = 1 \otimes x,$$

for all x in Λ_1. For a monomial u in $\mathbf{A}$ such that the normal form with respect to Λ is written $\widehat{u} = x_1 x_2 \ldots x_k$ in Λ_1^ℓ, we define

$$\iota_0(1 \otimes u) = x_1 \otimes x_2 \ldots x_k. \tag{9}$$

Then we extend ι_0 to any element of $\mathbf{A}$ by linearity. The map ι_0 is well defined by uniqueness of the normal form due to the convergence of the linear 2-polygraph Λ.

The exactness, $\operatorname{Im} d_0 = \operatorname{Ker} \varepsilon$, in $\mathbf{A}$ is a consequence of the two equalities:

$$\varepsilon d_0(x \otimes 1) = 0 \quad \text{and} \quad d_0 \iota_0 = \mathrm{id}_{\operatorname{Ker}(\varepsilon)}.$$

4.3.4. For $n \geqslant 0$, we define the pair (d_n, ι_n):

$$\mathbf{A}[\Omega_n(\Lambda)] \underset{\iota_n}{\overset{d_n}{\rightleftarrows}} \mathbf{A}[\Omega_{n-1}(\Lambda)] \underset{\iota_{n-1}}{\overset{d_{n-1}}{\rightleftarrows}} \mathbf{A}[\Omega_{n-2}(\Lambda)]$$

by induction on n. We suppose that the maps d_{n-1} and $\iota_{n-1} : \operatorname{Ker} d_{n-2} \longrightarrow \mathbf{A}[\Omega_{n-1}(\Lambda)]$, constructed such that the following equalities

$$d_{n-2} d_{n-1} = 0 \quad \text{and} \quad d_{n-1} \iota_{n-1} = \mathrm{Id}_{\operatorname{Ker} d_{n-2}}$$

hold. We define inductively d_n on a n-chain $u = v|t$ with tail t by

$$d_n(v|t \otimes 1) = v \otimes t - \iota_{n-1} d_{n-1}(v \otimes t). \tag{10}$$

In the right-hand side of (10), the term $v \otimes t$ will be the leading term with respect to $\prec_{\Omega_{n-1}}$.

4.3.5. Let us define recursively the map

$$\iota_n : \operatorname{Ker} d_{n-1} \longrightarrow \mathbf{A}[\Omega_n(\Lambda)].$$

Let $h \in \operatorname{Ker} d_{n-1} \subset \mathbf{A}[\Omega_{n-1}(\Lambda)]$. Denote by $u_{n-1} \otimes t$ the leading term of h with respect to $\prec_{\Omega_{n-1}}$, that is

$$h = \lambda u_{n-1} \otimes t + \text{lower terms},$$

where $\lambda \in \mathbb{K} \setminus \{0\}$. By Proposition 4.2.3, the $(n-1)$-chain u_{n-1} can be uniquely decomposed in

$$u_{n-1} = u_{n-2}|t',$$

where u_{n-2} is an $(n-2)$-chain and t' is the tail of u_{n-1}. By induction, we have

$$d_{n-1}(u_{n-1} \otimes 1) = u_{n-2} \otimes t' + \text{lower terms}.$$

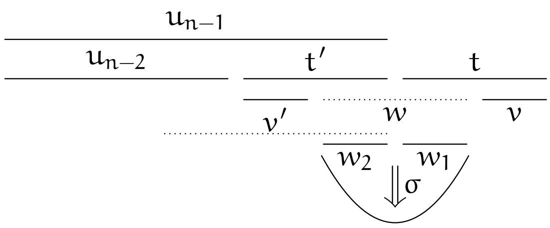
As d_{n-1} is a morphism of right $\mathbf{A}$-modules, we have

$$\begin{aligned} d_{n-1}(h) &= \lambda d_{n-1}(u_{n-1} \otimes t) + d_{n-1}(\text{lower terms}) \\ &= \lambda u_{n-2} \otimes t't + \text{lower terms}. \end{aligned}$$

Suppose that $t't$ is reduced, then $u_{n-2} \otimes t't$ remains the leading term of $d_{n-1}(h)$ and h cannot be in $\operatorname{Ker} d_{n-1}$ thus contradicting the hypothesis. It follows that $t't$ can be reduced, we set

$$t't = v'wv,$$

where w is the 1-source of the leftmost reduction σ that can be applied on the monomial $t't$:



Then $u_{n-2}v'w = u_{n-2}|t'|w_1$ forms an n-chain, it follows that $u_{n-2}v'w \otimes v \in \mathbf{A}[\Omega_n(\Lambda)]$. We set

$$\begin{aligned} \iota_n(h) &= \iota_n(\lambda u_{n-1} \otimes t + \text{lower terms}) \\ &= \lambda u_{n-2}v'w \otimes v + \iota_n(h - \lambda d_n(u_{n-2}v'w \otimes v)). \end{aligned} \tag{11}$$

This is well defined, because $h - \lambda d_n(u_{n-2}v'w \otimes v) \prec h$ by construction. Indeed

$$\begin{aligned} d_n(u_{n-2}v'w \otimes v) = d_n(u_{n-2}v'w_1w_2 \otimes v) &= u_{n-2}v'w_2 \otimes w_1v + \text{lower terms} \\ &= u_{n-1} \otimes t + \text{lower terms}. \end{aligned}$$

Moreover, $d_{n-1}(h - \lambda d_n(u_{n-2}v'w \otimes v)) = 0$.

From this construction, we deduce the following result.

4.3.6. Theorem ([Ani86, Thm 1.4]). *Let* **A** *be an algebra presented by a convergent reduced left-monomial linear 2-polygraph* Λ*, compatible with a given monomial order* $\prec$*. The complex of right* **A***-modules* $\mathcal{A}(\Lambda)$ *defined by*

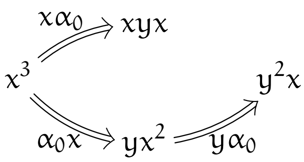
$$\cdots \longrightarrow \mathbf{A}[\Omega_n(\Lambda)] \xrightarrow{d_n} \mathbf{A}[\Omega_{n-1}(\Lambda)] \longrightarrow \cdots \longrightarrow \mathbf{A}[\Omega_1(\Lambda)] \xrightarrow{d_1} \mathbf{A}[\Lambda_1] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

where, for any $n \geqslant 0$*, the morphism* d_n *is defined on a* n*-chain* $v|t$ *by*

$$d_n(v|t \otimes 1) = v \otimes t + h,$$

where $\mathrm{lt}(h) \prec v|t \otimes 1$*, if* $h \neq 0$*, is a resolution of the trivial right* **A***-module* $\mathbb{K}$.

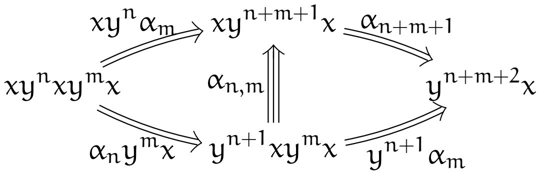
4.3.7. Example. Let consider the algebra **A** presented by the linear 2-polygraph Λ of Example 2.1.10 and denote by α_0 the 2-cell β. It appears one critical branching



We complete the linear 2-polygraph Λ with the 2-cells

$$\alpha_n : xy^n x \Longrightarrow y^{n+1}x,$$

for all $n > 0$. We note that, for any integers $n, m \geqslant 0$, we have a critical branching



The linear 2-polygraph Λ', whose set of 1-cell is Λ_1 and $\Lambda'_2 = \{\alpha_n \mid n \geqslant 0\}$ is convergent, compatible with the monomial order $\prec$ and Tietze equivalent to Λ. Equivalently, the set $\{xy^n x - y^{n+1}x \mid n \geqslant 0\}$ forms a Gröbner basis for the ideal $I(\Lambda)$. Anick's 1-chains are of the form $x|y^n x$ with $n \geqslant 0$ and Anick's 2-chains are of the form $x|y^n x|y^m x$ with $n, m \geqslant 0$. More generally, for any $k \geqslant 2$, we have

$$\Omega_k(\Lambda') = \{x|y^{n_1}x|y^{n_2}x|\dots|y^{n_k}x \ \text{ for } \ n_1, \dots, n_k \geqslant 0\},$$

Let us compute the boundary maps d_0, d_1, d_2 and d_3. We have $d_0(x \otimes 1) = x$, $d_0(y \otimes 1) = y$ and

$$\begin{aligned} d_1(x|y^n x \otimes 1) &= x \otimes y^n x - \iota_0 d_0(x \otimes y^n x), \\ &= x \otimes y^n x - \iota_0(1 \otimes xy^n x), \\ &= x \otimes y^n x - \iota_0(1 \otimes y^{n+1}x), \\ &= x \otimes y^n x - y \otimes y^n x. \end{aligned}$$

The last equality is consequence of the definition of the map ι_0 in (9).

$$\begin{aligned}
d_2(x|y^n x|y^m x \otimes 1) &= x|y^n x \otimes y^m x - \iota_1 d_1(x|y^n x \otimes y^m x),\\
&= x|y^n x \otimes y^m x - \iota_1(x \otimes y^n x y^m x - y \otimes y^n x y^m x),\\
&= x|y^n x \otimes y^m x - \iota_1(x \otimes y^{n+m+1} x - y \otimes y^{n+m+1} x),
\end{aligned}$$

By (11), we have

$$\iota_1(x\otimes y^{n+m+1}x - y\otimes y^{n+m+1}x) = x|y^{n+m+1}x\otimes 1 - \iota_1\big(x\otimes y^{n+m+1}x - y\otimes y^{n+m+1}x - x\otimes y^{n+m+1}x + y\otimes y^{n+m+1}x\big).$$

Hence

$$d_2(x|y^n x|y^m x \otimes 1) = x|y^n x \otimes y^m x - x|y^{n+m+1} x \otimes 1.$$

$$\begin{aligned}
d_3(x|y^n x|y^m x|y^k x \otimes 1) &= x|y^n x|y^m x \otimes y^k x - \iota_2\big(x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+1} x \otimes y^k x\big),\\
&= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1\\
&\quad - \iota_2\big(x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+1} x \otimes y^k x - x|y^n x \otimes y^{m+k+1} x - x|y^{n+m+k+2} x \otimes 1\big),\\
&= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 - \iota_2\big(-x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1\big),\\
&= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 + x|y^{n+m+1} x|y^k x \otimes 1\\
&\quad + \iota_2(x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1 - d_2(x|y^{n+m+1} x|y^k x \otimes 1)),\\
&= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 + x|y^{n+m+1} x|y^k x \otimes 1\\
&\quad + \iota_2(x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1 - x|y^{n+m+1} x \otimes y^k x - x|y^{n+m+k+1} x \otimes 1)),\\
&= x|y^n x|y^m x \otimes y^k x - x|y^n x|y^{m+k+1} x \otimes 1 + x|y^{n+m+1} x|y^k x \otimes 1.
\end{aligned}$$

4.3.8. Example. Let consider the algebra **A** given in 4.3.7, but with the presentation by the linear 2-polygraph Λ' of Example 2.1.10, compatible with the deglex order induced by the alphabetic order $x \prec y$. This polygraph does not have critical branching, thus the sets of Anick's n-chains are empty for $n \geqslant 2$. It follows that the associated Anick's resolution is

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[y|x] \xrightarrow{d_1} \mathbf{A}[x,y] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

with $d_0(x \otimes 1) = x$, $d_0(y \otimes 1) = y$ and $d_1(y|x \otimes 1) = y \otimes x - x \otimes x$.

4.3.9. Example. Consider the algebra **A** presented by the linear 2-polygraph Λ of Example 2.1.9. With the Gröbner basis computed in 3.6.7:

$$z^3 \overset{\alpha_f}{\Longrightarrow} xyz - x^3 - y^3 \qquad zy^3 \overset{\beta}{\Longrightarrow} zxyz - zx^3 + y^3z + x^3z - xyz^2$$

Anick's chains are of the form z^n and $z^n y^3$, for $n \geqslant 0$, so that Anick's resolution for the algebra **A** with respect to this Gröbner basis is infinite.

4.3.10. Exercise, [Ani86, Section 3]. Compute Anick's resolution for the algebra presented by the linear 2-polygraph $\langle * \mid x, y \mid xyxyx \Rightarrow xyx \rangle$.

4.4. Anick's resolution for a monomial algebra

4.4.1. Anick's chains for a monomial algebra. We construct Anick's resolution in the case of a monomial algebra $\mathbf{A}$. Recall from 2.1.20, that such an algebra can be presented by a monomial linear 2-polygraph Λ, that is, left-monomial and $t_1(\alpha) = 0$ for all α in Λ_2. Note that such a presentation is always convergent. Suppose that the polygraph Λ is reduced. The sets of chains for Λ are $\Omega_0(\Lambda) = \Lambda_1$, $\Omega_1(\Lambda) = s_1(\Lambda)$ and for any $n \geqslant 2$, $\Omega_n(\Lambda)$ is the set of n-overlapping $x|t_1|\ldots|t_{n-1}|t_n$ of branchings of Λ with $x, t_1, \ldots, t_n$ in Λ_1 and xt_1, $t_i t_{i+1}$ in $s_1(\Lambda)$ for any $1 \leqslant i \leqslant n-1$. We have

$$\widehat{xt_1} = 0 \qquad \text{and} \qquad \widehat{t_{i-1}t_i} = 0, \text{for all } 1 \leqslant i \leqslant n. \tag{12}$$

Consider the boundary map

$$d_n : \mathbf{A}[\Omega_n(\Lambda)] \longrightarrow \mathbf{A}[\Omega_{n-1}(\Lambda)]$$

defined by

$$d_n(x|t_1|\ldots|t_{n-1}|t_n \otimes 1) = x|t_1|\ldots|t_{n-1} \otimes t_n - \iota_{n-1} d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n).$$

By definition of d_{n-1}, we have

$$d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n) = x|t_1|\ldots|t_{n-2} \otimes t_{n-1}t_n - \iota_{n-2} d_{n-2}(x|t_1|\ldots|t_{n-2} \otimes t_{n-1}t_n)$$

Using relation in (12), we have $d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n) = 0$, hence

$$d_n(x|t_1|\ldots|t_{n-1}|t_n \otimes 1) = x|t_1|\ldots|t_{n-1} \otimes t_n.$$

As consequence, the map $d_n \otimes_{\mathbf{A}} 1_{\mathbb{K}}$ is zero, for all $n \geqslant 0$. This proves that Anick's resolution of a monomial algebras is minimal.

4.4.2. Proposition. *Let Λ be a monomial linear 2-polygraph, and $\mathbf{A}$ be the monomial algebra presented by Λ. The following statements hold.*

- **i)** *Anick's resolution $\mathcal{A}(\Lambda)$ is a minimal resolution.*
- **ii)** *There is an isomorphism $\mathrm{Tor}_n^{\mathbf{A}}(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}\Omega_{n-1}(\Lambda)$, for all $n \geqslant 0$.*

4.5. Computing homology with Anick's resolution

Given an algebra $\mathbf{A}$ presented by a convergent reduced left-monomial linear 2-polygraph Λ, compatible with a monomial order, Anick's resolution $\mathcal{A}(\Lambda)$ gives a method to compute the homology groups of $\mathbf{A}$ with coefficient in a $\mathbf{A}$-module M. In particular, Anick's resolution can be used to calculate Poincaré series. In this section, we give several examples of computations of homology groups with coefficients in $\mathbb{K}$.

4.5.1. Computing homology. From the resolution $\mathcal{A}(\Lambda)$, we compute the complex $\mathcal{A}(\Lambda) \otimes_{\mathbf{A}} \mathbb{K}$ given by

$$\cdots \longrightarrow \mathbb{K}[\Omega_n(\Lambda)] \xrightarrow{\overline{d}_n} \mathbb{K}[\Omega_{n-1}(\Lambda)] \longrightarrow \cdots \longrightarrow \mathbb{K}[\Omega_1(\Lambda)] \xrightarrow{\overline{d}_1} \mathbb{K}[\Lambda_1] \xrightarrow{\overline{d}_0} \mathbb{K} \longrightarrow 0$$

where $\mathbb{K}[\Omega_n(\Lambda)]$ denotes the free vector space on $\Omega_n(\Lambda)$ and $\overline{d}_n$ denotes the map $d_n \otimes \mathrm{Id}_{\mathbb{K}}$. These maps satisfy $\overline{d}_n \overline{d}_{n+1} = 0$, for all $n \geqslant 0$, and we have

$$H_0(\mathbf{A}, \mathbb{K}) = \mathbb{K}, \quad \text{and} \quad H_n(\mathbf{A}, \mathbb{K}) = \mathrm{Ker}\, \overline{d}_{n-1} / \mathrm{Im}\, \overline{d}_n.$$

As a first application, we have the following finiteness properties.

4.5.2. Proposition. *Let* **A** *be an algebra presented by a finite convergent left-monomial linear* 2*-polygraph. The following statements hold.*

i) **A** *is of homological type right-*FP$_\infty$*, that is, there exists an infinite length free finitely generated resolution of the trivial right* **A***-module* $\mathbb{K}$*.*

ii) *For any* $n \geqslant 0$*, the vector space* $H_n(\mathbf{A}, \mathbb{K})$ *is finitely generated.*

iii) *[Ani86, Lemma 3.1] The algebra* **A** *has a Poincaré series*

$$P_{\mathbf{A}}(t) = \sum_{n=0}^{\infty} \dim_{\mathbb{K}}(H_n(\mathbf{A}, \mathbb{K}))t^n, \tag{13}$$

with exponential or slower growth, that is, there are constants $c_1, c_2 > 0$*, such that*

$$0 \leqslant \dim_{\mathbb{K}}(H_n(\mathbf{A}, \mathbb{K})) \leqslant c_2(c_1)^n.$$

Note that the finiteness conditions **i)** and **ii)** were obtained by Kobayashi for monoids. A monoid **M** is of *homological type right-*FP$_\infty$ *over* $\mathbb{K}$ if the monoid algebra $\mathbb{K}\mathbf{M}$ is of homological type right-FP$_\infty$. In [Kob90], by constructing a resolution similar to the Anick resolution, Kobayashi shows that a monoid **M** having a presentation by a finite convergent rewriting system is of homological type FP$_\infty$. Similar constructions of resolutions of monoids presented by convergent rewriting systems were also obtained by Brown [Bro92] and by Groves [Gro90]. The diferent constructions are based on distinct ways to describe the n-fold critical branchings of a convergent rewriting system.

4.5.3. Exercise. Prove the conditions **i)** and **ii)** in Proposition 4.5.2.

4.5.4. Low-dimensional homology. Let us explicit the first terms of the series (13). In the first dimensions, we have the following complex

$$\mathbb{K}[\Omega_2(\Lambda)] \xrightarrow{\overline{d}_2} \mathbb{K}[\Omega_1(\Lambda)] \xrightarrow{\overline{d}_1} \mathbb{K}[\Lambda_1] \xrightarrow{\overline{d}_0} \mathbb{K} \longrightarrow 0$$

The map $\overline{d}_0$ is zero, hence

$$H_1(\mathbf{A}, \mathbb{K}) = \mathbb{K}[\Lambda_1] / \mathrm{Im}\, \overline{d}_1.$$

A 1-cell x of Λ_1 in $\operatorname{Im}\overline{d}_1$ comes from a relation with source or target x. It follows that x is a redundant generator in the presentation. Indeed, a term $x \otimes 1$, with x in Λ_1 appears in $\operatorname{Im} d_1$ if and only if x is the source or the target of a 2-cell in Λ_2. Let $\alpha : x \Rightarrow y_1 \dots y_k$ be a 2-cell in Λ_2, where by hypothesis $y_1 \dots y_k$ is reduced. Thus we have

$$d_1(x|1 \otimes 1) = x \otimes 1 - y_1 \otimes y_2 \dots y_k.$$

Hence $\overline{d}_1(x) = x$. Suppose now that $x_1 \dots x_k \overset{\alpha}{\Longrightarrow} y$ is a 2-cell in Λ_2. We have

$$d_1(x_1|x_2 \dots x_k \otimes 1) = x_1 \otimes x_2 \dots x_k - y \otimes 1.$$

Hence $\overline{d}_1(x_1 \dots x_k) = -y$. Thus, we have $\overline{d}_1 = 0$ if and only if the number of generators is minimal. In this way, $\dim_{\mathbb{K}} H_1(\mathbf{A}, \mathbb{K})$ is equal to the minimal number of generators for a presentation of the algebra $\mathbf{A}$. For analogous reasons, we show that $\dim_{\mathbb{K}} H_2(\mathbf{A}, \mathbb{K})$ is the minimal required number of the defining relations, see [Ufn95, Sect. 3.9].

4.5.5. Example. Consider the algebra $\mathbf{A}$ from Example 4.3.8. Using Anick's resolution computed in 4.3.8, we deduce the complex

$$\cdots \longrightarrow 0 \longrightarrow \mathbb{K}[y|x] \xrightarrow{\overline{d}_1} \mathbb{K}[x, y] \xrightarrow{\overline{d}_0} \mathbb{K} \longrightarrow 0$$

whose boundary maps $\overline{d}_0$ and $\overline{d}_1$ are zero. We deduce

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, 2, \\ \mathbb{K}^2 & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 3. \end{cases}$$

4.5.6. Exercise [Ani86, Thm 3.2]. Let $\mathbf{A}$ be an algebra admitting a presentation by a left-monomial reduced linear 2-polygraph compatible with a monomial order and having no critical branching. Show that $H_n(\mathbf{A}, \mathbb{K}) = 0$, for any $n \geqslant 3$. A presentation without critical branching is called *combinatorially free* in [Ani86].

4.5.7. Exercise. Show that the Poincaré series of the algebra $\mathbf{A}$ presented by the linear 2-polygraph $\langle * \mid x, y \mid x^2 \Rightarrow 0 \rangle$ is

$$P_{\mathbf{A}}(t) = 1 + 2t + \sum_{k=2}^{\infty} t^k.$$

4.5.8. Exercise. Let $\mathbf{B}_3^+$ be the monoid of positive braids on three strands given by the following Artin presentation:

$$\langle\, s, t \mid sts \Rightarrow tst \,\rangle.$$

Compute Anick's resolution and the Poincaré series of the monoid $\mathbf{B}_3^+$.

4.6. Minimality of Anick's resolution

4.6.1. Example. Let **A** be the algebra presented by the linear 2-polygraph $\langle * \mid x, y \mid x \Rightarrow y \rangle$, which is compatible with the deglex order induced by $y \prec x$. The Anick resolution is

$$0 \longrightarrow \mathbf{A}[x|1] \xrightarrow{d_1} \mathbf{A}[x,y] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

with

$$d_0(x \otimes 1) = x, \quad d_0(y \otimes 1) = y, \quad d_1(x|1 \otimes 1) = x \otimes 1 - 1 \otimes y.$$

This resolution is not minimal because $\overline{d}_1 \neq 0$. A minimal resolution for the algebra **A** can be constructed from the polygraph $\langle * \mid x \mid \emptyset \rangle$ with no 2-cell.

4.6.2. Example. Let consider the algebra **A** presented by the linear 2-polygraph

$$\Lambda = \langle * \mid x, y, z, r, s \mid xy \overset{\alpha}{\Longrightarrow} s,\ yz \overset{\beta}{\Longrightarrow} r \rangle$$

compatible with the deglex order induced by the alphabetic order $s \prec r \prec z \prec y \prec x$. There is a critical branching:

$$\begin{array}{ccc} & \overset{x\beta}{\Longrightarrow} & xr \\ xyz & & \Downarrow \gamma \\ & \underset{\alpha z}{\Longrightarrow} & sz \end{array}$$

which is confluent by adding the rule $xr \overset{\gamma}{\Longrightarrow} sz$. The linear 2-polygraph $\Lambda' = \langle \Lambda_1 \mid \alpha, \beta, \gamma \rangle$ is compatible with the deglex order considered above, convergent and Tietze equivalent to Λ. The induced the Anick resolution $\mathcal{A}(\Lambda')$ is

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[xy|z] \xrightarrow{d_2} \mathbf{A}[x|y, x|r, y|z] \xrightarrow{d_1} \mathbf{A}[x,y,z,r,s] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

with

$$d_1(x|y \otimes 1) = x \otimes y - s \otimes 1, \quad d_1(x|r \otimes 1) = x \otimes r - s \otimes z, \quad d_1(y|z \otimes 1) = y \otimes z - r \otimes 1,$$

and $d_2(x|y|z \otimes 1) = xy \otimes z - xr \otimes 1$. This resolution is not minimal, because the maps $\overline{d}_1$ and $\overline{d}_2$ are non zero. Note that

$$H_n(\mathbf{A}, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, \\ \mathbb{K}^3 & \text{if } n = 1, \\ 0 & \text{if } n \geqslant 2. \end{cases}$$

and a minimal resolution for the algebra **A** can be constructed from the linear 2-polygraph $\langle * \mid x, y, z \mid \emptyset \rangle$ which produces the following resolution

$$\cdots \longrightarrow 0 \longrightarrow \mathbf{A}[x,y,z] \xrightarrow{d_0} \mathbf{A} \xrightarrow{\varepsilon} \mathbb{K} \longrightarrow 0$$

4.6.3. Exercise. Consider the linear 2-polygraph

$$\Lambda = \langle * \mid x, y, z, r, s \mid xy \overset{\alpha}{\Longrightarrow} ss,\ yz \overset{\beta}{\Longrightarrow} sr \rangle.$$

1) Complete the polygraph Λ into a convergent polygraph Λ'.

2) Show that the Anick resolution of Λ' is not minimal.

3) Compute the homology of the algebra **A** presented by Λ.

4) Compute a minimal Anick's resolution of the algebra **A**.

4.6.4. Exercise. Let consider the algebra presented by

$$\langle * \mid x, y, z, r, s \mid xy = ss,\ yz = rr \rangle.$$

Show that there is no orientation of rules of this presentation giving a convergent linear 2-polygraph, and thus there is no minimal Anick's resolution for this algebra.

4.6.5. Proposition. *Let* **A** *be an algebra and let* Λ *be a left-monomial reduced convergent linear* 2-*polygraph compatible with a monomial order that presents* **A**. *If the Anick resolution* $\mathcal{A}(\Lambda)$ *is minimal, then, for any* $n \geqslant 0$, *there is an isomorphism of spaces*

$$H_n(\mathbf{A}, \mathbb{K}) \simeq \mathbb{K}[\Omega_{n-1}(\Lambda)].$$

4.6.6. Exercise. Prove Proposition 4.6.5.

4.6.7. When Anick's resolution is minimal. We have seen in Proposition 4.4.2 that the Anick resolution $\mathcal{A}(\Lambda)$ is minimal when the presentation is monomial. Following exercise gives an other situation for which the Anick resolution is minimal.

4.6.8. Exercise. Let Λ be a left-monomial reduced linear 2-polygraph compatible with a monomial order. Suppose that Λ is convergent and quadratic, that is, any 2-cell in Λ_2 is of the form $x_{i_1}x_{i_2} \Rightarrow y_{i_1}y_{i_2}$ with $x_{i_1}, x_{i_2}, y_{i_1}, y_{i_2}$ in Λ_1. Show that the Anick resolution $\mathcal{A}(\Lambda)$ is minimal.

4.6.9. Exercise. A linear 2-polygraph is *cubical* if its 2-cells are of the form $x_{i_1}x_{i_2}x_{i_3} \Rightarrow y_{i_1}y_{i_2}y_{i_3}$. Is the result of Exercise 4.6.8 can be extended to cubical convergent linear 2-polygraphs ?

4.6.10. Exercises. Compute homology spaces of the algebras presented by the following linear 2-polygraphs

1) $\langle * \mid x, y \mid xy \Rightarrow yx \rangle$. **2)** $\langle * \mid x, y \mid x^2 \Rightarrow 0 \rangle$. **3)** $\langle * \mid x, y \mid x^2 \Rightarrow y^2 \rangle$.
4) $\langle * \mid x, y \mid x^2 \Rightarrow xy \rangle$. **5)** $\langle * \mid x, y \mid x^2 \Rightarrow xy - y^2 \rangle$. **6)** $\langle * \mid x, y \mid xyx \Rightarrow yxy \rangle$.

5. Higher-dimensional linear rewriting

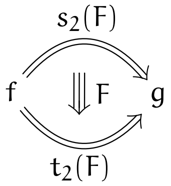
In this section, we recall the notion of coherent presentation for an algebra as a presentation of the algebra extended by a family of generating syzygies. We explain how to generate syzygies when the presentation is convergent. Finally, we recall from [GHM17] the notion of polygraphic resolution for an algebra as an acyclic polygraphic extension of a presentation of the algebra.

5.1. Coherent presentations of algebras

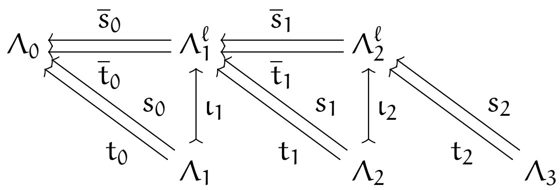
5.1.1. Linear 3-polygraph. Let Λ be a linear 2-polygraph. A *cellular extension* of the free 2-algebroid Λ_2^ℓ is a set Λ_3 equipped with maps

$$\Lambda_2^\ell \underset{t_2}{\overset{s_2}{\Longleftarrow}} \Lambda_3$$

such that, for every F in Λ_3, the pair $(s_2(F), t_2(F))$ is a 2-sphere in Λ_2^ℓ, that is, $s_1 s_2(F) = s_1 t_2(F)$ and $t_1 s_2(F) = t_1 t_2(F)$ hold in Λ_2^ℓ. The elements of Λ_3 are the 3-*cells* of the cellular extension and graphically represented by



A *linear* 3-*polygraph* is a data $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$, where $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a linear 2-polygraph and Λ_3 is a cellular extension of the free 2-algebroid Λ_2^ℓ:



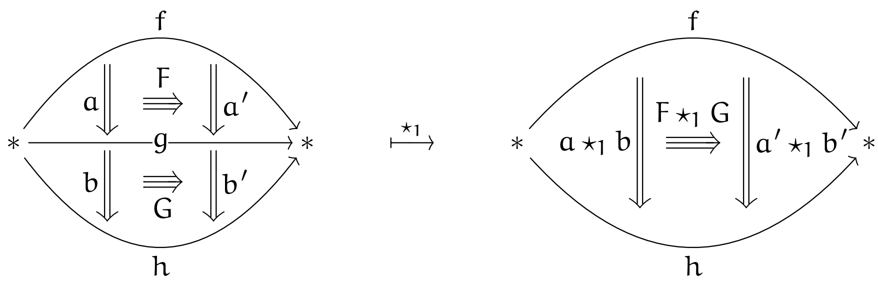
5.1.2. Three-dimensional algebras. We define a 3-*algebra* as an internal 2-category in the category **Alg**:

$$\mathbf{A}_1 \underset{t_1}{\overset{s_1}{\Longleftarrow}} \mathbf{A}_2 \underset{t_2}{\overset{s_2}{\Longleftarrow}} \mathbf{A}_3$$

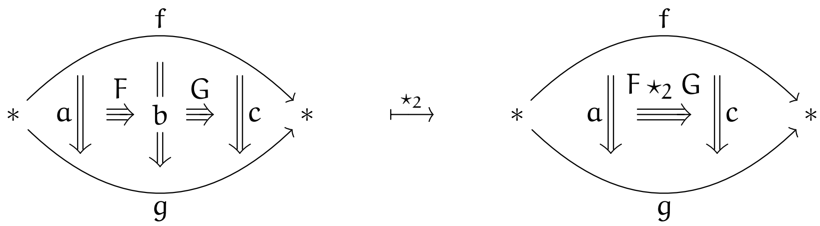
In particular, the algebras $\mathbf{A}_1$ and $\mathbf{A}_2$ with composition $\mathbf{A}_2 \times_{\mathbf{A}_1} \mathbf{A}_2 \xrightarrow{\star_1} \mathbf{A}_2$ form a 2-algebra. The 3-cells can be composed in two different ways:

$$\mathbf{A}_3 \times_{\mathbf{A}_1} \mathbf{A}_3 \xrightarrow{\star_1} \mathbf{A}_3 \qquad \mathbf{A}_3 \times_{\mathbf{A}_2} \mathbf{A}_3 \xrightarrow{\star_2} \mathbf{A}_3$$

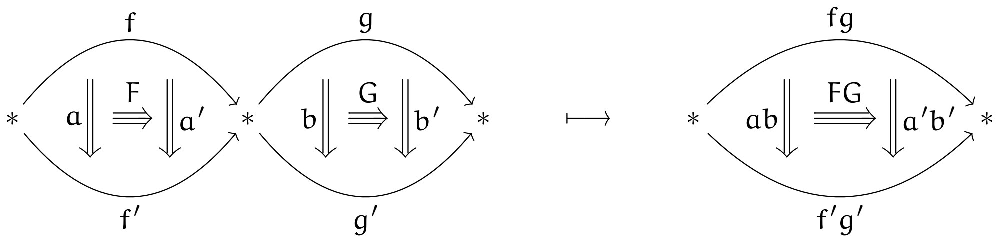
by $\star_1$, along their 1-dimensional boundary:



by $\star_2$, along their 2-dimensional boundary:



The source and target maps s_1, s_2 and t_1, t_2 being morphisms of algebras, the product of 3-cells F and G satisfies:



These compositions and the product satisfy remarkable properties similar to those given in 2.1.14 for 2-algebras.

5.1.3. Free 3-algebras. The *free 3-algebra over a linear 3-polygraph* Λ is constructed similarly to the free 2-algebra given in 2.1.16. It is the 3-algebra, denoted by Λ_3^ℓ, whose underlying 2-algebra is the free 2-algebra Λ_2^ℓ, and its 3-cells are all the formal 1-composition, 2-composition and product of 3-cells of Λ_3, of identities of 2-cells, up to associativity, identity, exchange and inverse relations, see [GHM17, 2.1.3] for more details.

5.1.4. Coherent presentations of algebras. A *coherent presentation* of an algebra **A** is a linear 3-polygraph Λ such that

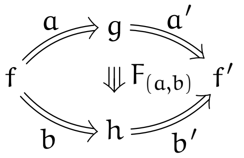
i) the linear 2-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a presentation of **A**,

ii) Λ_3 is a *homotopy basis* of the free 2-algebra Λ_2^ℓ, that is, a cellular extension

$$\Lambda_2^\ell \underset{t_2}{\overset{s_2}{\leftleftarrows}} \Lambda_3$$

such that for every 2-sphere (a, b) of the free 2-algebra Λ_2^ℓ, there exists a 3-cell A in the free 3-algebra Λ_3^ℓ such that $s_2(A) = a$ and $t_2(A) = b$.

5.1.5. Squier's completion. Let Λ be a left-monomial linear 2-polygraph. Suppose that all critical branching of Λ are confluent. For every critical branching (a, b) in Λ, we choose two positive 2-cells a' and b' making the branching confluent:



(14)

For any such a confluent branching, we consider a 3-cell $F_{(a,b)} : a \star_1 a' \Rrightarrow b \star_1 b'$. The set of such 3-cells

$$\Lambda_3 = \{ F_{(a,b)} \mid (a,b) \text{ is a critical branching } \}$$

forms a cellular extension of the free 2-algebra Λ_2^{ℓ}. The linear 3-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ is a *Squier's completion* of Λ. When the polygraph is confluent, there exists such a Squier's completion. However, the cellular extension Λ_3 is not unique in general. Indeed, the 3-cells can be directed in the reverse way and a branching (a, b) can have several possible positive 2-cells a' and b' making the branching confluent.

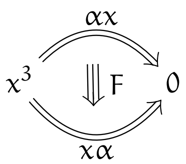
The following result is a formulation of Squier's Theorem, [SOK94], in the setting of linear 2-polygraphs:

5.1.6. Theorem (Squier's Theorem, [GHM17, Thm. 4.3.2]). *Let* **A** *be an algebra and let* Λ *be a convergent left-monomial presentation of* **A**. *Any Squier's completion of* Λ *is a coherent presentation of* **A**.

5.1.7. Linear oriented syzygies. Let Λ be a presentation of an algebra **A**. Any nontrivial 2-sphere (a, b) in the free 2-algebra Λ_2^{ℓ} is called a *linear oriented* 3*-syzygy* of the presentation Λ. If Λ is extended into a coherent presentation (Λ, Λ_3) of the algebra **A**, the quotient 2-algebra $\Lambda_2^{\ell}/\Lambda_3$ is *aspherical*, that is, for any 2-sphere (a, b) in $\Lambda_2^{\ell}/\Lambda_3$, we have $a = b$. In other words, the cellular extension Λ_3 forms a generating set of linear 3-syzygies of the presentation Λ. Theorem 5.1.6 say that, when the presentation Λ is convergent the 3-cells defined by confluence diagrams of the critical branchings, as in (14), form a family of generator for 3-syzygies.

5.1.8. Exercice. Let $\{F_1, \ldots, F_k\}$ be a generating set for linear 3-syzygies of a linear 2-polygraph Λ. Prove that $\{F_1^-, \ldots, F_k^-\}$ is also a generating set for linear 3-syzygies of Λ.

5.1.9. Example. The linear 2-polygraph $\langle * \mid x \mid x^2 \overset{\alpha}{\Longrightarrow} 0 \rangle$ has one critical branching



which is confluent. The polygraph being convergent the 3-cell $F : \alpha x \Rrightarrow x\alpha$ generates all linear 3-syzygies of this presentation.

5.1.10. Example. Consider the algebra **A** presented by the linear 2-polygraph Λ given in Example 2.1.9. It does not have critical branching, hence any Squier's completion of Λ is empty. As a consequence, Λ can be extended into a coherent presentation with an empty homotopy basis. That is, there is no 3-syzygy for this presentation.

The linear 2-polygraph $\langle * \mid x, y, z \mid \alpha_f, \beta \rangle$ considered in Example 3.6.7 is Tietze equivalent to Λ, convergent and compatible with a monomial order. It has three critical branchings, as shown in Example 3.6.7. It can be extended into a coherent presentation of **A** with three generating 3-syzygies.

5.1.11. Exercise. Give an explicit description of the 3-cells of a coherent presentation on the linear 2-polygraph Λ' of Example 5.1.10.

5.1.12. Exercise. Compute a coherent presentation for the algebras presented by the following linear 2-polygraphs

1) $\langle\, * \mid x, y \mid xyx \Rightarrow y^2 \,\rangle$.

2) $\langle\, * \mid x, y, z \mid yz \overset{\alpha}{\Longrightarrow} -x^2,\ zy \overset{\beta}{\Longrightarrow} -\lambda^{-1}x^2 \,\rangle$, where $\lambda \in \mathbb{K} \setminus \{0, 1\}$, see [PP05, 4.3].

5.1.13. Exercise. Compute a minimal coherent presentation for the algebra presented by the linear 2-polygraph $\langle\, * \mid x \mid x^3 = 0 \,\rangle$.

5.2. Polygraphic resolutions of algebras

In this subsection, we summarize the notion of polygraphic resolution for algebras as introduced in [GHM17]. Such a resolution can be computed for an algebra given by a convergent linear 2-polygraph. The first three steps of the resolution are generated by the cells of the 2-polygraph. For $n \geqslant 3$, the n-cells are generated by confluences diagrams induced by n-fold branchings.

5.2.1. Higher-dimensional algebras. Let n be a nonzero natural number. An n-*algebra* $\mathbf{A}$ is an internal $(n-1)$-category in the category **Alg**:

$$\mathbf{A}_1 \underset{t_1}{\overset{s_1}{\Longleftarrow}} \mathbf{A}_2 \underset{t_2}{\overset{s_2}{\Longleftarrow}} \mathbf{A}_3 \Longleftarrow \quad \cdots \quad \Longleftarrow \mathbf{A}_{n-1} \underset{t_{n-1}}{\overset{s_{n-1}}{\Longleftarrow}} \mathbf{A}_n$$

The elements of the algebra $\mathbf{A}_k$, for $1 \leqslant k \leqslant n$, are the k-*cells* of the n-algebra $\mathbf{A}$. A *cellular extension* of $\mathbf{A}$ is a set Γ equipped with maps

$$\mathbf{A}_n \underset{t_n}{\overset{s_n}{\Longleftarrow}} \Gamma$$

such that, for any γ in Γ, the pair $(s_n(\gamma), t_n(\gamma))$ is an n-sphere of $\mathbf{A}$, that is, $s_{n-1}s_n(\gamma) = s_{n-1}t_n(\gamma)$ and $t_{n-1}s_n(\gamma) = t_{n-1}t_n(\gamma)$.

In these notes, we will do not develop the construction of the free k-algebra $\mathbf{A}[\Gamma]$ on a pair of a $(k-1)$-algebra $\mathbf{A}$ and a cellular extension Γ of it, for $k \geqslant 3$. The construction is the same as in the case of 2-algebras given in 2.1.5. For more details we refer the reader to [GHM17, 2.1.3]. It has the $(k-1)$-algebra $\mathbf{A}$ as underlying $(k-1)$-algebra and its k-cells are all formal compositions by $\star_i$ for $1 \leqslant i \leqslant k$ and product of k cells in Γ and identities of $(k-1)$-cells, up to associativity, identity, exchange and inverse relation.

5.2.2. Linear polygraphs. A *linear* n-*polygraph* is a sequence $\Lambda = (\Lambda_0, \Lambda_1, \ldots, \Lambda_n)$ made of

i) a 1-polygraph (Λ_0, Λ_1),

ii) for any $k \geqslant 2$, a cellular extension Λ_k of the free $(k-1)$-algebra

$$\Lambda_{k-1}^{\ell} = \Lambda_1^{\ell}[\Lambda_2]\cdots[\Lambda_{k-1}],$$

The elements of Λ_k are called the k-*cells* of Λ.

5.2.3. A linear n-polygraph can be defined explicitly as a diagram

$$\begin{array}{ccccccccccc}
\Lambda_0 & \overset{\overline{s}_0}{\underset{\overline{t}_0}{\Longleftarrow}} & \Lambda_1^\ell & \overset{\overline{s}_1}{\underset{\overline{t}_1}{\Longleftarrow}} & \Lambda_2^\ell & \overset{\overline{s}_2}{\underset{\overline{t}_2}{\Longleftarrow}} & (\cdots) & \overset{\overline{s}_{n-2}}{\underset{\overline{t}_{n-2}}{\Longleftarrow}} & \Lambda_{n-1}^\ell & & \\
& {\scriptstyle s_0,\, t_0} \nwarrow\!\!\nwarrow & \uparrow {\scriptstyle \iota_1} & {\scriptstyle s_1,\, t_1} \nwarrow\!\!\nwarrow & \uparrow {\scriptstyle \iota_2} & {\scriptstyle s_2,\, t_2} \nwarrow\!\!\nwarrow & & {\scriptstyle s_{n-2},\, t_{n-2}} \nwarrow\!\!\nwarrow & \uparrow {\scriptstyle \iota_{n-1}} & {\scriptstyle s_{n-1},\, t_{n-1}} \nwarrow\!\!\nwarrow & \\
& & \Lambda_1 & & \Lambda_2 & & (\cdots) & & \Lambda_{n-1} & & \Lambda_n
\end{array}$$

where the maps $\overline{s}_k, \overline{t}_k : \Lambda_{k+1}^\ell \longrightarrow \Lambda_k^\ell$ are the extensions of the source and target maps s_k and t_k, defined by the universal property of the free k-algebra Λ_k^ℓ, and such that, for any $1 \leqslant k \leqslant n-1$, the following two conditions hold:

i) there is a structure of k-algebra on the following k-graph

$$\Lambda_0 \overset{\overline{s}_0}{\underset{\overline{t}_0}{\Longleftarrow}} \Lambda_1^\ell \overset{\overline{s}_1}{\underset{\overline{t}_1}{\Longleftarrow}} \Lambda_2^\ell \overset{\overline{s}_2}{\underset{\overline{t}_2}{\Longleftarrow}} (\cdots) \overset{\overline{s}_{k-1}}{\underset{\overline{t}_{k-1}}{\Longleftarrow}} \Lambda_k^\ell$$

ii) Λ_{k+1} is a cellular extension of the free k-algebra Λ_k^ℓ.

The *free n-algebra* over a linear n-polygraph Λ is the n-algebra $\Lambda_n^\ell = \Lambda_1^\ell[\Lambda_2] \cdots [\Lambda_n]$

5.2.4. Polygraphic resolutions of algebras. A *polygraphic resolution* of an algebra **A** is a linear ∞-polygraph Λ such that

i) the linear 2-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a presentation of **A**,

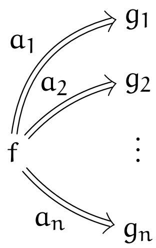
ii) for every $n \geqslant 2$, Λ_{n+1} is a *homotopy basis* of the free n-algebra Λ_n^ℓ, that is a cellular extension

$$\Lambda_n^\ell \overset{s_n}{\underset{t_n}{\Longleftarrow}} \Lambda_{n+1}$$

such that for every n-sphere (a, b) of Λ_n^ℓ, there exists an $(n+1)$-cell A in the free $(n+1)$-algebra Λ_{n+1}^ℓ such that $s_n(A) = a$ and $t_n(A) = b$.

As a consequence of this definition, for every $n \geqslant 2$, the quotient n-algebra $\Lambda_n^\ell/\Lambda_{n+1}$ of the free n-algebra Λ_n^ℓ by the congruence generated by the $(n+1)$-cells of Λ_{n+1} is *aspherical*, that is, any of its n-sphere γ is trivial: $s_n(\gamma) = t_n(\gamma)$. A linear ∞-polygraph satisfying this property for all n is said to be *acyclic*.

5.2.5. Higher-dimensional branchings. Let Λ be a reduced linear 2-polygraph. An *n-fold branching* of Λ is a family $(a_1, \ldots, a_n)$ of positive 2-cells of Λ_2^ℓ with a common source:

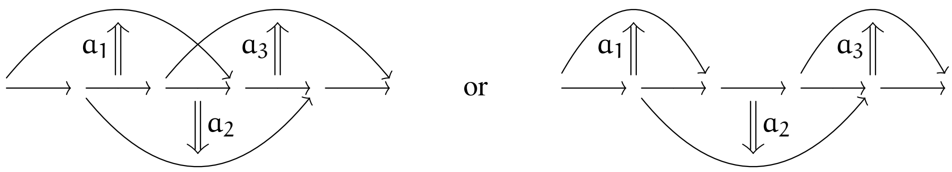


An n-fold branching $(a_1, \dots, a_n)$ is *local* when $a_1, \dots, a_n$ are rewriting steps. A local n-fold branching $(a_1, \dots, a_n)$ is *aspherical* when there is $1 \leqslant i \leqslant n-1$ such that (a_i, a_{i+1}) is aspherical, (resp. *additive*) *Peiffer* when there is $1 \leqslant i \leqslant n-1$ such that (a_i, a_{i+1}) is (resp. additive) Peiffer. In all the other cases, it is said *overlapping*.

A *critical n-fold branching* of Λ is an overlapping local n-fold branching of Λ with a monomial source and that is minimal for the relation on n-fold branchings defined by

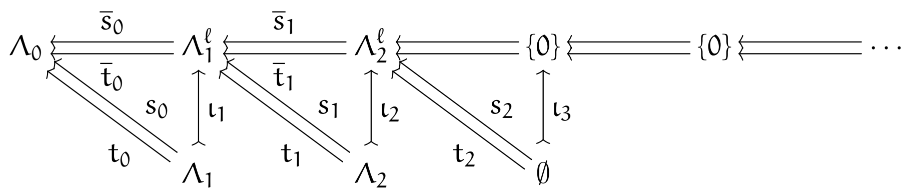
$$(a_1, \dots, a_n) \sqsubseteq (wa_1w', \ \dots \ , wa_nw')$$

for any monomials w, w' in Λ_1^*. For instance, a 3-fold critical branching can have two different shapes:



5.2.6. Theorem ([GHM17, Thm. 6.2.4]). *Any convergent linear 2-polygraph Λ extends to a Tietze-equivalent acyclic linear ∞-polygraph whose n-cells, for $n \geqslant 3$, are indexed by the critical $(n-1)$-fold branchings of Λ.*

5.2.7. Example. Consider the algebra **A** presented by the linear 2-polygraph given in Example 2.1.9. We have seen in Example 5.1.10 that any Squier's completion of Λ is empty. In particular, the polygraph Λ can be extended into a coherent presentation with an empty homotopy bases, and as a consequence, into a polygraphic resolution with an empty set of k-cell, for $k \geqslant 3$:



5.2.8. A free bimodules resolution. Let Λ be a linear ∞-polygraph whose underlying 2-polygraph is a presentation of an algebra **A**. For $k \geqslant 1$, we denote by $\mathbf{A}^e[\Lambda_k]$ the free **A**-bimodule on Λ_k, given by the linear combinations of $f[\alpha]g$, where f and g are 1-cells in **A** and α is a k-cell in Λ_k.

The mapping of every 1-cell x in Λ_1 to the element $[x]$ in $\mathbf{A}^e[\Lambda_1]$ is uniquely extended into a derivation, denoted by $[\cdot]$, from Λ_1^ℓ with values in the **A**-bimodule $\mathbf{A}^e[\Lambda_1]$, sending a 1-cell f in Λ_1^ℓ on the element $[f]$ in $\mathbf{A}^e[\Lambda_1]$, defined by linearity and by induction on the length of monomials as follows

$$[1] = 0, \qquad [u+v] = [u] + [v], \qquad [uv] = [u]\overline{v} + \overline{u}[v], \qquad [\lambda u] = \lambda[u],$$

for any monomials u and v in Λ_1^ℓ and scalar λ in $\mathbb{K}$. We extend the bracket notation to **A**-bimodules $\mathbf{A}^e[\Lambda_k]$, for $k > 1$ as follows. The mapping of every k-cell α of Λ_k to the element $[\alpha]$ in $\mathbf{A}^e[\Lambda_k]$ is extended to any k-cell a of Λ_k^ℓ by induction on the size of a. For any $(k-1)$-cell u, any k-cells a and b in Λ_k^ℓ and scalar λ, we set

$$[1_u] \ = \ 0, \qquad [a+b] \ = \ [a] + [b], \qquad [ab] \ = \ [a]\overline{b} + \overline{a}[b], \qquad [\lambda a] \ = \ \lambda[a].$$

To the linear ∞-polygraph Λ, we associate a complex of **A**-bimodules

$$0 \longleftarrow \mathbf{A} \overset{\mu}{\longleftarrow} \mathbf{A}^e[\Lambda_0] \overset{\delta_0}{\longleftarrow} \mathbf{A}^e[\Lambda_1] \longleftarrow \cdots \longleftarrow \mathbf{A}^e[\Lambda_k] \overset{\delta_k}{\longleftarrow} \mathbf{A}^e[\Lambda_{k+1}] \longleftarrow \cdots$$

where the boundary maps are defined as follows. The maps μ is defined by $\mu(f \otimes g) = fg$, for any 1-cells f and g in **A**. For any triple $f[x]g$ in $\mathbf{A}^e[\Lambda_1]$, we define

$$\delta_0(f[x]g) = f \otimes xg - fx \otimes g.$$

For $k \geqslant 1$, for any triple $f[\alpha]g$ in $\mathbf{A}^e[\Lambda_{k+1}]$, we define

$$\delta_k(f[\alpha]g) = f[s_k(\alpha)]g - f[t_k(\alpha)]g.$$

By induction on the length of f, we prove that $\delta_0([f]) = 1 \otimes f - f \otimes 1$, for all 1-cell f in Λ_1^ℓ. We have $\mu\delta_0 = 0$, and for any k-cell α in Λ_k with $k \geqslant 2$, we have

$$\delta_{k-1}\delta_k[\alpha] = [s_{k-1}s_k(\alpha)] + [t_{k-1}s_k(\alpha)] - [s_{k-1}t_k(\alpha)] - [t_{k-1}t_k(\alpha)].$$

It follows from the globular relations that $\delta_{k-1}\delta_k = 0$. Moreover, we prove that the acyclicity of the polygraph induces the acyclicity of the complex $\mathbf{A}^e[\Lambda]$.

5.2.9. Theorem ([GHM17, Thm. 7.1.3]). *If Λ is a (finite) polygraphic resolution of an algebra* **A***, then the complex $\mathbf{A}^e[\Lambda]$ is a (finite) free resolution of the* **A***-bimodule* **A**.

5.2.10. Example. Consider the algebra **A** presented by the linear 2-polygraph given in Example 2.1.9. The resolution of **A**-bimodules induced by the polygraphic resolution of Λ given in Example 5.2.7 is

$$0 \longleftarrow \mathbf{A} \overset{\mu}{\longleftarrow} \mathbf{A}^e \overset{\delta_0}{\longleftarrow} \mathbf{A}^e[x,y,z] \overset{\delta_1}{\longleftarrow} \mathbf{A}^e[\gamma] \longleftarrow 0 \longleftarrow \cdots$$

It follows that this algebra is of cohomological dimension 2. Note that the Anick resolution for the algebra **A** computed with the same presentation is of infinite length.

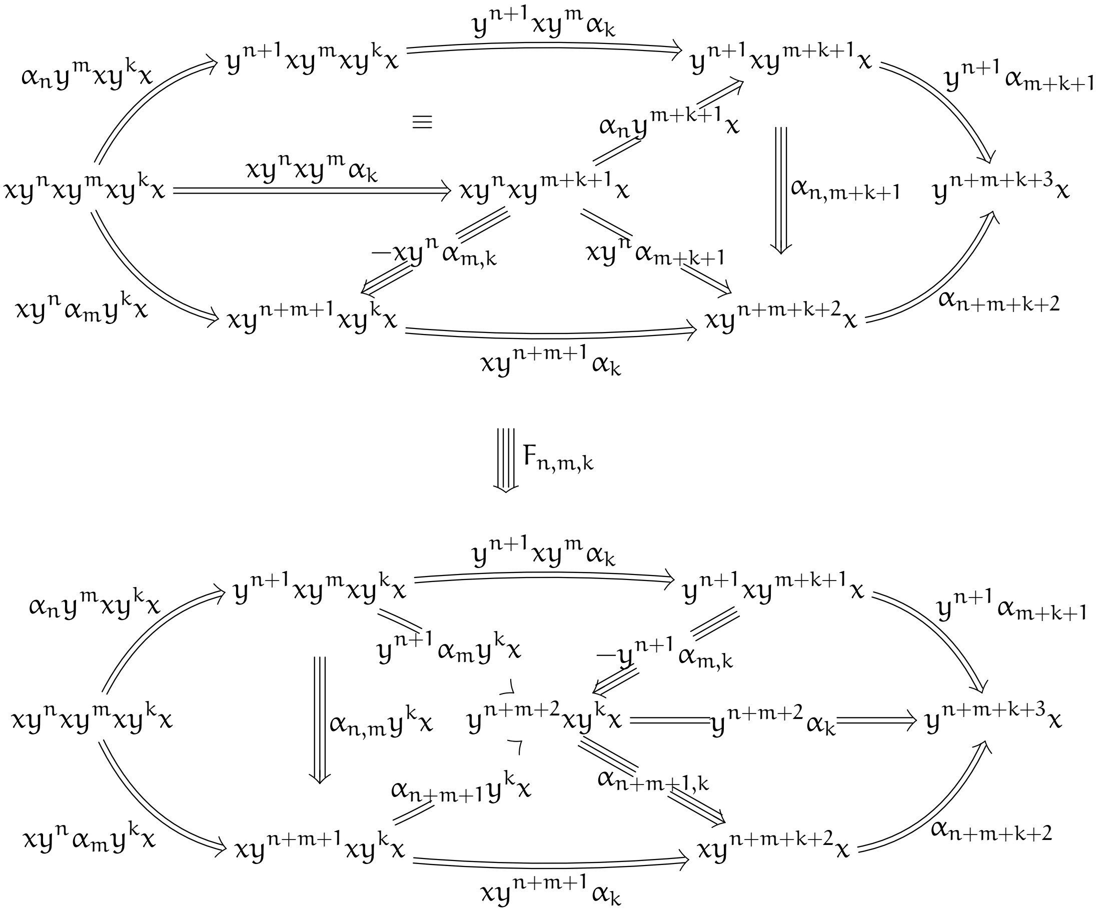
5.2.11. Exercise. Consider the algebra **A** presented by the linear 2-polygraph $\Lambda = \langle\, * \mid x, y \mid x^2 \overset{\alpha_0}{\Longrightarrow} yx \,\rangle$.

1) Compute the first four steps of a polygraphic resolution of the algebra **A** starting with Λ.

2) Compare the resolution of **A**-bimodules induced by this resolution with the Anick resolution $\mathcal{A}(\Lambda)$ computed in Example 4.3.7.

3) Compute a polygraphic resolution of the algebra **A** using the linear 2-polygraph $\langle\, * \mid x, y \mid yx \Rightarrow x^2 \,\rangle$.

[Hint. Here a 4-cell



6. Confluence and Koszulness

In this section we recall the notion of Koszulness for graded associative algebras. We show how Anick's resolution leads to relate this property for an algebra to the existence of a quadratic Gröbner basis for its ideal of relations. Finally, we show how polygraphic resolutions can be used to prove this property, allowing to relate Koszulness with polygraphic convergence.

6.1. Koszulness of associative algebras

6.1.1. Koszulness of quadratic algebras. Recall that a connected graded algebra **A** is *Koszul* if the Tor spaces $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K},\mathbb{K})$ vanish for $i \neq n$, where the grading n is the homological degree and the grading i corresponds to the internal grading of the algebra **A**. Koszul algebras were introduced by Priddy, [Pri70]. In particular, Priddy proved that quadratic algebras having a Poincaré-Birkhoff-Witt basis are Koszul, [Pri70]. The property can be also be stated in terms of existence of a linear minimal graded free resolution of $\mathbb{K}$ seen as a **A**-module, see [PP05]. Backelin gave a characterization of the Koszul property in term of lattice, [Bac83, BF85], and the Backelin condition were interpreted in term of confluence by Berger, [Ber98], using reduction operator theory.

6.1.2. Koszulness of N-homogeneous algebras. Koszulness was generalized by Berger to the case of N-homogeneous algebras, [Ber01, Def. 2.10.]. A graded N-homogeneous algebra **A**, with $N \geqslant 2$, is *left-Koszul* if the ground field $\mathbb{K}$ considered as a graded left **A**-module admits a graded projective resolution of the form

$$0 \longleftarrow \mathbb{K} \longleftarrow P_0 \longleftarrow P_1 \longleftarrow P_2 \longleftarrow \cdots$$

such that every P_i is generated (as a graded left **A**-module) by $P_i^{(\ell_N(i))}$, where $\ell_N : \mathbb{N} \longrightarrow \mathbb{N}$ is a map defined by

$$\ell_N(i) = \begin{cases} pN & \text{if } i = 2p, \\ pN + 1 & \text{if } i = 2p + 1. \end{cases}$$

Similarly, one can define the properties *right-Koszul* and *bi-Koszul* by considering projective resolutions of right and bi-modules respectively. The graduation on the algebra **A** induces a graduation on the vector spaces $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K}, \mathbb{K})$. The spaces $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K}, \mathbb{K})$ for a left-Koszul (or right-Koszul) algebra **A** vanish for $i \neq \ell_N(n)$ This property of the Tor groups is an equivalent definition of Koszul algebras, as Berger proved in [Ber01, Thm. 2.11]. Finally, the following result shows that the Koszul property corresponds to a limit case.

6.1.3. Proposition ([BM06, Prop. 2.1]). *Let* **A** *be an N-homogeneous algebra. The graded vector space $\mathrm{Tor}^{\mathbf{A}}_{n,(i)}(\mathbb{K}, \mathbb{K})$ always vanish for $i < \ell_N(n)$, for $n \geqslant 0$.*

6.2. Confluence and Koszulness

6.2.1. Koszulness of monomial algebras. Given a monomial linear 2-polygraph Λ which is quadratic, that is its 2-cells are of the form $x_i x_j \Rightarrow 0$, with x_i, x_j in Λ_1. Then the Anick resolution $\mathcal{A}(\Lambda)$ is concentrated in the diagonal in the following sense. The set of 0-chains is Λ_1 and they are of degree 1. The set of 1-chains is $s_1(\Lambda)$ and they are of degree 2. More generally, an n-chains $x|t_1 \ldots |t_{n-1}|t_n$ is of degree $n + 1$. As a consequence, we have the following result.

6.2.2. Theorem. *A quadratic monomial algebra is Koszul.*

More generally, the Anick resolution can be used to prove Koszulness of an algebra whose set of relations forms a quadratic Gröbner basis. In that case, the Anick resolution is concentrated in the right bidegree. Hence we have the following sufficient condition for Koszulness of quadratic algebras.

6.2.3. Theorem ([Ani86, Sect. 3]). *An algebra presented by a quadratic Gröbner basis is Koszul.*

Another way to prove this result is that the existence of a quadratic Gröbner basis implies the existence of a Poincaré-Birkhoff-Witt basis of **A**, [Gre99].

6.2.4. Example. The algebra $\mathbb{K}[x_1, \ldots, x_k]$ of commutative polynomials on k variables can be presented by the following linear 2-polygraph:

$$\Lambda = \langle\, * \mid x_1, \ldots, x_k \mid x_{i_1}x_{i_2} \overset{\tau_{i_1 i_2}}{\Longrightarrow} x_{i_2}x_{i_1}, \quad 1 \leqslant i_1 < i_2 \leqslant k \,\rangle.$$

For any triple (i_1, i_2, i_3) such that $1 \leqslant i_1 < i_2 < i_3 \leqslant k$, there is a critical branching on the monomial $x_{i_1}x_{i_2}x_{i_3}$ which is confluent

$$\begin{array}{ccccccc} & \overset{\tau_{i_1 i_2}x_{i_3}}{\nearrow} & x_{i_2}x_{i_1}x_{i_3} & \overset{x_{i_2}\tau_{i_1 i_3}}{\Longrightarrow} & x_{i_2}x_{i_3}x_{i_1} & \overset{\tau_{i_2 i_3}x_{i_1}}{\searrow} & \\ x_{i_1}x_{i_2}x_{i_3} & & & & & & x_{i_3}x_{i_2}x_{i_1} \\ & \underset{x_{i_1}\tau_{i_2 i_3}}{\searrow} & x_{i_1}x_{i_3}x_{i_2} & \underset{\tau_{i_1 i_3}x_{i_2}}{\Longrightarrow} & x_{i_3}x_{i_1}x_{i_2} & \underset{x_{i_3}\tau_{i_1 i_2}}{\nearrow} & \end{array}$$

It follows that the linear 2-polygraph Λ is convergent and quadratique, hence the algebra $\mathbb{K}[x_1, \ldots, x_k]$ is Koszul.

6.2.5. Example, [DC17]. Dotsenko and Roy Chowdhury show that the algebra **A** presented by

$$\langle\, * \mid x, y, z \mid yx + x^2, zy, xz \,\rangle$$

is Koszul. Their proof in [DC17] is based on the computation of Anick's resolution with respect to the degree-lexicographic ordering induced by the alphabetic order $x > y > z$. The three quadratic relations can be completed into the following infinite Gröbner basis:

$$xz \Rightarrow 0, \quad zy \Rightarrow 0, \quad xy^k x \Rightarrow y^{k+1}x, \quad \text{for } k \geqslant 0$$

Using Anick's resolution they show that the homology of the algebra **A** is concentrated on the diagonal, proving that the algebra **A** is Koszul.

6.2.6. A sufficient polygraphic condition. In [GHM17], a graded version of Theorem 5.2.9 is given. For that, a notion of graded linear polygraph is introduced, that generalizes in higher dimensions the notion of graded presentation for a graded algebra. As an application, one deduces the following polygraphic condition of Koszulness of graded algebras.

6.2.7. Theorem ([GHM17, Prop. 7.2.2]). *Let* **A** *be an* N*-homogeneous algebra. If* **A** *has a* ℓ_N*-concentrated polygraphic resolution, then* **A** *is bi-Koszul (resp. left-Koszul, resp. right-Koszul).*

From this sufficient condition, one deduces the following consequence. Suppose that an algebra **A** has a polygraphic resolution Λ such that $(\Lambda_0, \Lambda_1, \ldots, \Lambda_{k-1})$ is ℓ_N-concentrated, for some $k \geqslant 3$, and such that for some $i > \ell_N(k)$ the number of $(k+1)$-cells in $\Lambda_{k+1}^{(i)}$ is strictly less than the number of k-cells in $\Lambda_k^{(i)}$. Then the algebra **A** is not Koszul, [GHM17, Prop. 7.2.7].

Theorem 6.2.7 can be also used to extend the sufficient condition of Theorem 6.2.3 to linear 2-polygraph with an orientation that is not compatible with a monomial order.

6.2.8. Corollary ([GHM17]). *Let* **A** *be an algebra presented by a quadratic left-monomial convergent linear 2-polygraph* Λ*. Then* Λ *can be extended into a* ℓ_2*-concentrated polygraphic resolution and the algebra* **A** *is Koszul.*

6.2.9. Exercise. Let **A** be the algebra presented by $\langle * \mid x, y \mid x^2 = y^2 = xy \rangle$. Prove that **A** is not Koszul. [Hint. Consider the rules $xy \Rightarrow x^2$ and $y^2 \Rightarrow x^2$, compute a convergent presentation of **A** and its set of critical triple branchings.]

6.2.10. Remark. Note that, for an N-homogeneous algebra, that is whose relations are concentrated in degree N, the existence of a Gröbner basis concentrated in degree N is not enough to imply Koszulness. Indeed, an extra condition has to be checked as shown by Berger in [Ber01].

6.2.11. Homogeneous coherent presentations. A coherent ℓ_N-concentrated presentation of an algebra **A** having an empty homotopy basis can be extended into a polygraphic resolution with an empty set of k-cells for $k \geqslant 3$, thus a ℓ_N-concentrated polygraphic resolution. Hence, by Theorem 6.2.7, we have

6.2.12. Corollary ([GHM17]). *If a N-homogeneous algebra has a coherent ℓ_N-concentrated presentation with an empty homotopy basis, then it is Koszul. In particular, an algebra having a terminating presentation by a N-homogeneous polygraph without any critical branching is Koszul.*

The second statement is a consequence of Squier's Theorem 5.1.6. Indeed, if Λ is a convergent left-monomial linear 2-polygraph, then it can be extended into a coherent presentation whose homotopy basis is made of generating confluences. In particular, when the polygraph Λ has no critical branching, this homotopy basis is empty, and thus trivially ℓ_N-concentrated.

6.2.13. Example, [GHM17, Ex. 7.2.5]. Consider the algebra **A** presented by the linear 2-polygraph given in Example 2.1.9. From the resolution computed in Example 5.2.10, we have $\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$, $\mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3$, $\mathrm{Tor}^{\mathbf{A}}_{2,(3)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$ and $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanishes for other values of k and i. It follows that the algebra **A** is Koszul.

6.2.14. Exercice [PP05, 4.3]. Show that the algebra presented by the following linear 2-polygraph, see 5.1.12,

$$\langle * \mid x, y, z \mid yz \overset{\alpha}{\Longrightarrow} -x^2,\ zy \overset{\beta}{\Longrightarrow} -\lambda^{-1}x^2 \rangle,$$

where $\lambda \in \mathbb{K} \setminus \{0, 1\}$, is Koszul. In particular, show that $\mathrm{Tor}^{\mathbf{A}}_{0,(0)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}$, $\mathrm{Tor}^{\mathbf{A}}_{1,(1)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^3$, $\mathrm{Tor}^{\mathbf{A}}_{2,(2)}(\mathbb{K},\mathbb{K}) \simeq \mathbb{K}^2$ and $\mathrm{Tor}^{\mathbf{A}}_{k,(i)}(\mathbb{K},\mathbb{K})$ vanishes otherwise.

References

[AG87] David J. Anick and Edward L. Green. On the homology of quotients of path algebras. *Comm. Algebra*, 15(1-2):309–341, 1987.

[Ani85] David J. Anick. On monomial algebras of finite global dimension. *Trans. Amer. Math. Soc.*, 291(1):291–310, 1985.

[Ani86] David J. Anick. On the homology of associative algebras. *Trans. Amer. Math. Soc.*, 296(2):641–659, 1986.

[Bac78] Jörgen Backelin. La série de Poincaré-Betti d'une algèbre graduée de type fini à une relation est rationnelle. *C. R. Acad. Sci. Paris Sér. A-B*, 287(13):A843–A846, 1978.

[Bac83] Jörgen Backelin. A distributiveness property of augmented algebras, and some re- lated homological results. Ph.D. thesis, Stockholm University, 1983.

[BC14] Leonid A. Bokut and Yuqun Chen. Gröbner-Shirshov bases and their calculation. *Bull. Math. Sci.*, 4(3):325–395, 2014.

[BD16] Murray R. Bremner and Vladimir Dotsenko. *Algebraic operads*. CRC Press, Boca Raton, FL, 2016. An algorithmic companion.

[Ber78] George M. Bergman. The diamond lemma for ring theory. *Adv. in Math.*, 29(2):178–218, 1978.

[Ber98] Roland Berger. Confluence and Koszulity. *J. Algebra*, 201(1):243–283, 1998.

[Ber01] Roland Berger. Koszulity for nonquadratic algebras. *J. Algebra*, 239(2):705–734, 2001.

[BF85] Jörgen Backelin and Ralf Fröberg. Koszul algebras, Veronese subrings and rings with linear resolutions. *Rev. Roumaine Math. Pures Appl.*, 30(2):85–97, 1985.

[BM06] Roland Berger and Nicolas Marconnet. Koszul and Gorenstein properties for homogeneous algebras. *Algebr. Represent. Theory*, 9(1):67–97, 2006.

[BN98] Franz Baader and Tobias Nipkow. *Term rewriting and all that*. Cambridge University Press, 1998.

[BO93] Ronald Book and Friedrich Otto. *String-rewriting systems*. Texts and Monographs in Computer Science. Springer-Verlag, 1993.

[Bok76] Leonid A. Bokut. Imbeddings into simple associative algebras. *Algebra i Logika*, 15(2):117–142, 245, 1976.

[Bro92] Kenneth S. Brown. The geometry of rewriting systems: a proof of the Anick-Groves-Squier theorem. In *Algorithms and classification in combinatorial group theory (Berkeley, CA, 1989)*, volume 23 of *Math. Sci. Res. Inst. Publ.*, pages 137–163. Springer, New York, 1992.

[Buc65] Bruno Buchberger. *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal (An Algorithm for Finding the Basis Elements in the Residue Class Ring Modulo a Zero Dimensional Polynomial Ideal)*. PhD thesis, Mathematical Institute, University of Innsbruck, Austria, 1965. English translation in J. of Symbolic Computation, Special Issue on Logic, Mathematics, and Computer Science: Interactions. Vol. 41, Number 3-4, Pages 475–511, 2006.

[Buc70] Bruno Buchberger. Ein algorithmisches Kriterium für die Lösbarkeit eines algebraischen Gleichungssystems. *Aequationes Math.*, 4:374–383, 1970.

[Buc87] Bruno Buchberger. History and basic features of the critical-pair/completion procedure. *J. Symbolic Comput.*, 3(1-2):3–38, 1987. Rewriting techniques and applications (Dijon, 1985).

[Buc06] Bruno Buchberger. An algorithm for finding the basis elements of the residue class ring of a zero dimensional polynomial ideal. *J. Symbolic Comput.*, 41(3-4):475–511, 2006. Translated from the 1965 German original by Michael P. Abramson.

[Bur93] Albert Burroni. Higher-dimensional word problems with applications to equational logic. *Theoret. Comput. Sci.*, 115(1):43–62, 1993. 4th Summer Conference on Category Theory and Computer Science (Paris, 1991).

[Bur01] Serge Burckel. Syntactical methods for braids of three strands. *J. Symbolic Comput.*, 31(5):557–564, 2001.

[BW93] Thomas Becker and Volker Weispfenning. *Gröbner bases*, volume 141 of *Graduate Texts in Mathematics*. Springer-Verlag, New York, 1993. A computational approach to commutative algebra, In cooperation with Heinz Kredel.

[Coh65] Paul Moritz Cohn. *Universal algebra*. Harper & Row, Publishers, New York-London, 1965.

[DC17] Vladimir Dotsenko and Soutrik Roy Chowdhury. Anick resolution and Koszul algebras of finite global dimension. *Comm. Algebra*, 45(12):5380–5383, 2017.

[DK09] Vladimir Dotsenko and Anton Khoroshkin. Free resolutions via Gröbner bases. *ArXiv e-prints*, 2009.

[DK10] Vladimir Dotsenko and Anton Khoroshkin. Gröbner bases for operads. *Duke Math. J.*, 153(2):363–396, 2010.

[DK13] Vladimir Dotsenko and Anton Khoroshkin. Quillen homology for operads via Gröbner bases. *Doc. Math.*, 18:707–747, 2013.

[GGM15] Stéphane Gaussent, Yves Guiraud, and Philippe Malbos. Coherent presentations of Artin monoids. *Compos. Math.*, 151(5):957–998, 2015.

[GHM17] Yves Guiraud, Eric Hoffbeck, and Philippe Malbos. Convergent presentations and polygraphic resolutions of associative algebras. arXiv:1406.0815v2, December 2017.

[GM12a] Yves Guiraud and Philippe Malbos. Coherence in monoidal track categories. *Math. Structures Comput. Sci.*, 22(6):931–969, 2012.

[GM12b] Yves Guiraud and Philippe Malbos. Higher-dimensional normalisation strategies for acyclicity. *Adv. Math.*, 231(3-4):2294–2351, 2012.

[GM18] Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. *Math. Structures Comput. Sci.*, 28(2):155–201, 2018.

[Gra72] Hans Grauert. Über die Deformation isolierter Singularitäten analytischer Mengen. *Invent. Math.*, 15:171–198, 1972.

[Gre99] Edward L. Green. Noncommutative Gröbner bases, and projective resolutions. In *Computational methods for representations of groups and algebras (Essen, 1997)*, volume 173 of *Progr. Math.*, pages 29–60. Birkhäuser, Basel, 1999.

[Gro90] John R. J. Groves. Rewriting systems and homology of groups. In *Groups—Canberra 1989*, volume 1456 of *Lecture Notes in Math.*, pages 114–141. Springer, Berlin, 1990.

[Hir64] Heisuke Hironaka. Resolution of singularities of an algebraic variety over a field of characteristic zero. I, II. *Ann. of Math. (2) 79 (1964), 109–203; ibid. (2)*, 79:205–326, 1964.

[HM17] Nohra Hage and Philippe Malbos. Knuth's coherent presentations of plactic monoids of type A. *Algebr. Represent. Theory*, 20(5):1259–1288, 2017.

[Hue80] Gérard Huet. Confluent reductions: abstract properties and applications to term rewriting systems. *J. Assoc. Comput. Mach.*, 27(4):797–821, 1980.

[Jan20] Maurice Janet. Sur les systèmes d'équations aux dérivées partielles. *Journal de mathématiques pures et appliquées*, 8(3):65–151, 1920.

[JW09] Michael Jöllenbeck and Volkmar Welker. Minimal resolutions via algebraic discrete Morse theory. *Mem. Amer. Math. Soc.*, 197(923):vi+74, 2009.

[KB70] Donald Knuth and Peter Bendix. Simple word problems in universal algebras. In *Computational Problems in Abstract Algebra (Proc. Conf., Oxford, 1967)*, pages 263–297. Pergamon, Oxford, 1970.

[Klo92] Jan Willem Klop. Term rewriting systems. In *Handbook of Logic in Computer Science*, volume 2, chapter 1, pages 1–117. Oxford University Press, 1992.

[Kob90] Yuji Kobayashi. Complete rewriting systems and homology of monoid algebras. *J. Pure Appl. Algebra*, 65(3):263–275, 1990.

[Kob05] Yuji Kobayashi. Gröbner bases of associative algebras and the Hochschild cohomology. *Trans. Amer. Math. Soc.*, 357(3):1095–1124 (electronic), 2005.

[KR00] Martin Kreuzer and Lorenzo Robbiano. *Computational commutative algebra. 1*. Springer-Verlag, Berlin, 2000.

[Mit72] Barry Mitchell. Rings with several objects. *Advances in Math.*, 8:1–161, 1972.

[MM16] Philippe Malbos and Samuel Mimram. Homological computations for term rewriting systems. In *1st International Conference on Formal Structures for Computation and Deduction*, volume 52 of *LIPIcs. Leibniz Int. Proc. Inform.*, pages Art. No. 27, 17. Schloss Dagstuhl. Leibniz-Zent. Inform., Wadern, 2016.

[Mor94] Teo Mora. An introduction to commutative and noncommutative Gröbner bases. *Theoret. Comput. Sci.*, 134(1):131–173, 1994. Second International Colloquium on Words, Languages and Combinatorics (Kyoto, 1992).

[New42] Maxwell Newman. On theories with a combinatorial definition of "equivalence". *Ann. of Math. (2)*, 43(2):223–243, 1942.

[Pom78] Jean-Francois Pommaret. *Systems of partial differential equations and Lie pseudogroups*, volume 14 of *Mathematics and its Applications*. Gordon & Breach Science Publishers, New York, 1978. With a preface by André Lichnerowicz.

[PP05] Alexander Polishchuk and Leonid Positselski. *Quadratic algebras*, volume 37 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2005.

[Pri70] Stewart B. Priddy. Koszul resolutions. *Trans. Amer. Math. Soc.*, 152:39–60, 1970.

[Shi62] Anatoly Illarionovich Shirshov. Some algorithmic problems for Lie algebras. *Sib. Mat. Zh.*, 3:292–296, 1962.

[Skö06] Emil Sköldberg. Morse theory from an algebraic viewpoint. *Trans. Amer. Math. Soc.*, 358(1):115–129, 2006.

[SOK94] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

[SST00] Mutsumi Saito, Bernd Sturmfels, and Nobuki Takayama. *Gröbner deformations of hypergeometric differential equations*, volume 6 of *Algorithms and Computation in Mathematics*. Springer-Verlag, Berlin, 2000.

[Str76] Ross Street. Limits indexed by category-valued 2-functors. *J. Pure Appl. Algebra*, 8(2):149–181, 1976.

[Str87] Ross Street. The algebra of oriented simplexes. *J. Pure Appl. Algebra*, 49(3):283–335, 1987.

[Ter03] Terese. *Term rewriting systems*, volume 55 of *Cambridge Tracts in Theoretical Computer Science*. Cambridge University Press, 2003.

[Tho37] Joseph Miller Thomas. Differential systems. IX + 118 p. New York, American Mathematical Society (American Mathematical Society Colloquium Publications Vol. XXI) (1937)., 1937.

[Thu14] Axel Thue. Probleme über Veränderungen von Zeichenreihen nach gegebenen Regeln. *Kristiania Vidensk. Selsk, Skr.*, 10:493–524, 1914.

[Ufn95] Victor A. Ufnarovskij. Combinatorial and asymptotic methods in algebra. In *Algebra, VI*, volume 57 of *Encyclopaedia Math. Sci.*, pages 1–196. Springer, Berlin, 1995.

[Ufn98] Victor A. Ufnarovskij. Introduction to noncommutative Gröbner bases theory. In *Gröbner bases and applications (Linz, 1998)*, volume 251 of *London Math. Soc. Lecture Note Ser.*, pages 259–280. Cambridge Univ. Press, Cambridge, 1998.

— July 25, 2017 —

Philippe Malbos
malbos@math.univ-lyon1.fr
Univ Lyon, Université Claude Bernard Lyon 1
CNRS UMR 5208, Institut Camille Jordan
43 blvd. du 11 novembre 1918
F-69622 Villeurbanne cedex, France