

Diagrammatic bases from stratified normalization

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Abstract – In this article, we introduce a new rewriting method for computing hom-bases of diagrammatic categories. Our approach is based on stratified presentations, obtained by partitioning an equational presentation into strata according to the rewriting behavior of its relations. We establish modularity principles for termination and confluence in stratified rewriting systems with coefficients in a ring. We then introduce a completion procedure that transforms a stratified presentation into a normalized one, in which mixed interactions between successive strata are absorbed into the normalized rewriting rules, thereby inducing modularity of confluence. This leads to a stratified hom-basis theorem for diagrammatic categories. We illustrate the method by applying it to the q -Schur category, for which we construct a new hom-basis.

Keywords – Diagrammatic rewriting, q -Schur category, Monoidal categories.

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1. INTRODUCTION

The determination of explicit linear bases for morphism spaces in diagrammatic categories is a central problem in representation theory and higher algebra. Such bases provide concrete descriptions of morphisms, make computations effective, and often reveal structural properties of the categories under consideration. This problem appears prominently in many categories arising from categorification. For instance, the q -Schur category, as presented by Brundan, admits distinguished bases for its morphism spaces, closely related to classical bases of q -Schur algebras and to bases in $U_q(\mathfrak{gl}_n)$ -web categories [6]. Diagrammatic integral bases for the Hom-spaces of affine and cyclotomic web and q -Schur categories have been constructed in [37]. Another fundamental example is provided by Soergel bimodules: Libedinsky’s light leaves basis, and subsequently the double leaves basis, give explicit combinatorial bases for morphism spaces between Bott–Samelson bimodules. Similar questions arise in the diagrammatic categorification of quantum groups, where understanding spaces of 2-morphisms is an essential part of the theory.

A natural way to construct such bases from presentations by generators and relations is to orient the defining relations into rewriting rules and study the resulting rewriting system. When the system is terminating and confluent, normal forms provide canonical representatives and, under suitable monicity assumptions in the linear setting, yield bases of the corresponding hom-spaces. This principle underlies Gröbner–Shirshov basis method, originating in the work of Buchberger [8] and Shirshov [36], and has subsequently been extended to many algebraic structures, including operads [12, 13, 29].

In the categorical and higher-dimensional setting, rewriting has developed into a diagrammatic theory adapted to presentations of monoidal categories and 2-categories. Elias [17] developed a version of Bergman’s diamond lemma [5] for monoidal categories presented by generators and relations, with applications to Hecke-type algebras and categories. In a higher-dimensional linear setting, Dupont [16] developed rewriting modulo isotopy relations for pivotal linear 2-categories and used it to compute bases of spaces of 2-cells. More recently, Schelstraete developed rewriting modulo for linear 2-categories, and applied it to a basis theorem for a super-2-category related to odd Khovanov homology [35] and to categories of affine Brauer type [4]. Diagrammatic rewriting has now developed into an active line of research, with several complementary techniques and formalisms; see, for instance, [2].

Rewriting in linear monoidal categories and, more generally, in linear 2-categories can be formalized using higher-dimensional rewriting structures such as linear polygraphs [1, 16, 22, 35]. In the approach

adopted here, rewriting is formulated using 3-prepolygraphs, whose underlying 2-dimensional structure is a precategory rather than a strict 2-category. Rewriting acts on morphisms, viewed as 2-cells, while termination and confluence control the existence and uniqueness of normal forms.

These rewriting methods have been used to construct hom-bases in several diagrammatic settings. In particular, rewriting without relying on global monomial orders was developed for associative algebras in [22] and subsequently adapted to diagrammatic categories, yielding constructive hom-bases for the affine oriented Brauer category [1, 16], the Khovanov–Lauda–Rouquier category [15], the loop Hecke algebras [25], and the graded 2-category of gl_2 -foams [35].

Despite these developments, constructing convergent diagrammatic presentations remains difficult for two main reasons. The first concerns the application of the critical branching lemma, which characterizes confluence of the whole rewriting system in terms of local confluence on critical branchings. The first difficulty is combinatorial: in the setting of rewriting in 2-categories, the analysis of critical branchings gives rise to intricate families of confluence obstructions [23, 24, 31]. The second difficulty concerns termination, which requires the construction of a well-founded measure compatible with the orientation of all rewriting rules.

The purpose of this article is to address these two difficulties through a stratified approach to diagrammatic rewriting. Instead of treating all rewriting rules simultaneously, we partition them into strata according to their rewriting behavior. On the termination side, we introduce *stratified termination functions*, which allow termination to be proved stratum by stratum by combining different termination methods, in particular derivation-based and monomial-order-based arguments. On the confluence side, we introduce a normalization procedure in which the interactions between a new stratum and the preceding ones are absorbed into normalized rewriting rules. This makes confluence modular with respect to the stratification and reduces its verification to confluence within the individual strata.

A further motivation for the present work comes from the coefficient rings occurring naturally in diagrammatic algebra. Many diagrammatic algebras are defined over a commutative ring, and it is therefore desirable to perform rewriting directly over the ground ring, without passing to a field extension in which the relevant coefficients become invertible. Rewriting with coefficients in a ring has already been developed in Gröbner basis theory and reduction rings, notably in the works of Buchberger [7], Möller [32], and Stifter [38, 39].

The absence of inverses requires a refinement of the usual reduction relation, since the applicability of a rewriting rule depends on divisibility properties of coefficients. In the higher-dimensional setting considered here, this arithmetic becomes part of the diagrammatic rewriting process itself: coefficient reduction affects both admissible rewriting steps and the structure of local and critical branchings. We therefore develop rewriting directly over the ground ring for linear monoidal categories and 2-categories presented by 3-prepolygraphs, together with the corresponding notions of reduction, critical branching, and completion.

As an application, we study Brundan’s presentation of the q -Schur category from [6, Thm. 2]. Applying the normalization procedure to this presentation, we construct a new hom-basis for the q -Schur category, called the *normal form basis*. In Section 6, we compare this basis with other known hom-bases for the q -Schur category, including the standard, canonical, and codeterminant bases.

The paper is organized as follows. In Section 2, we introduce a notion of diagrammatic rewriting over a commutative ring, formulated for linear monoidal categories and 2-categories using linear 3-prepolygraphs. We then recall the notions of termination, confluence, and normal forms, and explain under the relevant monicity assumptions how convergent presentations give rise to hom-bases. We also develop the rewriting-theoretic machinery used throughout the article.

Section 3 introduces stratified 3-prepolygraphs modulo and stratified termination functions. Theorem 3.1.5 states that any 3-prepolygraph modulo admitting a stratified termination function is ter-

minating modulo. This approach provides a modular proof of termination, carried out stratum by stratum and allowing different strata to be controlled by different termination methods, in particular derivation-based and monomial-order-based arguments. As an illustration, we revisit the presentation of Khovanov–Lauda–Rouquier algebras and obtain a streamlined proof of termination compared with that of [15].

Section 4 introduces a normalized completion procedure for stratified presentations and shows that, under suitable compatibility assumptions, confluence becomes a modular property that can be analysed stratum by stratum. Combined with the monicity assumptions required for linear normal forms, this leads to our main hom-basis theorem:

Theorem 4.1.9. *Let C be a linear monoidal category presented by a stratified 3-prepolygraph F^*X . If there exists an F_s^1X -compatible monomial order $\leq$ such that every stratum $F_s^i\widehat{X}$ of the associated $\leq$ -normalization $F^*\widehat{X}$ is left-monomial and confluent, then $\text{Nf}_m(\widehat{X})$ forms a hom-basis of C .*

Subsection 4.2 develops a completion procedure for computing hom-bases from stratified presentations whenever the procedure terminates.

In Section 5, we apply Theorem 4.1.9 to Brundan’s presentation of the q -Schur category. We construct a stratified presentation, prove its termination, perform its normalization, and derive a new hom-basis, called the *normal form basis*.

Finally, Section 6 gives an explicit diagrammatic description of the normal form basis. We develop a picture calculus for the normal forms associated with the stratified presentation and use it to compare the resulting basis with the standard, canonical, and codeterminant bases of the q -Schur category.

2. HOM-BASES BY DIAGRAMMATIC REWRITING

This section recalls the foundations of rewriting for diagrammatic categories formulated in the framework of linear 2-categories. Higher-dimensional rewriting originates from a series of works on equational presentations of higher categories [9, 33, 40, 41]. More recently, these methods have been developed through several approaches, depending on whether rewriting is performed modulo the interchange relations [1, 16, 23], or whether these relations are made explicit as rewriting rules within the system itself [18, 35].

2.1. (Pre)polygraphs and diagrammatic rewriting

We shall work within the framework of precategories [18], which generalize the structure of sesquicategories [42] to arbitrary dimensions. Although we focus here on rewriting in two-dimensional precategories, most of the constructions developed below extend to arbitrary higher dimensions.

2.1.1. Linear categories. Let C be a 2-category. For each $0 \leq k \leq 2$, we use C_k to denote both the set of k -cells of C and the corresponding k -category. For $f \in C_k$ and $i < k$, we write $s_i(f)$ and $t_i(f)$ for the i -source and i -target of f . We further set $sf := s_{k-1}(f)$ and $tf := t_{k-1}(f)$ for $k > 0$.

For $k > 0$, a k -sphere of C is a pair (f, g) of parallel k -cells, that is, such that $sf = sg$ and $tf = tg$. For $f \in C_k$, we denote by $f^\partial := (sf, tf)$ the *boundary* $(k-1)$ -sphere of f . The set of k -spheres of C is denoted by $\text{Sph}_k(C)$. A (cellular) extension of C_k is a set X equipped with a map $\rho : X \rightarrow \text{Sph}_k(C)$. We denote by $\mathcal{E}(C_k)$ the set of extensions of C_k . If X and Y are two such extensions, we denote their disjoint union by XY .

By regarding the elements of an extension X of C_k as generating $(k+1)$ -cells with sources and targets in C , and by forming all their formal composites, one obtains the *free* $(k+1)$ -category generated

by X over C , denoted by $C[X]$. The size of a $(k+1)$ -cell f in $C[X]$ is the number of occurrences of generating $(k+1)$ -cells from X in f , denoted by $|f|$. The quotient of C_k by X , denoted by C_k/X , is the k -category obtained from C by identifying $s(\alpha)$ with $t(\alpha)$ for every $\alpha \in X$.

Recall that a *linear 2-category* over a commutative ring $\mathbf{k}$ with unity is a 2-category C such that, for every pair of 1-cells $p, q \in C_1$, the hom-set $C_2(p, q)$ is endowed with a structure of a $\mathbf{k}$ -module, and such that, for all $p, q, r \in C_1$, the composition functor

$$\circ_1 : C_2(p, q) \otimes_{\mathbf{k}} C_2(q, r) \rightarrow C_2(p, r) \quad (2.1.2)$$

is $\mathbf{k}$ -linear. A *linear 2-precategory* is defined in the same way, except that the interchange law, or equivalently the functoriality of the composition §2.1.2, is not required to hold.

A *linear monoidal category* is a linear 2-category with a single 0-cell. Since this article is concerned with presentations of linear monoidal categories, all the 2-categorical structures considered below are assumed to have a single 0-cell. Nevertheless, all the constructions and results extend straightforwardly to the multi-object setting.

The 2-categories and 2-precategories considered here, as well as their linear counterparts, are generated by *2-polygraphs*. A *2-polygraph* is a datum $X_{\leq 2} = (X_0, X_1, X_2)$ consisting of a directed graph (X_0, X_1) , whose elements are called generating 0-cells and generating 1-cells, respectively, together with a cellular extension X_2 of the free category X_1^* generated by this graph. In accordance with the convention above, we assume throughout that X_0 is a singleton.

We denote by X_2^* , X_2^ℓ , X_2° , and $X_2^{\circ\ell}$ the 2-category, linear 2-category, 2-precategory, and linear 2-precategory freely generated by X , respectively. These structures have the same underlying 1-category, namely X_1^* .

A 2-cell of X_2° is called a *2-monomial*. A 2-cell of $X_2^{\circ\ell}$ of the form ah , where h is a 2-monomial and $a \in \mathbf{k} \setminus \{0\}$, is called a *2-term*. Every 2-monomial h admits a unique decomposition $h = h_1 \circ_1 \dots \circ_1 h_{|h|}$, where each h_i is a 0-composition $h_i = u_i \varphi_i v_i$, with $\varphi_i \in X_2$ and $u_i, v_i \in X_1^*$.

In the sequel, we will use a diagrammatic representation of 2-cells. The 2-monomial h_i and the composition $h_i \circ_1 h_{i+1}$ will be respectively denoted by the following diagram

$$h_i = \begin{array}{c} | \quad \dots \quad | \\ | \quad \varphi_i \quad | \\ | \quad \dots \quad | \end{array}, \quad \text{and} \quad h_i \circ_1 h_{i+1} = \begin{array}{c} | \quad \dots \quad | \\ | \quad h_{i+1} \quad | \\ | \quad \dots \quad | \\ | \quad h_i \quad | \\ | \quad \dots \quad | \end{array}.$$

The 2-monomials of $X_2^\circ(p, q)$ form a basis of the $\mathbf{k}$ -modules $X_2^{\circ\ell}(p, q)$ of 2-cells, for all 1-cells p and q . Equivalently, every 2-cell f of $X_2^{\circ\ell}$ admits a unique decomposition $f = \sum_i a_i h_i$, where the h_i are pairwise distinct 2-monomials and $a_i \in \mathbf{k} \setminus \{0\}$. The coefficient a_i of the monomial h_i in the decomposition of f is denoted by $\text{cf}_{h_i}(f)$. The set of 2-terms $\{a_i h_i\}_i$ forms the *support* of f , denoted by $\text{Supp}(f)$, while the set of 2-monomials $\{h_i\}_i$ forms its *monomial support*, denoted by $\text{MSupp}(f)$.

The 2-category X_2^* (resp. linear 2-category X_2^ℓ) is obtained as the quotient of the 2-precategory X_2° (resp. linear 2-precategory $X_2^{\circ\ell}$) by the *interchange extension* $\text{Int}(X_2)$. This extension consists of 3-generators

$$\alpha_{\varphi, u, \psi} : (\varphi u s_1(\psi)) \circ_1 (t_1(\varphi) u \psi) \rightarrow (s_1(\varphi) u \psi) \circ_1 (\varphi t_1(\psi)), \quad (2.1.3)$$

for all $\varphi, \psi \in X_2$ and $u \in X_1^*$, diagrammatically represented as

$$\alpha_{\varphi, u, \psi} : \begin{array}{c} | \quad \dots \quad | \\ | \quad \varphi \quad | \\ | \quad \dots \quad | \end{array} u \begin{array}{c} | \quad \psi \quad | \\ | \quad \dots \quad | \\ | \quad \dots \quad | \end{array} \rightarrow \begin{array}{c} | \quad \dots \quad | \\ | \quad \varphi \quad | \\ | \quad \dots \quad | \end{array} u \begin{array}{c} | \quad \dots \quad | \\ | \quad \psi \quad | \\ | \quad \dots \quad | \end{array}.$$

A *(one-hole) context*, or simply a *context*, of the 2-precategory $X_2^{\circ\ell}$ is a pair $(\square, C)$, denoted by $C[\square]$, or simply by C , where $\square$ is a 1-sphere of X_1^* and C is a 2-term *ag*, with $a \in \mathbf{k} \setminus \{0\}$ and $g \in (X_2 \sqcup \{\square\})^\circ$

containing exactly one occurrence of the generating 2-cell $\square$. Every context can be written uniquely in the form

$$C[\square] = a(h \circ_1 u \square v \circ_1 k),$$

where $a \in \mathbf{k}$, $h, k \in X_2^\circ$ and $u, v \in X_1^*$. The size of C is defined by $|C| = |h| + |k|$. A context $C[\square]$ is compatible with $f \in X_2^\circ$ if $\square = f^\partial$. A context of the form $C = au\square v$ is called a *whisker context*.

Similarly, a *two-hole context* of $X_2^{\circ\ell}$ is a triple $(\square_1, \square_2, D)$, denoted by $D[\square_1, \square_2]$, where $\square_1$ and $\square_2$ are 1-spheres of X_1^* and D is a 2-term ag , with $a \in \mathbf{k} \setminus \{0\}$ and $g \in (X_2 \sqcup \{\square_1, \square_2\})^\circ$ containing exactly one occurrence of each of the 2-generators $\square_1$ and $\square_2$.

The category of contexts is denoted by $\mathbf{Cont}(X_2^{\circ\ell})$. Its objects are the 2-terms of $X_2^{\circ\ell}$, and a morphism from f to g is a compatible context C such that $C[f] = g$. Composition is given by the nesting of contexts: for contexts C and D , we set $C_1C_2[\square] := C_1[C_2[\square]]$. When $C[f] = g$, we say that f *divides* g and write $f | g$. In a same way we define the category of contexts on 2-monomials, denoted by $\mathbf{Cont}(X_2^\circ)$.

2.1.4. Three-dimensional prepolygraphs. An equational presentation of a linear monoidal category can be studied from a rewriting perspective by orienting its defining relations. This leads to the notion of a *linear 3-prepolygraph*, defined as a pair $X = (X_{\leq 2}, X_3)$, where $X_{\leq 2}$ is a 2-polygraph and $X_3 \in \mathcal{E}(X_2^{\circ\ell})$.

A linear 3-prepolygraph is *modulo* if it is defined as $(X_{\leq 2}, X_3 \sqcup E)$ where $E \in \mathcal{E}(X_2^{\circ\ell})$ is symmetric, it is then denoted by (X, E) . For 2-cells f, g in $X_2^{\circ\ell}$, we denote $f \sim_E g$ if there is a 3-cell in the free 3-category $E^{\circ\ell}$ with source f and target g . We denote ${}_E X_E$ the extension of $X_2^{\circ\ell}$ consisting of spheres (f, g) such that $f \sim_E f' \xrightarrow[X]{\quad} g' \sim_E g$. For instance, when E is the symmetric closure of the extension $\text{Int}(X_2)$, we recover the structure of linear 3-polygraph [2, Sec. 18.3].

We denote by $X_3^{\circ\ell}$ the free linear 3-precategory generated by X , and by $\bar{X} := X_2^{\circ\ell} / (X_3 \sqcup \text{Int}(X_2))$ the linear 2-category presented by X . Since all the 2-categorical structures considered here have a single 0-cell, $\bar{X}$ can equivalently be regarded as a linear monoidal category. Two linear 3-prepolygraphs are *Tietze equivalent* if they present isomorphic linear 2-categories.

A linear 3-prepolygraph is *left-term* if the source of each of its 3-generators is a 2-term. Up to Tietze equivalence, any linear 3-prepolygraph may be assumed to be left-term. Unless otherwise stated, all 3-prepolygraphs considered in what follows are assumed to be linear and left-term.

The elements of the extension X_3 are called *X-rewriting rules*, denoted by $\alpha : s\alpha \rightarrow t\alpha$, and diagrammatically written as

$$\alpha : a \begin{array}{|c|} \hline \cdots \\ \hline h \\ \hline \cdots \\ \hline \end{array} \rightarrow \sum_i a_i \begin{array}{|c|} \hline \cdots \\ \hline h_i \\ \hline \cdots \\ \hline \end{array},$$

when $s\alpha = ah$ and $t\alpha = \sum_i a_i h_i$. We assume throughout that $h \notin \text{MSupp}(t\alpha)$. Whenever a is invertible, we normalize the rule by replacing $ah \rightarrow t\alpha$ with $h \rightarrow a^{-1}t\alpha$. Thus, the coefficient a is either 1 or a nonzero noninvertible element of $\mathbf{k}$. We call a the *left coefficient* and h the *left monomial* of α , and write $\text{lc}(\alpha) = a$ and $\text{lm}(\alpha) = h$. Finally, we set $\partial\alpha := s\alpha - t\alpha$.

2.1.5. Rings with remainder system. When we rewrite over a ring, we have to take into account the interaction of the context of rewriting with that divisibility properties of the ring in order to guaranty a decreasing reduction order and to do not introduce any cycle of reductions just induced by this interaction and not due to the rule system itself [7].

We define a *normalized remainder system* (NRS) on a commutative integral domain $\mathbf{k}$ as a family $\mathcal{R} = (\text{Rem}(a))_{a \in \mathbf{k} \setminus \{0\}}$, where $\text{Rem}(a) \subseteq \mathbf{k}$, such that, for every $a \in \mathbf{k} \setminus \{0\}$, one has $0 \in \text{Rem}(a)$ and the canonical map

$$\text{Rem}(a) \longrightarrow \mathbf{k}/(a),$$

sending r to $r + (a)$ is bijective. Equivalently, $\text{Rem}(a)$ is a fixed set of representatives in $\mathbf{k}$ for the residue classes modulo the ideal (a) . For every $b \in \mathbf{k}$, there are therefore unique elements $\text{quo}_a(b)$ in $\mathbf{k}$ and

$\text{rem}_a(b)$ in $\text{Rem}(a)$ satisfying

$$b = a \text{ quo}_a(b) + \text{rem}_a(b).$$

These maps satisfy $\text{rem}_a(b) = 0$ if and only if $b \in (a)$ and $\text{quo}_a(b) = 0$ if and only if $b \in \text{Rem}(a)$. We have also $\text{rem}_a(b + ca) = \text{rem}_a(b)$ for every $c \in \mathbf{k}$. In particular, for every unit $u \in \mathbf{k}$, we have $\text{Rem}(u) = \{0\}$, $\text{rem}_u(b) = 0$, and $\text{quo}_u(b) = u^{-1}b$.

A NRS is *decreasing* if there exists a well-order $\leq_{\mathbf{k}}$ on $\mathbf{k}$, with 0 as its least element, such that

$$\text{rem}_a(b) \neq b \quad \text{implies} \quad \text{rem}_a(b) <_{\mathbf{k}} b,$$

for all $b \in \mathbf{k}$ and $a \in \mathbf{k} \setminus \{0\}$. Such a system can be constructed by fixing a well-order $\leq_{\mathbf{k}}$ on $\mathbf{k}$ with least element 0 and setting

$$\text{rem}_a(b) = \min_{\leq_{\mathbf{k}}} (b + (a)), \quad a \text{ quo}_a(b) = b - \text{rem}_a(b), \quad \text{and} \quad \text{Rem}(a) = \{\text{rem}_a(b) \mid b \in \mathbf{k}\}. \quad (2.1.6)$$

The resulting system is decreasing because $b \in b + (a)$. Consequently, there is no infinite decreasing sequence $(b_i)_i$, with $b_{i+1} = \text{rem}_{a_i}(b_i)$ for some $a_i \in \mathbf{k}$.

The existence of a decreasing NRS is weaker than Euclidean. A Euclidean structure controls remainders relative to the divisor, whereas a decreasing NRS only provides well-founded normalization inside each residue class.

2.1.7. Examples. When $\mathbf{k}$ is a field, every nonzero element is a unit. Hence the unique normalized remainder system is given by $\text{Rem}(a) = \{0\}$, $\text{rem}_a(b) = 0$, and $\text{quo}_a(b) = a^{-1}b$. It is decreasing for any well-order on $\mathbf{k}$ having 0 as its least element.

When $\mathbf{k}$ is an Euclidean domain equipped with a Euclidean function $\varphi : \mathbf{k} \setminus \{0\} \rightarrow \mathbb{N}$. Choose a well-order $\leq_{\mathbf{k}}$ with least element 0 such that, for nonzero a, b , $\varphi(a) < \varphi(b)$ implies that $a <_{\mathbf{k}} b$. Such an order can be constructed by ordering the nonzero elements first by their Euclidean value and then by a fixed well-order within each level. Euclidean division ensures that every coset modulo (a) contains either 0 or an element of Euclidean value strictly smaller than $\varphi(a)$. Therefore, we set $\text{rem}_a(b) = 0$ or $\varphi(\text{rem}_a(b)) < \varphi(a)$.

For example, when $\mathbf{k} = \mathbb{Z}$, for every $a \in \mathbb{Z} \setminus \{0\}$, we set $\text{Rem}(a) = \{0, 1, \dots, |a| - 1\}$ and for every $b \in \mathbf{k}$, we define the remainder $\text{rem}_a(b)$ and quotient $\text{quo}_a(b)$ by Euclidian division as follows

$$\text{rem}_a(b) = b - |a| \left\lfloor \frac{b}{|a|} \right\rfloor, \quad \text{and} \quad a \text{ quo}_a(b) = b - \text{rem}_a(b).$$

This NRS is decreasing for the well-order $0 < \dots < n < \dots < -1 < \dots < -n < \dots$.

When $\mathbf{k} = \mathbb{Z}[q, q^{-1}]$, for $f = \sum_i a_i q^i$, we set $H(f) = \sum_i (1 + |i|)|a_i|$. We order $\mathbf{k}$ first by H , breaking ties lexicographically on $(a_0, a_1, a_{-1}, a_2, a_{-2}, \dots)$, with the coefficient order $0 < 1 < -1 < 2 < -2 < \dots$. Since H has finite sublevel sets, this defines a well-order $\leq_{\mathbf{k}}$ with least element 0. We then define a decreasing NRS by using definitions (2.1.6).

2.1.8. Rewriting steps. From now on, we assume that $\mathbf{k}$ is a commutative integral domain equipped with a NRS $\mathcal{R}$ with associated quotient and remainder maps quo and Rem .

Let X be a 3-prepolygraph. An X -rewriting step is a 3-cell of $X_3^{\circ\ell}$ of size 1, of the form

$$\sigma = aC[\alpha] + 1_f, \quad (2.1.9)$$

where $a \in \mathbf{k} \setminus \{0\}$, $\alpha \in X_3$, $f \in X_2^{\circ\ell}$, and $C \in \text{Cont}(X_2^{\circ})$, subject to the following condition. Writing

$$f = bC[\text{lm}(\alpha)] + f' \quad \text{and} \quad C[\text{lm}(\alpha)] \notin \text{MSupp}(f'),$$

we require $\text{quo}_{\text{lc}(\alpha)}(b) = 0$. The 3-cell σ is diagrammatically written

$$\sigma : a \cdot \begin{array}{c} \text{---} \\ | \\ \boxed{k} \\ | \\ \text{---} \end{array} u \begin{array}{c} \text{---} \\ | \\ \boxed{s\alpha} \\ | \\ \text{---} \end{array} v + \begin{array}{c} \text{---} \\ | \\ \boxed{f} \\ | \\ \text{---} \end{array} \rightarrow a \cdot \begin{array}{c} \text{---} \\ | \\ \boxed{k} \\ | \\ \text{---} \end{array} u \begin{array}{c} \text{---} \\ | \\ \boxed{t\alpha} \\ | \\ \text{---} \end{array} v + \begin{array}{c} \text{---} \\ | \\ \boxed{f} \\ | \\ \text{---} \end{array}.$$

Finally, when $f = 0$, we call σ *left-term*.

2.1.10. Remark. When X is left-monomial, the condition in the definition of an X -rewriting step σ (2.1.9) reduces to $C[\text{lm}(\alpha)] \notin \text{MSupp}(f)$. Indeed, put $h = C[\text{lm}(\alpha)]$ and let $b = \text{cf}_h(f)$. Since X is left-monomial, we have $\text{lc}(\alpha) = 1$. It follows from $\text{Rem}(1) = \{0\}$ that $\text{quo}_1(b) = b$ and $\text{rem}_1(b) = 0$. By definition, the 3-cell σ is an X -rewriting step if and only if $\text{quo}_1(b) = b = 0$, or equivalently, if and only if $C[\text{lm}(\alpha)] \notin \text{MSupp}(f)$.

In particular, over a field, this recovers the standard notion of a rewriting step for a left-monomial linear polygraph, as formulated in the associative-algebra setting in [22, Sec. 3.1] and in the higher-dimensional setting in [1, 15, 35].

2.1.11. Example. Let $\mathbf{k} = \mathbb{Z}$ be an NRS as in §2.1.7. Let consider the 3-prepolygraph X with two 2-generators ϕ and $\bowtie$ and the 3-generator $\alpha : 7\phi \rightarrow 4\bowtie$. Then the 3-cell

$$\sigma := C[\alpha] + 8h : 15h \rightarrow 4\bowtie + 8h,$$

where $C = \begin{array}{c} \boxed{\phi} \\ | \\ \bowtie \end{array}$, and $h = \begin{array}{c} \phi \\ | \\ \bowtie \end{array}$ is not an X -rewriting step, since $\text{rem}_7(8) = 1$ and $\text{quo}_7(8) = 1$. However, it decomposes as a composite of two 3-cells $\sigma = \tau \circ_2 \rho^{-1}$, where

$$\tau := 2C[\alpha] + h : 15h = 2 \cdot 7h + h \rightarrow 2 \cdot 4\bowtie + h = 8\bowtie + h,$$

and

$$\rho := C[\alpha] + h + 4\bowtie : 4\bowtie + 8h = 4\bowtie + 7h + h \rightarrow 4\bowtie + 4\bowtie + h = 8\bowtie + h.$$

Indeed, both τ and ρ are X -rewriting steps, since the coefficient of h in their additive parts is 1, with $\text{quo}_7(1) = 0$ and $\text{rem}_7(1) = 1$.

This observation leads to the following lemma, analogous to [22, Lem. 3.1.3], which was established in the setting of rewriting in associative algebras over a field.

2.1.12. Lemma. *Let X be a 3-prepolygraph. Every 3-cells σ of size 1 in X_3^{of} admits a decomposition $\sigma = \tau \circ_2 \rho^{-1}$, where each of τ and ρ is either an identity or an X -rewriting step.*

Proof. We write $\sigma = aC[\alpha] + 1_g$, where $a \in \mathbf{k} \setminus \{0\}$, $\alpha \in X_3$, $g \in X_2^{\text{of}}$, and $C \in \mathbf{Cont}(X_2^{\circ})$. We set $\lambda = \text{lc}(\alpha)$ and $h = C[\text{lm}(\alpha)]$, and we write $g = bh + g'$ with $h \notin \text{MSupp}(g')$. Using the NRS, we consider $q = \text{quo}_{\lambda}(b)$ and $r = \text{rem}_{\lambda}(b)$. Thus $b = q\lambda + r$, with $r \in \text{Rem}(\lambda)$ and $\text{quo}_{\lambda}(r) = 0$.

Let us define the following 3-cells in X_3^{of}

$$\tau = (a + q)C[\alpha] + 1_{rh+g'} \quad \text{and} \quad \rho = qC[\alpha] + 1_{aC[t\alpha]+rh+g'},$$

that form the following diagram

$$\begin{array}{ccc} ((a + q)\lambda + r)h + g' & \xrightarrow{\sigma} & aC[t\alpha] + (q\lambda + r)h + g' \\ & \searrow \tau \quad \swarrow \rho & \\ & (a + q)C[t\alpha] + rh + g'. & \end{array}$$

Since $\text{lm}(\alpha) \notin \text{MSupp}(t\alpha)$ and monomial contexts in the free precategory X_2° are injective on monomials, we have $h \notin \text{MSupp}(C[t\alpha])$.

Consequently, the coefficient of h in each of $rh + g'$ and $aC[t\alpha] + rh + g'$ is r . As $\text{quo}_\lambda(r) = 0$, by definition of an X -rewriting step, if $a + q \neq 0$, then τ is an X -rewriting step, and if $q \neq 0$, then ρ is an X -rewriting step. Otherwise, σ and τ are identities. Finally, the linear composition identity gives

$$\sigma \circ_2 \rho = \sigma + \rho - 1_{t\sigma} = (a + q)C[\alpha] + 1_{rh+g'} = \tau.$$

Hence $\sigma = \tau \circ_2 \rho^{-1}$. □

2.1.13. Termination and normalization. A X -rewriting path of length n is a 3-cell of $X_3^{\circ\ell}$ defined as a $\circ_2$ -composite $\sigma = \sigma_1 \circ_2 \dots \circ_2 \sigma_n$, where each σ_i is an X -rewriting step.

If σ is an X -rewriting step (resp. X -rewriting path) from f to g , we write $f \xrightarrow{\sigma} g$ (resp. $f \xrightarrow{\sigma} g$). When there exists an X -rewriting step (resp. X -rewriting path) with source f and target g , we write $f \xrightarrow{X} g$ (resp. $f \xrightarrow{X} g$).

A 2-cell f of $X_2^{\circ\ell}$ is X -reducible if there is a rule α in X_3 and a context C such that $h = C[\text{lm}(\alpha)] \in \text{MSupp}(f)$, and writing

$$f = bh + f' \quad \text{and} \quad h \notin \text{MSupp}(f'),$$

we have $q = \text{quo}_{\text{lc}(\alpha)}(b) \neq 0$. The corresponding X -rewriting step is thus:

$$\sigma = qC[\alpha] + 1_{f-q\text{lc}(\alpha)h} : f \rightarrow f - qC[\partial\alpha].$$

A 3-prepolygraph X is *terminating*, if the induced rewrite relation $\xrightarrow{X}$ is contained in a well-founded ordering. A 3-prepolygraph modulo (X, E) is *E-terminating* if the relation $\sim_E \xrightarrow{X} \sim_E$ is contained in a well-founded ordering.

A 2-cell f of $X_2^{\circ\ell}$ is in X -normal form if it cannot be reduced by any X -rewriting step. We denote by $\text{Nf}(X)$ the set of X -normal forms, and by $\text{Nf}_m(X)$ its subset of monomials. A 2-cell g is a *normal form of a 2-cell f* if $f \xrightarrow{X} g$ and g is in X -normal form, and we write $g = f \downarrow_X$. The polygraph X is *normalizing* if every 2-cell of $X_2^{\circ\ell}$ has a normal form. In that case, a *X-normalization strategy* is a map $(-)\downarrow_X$ sending every 2-cell f of $X_2^{\circ\ell}$ to an X -normal form $f \downarrow_X$.

A context $C = a(h \circ_1 u \square v \circ_1 k)$ in $X_2^{\circ\ell}$ is X -normal, denoted by $C \in \text{Nf}(X)$, if $ah, ak \in \text{Nf}(X)$.

Let Y be a normalizing extension of $X_2^{\circ\ell}$, together with a Y -normalization strategy $(-)\downarrow_Y$, and let α be a rule in X_3 . A context C compatible with α is said to be *Y-active on α* if either $\alpha \notin \text{Nf}(Y)$ and $|C| = 0$, or $\alpha \in \text{Nf}(Y)$, $C \in \text{Nf}(Y)$, and $C[\alpha] \notin \text{Nf}(Y)$. We denote by $\text{Act}(\alpha, Y)$ the set of Y -active contexts on α . For instance, let $Y := \text{Int}(X_2)$. If $\alpha \in \text{Nf}(Y)$, we have

$$\text{Act}(\alpha, Y) = \left(\begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \boxed{g} \end{array} \middle| \begin{array}{c} \vdots \\ \vdots \\ \vdots \\ \square \end{array} \middle| \begin{array}{c} \boxed{f} \\ \vdots \\ \vdots \\ \vdots \end{array} \middle| f, g \in \text{Nf}(Y) \right).$$

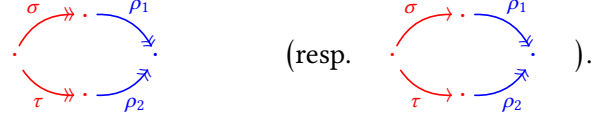
When Y is left-monomial, every Y -active context on α has coefficient 1.

We define the *context relation* $\sqsubseteq$ on the 3-cells of $X_3^{\circ\ell}$ by setting, for all $\beta_1, \beta_2 \in X_3^{\circ\ell}$, $\beta_1 \sqsubseteq \beta_2$ if there exists a context C in $X_2^{\circ\ell}$ such that $C[\beta_1] = \beta_2$. For an extension Y of $X_2^{\circ\ell}$, we denote by $\min_{\sqsubseteq} Y$ the set of minimal 3-generators in Y with respect to the relation $\sqsubseteq$.

2.2. Confluence and hom-bases by confluence

We now introduce the rewriting notions used throughout this work, together with the rewriting machinery allowing one to construct hom-bases from convergent presentations of categories.

2.2.1. Confluence. A *branching* of a 3-prepolygraph X is a pair (σ, τ) of X -rewriting paths with the same source, that is, $s\sigma = s\tau$. It is called *local* if both σ and τ are X -rewriting steps. The 3-prepolygraph X is *confluent* (resp. *local confluent*) if for any branching (resp. local branching) (σ, τ) , there exists a pair of X -rewriting paths (ρ_1, ρ_2) , called a *confluence*, as in the following diagram



We call X *convergent* if it is both terminating and confluent. In this case, every 2-cell f admits a unique normal form, denoted by $f \downarrow_X$. Thus, the normal-form map $f \rightarrow f \downarrow_X$ is uniquely determined. However, the X -rewriting paths from f to $f \downarrow_X$ need not be unique. A normalization rewriting strategy amounts to choosing such a path for each 2-cell f .

2.2.2. Classification of local branchings. Let X be a left-term 3-prepolygraph, and let (σ, τ) be a local branching of X with common source f . We write

$$\sigma = q_1 C_1[\alpha_1] + 1_{f - q_1 \lambda_1 h_1}, \quad \tau = q_2 C_2[\alpha_2] + 1_{f - q_2 \lambda_2 h_2},$$

where, for $i = 1, 2$, we have $\alpha_i \in X_3$, $C_i \in \mathbf{Cont}(X_2^\circ)$, $\lambda_i = \text{lc}(\alpha_i)$, $h_i = C_i[\text{lm}(\alpha_i)] \in \text{MSupp}(f)$, $b_i = \text{cf}_{h_i}(f)$, $q_i = \text{quo}_{\lambda_i}(b_i) \neq 0$, and $r_i = \text{rem}_{\lambda_i}(b_i)$, so that $b_i = q_i \lambda_i + r_i$. We distinguish the following four families of local branchings.

i) *Aspherical branchings*, when the two rewriting steps coincide, that is, $\sigma = \tau$.

ii) *Additive branchings*, when the two rules α_1 and α_2 are applied to distinct terms in $\text{Supp}(f)$, that is, $h_1 \neq h_2$. The common source is $f = b_1 h_1 + b_2 h_2 + g$, where $h_1, h_2 \notin \text{MSupp}(g)$. More explicitly,

$$\sigma = q_1 C_1[\alpha_1] + 1_{r_1 h_1 + b_2 h_2 + g}, \quad \tau = q_2 C_2[\alpha_2] + 1_{b_1 h_1 + r_2 h_2 + g}.$$

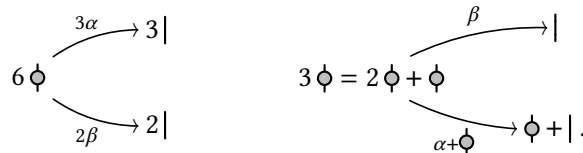
iii) *Peiffer branchings*, when the two rules α_1 and α_2 are applied to disjoint occurrences within the same monomial $h = h_1 = h_2$, and the product of their leading coefficients divides the coefficient $b = b_1 = b_2$ of h in f , that is, $\lambda_1 \lambda_2 \mid b$. Thus, there exist a monomial two-hole context D and a nonzero scalar μ such that $h = D[\text{lm}(\alpha_1), \text{lm}(\alpha_2)]$ and $b = \mu \lambda_1 \lambda_2$. The common source is $f = \mu \lambda_1 \lambda_2 h + g$, where $h \notin \text{MSupp}(g)$. More explicitly,

$$\sigma = \mu \lambda_2 D[\alpha_1, \text{lm}(\alpha_2)] + 1_g, \quad \tau = \mu \lambda_1 D[\text{lm}(\alpha_1), \alpha_2] + 1_g.$$

iv) *Overlapping branchings*, when the two rewriting steps are distinct and apply the rules α_1 and α_2 to the same monomial $h = h_1 = h_2$. Moreover, either the occurrences of $\text{lm}(\alpha_1)$ and $\text{lm}(\alpha_2)$ in h overlap, or they are disjoint but the product of their leading coefficients does not divide the coefficient $b = b_1 = b_2$ of h in f , that is, $\lambda_1 \lambda_2 \nmid b$. The common source is $f = b h + g$, where $h \notin \text{MSupp}(g)$. More explicitly,

$$\sigma = q_1 C_1[\alpha_1] + 1_{r_1 h + g}, \quad \tau = q_2 C_2[\alpha_2] + 1_{r_2 h + g}.$$

The *critical branchings* of X are the overlapping branchings (σ, τ) of X whose common source $s\sigma = s\tau$ is a 2-term and which are minimal with respect to the context relation $\sqsubseteq$ on X_3^{of} . For instance, the rules $\alpha : 2\phi \rightarrow |$ and $\beta : 3\phi \rightarrow |$ give rise to the following two critical branchings:



The first branching arises from the common multiple 6 of the leading coefficients, whereas the second arises from the nonzero remainder in $3 = 1 \cdot 2 + 1$; in particular, $\alpha + \phi$ is an X -rewriting step.

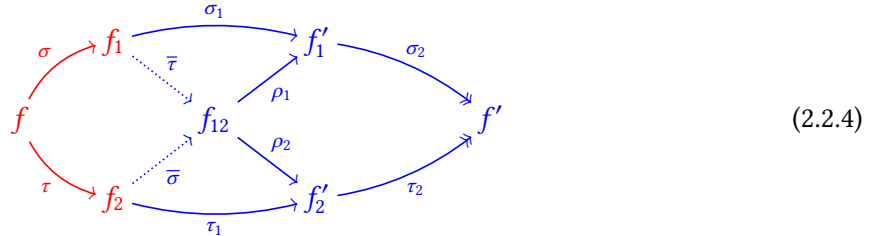
A key feature of the precategorical setting is that overlaps between 2-monomials in X_2° can be decomposed into overlaps between strings. Indeed, every 2-monomial $h \in X_2^\circ$ admits a unique decomposition $h = h_1 \circ_1 \cdots \circ_1 h_{|h|}$, where $h_i = C_i[\varphi_i]$, with $\varphi_i \in X_2$ and C_i is a whisker context. An occurrence of a 2-monomial k inside u , given by a monomial 2-context C such that $u = C[v]$, corresponds, up to whiskering by 1-cells, to a contiguous factor in the decomposition of h . Hence, if two occurrences of leading monomials $\text{lm}(\alpha_1)$ and $\text{lm}(\alpha_2)$ overlap in a 2-monomial h , the corresponding factors in the canonical decomposition of h overlap. The study of overlapping branchings is therefore reduced to the study of compatible overlaps of finite strings. In particular, for a fixed pair of rules, their possible overlaps can be determined combinatorially from the string decompositions of their leading monomials. This reduction to string overlaps is one of the main advantages of the precategorical formulation of 3-dimensional rewriting introduced in [18].

We now state the critical branching lemma.

2.2.3. Lemma. *Assume that X is a terminating 3-prepolygraph. If every critical branching of X is confluent, then X is locally confluent. Moreover, X is confluent.*

Proof. Following [22, 3.2.2 and Thm. 4.2.1], we proceed by noetherian induction on the sources of local branchings, with respect to the well-founded rewrite order $<_X$ of X . Let f be a reducible 2-cell, and assume that X is locally confluent at every 2-cell $f' <_X f$. By Newman's lemma, it follows that X is confluent at every $f' <_X f$.

Let (σ, τ) be a local branching with source f . If (σ, τ) is aspherical, then it is trivially confluent. We consider the remaining three cases. For each case, we use the notation of the classification in §2.2.2, and construct a confluence diagram of the following form



Case 1. Suppose that (σ, τ) is additive. By definition of σ and τ , we have $f = b_1 h_1 + b_2 h_2 + g$. We set

$$f_1 = t(\sigma) = q_1 \bar{h}_1 + r_1 h_1 + b_2 h_2 + g, \quad f_2 = t(\tau) = b_1 h_1 + q_2 \bar{h}_2 + r_2 h_2 + g, \quad f_{12} = q_1 \bar{h}_1 + r_1 h_1 + q_2 \bar{h}_2 + r_2 h_2 + g,$$

with $\bar{h}_i = C_i[t(\alpha_i)]$. Let consider the 3-cells

$$\bar{\sigma} = q_1 C_1[\alpha_1] + 1_{r_1 h_1 + q_2 \bar{h}_2 + r_2 h_2 + g} : f_2 \rightarrow f_{12}, \quad \text{and} \quad \bar{\tau} = q_2 C_2[\alpha_2] + 1_{q_1 \bar{h}_1 + r_1 h_1 + r_2 h_2 + g} : f_1 \rightarrow f_{12}.$$

They may be not X -rewriting steps, since we may have $h_1 \in \text{MSupp}(\bar{h}_2)$ or $h_2 \in \text{MSupp}(\bar{h}_1)$ and the coefficients of h_1 and h_2 in the additive parts of $\bar{\sigma}$ and $\bar{\tau}$ need not belong to $\text{Rem}(\lambda_1)$ and $\text{Rem}(\lambda_2)$, respectively. However, they have size 1. Lemma 2.1.12 therefore yields 2-cells f'_1 and f'_2 , together with X -rewriting paths σ_1, ρ_1, τ_1 and ρ_2 as listed in (2.2.4). Hence X is confluent at $f_{12} <_X f$ by hypothesis, so the branching (ρ_1, ρ_2) admits a confluence (σ_2, τ_2) .

Case 2. Suppose next that (σ, τ) is Peiffer. Set $\bar{h}_i = t(\alpha_i)$ for $i = 1, 2$. The common source is $f = \mu \lambda_1 \lambda_2 h + g$, where $h = D[\text{lm}(\alpha_1), \text{lm}(\alpha_2)]$. Put

$$f_1 = \mu \lambda_2 D[\bar{h}_1, \text{lm}(\alpha_2)] + g, \quad f_2 = \mu \lambda_1 D[\text{lm}(\alpha_1), \bar{h}_2] + g, \quad f_{12} = \mu D[\bar{h}_1, \bar{h}_2] + g.$$

Then $f_1 = t(\sigma)$ and $f_2 = t(\tau)$. There are 3-cells

$$\bar{\sigma} = \mu D[\alpha_1, \bar{h}_2] + 1_g : f_2 \rightarrow f_{12}, \quad \bar{\tau} = \mu D[\bar{h}_1, \alpha_2] + 1_g : f_1 \rightarrow f_{12},$$

which need not be X -rewriting paths, since we may have $\text{MSupp}(D[\bar{h}_1, \text{lm}(\alpha_2)]) \cap \text{MSupp}(g) \neq \emptyset$ or $\text{MSupp}(D[\text{lm}(\alpha_1), \bar{h}_2]) \cap \text{MSupp}(g) \neq \emptyset$, and the corresponding coefficients in g need not belong to $\text{Rem}(\lambda_2)$ and $\text{Rem}(\lambda_1)$, respectively. We decompose them into 3-cells of size 1 and apply Lemma 2.1.12 successively, as in [22, Lem. 4.1.2]. This yields 2-cells f'_1 and f'_2 , together with X -rewriting paths σ_1, ρ_1, τ_1 and ρ_2 as listed in §2.2.4. Hence X is confluent at $f_{12} <_X f$ by hypothesis, so the branching (ρ_1, ρ_2) admits a confluence (σ_2, τ_2) .

Case 3. Finally, suppose that (σ, τ) is overlapping. The common source is $f = bh + g$, where $h = C_1[\text{lm}(\alpha_1)] = C_2[\text{lm}(\alpha_2)]$ and $b = q_i \lambda_i + r_i$ for $i = 1, 2$. Set

$$\bar{h}_i = C_i[t(\alpha_i)], \quad f_i = q_i \bar{h}_i + r_i h + g, \quad i = 1, 2.$$

Then $f_1 = t(\sigma)$ and $f_2 = t(\tau)$. By minimality, there exist a critical branching (σ_0, τ_0) and a context C such that $(q_1 C_1[\alpha_1] + 1_{r_1 h}, q_2 C_2[\alpha_2] + 1_{r_2 h}) = C[\sigma_0, \tau_0]$. The critical branching is confluent by hypothesis, it admits a confluence (σ'_0, τ'_0) with common target e_0 . Set $f_{12} = C[e_0] + 1_g$. Applying C gives the 3-cells

$$\bar{\sigma} = C[\tau'_0] + 1_g : f_2 \rightarrow f_{12}, \quad \bar{\tau} = C[\sigma'_0] + 1_g : f_1 \rightarrow f_{12},$$

which may not be X -rewriting paths, since $\text{MSupp}(g)$ may intersect $\text{MSupp}(C[\text{lm}(\sigma'_0)])$ or $\text{MSupp}(C[\text{lm}(\tau'_0)])$, and the corresponding coefficients in g need not belong to $\text{Rem}(\lambda_2)$ and $\text{Rem}(\lambda_1)$, respectively. We decompose them into 3-cells of size 1 and apply Lemma 2.1.12 successively. This yields 2-cells f'_1 and f'_2 , together with X -rewriting paths σ_1, ρ_1, τ_1 and ρ_2 as listed in §2.2.4. Hence X is confluent at $f_{12} <_X f$ by hypothesis, so the branching (ρ_1, ρ_2) admits a confluence (σ_2, τ_2) .

Therefore, X is locally confluent. Since X is terminating, it is confluent by Newman's Lemma. $\square$

We denote by $\text{CB}(X)$ the set of critical branchings of X . For two 3-prepolygraphs X and Y sharing the same underlying 2-polygraph, we denote by $\text{CB}(X, Y)$ the set of critical branchings involving both an X -rewriting step and a Y -rewriting step.

In [23, Sec. 5.1], critical branchings of 3-polygraphs are classified into three families: regular, inclusion, and indexed branchings. The classification for 3-prepolygraphs is similar, except that indexed branchings do not appear, since interchange relations are not available in this setting. This significantly simplifies the structure of critical branchings. On the other hand, interchange relations become explicit rewriting rules, generating new families of critical branchings that must be analyzed. The stratification approach introduced in Subsection 3.1 is precisely designed to handle this additional complexity.

2.2.5. Hom-bases by confluence over rings. Following [1, 15, 16, 35], in the case of rewriting over a field, if X is convergent, then $\text{Nf}_m(X)$ forms a *hom-basis* of the category $\mathcal{C}$ presented by X , that is, a family of linear bases $B(u, v)$ of the vector spaces $C_2(u, v)$, for all $u, v \in X_1^*$. Indeed, convergence ensures that every monomial has a unique normal-form representative, and the result follows from the equality $\text{Nf}(X) = \mathbf{k} \text{Nf}_m(X)$, that is, the module of normal forms need not be the free $\mathbf{k}$ -module spanned by the monomial normal forms.

When $\mathbf{k}$ is a ring, however, this equality need not hold, and hence the hom-basis theorem may fail for rewriting over a ring. For instance, let $\mathbf{k} = \mathbb{Z}$, and consider the 3-prepolygraph X with one 1-cell i , one 2-generator $\phi : i \rightarrow i$, and one 3-generator $2\phi \rightarrow 0$. Then, in $X_2^{\circ\ell}(i, i)$, every monomial $\phi^n := \phi \circ_1 \dots \circ_{n\text{-times}} \phi$ is X -irreducible. Hence $\text{Nf}_m(X)$ is the set of 2-monomial ϕ^n , for $n \geq 1$. On the other hand, a linear combination is irreducible precisely when all its nonzero coefficients are odd. Thus,

the set $\text{Nf}(X)$ of 2-cells in X -normal forms is made of 2-cells $\sum_{n \geq 1} \lambda_n \phi^n$ where λ_n is odd. In particular, $\text{Nf}(X)$ is not closed under addition. Consequently, we have $\text{Nf}(X) \neq \mathbf{k} \text{Nf}_m(X)$. This failure stems from the fact that the underlying $\mathbf{k}$ -module of $\overline{X}$ is not free. However, when X is left-monomial, we have the following result.

2.2.6. Theorem. *Let C be a linear monoidal category presented by a left-monomial and convergent 3-prepolygraph X . Then $\text{Nf}_m(X)$ forms a hom-basis of C .*

Proof. First we prove that under the hypothesis that we have $\text{Nf}(X) = \mathbf{k} \text{Nf}_m(X)$, or equivalently for every 2-cell f , we have $f \in \text{Nf}(X)$ if and only if $\text{MSupp}(f) \subseteq \text{Nf}_m(X)$.

Suppose first that $f \in \text{Nf}(X)$, and assume that some $u \in \text{MSupp}(f)$ is reducible. Then $u = C[\text{lm}(\alpha)]$ for some $\alpha \in X_3$ and some monomial context C . Let $a = \text{cf}_u(f) \neq 0$. Since X is left-monomial, we have $\text{lc}(\alpha) = 1$ and $\text{quo}_1(a) = a \neq 0$. Hence, the 3-cell $aC[\alpha] + 1_{f-au}$ is an X -rewriting step with source f , so f is reducible. This contradicts $f \in \text{Nf}(X)$. Therefore every monomial in $\text{MSupp}(f)$ is irreducible, and thus $\text{MSupp}(f) \subseteq \text{Nf}_m(X)$.

Conversely, suppose that f is reducible. Then there exists an X -rewriting step with source f . Such a step rewrites a monomial $C[\text{lm}(\alpha)]$ occurring in $\text{MSupp}(f)$, for some $\alpha \in X_3$ and some monomial context C . Hence $\text{MSupp}(f)$ contains a reducible monomial.

We have thus proved that a 2-cell is in normal form if and only if all the monomials in its support are in normal form. Equivalently, $\text{Nf}(X) = \mathbf{k} \text{Nf}_m(X)$.

We show that $\text{Nf}_m(X)$ is a hom-basis of C by considering $u, v \in X_1^*$ and $\pi_{u,v}$ the canonical projection of $X_2^{\circ\ell}(u, v)$ on $C_2(u, v)$. Since X is convergent, every 2-cell admits a unique normal form. It follows that $\pi_{u,v}$ restricts to a bijection

$$\text{Nf}(X) \cap X_2^{\circ\ell}(u, v) \longrightarrow C_2(u, v).$$

As its domain is the free $\mathbf{k}$ -module generated by the monomial normal forms from u to v . Hence the images of these monomial normal forms form a $\mathbf{k}$ -basis of $C_2(u, v)$. Since this holds for all $u, v \in X_1^*$, the family $\text{Nf}_m(X)$ is a hom-basis of C . $\square$

In order to apply the above result when X is terminating but not confluent, one may apply a completion procedure, in the spirit of those introduced for algebras in [8] and for term rewriting systems in [27]. In many situations, however, this completion procedure does not terminate, due to the large number of rules and the interactions between them. To reduce the number of interactions that need to be considered and to obtain a more tractable completion procedure, we introduce in the next sections a modular stratification of the set of rules. The aim is to decompose the rewriting system into strata in such a way that confluence can be established stratum by stratum and then lifted to the whole rewriting system.

2.3. Termination by orderings and derivations

Termination techniques have been extensively developed in the setting of term rewriting systems [10, 43]. These techniques often rely on interpretation methods, which assign to each term a value in a well-founded ordered structure so that every rewriting step strictly decreases this value. In the higher-dimensional setting, however, classical interpretation methods do not directly provide suitable reduction orders for 3-polygraphs [20, 21]. This has led to the development of derivation-based methods, which adapt the principle of interpretations to the structure of 3-polygraphs. In the linear setting, termination can also be established by means of monomial orders.

In this subsection, we recall these two approaches, which will be combined through stratification in the next section.

2.3.1. Monomial orders. A *total preorder* on a set Z is a transitive relation $\leq_Z$ on Z such that either $x \leq_Z y$ or $y \leq_Z x$ for all $x, y \in Z$. For distinct x and y , if $x \leq_Z y$, we write $x <_Z y$. Such an order is a *total order* if it is moreover antisymmetric, that is, if $x \leq_Z y$ and $y \leq_Z x$ imply $x = y$.

A (*hom-set*) *relation* $<$ on a free 2-precategory X_2° is a family of relations $<_{u,v}$ indexed by pairs (u, v) of 1-cells of X_1^* , where each relation $<_{u,v}$ is defined on the hom-set $X_2^\circ(u, v)$. The notions of *total preorder* and *total order* on X_2° are then defined pointwise on each hom-set. A relation $<$ is *monotone* if, for any parallel 2-cells $f, g : u \rightarrow v$ and compatible context C of X_2° , we have

$$f <_{u,v} g \quad \text{implies} \quad C[f] <_{C[u], C[v]} C[g].$$

It is *well-founded* if each relation $<_{u,v}$ is well-founded, that is, admits no infinite descending chain. A total order $\leq$ on a precategory X_2° is a *monomial order* if it is monotone and well-founded.

For every nonzero 2-cell f of $X_2^{\circ\ell}$, the $\leq$ -*leading monomial* of f , denoted by $\text{lm}_\leq(f)$, is the $\leq$ -maximum element of $\text{Supp}(f)$. The $\leq$ -*leading coefficient* of f , denoted by $\text{lc}_\leq(f)$, is the coefficient of $\text{lm}_\leq(f)$ in f . Finally, the $\leq$ -*leading term* of f is defined by $\text{lt}_\leq(f) = \text{lc}_\leq(f) \text{lm}_\leq(f)$.

Let X be a 3-prepolygraph. A monomial order $\leq$ on X_2° is X -*compatible* if $v < \text{lm}(\alpha)$, for all $\alpha \in X_3$ and $v \in \text{MSupp}(t\alpha)$. For every 3-cell $\beta \in X_3^{\circ\ell}$ such that $\partial\beta \neq 0$, set

$$\lambda_\beta := \begin{cases} \text{lc}_\leq(\partial\beta)^{-1}, & \text{if } \text{lc}_\leq(\partial\beta) \text{ is invertible,} \\ 1, & \text{otherwise.} \end{cases}$$

The $\leq$ -*oriented left-term 3-cell* associated with β , denoted by $\beta^\leq$, is defined by setting

$$s(\beta^\leq) := \lambda_\beta \text{lt}_\leq(\partial\beta), \quad t(\beta^\leq) := \begin{cases} \lambda_\beta \text{lt}_\leq(\partial\beta) + \lambda_\beta \partial\beta, & \text{if } \text{lm}_\leq(\partial\beta) \in \text{MSupp}(t\beta) \setminus \text{MSupp}(s\beta), \\ \lambda_\beta \text{lt}_\leq(\partial\beta) - \lambda_\beta \partial\beta, & \text{otherwise.} \end{cases}$$

The existence of an X -compatible monomial order implies that X is termination, see [22, Sec. 3.2].

2.3.2. Remark. Compared with 3-prepolygraphs, proving termination of a 3-polygraph by constructing a monomial order is often difficult [1, 14, 23]. For instance, consider a 3-polygraph X with a single 2-generator $\cap$ and a single 3-generator

$$| \mid \cap \rightarrow \cap \mid |.$$

If there exists an X -compatible monomial order $\leq$, then necessarily $\cap \mid | < \mid | \cap$. Composing both sides on top with the 2-cell $\cap$ yields

$$\cap \cap < \cap \cap.$$

This contradicts the interchange relations.

We now construct a monomial order for 3-prepolygraphs, which will be used in Section 5.

2.3.3. Lexicographic orders. Given a family $(\leq_{o_i})_{1 \leq i \leq N}$ of total preorders on a free 2-precategory X_2° , with $N \geq 2$, we define a new total preorder on X_2° by induction on i , denoted by $\leq_{o_1, \dots, o_N}$, as follows:

$$u \leq_{o_1, \dots, o_N} v \quad \text{if} \quad u <_{o_1} v \text{ or } (u =_{o_1} v \text{ and } u \leq_{o_2, \dots, o_N} v), \quad \text{for } u, v \in X_2^\circ.$$

If the last order $\leq_{o_N}$ is a total order, then the order $\leq_{o_1, \dots, o_N}$ is itself a total order [28, Lem. 1.15].

Similarly, the orders $(\leq_{o_i})_{1 \leq i \leq N}$ also induce a total preorder on the Cartesian product $(X_2^\circ)^N$, denoted by $\leq_{[o_1, \dots, o_N]}$, defined as follows: for $(u_1, \dots, u_N)$ and $(v_1, \dots, v_N)$ in $(X_2^\circ)^N$,

$$(u_1, \dots, u_N) \leq_{[o_1, \dots, o_N]} (v_1, \dots, v_N) \quad \text{if} \quad u_1 <_{o_1} v_1 \text{ or } (u_1 =_{o_1} v_1 \text{ and } (u_2, \dots, u_N) \leq_{[o_2, \dots, o_N]} (v_2, \dots, v_N)).$$

Moreover, if each total preorder $\leq_{o_i}$ is well-founded, then the order $\leq_{[o_1, \dots, o_N]}$ is also well-founded [28, Lem. 1.15].

Suppose that there exists a well-founded total order $\leq_o$ on the set $X_1 \sqcup X_2$. We define a total order $\leq_{\text{lex}}$ on the free 2-precategory X_2° as follows.

- i) Let f and g be two parallel 2-cells of X_2° of size 1, written as $f = x_n \cdots x_1 \circ_0 \varphi \circ_0 y_1 \cdots y_m$ and $g = x'_{n'} \cdots x'_1 \circ_0 \varphi' \circ_0 y'_1 \cdots y'_{m'}$, where $\varphi, \varphi' \in X_2$ and $x_i, y_i, x'_i, y'_i \in X_1$. We define $f \leq_{\text{lex}} g$ if either $m + n < m' + n'$, or $m + n = m' + n'$ and

$$(x_n, \dots, x_1, \varphi, y_1, \dots, y_m) \leq_{[o, \dots, o]} (x'_{n'}, \dots, x'_1, \varphi', y'_1, \dots, y'_{m'}).$$

For convenience, we still write $\leq_o := \leq_{[o, \dots, o]}$ for the order on parallel 2-cells of size 1.

- ii) In general, we consider $|f| = n$ and $|g| = m$, and write their unique $\circ_1$ -decompositions as $f = C_1[\varphi_1] \circ_1 \dots \circ_1 C_n[\varphi_n]$ and $g = D_1[\psi_1] \circ_1 \dots \circ_1 D_m[\psi_m]$, where $\varphi_i, \psi_i \in X_2$. We define $f \leq_{\text{lex}} g$ if either $n < m$, or $n = m$ and

$$(C_1[\varphi_1], \dots, C_n[\varphi_n]) \leq_{[o, \dots, o]} (D_1[\psi_1], \dots, D_m[\psi_m]).$$

We prove that the relation $\leq_{\text{lex}}$ is a monomial order on X_2° , called the *lexicographic order*, using the same arguments as for the lexicographic order on free words [28, Lem. 1.15].

2.3.4. Derivations. Given a 2-category C , a C -module O is a functor from the category of contexts $\text{Cont}(C)$ to the category $\mathbf{Ab}$ of abelian groups. Denote by $\mathbf{Ord}$ the 2-category with a single 0-cell, whose 1-cells are partially ordered sets and whose 2-cells are monotone maps. The 0-composition is given by the cartesian product while the 1-composition is given by the composition of monotone maps. Given a 2-functor $Z : C \rightarrow \mathbf{Ord}$, and an ordered abelian group $(G, \leq_G)$, the following assignments define a C -module, denoted by $O_{Z,G}$:

- i) Every 2-cell f is sent to the ordered abelian group of morphisms $O_{Z,G}(f) = \mathbf{Ord}(Z(sf), G)$, consisting of monotone maps from $Z(sf)$ to G , provided with the natural order define by $a \leq b$, if $a(y) \leq_G b(y)$, for all $a, b \in O_{Z,G}(f)$ and $y \in Z(sf)$.
- ii) If $C = w \square w'$ is a context of C , with a 1-sphere $\square$ and $w, w' \in C_1$, then $O_{Z,G}(C)$ sends a monotone map $a : Z(s\square) \rightarrow G$ to the map

$$Z(w) \times Z(s\square) \times Z(w') \rightarrow G$$

sending (y, z, y') to $a(z)$.

- iii) If $C = g \circ_1 \square \circ_1 h$ is a context of C , with a 1-sphere $\square$ and $g, h \in C_2$, then $O_{Z,G}(C)$ sends a monotone map $a : Z(s\square) \rightarrow G$ to the map $Z(sg) \rightarrow G$ sending y to $a(Z(g)(y))$.

Suppose now that the 2-category C is presented by a 3-polygraph X . Then the C -module $O_{Z,G}$ is uniquely determined by the partially ordered sets $Z(x)$, for $x \in X_1$, together with the monotone maps $Z(\varphi) : Z(s\varphi) \rightarrow Z(t\varphi)$, for $\varphi \in X_2$.

A *derivation* of C with values in a C -module O is a map $d : C \rightarrow O$ sending each 2-cell f to an element $d(f) \in O(f)$ such that, for every $i \in \{0, 1\}$ and every i -composable pair (f, g) of 2-cells in C ,

$$d(f \circ_i g) = f \circ_i d(g) + d(f) \circ_i g.$$

Following [23, Thm. 4.2.1], a 3-polygraph X is terminating if there exist a 2-functor $Z : X_2^* \rightarrow \mathbf{Ord}$, an ordered abelian group G , and a derivation $d : X_2^* \rightarrow O_{Z,G}$ satisfying the following two conditions:

- i) For any 1-cell c in X_1 , the set $Z(c)$ is non-empty and, $Z(s\alpha) \geq Z(t\alpha)$ for every $\alpha \in X_3$.
- ii) For any 2-cell f in X_2^* , we have $d(f) > 0$ and, $d(s\alpha) \geq d(t\alpha)$ for every $\alpha \in X_3$.

This termination criterion was extended to linear 3-polygraphs in [16, §1.3.3] and applies directly to 3-prepolygraphs. However, it is not sufficiently flexible for certain diagrammatic algebras involving large rewriting systems, such as the KLR algebras considered in Subsection 3.3 and the q -Schur categories studied in Section 5. In such cases, it is difficult to construct functors Z and derivations d satisfying the above conditions on the entire set of rewriting rules, which motivates the idea of stratifying the rewriting system.

3. STRATIFIED PREPOLYGRAPHS AND TERMINATION BY STRATIFICATION

This section introduces stratifications of 3-prepolygraphs modulo together with stratified termination functions, generalizing both derivation-based and monomial-order-based methods. We also illustrate this approach on Khovanov-Lauda-Rouquier algebras, obtaining a more efficient proof of termination than the one given in [15].

3.1. Stratified prepolygraphs and termination

3.1.1. Stratified prepolygraphs modulo. A k -stratified 3-prepolygraph modulo $F^*(X, E)$ is a 3-prepolygraph modulo (X, E) together with a sequence $(F^i X)_{1 \leq i \leq k}$ in $\mathcal{E}(X_2^{\circ\ell})$ such that

$$F^0 X = E \subseteq F^1 X \subseteq \dots \subseteq F^k X = X_3 E,$$

which we call a k -stratification of the 3-prepolygraph modulo (X, E) . The sequence $(F_s^i X)_{1 \leq i \leq k}$, where $F_s^i X := F^i X \setminus F^0 X$, is called the *reduced stratification* of the 3-prepolygraph (X, E) . For $i \geq 1$, the extension $F_s^i X := F^i X \setminus F^{i-1} X$ is called the *stratum of depth i* of the extension X_3 . When E is empty, we call it a k -stratified 3-prepolygraph and write $F^* X$.

3.1.2. Weighted functions. A *weighted function* on X_2° is a hom-set-wise function

$$\delta = (\delta_{u,v} : X_2^\circ(u, v) \rightarrow (Z_{u,v}, \leq_{u,v}))_{u,v \in X_1^*},$$

where each $(Z_{u,v}, \leq_{u,v})$ is a well-founded totally preordered set. We require that δ be monotone with respect to contexts, that is for all 2-cells f, g in $X_2^\circ(u, v)$ and every compatible context C , one has

$$\delta(f) \leq_{u,v} \delta(g) \quad \text{implies} \quad \delta(C[f]) \leq_{u,v} \delta(C[g]).$$

Given an extension Y of $X_2^{\circ\ell}$, we say that δ is Y -compatible (resp. *strictly Y -compatible*; an Y -identity) if, for every $\alpha \in Y$, one has $\delta(v) \leq \delta(\text{lm}(\alpha))$ (resp. $\delta(v) < \delta(\text{lm}(\alpha))$; $\delta(v) = \delta(\text{lm}(\alpha))$), for every $v \in \text{MSupp}(t\alpha)$. Note that the preorders $\leq_{u,v}$ need not be antisymmetric.

3.1.3. Termination function. A *termination function* of a k -stratified 3-prepolygraph modulo $F^*(X, E)$ is a family of weighted functions $(\delta_i)_{1 \leq i \leq k}$, satisfying the following conditions,

- i) δ_i is strictly $F_s^i X$ -compatible, for every $1 \leq i \leq k$,
- ii) δ_i is $F_s^{i-1} X$ -compatible, for every $2 \leq i \leq k$,
- iii) δ_i is E -identity, for every $1 \leq i \leq k$.

Condition **i)** ensures termination within each stratum, while Condition **ii)** provides the compatibility between successive strata required for the modularity of termination established in Theorem 3.1.5. The proof relies on a reduction from linear rewriting to monomial rewriting, which we formulate in terms of the *monomial 3-prepolygraph* $mX = (X_{\leq 2}, mX_3)$ associated with a left-monomial 3-prepolygraph X , defined by

$$mX_3 := \{(\text{lm}(\alpha), v) \mid \alpha \in X_3, v \in \text{MSupp}(t\alpha)\}.$$

3.1.4. Lemma. *A 3-prepolygraph modulo (X, E) is E -terminating if and only if its associated monomial 3-prepolygraph modulo (mX, mE) is mE -terminating.*

Proof. This result follows from properties of multiset relations, as introduced in [11], that we adapt to our modulo setting. We define a relation $\succ_+$ on X_2° by setting

$$s\alpha \succ_+ t\alpha \quad \text{for every } \alpha \in mX_3, \quad s\beta =_+ t\beta \quad \text{for every } \beta \in mE.$$

Moreover, for any compatible context C , we require $C[f] \succ_+ C[g]$ if $f \succ_+ g$. This induces a binary relation $\vdash$ on X_2°/mE defined by

$$u \vdash v \quad \text{if there exist } u', v' \in X_2^\circ \text{ such that } u =_+ u' \succ_+ v' =_+ v.$$

We then define analogously the relation on $X_2^{\circ\ell}/E$ as the restriction to finite subsets of X_2°/mE , of the multiset relation generated by $\vdash$. We refer to [3, Sec. 2.5] for the definition and properties of the multiset relation generated by a binary relation $\vdash$. In particular, the relation $\vdash$ is well-founded on the 2-cells of $X_2^{\circ\ell}/mE$ if and only if it is well-founded on the monomials of X_2°/E , whose proof is based on König's lemma. $\square$

3.1.5. Theorem. *A k -stratified 3-prepolygraph modulo $F^*(X, E)$ having a termination function is E -terminating.*

Proof. We have $X_3 = F_s^1 X F_s^2 \dots X F_s^k X$. We first consider the case where X_3 is a monomial extension. Suppose, for a contradiction, that there exists an infinite $_E X E$ -rewriting path

$$f_1 \sim_E g_1 \xrightarrow{\sigma_1} f_2 \sim_E g_2 \xrightarrow{\sigma_2} f_3 \sim_E g_3 \xrightarrow{\sigma_3} \dots \xrightarrow{\sigma_{n-1}} f_n \sim_E g_n \xrightarrow{\sigma_n} \dots,$$

where each σ_i is a X -rewriting step. Since δ is a termination function, we have $\delta_k(f_i) = \delta_k(g_i)$, for every $1 \leq i$. Moreover, according to whether σ_i is a $F_s^k X$ -rewriting step or a $F_s^{k-1} X$ -rewriting step, we have respectively

$$\delta_k(g_i) > \delta_k(f_{i+1}) \quad \text{or} \quad \delta_k(g_i) \geq \delta_k(f_{i+1}).$$

By well-foundedness, the rewriting path can therefore contain only finitely many $F_s^k X$ -rewriting steps. Hence, there exists an index j_k such that σ_i is a $F_s^{k-1} X$ -rewriting step for every $i \geq j_k$.

Repeating the same argument for each function δ_r for $r \leq k-1$, we eventually obtain an index j_2 such that all σ_i is an $F_s^1 X$ -rewriting step for every $i \geq j_2$. This is sufficient to prove that the stratum $F_s^1 X$ is terminating. By definition of the termination function δ , the map δ_1 is strictly $F_s^1 X$ -compatible. Hence, $F_s^1 X$ is terminating, that is, the 3-prepolygraph modulo $(F_s^1 X, E)$ is E -terminating.

In the general case, if X_3 is an extension of $X_2^{\circ\ell}$, the stratification $F^*(X, E)$ and the termination function δ induce a stratification of $G^*(mX, mE)$ and an associated termination function δ' by setting

$$G_s^* mX = F_s^1 mX F_s^2 mX \dots F_s^k mX \quad \text{and} \quad \delta'(f) = \delta(f),$$

for every $f \in X_2^\circ$, where $F_s^i mX := mF_s^i X$. Following the previous monomial case, we deduce that the 3-prepolygraph modulo $G^*(mX, mE)$ is mE -terminating. By Lemma 3.1.4, we therefore conclude that $F^*(X, E)$ is E -terminating. $\square$

3.2. Stratified termination using monomial orders and derivations

In practice, termination functions can often be constructed from familiar termination measures. We first explain how a global monomial order gives rise to a termination function. We then formulate a stratified version of the classical derivation criterion for termination.

3.2.1. Proposition. *Let $F^*(X, E)$ be a k -stratified 3-prepolygraph modulo. If δ is a termination function of $F^*(X, E)$, such that δ_1 takes values in a well-founded totally ordered set, then the relation $\leq_\delta$ defined by*

$$f \leq_\delta g \quad \text{if} \quad (\delta_k(f), \dots, \delta_1(f)) \leq_{[k, \dots, 1]} (\delta_k(g), \dots, \delta_1(g)),$$

for all $f, g \in X_2^\circ$, is an X -compatible monomial order. Conversely, for any X -compatible monomial order $\leq$ on X_2° , the identity map on X_2° defines a termination function of F^*X .

Proof. Let δ be a termination function of $F^*(X, E)$. Since for each $2 \leq i \leq k$ the relation $\leq_i$ is a well-founded total preorder and $\leq_1$ is a well-founded total order, the lexicographic order $\leq_{[k, \dots, 1]}$ is a well-founded total order following §2.3.3, and hence so is $\leq_\delta$. Moreover, the X -compatibility of δ implies that $\leq_\delta$ is an X -compatible monomial order.

Conversely, assume that $\leq$ is an X -compatible monomial order on X_2° . Then $\leq$ is a well-founded total order on X_2° , and by X -compatibility $\text{id}_{X_2^\circ}(t\alpha) < \text{id}_{X_2^\circ}(s\alpha)$ holds, for every $\alpha \in X_3$. Therefore, $\text{id}_{X_2^\circ}$ is a termination function. $\square$

3.2.2. Bounded below derivations. Let (X, E) be a 3-prepolygraph modulo and $(G, \leq)$ be an ordered abelian group. A 2-functor $Z : X_2^\circ \rightarrow \mathbf{Ord}$ is X_3E -monotone if, for every 1-cell $c \in X_1$, the set $Z(c)$ is non-empty and, for all $\alpha \in mX_3$ and $\beta \in E$, we have

$$Z(s\alpha) \geq Z(t\alpha) \quad \text{and} \quad Z(s\beta) = Z(t\beta).$$

A derivation $d : X_2^\circ \rightarrow O_{Z, G}$ is *bounded below*, if there exists an element $m \in G$ such that, for all $\alpha \in X_2$ and $y \in Z(s\alpha)$, we have $d(\alpha)(y) \geq m$.

3.2.3. Proposition. *Let (X, E) be a 3-prepolygraph modulo. Then any X_3E -monotone 2-functor $Z : X_2^\circ \rightarrow \mathbf{Ord}$ and bounded below derivation $d : X_2^\circ \rightarrow O_{Z, G}$ induces a weighted function $\delta^{Z, d}$ defined, for all $u, v \in X_1^*$, by*

$$\delta_{u, v}^{Z, d} : X_2^\circ(u, v) \rightarrow (\mathbf{Ord}(Z(u), G), \leq),$$

sending f to $d(f)$, where $\leq$ is the natural order on $\mathbf{Ord}(Z(u), G)$.

Proof. For every $f \in X_2^\circ(u, v)$, we have $d(f) \in O_Z(f) = \mathbf{Ord}(Z(u), G)$, that is $\delta_{u, v}^{Z, d}$ is well-defined. Assume that there exist two 2-cells f and g in X_2° such that $\delta^{Z, d}(f) < \delta^{Z, d}(g)$. For all compatible context $C = h \circ_1 w \square w' \circ_1 h'$, by the definition of derivations, we have

$$d(C[f]) = d(h) \circ_1 wf w' \circ_1 h' + h \circ_1 wd(f) w' \circ_1 h' + h \circ_1 wf w' \circ_1 d(h')$$

and similarly for $d(C[g])$. By the hypothesis that $d(f) < d(g)$, we have

$$h \circ_1 wd(f) w' \circ_1 h' < h \circ_1 wd(g) w' \circ_1 h'$$

Since the functor Z is X_3E -monotone, it is decreasing on generators and thus

$$(d(h) \circ_1 wf w' \circ_1 h') \leq (d(h) \circ_1 wg w' \circ_1 h') \quad \text{and} \quad (h \circ_1 wf w' \circ_1 d(h')) \leq (h \circ_1 wg w' \circ_1 d(h')).$$

Therefore, we deduce that $d(C[f]) < d(C[g])$, that is $\delta^{Z, d}(C[f]) < \delta^{Z, d}(C[g])$.

Since the derivation d is bounded below, there exists $m \in G$ such that, for every $\alpha \in X_2$ and every $y \in Z(s\alpha)$, one has $d(\alpha)(y) \geq m$. For every $f \in X_2^\circ$, write its unique $\circ_1$ -decomposition $f = C_1[\alpha_1] \circ_1 \dots \circ_1 C_n[\alpha_n]$, where $\alpha_i \in X_2$. Then we have

$$d(f)(a) = \left(C_1[d(\alpha_1)] \cdot \dots \cdot C_n[\alpha_n] + C_1[\alpha_1] \cdot \dots \cdot C_n[d(\alpha_n)] \right)(a) \geq nm,$$

for every $a \in Z(sf)$. If the order $\leq$ is not well-founded, then there exists an infinite sequence of parallel 2-cells $(f_k)_k$ with the same source s , which yields a strictly decreasing sequence of integers $(d(f_k)(a_k))_k$, where $a_k \in Z(s)$. This contradicts the hypothesis that the derivation d is bounded below. Hence, $\delta^{Z,d}$ is a weighted function. $\square$

3.2.4. Remark. Stratified termination functions generalize both monomial-order-based and derivation-based termination criteria for a k -stratified 3-prepolygraph $F^*(X, E)$. Indeed, by Proposition 3.2.1, every X -compatible monomial order on X_2° induces a termination function on $F^*(X, E)$. Similarly, by Proposition 3.2.3, every X_3 -monotone 2-functor equipped with a bounded below derivation induces such a termination function. In both cases, Theorem 3.1.5 implies that X is terminating, thereby recovering the monomial-order-based and derivation-based termination criteria.

Consequently, stratified termination functions extend these classical termination criteria by making it possible to establish termination stratum by stratum in situations where suitable global monomial orders or derivations are difficult to construct due to the large number of rewriting rules.

3.3. Examples: termination of a presentation of KLR algebras

In this subsection, we illustrate how termination of a 3-prepolygraph modulo can be established by means of a stratification of its set of rules. As an application, we consider the challenging presentation of KLR algebras [26, 34], whose rewriting system involves a large number of rules. Throughout this subsection, we set $\mathbf{k} = \mathbb{Z}$.

3.3.1. Polygraphic presentation of KLR algebras. Let Γ be a finite tree (a connected graph with no loops, nor multiple edges) with set of vertices denoted by I . Let $\cdot$ be a bilinear form on $\mathbb{Z}[I]$ defined by $i \cdot i = 2$ for any $i \in I$ and for $i \neq j$ in I

$$i \cdot j = \begin{cases} -1 & \text{if there is an edge between } i \text{ and } j \text{ in } \Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

Let (X, E) be the 3-prepolygraph modulo with a single 0-cell, whose 1-generators are the elements of I and whose 2-generators are

$$\begin{array}{c} \times \\ i \quad j \end{array} : i \circ_0 j \longrightarrow j \circ_0 i, \quad \text{and} \quad \begin{array}{c} \phi \\ i \end{array} : i \longrightarrow i,$$

for any $i, j \in I$. We set $E := \text{Int}(X_2)$, and the 3-generators in X_3 are for all $i, j, k \in I$,

$$\begin{aligned} \alpha_{i,j}^L : \begin{array}{c} \phi \\ i \quad j \end{array} &\longrightarrow \begin{array}{c} \times \\ i \quad j \end{array}, \quad \text{when } i \neq j, & \alpha_{i,j}^R : \begin{array}{c} \times \\ i \quad j \end{array} &\longrightarrow \begin{array}{c} \phi \\ i \quad j \end{array}, \quad \text{when } i \neq j, \\ \alpha_i^L : \begin{array}{c} \phi \\ i \quad i \end{array} &\longrightarrow \begin{array}{c} \times \\ i \quad i \end{array} + \begin{array}{c} | \quad | \\ i \quad i \end{array}, & \alpha_i^R : \begin{array}{c} \times \\ i \quad i \end{array} &\longrightarrow \begin{array}{c} \phi \\ i \quad i \end{array} - \begin{array}{c} | \quad | \\ i \quad i \end{array}, & \beta_i : \begin{array}{c} \times \\ i \quad i \end{array} &\longrightarrow 0, \\ \beta_{j,k} : \begin{array}{c} \times \\ j \quad k \end{array} &\longrightarrow \begin{array}{c} | \quad | \\ j \quad k \end{array}, \quad \text{when } j \cdot k = 0, & \beta_{i,j} : \begin{array}{c} \times \\ i \quad j \end{array} &\longrightarrow \begin{array}{c} \phi \\ i \quad j \end{array} + \begin{array}{c} | \quad | \\ i \quad j \end{array}, \quad \text{when } i \cdot j = -1, \end{aligned}$$

$$\gamma_{i,j,k} : \begin{array}{c} \text{Diagram 1} \\ i \quad j \quad k \end{array} \rightarrow \begin{array}{c} \text{Diagram 2} \\ i \quad j \quad k \end{array}, \text{ when } i \neq k \text{ or } i \cdot j \neq -1, \quad \gamma_{i,j,i} : \begin{array}{c} \text{Diagram 3} \\ i \quad j \quad i \end{array} \rightarrow \begin{array}{c} \text{Diagram 4} \\ i \quad j \quad i \end{array} + \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ j \end{array} \begin{array}{c} | \\ i \end{array}, \text{ when } i \cdot j = -1.$$

The 3-prepolygraph modulo (X, E) is presenting the KLR monoidal category, seen as a 2-category with only one 0-cell, associated with Γ , see [15, Thm.2.3.2]. This monoidal category contains the KLR algebras as hom-spaces.

3.3.2. Stratified termination of (X, E) . Thanks to the Lemma 3.1.4, we prove E -termination of the 3-prepolygraph modulo (X, E) by proving that its associated monomial 3-prepolygraph modulo is E -terminating. For each $i, j \in I$, we set

$$\begin{aligned} \alpha_i^{L1} : \begin{array}{c} \text{Diagram 5} \\ i \quad i \end{array} &\rightarrow \begin{array}{c} \text{Diagram 6} \\ i \quad i \end{array}, & \alpha_i^{L2} : \begin{array}{c} \text{Diagram 7} \\ i \quad i \end{array} &\rightarrow \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ i \end{array}, & \alpha_i^{R1} : \begin{array}{c} \text{Diagram 8} \\ i \quad i \end{array} &\rightarrow \begin{array}{c} \text{Diagram 9} \\ i \quad i \end{array}, & \alpha_i^{R2} : \begin{array}{c} \text{Diagram 10} \\ i \quad i \end{array} &\rightarrow \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ i \end{array}, \\ \beta_{i,j}^1 : \begin{array}{c} \text{Diagram 11} \\ i \quad j \end{array} &\rightarrow \begin{array}{c} \text{Diagram 12} \\ i \quad j \end{array}, & \beta_{i,j}^2 : \begin{array}{c} \text{Diagram 13} \\ i \quad j \end{array} &\rightarrow \begin{array}{c} | \\ i \end{array} \begin{array}{c} \text{Diagram 14} \\ j \end{array}, & \gamma_{i,j,i}^1 : \begin{array}{c} \text{Diagram 15} \\ i \quad j \quad i \end{array} &\rightarrow \begin{array}{c} \text{Diagram 16} \\ i \quad j \quad i \end{array}, & \gamma_{i,j,i}^2 : \begin{array}{c} \text{Diagram 17} \\ i \quad j \quad i \end{array} &\rightarrow \begin{array}{c} | \\ i \end{array} \begin{array}{c} | \\ j \end{array} \begin{array}{c} | \\ i \end{array}, \end{aligned}$$

and define a 4-stratification of mX by setting

$$\begin{aligned} F_s^4 mX &= \{\alpha_i^{L2}, \alpha_i^{R2}, \beta_i, \beta_{j,k}, \beta_{i,j}^1, \beta_{i,j}^2, \gamma_{i,j,i}^2\}, & F_s^3 mX &= \{\alpha_{i,j}^L, \alpha_i^{L1}\}, \\ F_s^2 mX &= \{\gamma_{i,j,k}, \gamma_{i,j,i}^1\}, & F_s^1 mX &= \{\alpha_{i,j}^R, \alpha_i^{R1}\}. \end{aligned}$$

This defines a 4-stratified 3-prepolygraph modulo $F^*(mX, E)$, on which we now define a termination function $\delta = (\delta_k)_{1 \leq k \leq 4}$.

We define $\delta_4(f)$ to be the size of f for every $f \in X_2^\circ$. Since every source in $F_s^4 mX$ has strictly greater size than its target, whereas for every rule in $F_s^3 mX$ the source and the target have the same size, the weighted function δ_4 is strictly $F_s^4 mX$ -compatible and $F_s^3 mX$ -compatible.

For every $k \leq 3$, following Proposition 3.2.3, we construct a weighted map δ_k induced by a $F^k mX$ -monotone 2-functor $Z_k : X_2^\circ \rightarrow \mathbf{Ord}$ and a bounded below derivation $d_k : X_2^\circ \rightarrow \mathcal{O}_{Z_k, \mathbb{Z}}$.

We first set $Z_k(i) = \mathbb{N}$ for every simple root $i \in I$, so that $Z_k(i \circ_0 j) = \mathbb{N} \times \mathbb{N}$ for all $i, j \in I$. We then define the Z_k and d_k by setting (omitting the labels on the strands).

i) We define the functor Z_3 and the derivation d_3 by setting the following values on X_2

$$Z_3(\text{Diagram 18})(n, m) = (m + 1, n), \quad Z_3(\text{Diagram 19})(n) = n, \quad d_3(\text{Diagram 18})(n, m) = m, \quad d_3(\text{Diagram 19})(n) = n.$$

for all $n, m \in \mathbb{N}$. Since $m, n \geq 0$, the derivation d_3 is bounded below. Moreover, the functor Z_3 is $F^3 mX$ -monotone, since the following relations hold:

$$Z_3(\text{Diagram 20})(n, m) = Z_3(\text{Diagram 21})(n, m) = (m + 1, n), \quad Z_3(\text{Diagram 22})(n, m) = Z_3(\text{Diagram 23})(n, m) = (m + 1, n),$$

$$Z_3(\text{Diagram 24})(n, m, l) = Z_3(\text{Diagram 25})(n, m, l) = (l + 2, m + 1, n).$$

We set δ_3 to be the weighted function δ^{Z_3, d_3} induced by Z_3 and d_3 . It is strictly $F_s^3 mX$ -compatible and $F_s^2 mX$ -compatible, since the following relations hold:

$$d_3(\text{Diagram 26})(n, m) = 2m + 1 > 2m = d_3(\text{Diagram 27})(n, m), \quad d_3(\text{Diagram 28})(n, m) = m + n = d_3(\text{Diagram 29})(n, m),$$

$$d_3(\text{Diagram 30})(n, m, l) = m + 2l + 1 = d_3(\text{Diagram 31})(n, m, l).$$

ii) We define the functor Z_2 and the derivation d_2 by setting the following values on X_2

$$Z_2(\bowtie)(n, m) = (m + 1, n), \quad Z_2(\oplus)(n) = n, \quad d_2(\bowtie)(n, m) = n, \quad d_2(\oplus)(n) = n,$$

for all $n, m \in \mathbb{N}$. Since $n \geq 0$, the derivation d_2 is bounded below. Moreover, the functor Z_2 is F^2mX -monotone, since the following relations hold:

$$Z_2(\text{diagram 1})(n, m, l) = Z_2(\text{diagram 2})(n, m, l) = (l+2, m+1, n), \quad Z_2(\text{diagram 3})(n, m) = Z_2(\text{diagram 4})(n, m) = (m+1, n),$$

We set δ_2 to be the weighted function δ^{Z_2, d_2} induced by Z_2 and d_2 . It is strictly $F_s^2 mX$ -compatible and $F_5^1 mX$ -compatible, since the following relations hold:

$$d_2(\text{diagram 1})(n, m, l) = 2n+m+1 > 2n+m = d_2(\text{diagram 2})(n, m, l), \quad d_2(\text{diagram 3})(n, m) = 2n = d_2(\text{diagram 4})(n, m).$$

iii) We define the functor Z_1 and the derivation d_1 by setting the following values on X_2

$$Z_1(\bowtie)(n, m) = (m, n + 1), \quad Z_1(\circlearrowleft)(n) = n, \quad d_1(\bowtie)(n, m) = n, \quad d_1(\circlearrowleft)(n) = n,$$

for all $n, m \in \mathbb{N}$. Since $n \geq 0$, the derivation d_1 is bounded below. Moreover, the functor Z_1 is F^1mX -monotone, since the following relation hold:

$$Z_1(\text{diagram 1})(n, m) = Z_1(\text{diagram 2})(n, m) = (m, n + 1).$$

We set δ_1 to be the weighted function δ_{Z_1, d_1} induced by Z_1 and d_1 . It is strictly $F_s^1 mX$ -compatible, since the following relation hold:

$$d_1(\text{diagram 1})(n, m) = 2n + 1 > 2n = d_1(\text{diagram 2})(n, m).$$

Note that each δ_i is F^0mX -identity, since

$$d_i \left(\begin{array}{c} \dots \\ \dots \\ \varphi \\ \dots \end{array} \middle| u \middle| \begin{array}{c} \psi \\ \dots \\ \dots \end{array} \right) = d_i \left(\begin{array}{c} \dots \\ \dots \\ \varphi \\ \dots \end{array} \middle| u \middle| \begin{array}{c} \dots \\ \dots \\ \psi \\ \dots \end{array} \right)$$

for every $1 \leq i \leq 4$, all 1-cell $u \in X_1^*$, and 2-generators $\varphi, \psi \in X_2$. Therefore, δ is a termination function of the stratified 3-prepolygraph modulo $F^*(mX, E)$. By Theorem 3.1.5, the monomial 3-prepolygraph modulo (mX, E) is E -terminating. It then follows from Lemma 3.1.4 that the original 3-prepolygraph modulo (X, E) is also E -terminating.

3.3.3. Remarks. This example illustrates how stratified termination simplifies the construction of termination arguments by allowing monomial orders and derivations to be adapted to subsets of rewriting rules grouped into strata according to their termination behaviour. Rather than defining a single global measure on the whole rewriting system, one may choose, for each stratum, the termination method that is best suited to the corresponding family of rules. In particular, monomial orders and derivations can be combined within the same stratified proof, with different strata controlled by different types of termination measures. In this sense, the argument given here extends the derivation-based approach used for KLR algebras in [15] and provides a flexible method for treating rewriting systems with large and structurally heterogeneous sets of rules.

We have illustrated the stratification method for proving termination in the case of a one-object presentation of the KLR 2-category. In future work, it would be natural to extend this approach to the Kac–Moody 2-category $\mathcal{KLR}$, defined in [26, 34], which is its multi-object counterpart. The 3-cells of this presentation encode structural relations, including isotopy and bubble relations, whose termination raises difficulties inherent in the 2-dimensional structure. The main challenge will therefore be to construct a stratification adapted to the category as a whole.

4. MODULAR CONFLUENCE AND HOM-BASES FROM STRATIFIED NORMALIZATION

This section develops a modular approach to confluence for stratified rewriting systems and applies it to the construction of hom-bases. We first introduce confluence modularity and derive a general hom-basis theorem for terminating stratified presentations satisfying this property. We then construct a stratified normalization procedure that enforces modularity by normalizing each new stratum with respect to the preceding ones, thereby absorbing mixed rewriting interactions into the normalized rules. We prove that the resulting presentation is terminating, presents the same linear category, and satisfies confluence modularity. In this sense, stratified normalization refines classical completion procedures [27, 30] by organizing completion along the successive strata of the presentation. The hom-basis theorem then applies directly to the normalized presentation. We conclude the section by illustrating the method on the associative and coassociative category and on the Hecke category.

4.1. Modular confluence and stratified normalization

4.1.1. Modularity of confluence. Let F^*X be a k -stratified 3-prepolygraph. Confluence of the individual strata $F_s^i X$ does not imply confluence of their union, since rewriting steps belonging to different strata may interact and produce additional mixed branchings. We say that F^*X satisfies the *confluence modularity property* if, for every $1 \leq j \leq k$, the confluence of each stratum $F_s^i X$, for $1 \leq i \leq j$, implies the confluence of the extension $F^j X$.

This property separates two aspects of a confluence proof. On the one hand, confluence may be checked independently within each stratum; on the other hand, the compatibility between different strata is encoded globally by the modularity property. As a consequence, for a terminating stratified presentation satisfying confluence modularity, the confluence problem for the whole rewriting system reduces to the confluence analysis of its individual strata. By Theorem 2.2.6, we have thus the following hom-bases theorem from modular confluence.

4.1.2. Theorem. *Let C be a linear monoidal category presented by a left-monomial terminating stratified 3-prepolygraph F^*X , satisfying the confluence modularity property, and whose each stratum $F_s^i X$ is confluent. Then the set $\text{Nf}_m(X)$ forms a hom-basis of C .*

We now introduce a normalization procedure on stratified prepolygraphs in order to reach this modularity property holds without changing the presented monoidal category. Indeed, by normalizing each new stratum with respect to the preceding ones, we mix interactions between successive strata are incorporated into the normalized rewriting rules.

4.1.3. Stratified normalization. Let F^*X be a k -stratified 3-prepolygraph, and $\leq$ be a monomial order on X_2° . A $\leq$ -normalization of F^*X is a k -stratified 3-prepolygraph, denoted by $F^*\widehat{X}$, whose extensions $F_s^i \widehat{X}$ in $\mathcal{E}(X_2^{\circ\ell})$ are defined inductively.

We set $F^1 \widehat{X} := F^1 X$. Assume that $F^j \widehat{X}$ has been defined for all $1 \leq j \leq i$, and fix an $F^i \widehat{X}$ -normalization strategy $(-)\downarrow_i$. We then construct $F^{i+1} \widehat{X}$ as follows.

For every $\alpha \in F_s^{i+1} X$ and context C of X_2° , there is an induced 3-cell $C[\alpha]$ in $X_3^{\circ\ell}$. When $C[\partial\alpha]\downarrow_i \neq 0$ this yields a 3-cell

$$C[\alpha]\downarrow_i : C[s\alpha]\downarrow_i \rightarrow C[t\alpha]\downarrow_i$$

and, as defined in §2.3.1, a $\leq$ -oriented left-term associated 3-generator $C[\alpha]\downarrow_i \lessdot$ defined on $F^i \widehat{X}$ -normal forms. We denote by $\llbracket \alpha \rrbracket \downarrow_i$ the extension of $X_2^{\circ\ell}$ consisting of such 3-generators deduced from an $F^i \widehat{X}$ -active context on α , that is,

$$\llbracket \alpha \rrbracket \downarrow_i := \{C[\alpha]\downarrow_i \lessdot \mid C \in \text{Act}(\alpha, F^i \widehat{X})\}.$$

The 3-generators in $\llbracket \alpha \rrbracket \downarrow_i$ are called the $F^i \widehat{X}$ -normal reductions of α . Finally, we set

$$F_s^{i+1} \widehat{X} := \bigsqcup_{\alpha \in F_s^{i+1} X} \min_{\sqsubseteq} \llbracket \alpha \rrbracket \downarrow_i,$$

where $\min_{\sqsubseteq}$ is context relation, so that $F^{i+1} \widehat{X} = F^i \widehat{X} \sqcup F_s^{i+1} \widehat{X}$.

4.1.4. Remarks. A $\leq$ -normalization of a stratified prepolygraph F^*X need not be unique, since it depends on the normalization strategy $(-)\downarrow_i$ chosen at each stratum $1 \leq i \leq k$. If $F^i \widehat{X}$ is confluent, however, the strategy $(-)\downarrow_i$ yields the unique $F^i \widehat{X}$ -normal form of each 2-cell. In this case, the extension $F^{i+1} \widehat{X}$ obtained from the above construction is uniquely determined.

In addition, the construction of the extension $F^{i+1} \widehat{X}$ works as a completion of the critical branchings in $\text{CB}(F^i \widehat{X}, F_s^{i+1} X)$. Indeed, for each non-confluent critical branching $(\sigma, \tau) \in \text{CB}(F^i \widehat{X}, F_s^{i+1} X)$, the construction yields a rule making this branching confluent. More precisely, the $F_s^{i+1} X$ -rewriting step τ can be written as $\tau = C[\alpha]$ for some $\alpha \in F_s^{i+1} X$ yielding the following diagram:

$$\begin{array}{ccc} & \xrightarrow[\textcolor{red}{F^i \widehat{X}}]{\textcolor{red}{\sigma}} f & \xrightarrow[\textcolor{blue}{F^i \widehat{X}}]{\textcolor{blue}{C[s\alpha] \downarrow_i}} \\ \textcolor{red}{C[s\alpha]} & \xrightarrow[\textcolor{red}{F_s^{i+1} X}]{\textcolor{red}{\tau}} \textcolor{red}{C[t\alpha]} & \xrightarrow[\textcolor{blue}{F^i \widehat{X}}]{\textcolor{blue}{C[t\alpha] \downarrow_i}} \\ & & \textcolor{blue}{C[\alpha] \downarrow_i} \end{array}$$

Note that $f \downarrow_i = C[s\alpha] \downarrow_i$ since $F^i \widehat{X}$ is confluent. In this case, the normalization procedure adds a $F^i \widehat{X}$ -normal reduction $C[\alpha] \downarrow_i \leq$ to $F_s^{i+1} \widehat{X}$, connecting $C[s\alpha] \downarrow_i$ and $C[t\alpha] \downarrow_i$, thereby making this critical branching confluent. In particular, all critical branching in $\text{CB}(F^i \widehat{X}, F_s^{i+1} X)$ is confluent if and only if $F_s^{i+1} \widehat{X} = F_s^{i+1} X$.

4.1.5. Lemma. For every $F_s^{i+1} X$ -rewriting step $\sigma = bC[\alpha] + 1_f$, with $i \geq 0$, if $C[s\alpha]$ is $F^i \widehat{X}$ -reducible and $F_s^{i+1} \widehat{X}$ is confluent, there exist two $F_s^{i+1} \widehat{X}$ -rewriting paths τ and ρ (possibly identities) as in the following diagram

$$\begin{array}{ccc} s\sigma & \xrightarrow[\textcolor{red}{F^i \widehat{X}}]{\textcolor{red}{(s\sigma) \downarrow_i}} & \textcolor{blue}{(s\sigma) \downarrow_i} \\ \textcolor{red}{F_s^{i+1} X} \downarrow \sigma & & \textcolor{blue}{F_s^{i+1} \widehat{X}} \searrow \tau \\ t\sigma & \xrightarrow[\textcolor{blue}{F^i \widehat{X}}]{\textcolor{blue}{(t\sigma) \downarrow_i}} & \textcolor{blue}{(t\sigma) \downarrow_i} \\ & & \textcolor{blue}{F_s^{i+1} \widehat{X}} \nearrow \rho \end{array} \quad (4.1.6)$$

Moreover, the same statement holds for every $F_s^{i+1} \widehat{X}$ -rewriting step σ .

Proof. The proof relies on the $F^i \widehat{X}$ -normal reduction defined in §4.1.3 to connect $(s\sigma) \downarrow_i$ and $(t\sigma) \downarrow_i$.

Let $\sigma = bC[\alpha] + 1_f$ be a $F_s^{i+1} X$ -rewriting step, where $\alpha \in F_s^{i+1} X$ and $f \in X_2^{\circ \ell}$. We set the context $C := \lambda(g \circ_1 u \square v \circ_1 h)$. If $|C| = 0$, then C is $F^i \widehat{X}$ -active on α , since $s\alpha$ is reducible by hypothesis. Otherwise, if $\lambda g \notin \text{Nf}(F^i \widehat{X})$ or $\lambda h \notin \text{Nf}(F^i \widehat{X})$, we have the $F^i \widehat{X}$ -normal reduction $C[\alpha] \downarrow_i \leq C'[\alpha] \downarrow_i$, where $C' = \lambda g \downarrow_i \circ_1 u \square v \circ_1 h \downarrow_i$ or $C' = g \downarrow_i \circ_1 u \square v \circ_1 \lambda h \downarrow_i$. Hence, we only consider the case of $F^i \widehat{X}$ -active contexts C on α . Let $h = \text{lm}_{\leq}(C[\partial\alpha] \downarrow_i)$ and $a = \text{lc}_{\leq}((C[\partial\alpha] \downarrow_i))$. We assume that a is either 1 or noninvertible; the case where a is invertible is treated similarly. We distinguish three cases.

Case 1. If $h \in \text{MSupp}(C[s\alpha] \downarrow_i) \setminus \text{MSupp}(C[t\alpha] \downarrow_i)$, then there exists a $F^i \widehat{X}$ -normal reduction

$$C[\alpha] \downarrow_i \leq : ah \rightarrow C[t\alpha] \downarrow_i - C[s\alpha] \downarrow_i + ah$$

in the extension $\llbracket \alpha \rrbracket \downarrow_i$. If $C[\alpha] \downarrow_i \leq \in \min_{\sqsubseteq} \llbracket \alpha \rrbracket \downarrow_i$, we have $C[\alpha] \downarrow_i \leq \in F_s^{i+1} \widehat{X}$. Then there exists a 3-cell σ' of size 1 in $(F_s^{i+1} \widehat{X})^{\circ \ell}$

$$\sigma' := bC[\alpha] \downarrow_i \leq + bC[s\alpha] \downarrow_i - bah + f \downarrow_i$$

with source $(s\sigma)\downarrow_i$ and target $(t\sigma)\downarrow_i$. By Lemma 2.1.12, there exist identities or $F_s^{i+1}\widehat{X}$ -rewriting paths τ and ρ as in (4.1.6). If $C[\alpha]\downarrow_i \not\in \min_{\sqsubseteq} \llbracket \alpha \rrbracket\downarrow_i$, then by definition of the $\sqsubseteq$ -minimal set $\min_{\sqsubseteq} \llbracket \alpha \rrbracket\downarrow_i$, there exist a context D and a rule $\beta \in \min_{\sqsubseteq} \llbracket \alpha \rrbracket\downarrow_i$ such that $C[\alpha]\downarrow_i \preceq D[\beta]$. It follows that σ' can be written as

$$\sigma' = bD[\beta] + bC[s\alpha]\downarrow_i - bah + f\downarrow_i,$$

where $\beta \in F_s^{i+1}\widehat{X}$, so that σ' is again a 3-cell in $(F_s^{i+1}\widehat{X})^{\text{of}}$. Therefore, in following two cases, we may assume without loss of generality that $C[\alpha]\downarrow_i \in \min_{\sqsubseteq} \llbracket \alpha \rrbracket\downarrow_i$.

Case 2. If $h \in \text{MSupp}(C[t\alpha]\downarrow_i) \setminus \text{MSupp}(C[s\alpha]\downarrow_i)$, then there exists a $F^i\widehat{X}$ -normal reduction

$$C[\alpha]\downarrow_i \preceq : ah \rightarrow C[s\alpha]\downarrow_i - C[t\alpha]\downarrow_i + ah$$

in $\min_{\sqsubseteq} \llbracket \alpha \rrbracket\downarrow_i$. Then there exists a 3-cell σ' of size 1 in $(F_s^{i+1}\widehat{X})^{\text{of}}$

$$\sigma' := bC[\alpha]\downarrow_i \preceq + bC[t\alpha]\downarrow_i - bah + f\downarrow_i$$

with source $(t\sigma)\downarrow_i$ and target $(s\sigma)\downarrow_i$. By Lemma 2.1.12, there exist identities or $F_s^{i+1}\widehat{X}$ -rewriting paths τ and ρ as in (4.1.6).

Case 3. If h occurs in both $\text{MSupp}(C[s\alpha]\downarrow_i)$ and $\text{MSupp}(C[t\alpha]\downarrow_i)$, we write

$$C[s\alpha]\downarrow_i = a_1h + s_0, \quad \text{and} \quad C[t\alpha]\downarrow_i = a_2h + t_0,$$

where $h \notin \text{MSupp}(s_0) \sqcup \text{MSupp}(t_0)$. Note that $a = a_1 - a_2$. There exists an $F^i\widehat{X}$ -normal reduction

$$C[\alpha]\downarrow_i \preceq : ah \rightarrow t_0 - s_0$$

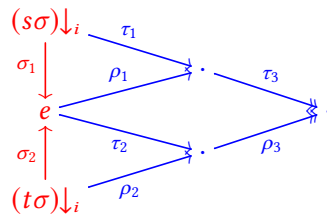
in $\min_{\sqsubseteq} \llbracket \alpha \rrbracket\downarrow_i$. For $j = 1, 2$, set $q_j := \text{quo}_a(a_j)$, and $r_j := \text{rem}_a(a_j)$. Since $a_1 - a_2 = a$, we have $r_1, r_2 \in \text{Rem}(a)$, that is $r_1 = r_2$. Following $a_1 = q_1a + r_1$ and $a_2 = q_2a + r_2$, we have

$$a = (q_1 - q_2)a + (r_1 - r_2) = (q_1 - q_2)a.$$

Since $\mathbf{k}$ is an integral domain and $a \neq 0$, we obtain $q_1 - q_2 = 1$. We define two 3-cells of size 1 in $(F_s^{i+1}\widehat{X})^{\text{of}}$ by

$$\sigma_1 := bq_1\beta + br_1h + bs_0 + f\downarrow_i : (s\sigma)\downarrow_i \rightarrow e_1 \quad \text{and} \quad \sigma_2 := bq_2\beta + br_2h + bt_0 + f\downarrow_i : (t\sigma)\downarrow_i \rightarrow e_2,$$

where $e_1 = bq_1(t_0 - s_0) + br_1h + bs_0 + f\downarrow_i$, and $e_2 = bq_2(t_0 - s_0) + br_2h + bt_0 + f\downarrow_i$. Using $r_1 = r_2$ and $q_1 - q_2 = 1$, we have $e_1 - e_2 = b(q_1 - q_2)(t_0 - s_0) + b(s_0 - t_0) = 0$. Thus, we obtain the diagram



where $e := e_1 = e_2$ and the identities or $F_s^{i+1}\widehat{X}$ -rewriting paths τ_1, ρ_1, τ_2 and ρ_2 are obtained from Lemma 2.1.12. Moreover, the paths τ_3 and ρ_3 exist since the branching (τ_2, ρ_1) is confluent by hypothesis. Hence, this yields the identities or $F_s^{i+1}\widehat{X}$ -rewriting paths $\tau = \tau_1 \circ_1 \tau_3$ and $\rho = \rho_1 \circ_1 \rho_3$, as required in (4.1.6).

Finally, if σ be a $F_s^{i+1}\widehat{X}$ -rewriting step, we write $\sigma = bD[C[\alpha]\downarrow_i \preceq] + f$, where $C[\alpha]\downarrow_i \preceq \in F_s^{i+1}\widehat{X}$ and $\alpha \in F_s^{i+1}X$. We have $(s\sigma)\downarrow_i = b(DC[s\alpha])\downarrow_i + f\downarrow_i$ and $(t\sigma)\downarrow_i = b(DC[t\alpha])\downarrow_i + f\downarrow_i$. The conclusion then follows exactly as in the case of an $F_s^{i+1}X$ -rewriting step $\sigma = b(DC[\alpha]) + f$. $\square$

4.1.7. Lemma. *If the monomial order $\leq$ is $F_s^1 X$ -compatible, then any $\leq$ -normalization $F^* \widehat{X}$ is terminating and Tietze equivalent to $F^* X$.*

Proof. The stratified normalization procedure to define normal reductions in §4.1.3 ensures that the monomial order $\leq$ is $F_s^{i+1} \widehat{X}$ -compatible for every $1 \leq i \leq k-1$. Together with the assumption that $\leq$ is $F_s^1 X$ -compatible, it follows that $\leq$ is $F^k \widehat{X}$ -compatible. Hence, the extension $F^* \widehat{X}$ is terminating. It remains to prove that $\widehat{X}_2^{\text{of}} / \langle \widehat{X}_3 \rangle = X_2^{\text{of}} / \langle X_3 \rangle$.

For each $1 \leq i \leq k-1$ and every rule $f \rightarrow h$ in $F_s^{i+1} X$, by Lemma 4.1.5 there exists two $F_s^{i+1} \widehat{X}$ -rewriting paths τ and ρ such that

$$f \xrightarrow{F^i \widehat{X}} f \downarrow_i \xrightarrow{F_s^{i+1} \widehat{X}} e \xleftarrow{F_s^{i+1} \widehat{X}} h \downarrow_i \xleftarrow{F^i \widehat{X}} h.$$

Hence we obtain $\langle F_s^{i+1} X \rangle \subseteq \langle F^{i+1} \widehat{X} \rangle$, that is, $X_2^{\text{of}} / \langle F^{i+1} \widehat{X} \rangle \subseteq X_2^{\text{of}} / \langle F_s^{i+1} X \rangle$. When $i = 0$, we have $F^1 \widehat{X} = F_s^1 X$ by the definition in §4.1.3. By induction on i , we have $\widehat{X}_2^{\text{of}} / \langle \widehat{X}_3 \rangle \subseteq X_2^{\text{of}} / \langle X_3 \rangle$.

For each $1 \leq i \leq k-1$ and every rule $f \rightarrow h$ in $F_s^{i+1} \widehat{X}$, by the construction of normalization procedure there exist $\alpha \in F_s^{i+1} X$, a 3-cell g in $(F^i \widehat{X})^{\text{of}}$, and a scalar μ such that

$$\mu f + g = C[s\alpha] \downarrow_i \quad \text{and} \quad \mu h + g = C[t\alpha] \downarrow_i,$$

that is

$$\mu f + g = C[s\alpha] \downarrow_i \xleftarrow{F^i \widehat{X}} C[s\alpha] \xrightarrow{F_s^{i+1} X} C[t\alpha] \xrightarrow{F^i \widehat{X}} C[t\alpha] \downarrow_i = \mu h + g.$$

By induction, we have

$$\langle F_s^{i+1} \widehat{X} \rangle \subseteq \langle F_s^{i+1} X \sqcup F^i \widehat{X} \rangle = \langle F_s^{i+1} X \sqcup F_s^i \widehat{X} \sqcup F^{i-1} \widehat{X} \rangle \subseteq \langle F_s^{i+1} X \sqcup F_s^i X \sqcup F^{i-1} \widehat{X} \rangle \subseteq \dots$$

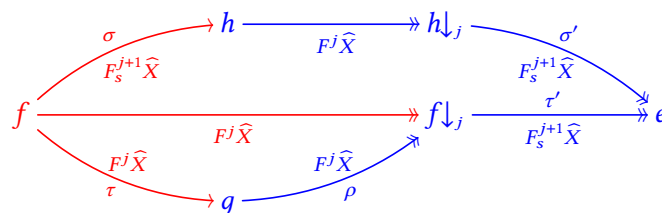
Since $F^1 \widehat{X} = F_s^1 X$, it follows that $\langle F_s^{i+1} \widehat{X} \rangle \subseteq \langle F^{i+1} X \rangle$, that is, $X_2^{\text{of}} / \langle F^{i+1} X \rangle \subseteq X_2^{\text{of}} / \langle F_s^{i+1} \widehat{X} \rangle$. By induction on i , we have $X_2^{\text{of}} / \langle X_3 \rangle \subseteq \widehat{X}_2^{\text{of}} / \langle \widehat{X}_3 \rangle$. $\square$

The main structural property of stratified normalization is that it makes confluence modular with respect to the stratification as stated by the following lemma.

4.1.8. Lemma. *Any $\leq$ -normalization $F^* \widehat{X}$ satisfies the confluence modularity property.*

Proof. For $j = 1$, the result follows from the hypothesis, since $F^1 \widehat{X} = F_s^1 \widehat{X}$. Assume now that $1 \leq j < k-1$ and that $F^j \widehat{X}$ is confluent. We prove that $F^{j+1} \widehat{X}$ is confluent. For every critical branching $(\sigma, \tau) \in \text{CB}(F^{j+1} \widehat{X})$, we distinguish two cases:

- i) If $(\sigma, \tau) \in \text{CB}(F_s^{j+1} \widehat{X})$, or $(\sigma, \tau) \in \text{CB}(F^j \widehat{X})$, then it is confluent by hypothesis.
- ii) If $(\sigma, \tau) \in \text{CB}(F_s^{j+1} \widehat{X}, F^j \widehat{X})$ with common source f , we have the following diagram:



where σ' and τ' are obtained from Lemma 4.1.5, since σ is an $F_s^{j+1} \widehat{X}$ -rewriting step, and ρ is obtained from the equality $f \downarrow_j = g \downarrow_j$, since $F^j \widehat{X}$ is confluent.

By Lemma 4.1.7, the extension $\widehat{X}$ is terminating, and so is $F^{j+1}\widehat{X}$. Hence $F^{j+1}\widehat{X}$ is locally confluent by Lemma 2.2.3. $\square$

4.1.9. Theorem. *Let C be a linear monoidal category presented by a stratified 3-prepolygraph F^*X . If there exists an F_s^1X -compatible monomial order $\leq$ such that every stratum $F_s^i\widehat{X}$ of the associated $\leq$ -normalization $F^*\widehat{X}$ is left-monomial and confluent, then $\text{Nf}_m(\widehat{X})$ forms a hom-basis of C .*

Proof. Let F^*X be a left-monomial k -stratified 3-prepolygraph for $k \geq 1$ satisfying conditions of the statement. By Lemma 4.1.7, the 3-prepolygraph $F^*\widehat{X}$ is terminating and presents the category C . By Lemma 4.1.8, it satisfies the confluence modularity property. Since each stratum $F_s^i\widehat{X}$ is confluent and left-monomial, Theorem 4.1.2 applies and yields that $\text{Nf}_m(\widehat{X})$ forms a hom-basis of C . $\square$

4.1.10. Generic branchings. Let F^*X be a stratified 3-prepolygraph. For $2 \leq i \leq k$, let $u, v \in X_1^\circ$ and C be a context. For every $1 \leq j < i$, the context C is called $F^j\widehat{X}$ -stable with (u, v) if, for every $F_s^i\widehat{X}$ -rewriting step ρ whose source and target belong to $X_2^\circ(u, v)$, there exists an $F_s^i\widehat{X}$ -rewriting step

$$\rho^{C,j} : C[s\rho]\downarrow_j \rightarrow C[t\rho]\downarrow_j.$$

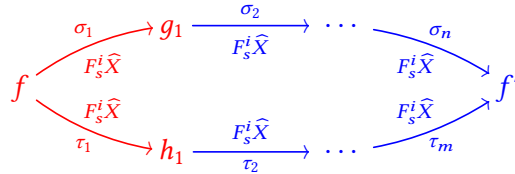
If moreover $\rho \not\sqsubseteq \rho^{C,j}$, with respect to the context relation $\sqsubseteq$, then C is called *strictly* $F^j\widehat{X}$ -stable with ρ .

Let (σ, τ) be an $F_s^i\widehat{X}$ -branching with source f . Any $F^j\widehat{X}$ -stable context C with f^∂ induces a branching $(\sigma^{C,j}, \tau^{C,j})$ with source $C[f]\downarrow_j$. This yields a well-founded relation $\vdash$ on the set of $F_s^i\widehat{X}$ -branchings, defined by

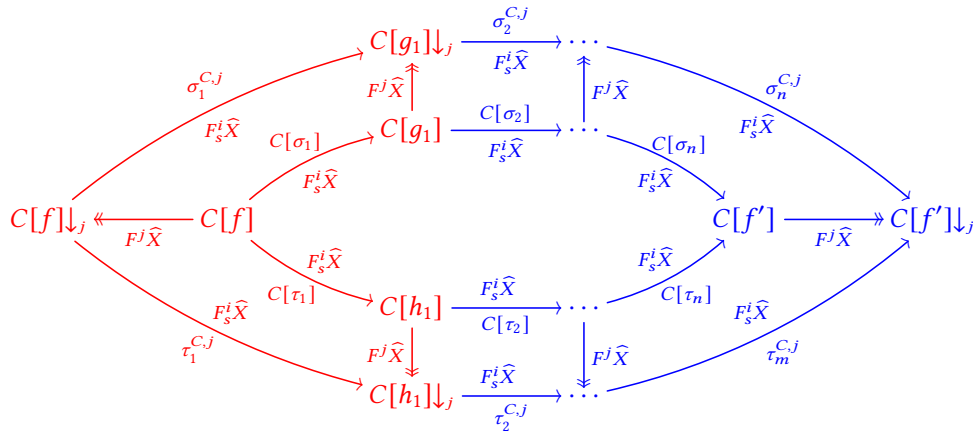
$$(\sigma, \tau) \vdash (\sigma^{C,j}, \tau^{C,j}).$$

A branching that is minimal with respect to $\vdash$ is called a *generic* $F_s^i\widehat{X}$ -branching.

Suppose that the branching (σ_1, τ_1) is confluent and admits a confluence diagram of the form



Then the induced branching $(\sigma_1^{C,j}, \tau_1^{C,j})$ is also confluent, as witnessed by the following diagram:



We therefore obtain the following criterion.

4.1.11. Lemma. *All $F_s^i\widehat{X}$ -critical branchings are confluent if and only if all generic $F_s^i\widehat{X}$ -critical branchings are confluent.*

4.2. Hom-basis computation procedure

This subsection presents a procedure for computing a hom-basis of a linear monoidal category from a stratified presentation and the normalization-completion procedure. We also introduce a stratified method for active contexts ensuring the termination of these procedures.

4.2.1. Normalization-completion procedure. We summarize the constructions of the normalization of stratified prepolygraphs in §4.1.3 into the following *stratified normalization procedure* $\text{SN}(F^*X, \leq)$.

Input: A k -stratified 3-prepolygraph F^*X , an F_s^1X -compatible monomial order $\leq$.

Output: The normalization $F^*\widehat{X}$.

$F^1\widehat{X} \leftarrow F^1X, i \leftarrow 1$

while $i + 1 \leq k$ **do**

$T \leftarrow \emptyset$

for each $\alpha \in F_s^{i+1}X$ **do**

for each $C \in \text{Act}(\alpha, F^iX)$ **do**

$d \leftarrow \text{Nf}_{F^i\widehat{X}}(C[s\alpha] - C[t\alpha])$

$w \leftarrow \text{lm}_{\leq}(d), \quad \lambda \leftarrow \text{lc}_{\leq}(d)$

if λ is invertible **then** $\mu \leftarrow \lambda^{-1}$

else $\mu \leftarrow 1$

if $w \in \text{MSupp}(\text{Nf}_{F^i\widehat{X}}(C[t\alpha])) \setminus \text{MSupp}(\text{Nf}_{F^i\widehat{X}}(C[s\alpha]))$ **then**

 Add the rule $\mu\lambda w \rightarrow \mu(d + \lambda w)$ to T

else

 Add the rule $\mu\lambda w \rightarrow \mu(-d + \lambda w)$ to T

$F_s^{i+1}\widehat{X} \leftarrow T \setminus \{ \beta \in T \mid \exists \gamma \in T \text{ and a context } C \text{ such that } |C| > 0 \text{ and } C[\gamma] = \beta \}$

$i \leftarrow i + 1$

$F_s^*\widehat{X} \leftarrow F_s^1\widehat{X} \dots F_s^k\widehat{X}$

return $F_s^*\widehat{X}$

If the set of active contexts is finite, the procedure terminates and yields the normalization. In general, however, this set is infinite. We restrict the set of normal reductions by filtering active contexts according to their size, as explained in 4.2.2.

4.2.2. Filtrations of normal reductions. Let F^*X be a k -stratified 3-prepolygraph and $\leq$ an F_s^1X -compatible monomial order. For $i \leq k$ and $\alpha \in F_s^{i+1}X$, we define a filtration of the extension $\llbracket \alpha \rrbracket_{\downarrow i}$ by the size of active contexts:

$$L^*\llbracket \alpha \rrbracket_{\downarrow i} : L^0\llbracket \alpha \rrbracket_{\downarrow i} \subseteq L^1\llbracket \alpha \rrbracket_{\downarrow i} \subseteq \dots \subseteq L^j\llbracket \alpha \rrbracket_{\downarrow i} \subseteq \dots,$$

where, for every $j \geq 0$,

$$L^j\llbracket \alpha \rrbracket_{\downarrow i} := \{ C[\alpha]_{\downarrow i} \mid C \in \text{Act}(\alpha, L^i\widehat{X}) \text{ and } |C| \leq j \}.$$

We define $L^j \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i}$ as the set of minimal elements of $L^j\llbracket \alpha \rrbracket_{\downarrow i}$ with respect to the context relation $\sqsubseteq$. This defines a filtration $L^* \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i}$ of the extension $\min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i}$. We say that $\min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i}$ is *bounded* if there exists $n \geq 0$ such that

$$\min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i} = L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i}.$$

4.2.3. Proposition. Let $\alpha \in F_s^{i+1}X$, and assume that $F^i\widehat{X}$ is left-monomial and that $|s\beta| = |t\beta|$ for every $\beta \in F^i\widehat{X}$. If there exists a natural number n such that

$$n \geq \max_{\beta \in F^i\widehat{X}} \{ |s\beta| - |\alpha| \} \quad \text{and} \quad L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i} = L^{n+1} \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow i},$$

then the extension $\min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i}$ is bounded.

Proof. For every context $C \in \text{Act}(\alpha, F^i \widehat{X})$ of size $(n+2)$, $C[\alpha]$ falls two cases:

$$f \circ_1 C'[\alpha] \quad \text{and} \quad C'[\alpha] \circ_1 g,$$

where $|f| = |g| = 1$ and $|C'| = n+1$. We consider the first case. The other case is treated similarly.

Since $|s\beta| = |t\beta|$ for every $\beta \in F^i \widehat{X}$, the source and target of $C'[\alpha]_{\downarrow_i}^{\leq}$ have the same size as those of $C'[\alpha]$. By the assumption $L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i} = L^{n+1} \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i}$, it follows that $f \circ_1 C'[\alpha]_{\downarrow_i}^{\leq}$ is of the following two forms:

$$f \circ_1 h_1 \circ_1 D[\alpha]_{\downarrow_i}^{\leq}, \quad \text{and} \quad f \circ_1 D[\alpha]_{\downarrow_i}^{\leq} \circ_1 h_2,$$

where $|h_1| = |h_2| = 1$ and $|D| = n$. Note that the 2-cells $f \circ_1 h_1$ and $f \circ_1 sD[\alpha]_{\downarrow_i}^{\leq}$ may be reducible. To compute the normal reduction $C[\alpha]_{\downarrow_i}$, one needs to further reduce their sources and targets to normal forms. Since the monomial order $\leq$ is X -compatible, this process will terminate. Consequently, $C[\alpha]_{\downarrow_i}^{\leq}$ is of one of the following three forms:

$$f' \circ_1 g' \circ_1 D'[\alpha]_{\downarrow_i}^{\leq}, \quad D'[\alpha]_{\downarrow_i}^{\leq} \circ_1 f' \circ_1 g', \quad \text{and} \quad f' \circ_1 D'[\alpha]_{\downarrow_i}^{\leq} \circ_1 g',$$

where $|f'| = |g'| = 1$, $|D'| = n$, and $D'[\alpha]_{\downarrow_i}^{\leq} \in L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i}$. Hence, by taking minimal rules with respect to $\sqsubseteq$, we obtain $L^{n+2} \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i} = L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i}$. By induction on the size of active contexts, one obtains

$$L^j \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i} = L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i}, \quad \text{for all } j \geq n.$$

Therefore, we have $\min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i} = L^n \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow_i}$. □

4.2.4. Hom-basis procedure. We denote by $\text{KB}(Y, \leq)$ the 3-prepolygraph obtained by applying the Knuth-Bendix completion procedure [27] to a 3-prepolygraph Y with respect to $\leq$. For every equational presentation $\mathcal{E}$ of a linear monoidal category C , we compute its hom-basis using the following *hom-basis procedure* $\text{HB}(\mathcal{E}, \leq)$:

Input: A presentation $\mathcal{E}$ of a linear monoidal category C , a monomial order $\leq$ on C

Output: A hom-basis of C .

$X_{\leq 2} \leftarrow \mathcal{E}_{\leq 2}; \quad X_3 \leftarrow \emptyset;$

for each (u, v) **in** $\mathcal{E}_3$ **do**

$\beta \leftarrow \{u \rightarrow v\};$
 $X_3 \leftarrow X_3 \sqcup \{\beta^{\leq}\};$

Fix a stratification $F_s^* X$ of length k of X

$F_s^* \widehat{X} \leftarrow \text{SN}(F_s^* X, \leq)$

while $F_s^* \widehat{X}$ **is not confluent** **do**

$I \leftarrow \{j \in \{1, \dots, k\} \mid F_s^j \widehat{X} \text{ is not confluent}\}$
for each $j \in I$ **do**
 $F_s^j \widehat{X} \leftarrow \text{KB}(F_s^j \widehat{X}, \leq)$
 $F_s^* \widehat{X} \leftarrow \text{SN}(F_s^* \widehat{X}, \leq)$

return $\text{Nf}(\widehat{X})$

If the stratum $F_s^j \widehat{X}$ is confluent for every $1 \leq j \leq k$, then the procedure stops. As a consequence of Theorem 4.1.9, we obtain the following result.

4.2.5. Proposition. *If the procedure $\text{HB}(\mathcal{E}, \leq)$ terminates and its output $\widehat{X}$ is left-monomial, then it computes a hom-basis of the category C .*

4.3. Examples of the hom-basis procedure

In this subsection, we illustrate the hom-basis computation procedure on two examples. The first computes a hom-basis of the associative-coassociative category, which provides the starting point for the q -Schur category in the next section. The second applies the procedure to the Hecke category, yielding bases for the Iwahori–Hecke algebras.

4.3.1. Example. The *associative-coassociative category*, denoted by $\mathcal{AC}$, is the linear monoidal category generated by 1-cells (r) , for $r \in \mathbb{N}^+$, together with 2-cells, called *merges* and *splits*, given by

$$\begin{array}{c} a+b \\ \text{merge} \\ a \quad b \end{array} : (a) \circ_0 (b) \rightarrow (a+b), \quad \begin{array}{c} a \quad b \\ \text{split} \\ a+b \end{array} : (a+b) \rightarrow (a) \circ_0 (b),$$

satisfying the following *associativity* and *coassociativity* relations

$$\begin{array}{c} \text{merge} \\ a \quad b \quad c \end{array} = \begin{array}{c} \text{merge} \\ a \quad \text{merge} \\ b \quad c \end{array} \quad \text{and} \quad \begin{array}{c} \text{split} \\ a \quad b \quad c \end{array} = \begin{array}{c} \text{split} \\ \text{split} \quad b \quad c \\ a \end{array}, \quad (4.3.2)$$

for all $a, b, c > 0$. Let $F^* \mathcal{AC}$ be the following 2-stratified 3-prepolygraph presenting $\mathcal{AC}$:

- i) $\mathcal{AC}_1$ has the 1-generators (r) , for each $r \in \mathbb{N}^+$, and $\mathcal{AC}_2$ has the following 2-generators, for all $a, b > 0$,

$$\begin{array}{c} a+b \\ \text{merge} \\ a \quad b \end{array} : (a) \circ_0 (b) \rightarrow (a+b), \quad \begin{array}{c} a \quad b \\ \text{split} \\ a+b \end{array} : (a+b) \rightarrow (a) \circ_0 (b),$$

- ii) $F_s^1 \mathcal{AC} = \text{Int}(\mathcal{AC}_2)$ as defined in (2.1.3) and $F_s^2 \mathcal{AC}$ consists of the 3-generators

$$\alpha_{a,b,c} : \begin{array}{c} \text{merge} \\ a \quad b \quad c \end{array} \rightarrow \begin{array}{c} \text{merge} \\ a \quad \text{merge} \\ b \quad c \end{array}, \quad \beta_{a,b,c} : \begin{array}{c} \text{split} \\ a \quad b \quad c \end{array} \rightarrow \begin{array}{c} \text{split} \\ \text{split} \quad b \quad c \\ a \end{array},$$

for all $a, b, c > 0$.

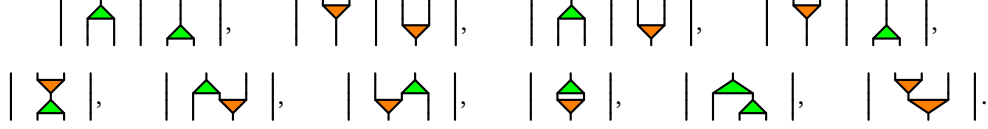
These rules will be written simply as α and β , omitting subscripts whenever no confusion arises. We now compute its normalization with respect to the monomial order $\leq_{\text{Sch}}$ defined in §5.2.1.

We set $F_s^1 \widehat{\mathcal{AC}} := F_s^1 \mathcal{AC}$. Since $s\alpha \in \text{Nf}(F_s^1 \widehat{\mathcal{AC}})$, we have $F^0 \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow 1} = \{\alpha\}$. For every $F_s^1 \widehat{\mathcal{AC}}$ -active context C of size 1, we have $C[s\alpha] \in \text{Nf}(F_s^1 \widehat{\mathcal{AC}})$, except in the following cases:

$$\begin{array}{ll} C_1[\alpha] : \begin{array}{c} \text{merge} \\ \text{merge} \end{array} \rightarrow \begin{array}{c} \text{merge} \\ \text{merge} \end{array} & \rightsquigarrow \quad C_1[\alpha]_{\downarrow 1}^{\leq} : \begin{array}{c} \text{merge} \\ \text{merge} \end{array} \rightarrow \begin{array}{c} \text{merge} \\ \text{merge} \end{array}, \\ C_2[\alpha] : \begin{array}{c} \text{split} \\ \text{merge} \end{array} \rightarrow \begin{array}{c} \text{split} \\ \text{merge} \end{array} & \rightsquigarrow \quad C_2[\alpha]_{\downarrow 1}^{\leq} : \begin{array}{c} \text{split} \\ \text{merge} \end{array} \rightarrow \begin{array}{c} \text{split} \\ \text{merge} \end{array}, \\ C_3[\alpha] : \begin{array}{c} \text{merge} \\ \text{split} \end{array} \rightarrow \begin{array}{c} \text{merge} \\ \text{split} \end{array} & \rightsquigarrow \quad C_3[\alpha]_{\downarrow 1}^{\leq} : \begin{array}{c} \text{merge} \\ \text{split} \end{array} \rightarrow \begin{array}{c} \text{merge} \\ \text{split} \end{array}, \\ C_4[\alpha] : \begin{array}{c} \text{split} \\ \text{split} \end{array} \rightarrow \begin{array}{c} \text{split} \\ \text{split} \end{array} & \rightsquigarrow \quad C_4[\alpha]_{\downarrow 1}^{\leq} : \begin{array}{c} \text{split} \\ \text{split} \end{array} \rightarrow \begin{array}{c} \text{split} \\ \text{split} \end{array}. \end{array}$$

Hence, $L^1 \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow 1} = \{\alpha\} = L^0 \min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow 1}$. Since $|\alpha| = |\beta| = 2$, we obtain $\min_{\sqsubseteq} \llbracket \alpha \rrbracket_{\downarrow 1} = \{\alpha\}$ by Proposition 4.2.3. Similarly, $\min_{\sqsubseteq} \llbracket \beta \rrbracket_{\downarrow 1} = \{\beta\}$. Therefore, we have $F_s^2 \widehat{\mathcal{AC}} = F_s^2 \mathcal{AC}$. It is straightforward to verify that $F_s^2 \widehat{\mathcal{AC}}$ and $F_s^1 \widehat{\mathcal{AC}}$ are both confluent. By Procedure 4.2.4, we obtain a hom-basis $\text{Nf}_m(\widehat{\mathcal{AC}})$

of the category $\mathcal{AC}$, in which each 2-cell admits a unique decomposition $f_1 \circ_1 f_2 \circ_1 \dots \circ_1 f_n$, where each f_i is a 2-cell of $\mathcal{AC}_2^\circ$ of size 1, and each composite $f_i \circ_1 f_{i+1}$ is one of the following forms:



Since the q -Schur category is obtained from the category $\mathcal{AC}$ by imposing additional relations, in the Section 5, we will extend the stratified prepolygraph $F^*\mathcal{AC}$ by adding further strata corresponding to the q -Schur relations, and apply Procedure 4.2.4 to compute a hom-basis of the q -Schur category.

4.3.3. Example. We set $\mathbf{k} = \mathbb{Z}[q, q^{-1}]$ and $z = q - q^{-1}$. For $n \geq 1$, the type A_{n-1} Iwahori–Hecke algebra $H_n(q)$ is the $\mathbf{k}$ -algebra generated by $T_1, \dots, T_{n-1}$, subject to the relations

$$T_i^2 = 1 + zT_i, \quad T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1}, \quad T_i T_j = T_j T_i \quad \text{if } |i - j| > 1.$$

Equivalently, the quadratic relation can be written as $T_i - T_i^{-1} = z$. The Hecke category, denoted by $\mathcal{H}$, is the linear monoidal categories, generated by one 1-generator 1 and two 2-generators

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}, \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} : 1 \circ_0 1 \longrightarrow 1 \circ_0 1,$$

corresponding to T_i and T_i^{-1} respectively, and subject to the *skein*, *inverse*, and *braid relations*

$$\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} - z \mid, \quad \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} = \mid, \quad \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} = \mid, \quad \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \diagdown \diagup \diagup \diagdown \end{array} = \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \diagdown \diagup \diagdown \diagup \end{array}.$$

For every $n \geq 1$, the algebra $\text{End}_{\mathcal{H}}(1^{\circ n})$ is isomorphic to $H_n(q)$, where T_i corresponds to the crossing of the i -th and $(i + 1)$ -st strands. The distant braid relations follow from the exchange law.

Let $F^*\mathcal{H}$ be the following 4-stratified 3-prepolygraph presenting the 2-category $\mathcal{H}$:

i) $\mathcal{H}_{\leq 2}$ has one 0-generator, one 1-generator, and the two crossing 2-generators $\begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$ and $\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$ above.

ii) Its strata are defined by $F_s^1 \mathcal{H} = \text{Int}(\mathcal{H}_2)$, $F_s^2 \mathcal{H} = \{\alpha\}$, $F_s^3 \mathcal{H} = \{\beta_1, \beta_2\}$, and $F_s^4 \mathcal{H} = \{\gamma\}$, where

$$\alpha : \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rightarrow \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} - z \mid, \quad \beta_1 : \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \rightarrow \mid, \quad \beta_2 : \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rightarrow \mid \mid \quad \text{and} \quad \gamma : \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \diagdown \diagup \diagup \diagdown \end{array} \rightarrow \begin{array}{c} \diagup \diagdown \diagup \diagdown \\ \diagdown \diagup \diagdown \diagup \end{array}.$$

We now compute its normalization with respect to a monomial order. Let $\leq$ be a well-founded total order on $\mathcal{H}_1 \sqcup \mathcal{H}_2$ satisfying $1 < \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} < \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array}$. By §2.3.3, this order $\leq$ induces a lexicographic monomial order $\leq_{\mathcal{H}}$ on $\mathcal{H}_2^\circ$. It is straightforward to verify that $\leq_{\mathcal{H}}$ is $\mathcal{H}$ -compatible.

The first two strata are already normalized, so that $F_s^1 \widehat{\mathcal{H}} = F_s^1 \mathcal{H}$ and $F_s^2 \widehat{\mathcal{H}} = F_s^2 \mathcal{H}$. For the normalization of the strata $F_s^3 \mathcal{H}$, we consider the $F_s^2 \widehat{\mathcal{H}}$ -active contexts of size 0 on β_1 and β_2 . We have the following normalization

$$\begin{array}{c} \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \xrightarrow{\beta_1} \mid \mid \\ \downarrow F_s^2 \widehat{\mathcal{H}} \\ \begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} - z \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \end{array} \quad \parallel \quad \begin{array}{c} \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \xrightarrow{\beta_2} \mid \mid \\ \downarrow F_s^2 \widehat{\mathcal{H}} \\ \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} - z \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \end{array} \quad \parallel \quad \begin{array}{c} \mid \mid \end{array}.$$

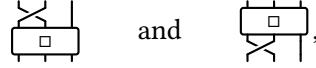
Thus, both inverse relations β_1 and β_2 yield the same $F_s^2 \widehat{\mathcal{H}}$ -normal reduction $\begin{array}{c} \diagdown \diagup \\ \diagup \diagdown \end{array} \rightarrow \mid \mid + z \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array}$. The $F_s^2 \widehat{\mathcal{H}}$ -active contexts of size 1 on β_1 and β_2 are of the forms

$$\begin{array}{c} \boxed{\diagup \diagdown} \mid \mid \quad \text{and} \quad \mid \mid \boxed{\diagdown \diagup}, \end{array}$$

which produce no further F_s^2 H-normalized reductions. Hence, by Proposition 4.2.3, we obtain

$$F_s^3 \widehat{\mathbf{H}} = \left\{ \beta' : \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \rightarrow \mid \mid + z \begin{array}{c} \diagup \diagdown \\ \diagdown \diagup \end{array} \right\}.$$

The $F_s^3\text{H}$ -active contexts of size 1 on β' are of the forms



which produce no additional $F_s^3 \widehat{\mathbb{H}}$ -normalized reductions. Hence, we obtain

$F_s^4 \widehat{\mathbb{H}} :$

It is straightforward to verify that the extensions $F_s^i \widehat{H}$, for $1 \leq i \leq 3$, are confluent, while the confluence of $F_s^4 \widehat{H}$ follows from [23, 5.4.4]. Therefore, by Procedure 4.2.4, the set $\text{Nf}_m(\widehat{H})$ forms a hom-basis of the Hecke category $\mathcal{H}$. Each 2-cell in $\text{Nf}_m(\widehat{H})$ admits a unique decomposition

$$\left| \begin{array}{c} \vdots \\ k_1 \\ \vdots \end{array} \right| \circ_1 \left| \begin{array}{c} \vdots \\ k_2 \\ \vdots \end{array} \right| \circ_1 \cdots \circ_1 \left| \begin{array}{c} \vdots \\ k_\ell \\ \vdots \end{array} \right|, \quad k_i \geq 1.$$

such that, for each $1 \leq i < \ell$, the part connecting  and  has one of the forms

$$| \text{---} \text{X} \text{---} | \quad | \text{---} \text{X} \text{---} | \quad | \text{---} \text{X} \text{---} |$$

or, when $k_i = 1$, the form

$$| \text{---} \text{X} \text{---} |.$$

Here, $\overset{j}{\downarrow}$ denotes an identity 2-cell, with $j \geq k_i - 2$. In particular, for every $n \geq 1$, the set

$$\mathrm{Nf}_m(\hat{H}) \cap \mathrm{End}_{\mathcal{H}}(1^{\circ 0n})$$

forms a $\mathbb{Z}[q, q^{-1}]$ -basis of the Iwahori–Hecke algebra $H_n(q)$.

When $z = 0$, the quadratic relation $T_i^2 = 1 + zT_i$ becomes $T_i^2 = 1$, and hence $H_n(q)$ specialises to the group algebra $\mathbb{Z}[S_n]$ of the symmetric group. Under this specialization, the hom-basis $\text{Nf}_m(\widehat{H}) \cap \text{End}_{\mathcal{H}}(1^{\circ n})$ of $H_n(q)$ obtained above specialises to a basis indexed by the permutations $w \in S_n$. For each reduced expression $w = s_{i_1} \cdots s_{i_r}$, the corresponding basis element T_w is represented by the reduced braid diagram $T_{i_1} \cdots T_{i_r}$.

5. THE NORMAL FORM BASIS OF THE q -SCHUR CATEGORY

In this section, we construct a new hom-basis of the q -Schur category, the *normal form basis*, by applying the stratified rewriting approach to Brundan’s presentation [6, Thm. 2]. In Subsection 5.1, we first derive from this presentation a stratified 3-prepolygraph, whose termination is established in Subsection 5.2 using a monomial-order-based criterion. In Subsection 5.3, we apply the stratified normalization procedure to this presentation and deduce the new hom-basis. Throughout this section, we set $\mathbf{k} = \mathbb{Z}[q, q^{-1}]$.

5.1. Presentations of q -Schur categories

We begin by recalling the presentation of the q -Schur category from [6, Thm. 2], and reformulate it as a 5-stratified 3-prepolygraph. For an integer $n > 0$, we will denote the *quantum integer* by

$$[n]_q := \frac{q^n - q^{-n}}{q - q^{-1}}.$$

In particular, $[0]_q = 0$ and $[-n]_q = -[n]_q$. The corresponding *quantum binomial coefficient* is given by

$$\begin{bmatrix} n \\ s \end{bmatrix}_q := \frac{[n]_q [n-1]_q \cdots [n-s+1]_q}{[s]_q [s-1]_q \cdots [1]_q},$$

with the convention that $\begin{bmatrix} n \\ 0 \end{bmatrix}_q = 1$ and $\begin{bmatrix} n \\ s \end{bmatrix}_q = 0$ whenever $s < 0$.

5.1.1. q -Schur category. The q -Schur category, denoted by qS , is the linear 2-category generated by $\mathcal{AC}_i$, for $0 \leq i \leq 2$, subject to the relations (4.3.2), together with the following *square-switch relation*:

$$\begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ d \\ a \\ b \end{array} = \sum_{s=\max(0, c-b)}^{\min(c, d)} \begin{bmatrix} a-b+c-d \\ s \end{bmatrix}_q \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d-s \\ c-s \\ a \\ b \end{array}$$

for $a, b, c, d \geq 0$ with $d \leq a$ and $c \leq b + d$, where $\begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ a \end{array} := \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ a \end{array}$ and $\begin{array}{c} \text{Orange triangle down} \\ \text{Green triangle up} \end{array} \begin{array}{c} b \\ 0 \end{array} := \begin{array}{c} \text{Orange triangle down} \\ \text{Green triangle up} \end{array} \begin{array}{c} b \\ 0 \end{array}$.

We present the q -Schur category qS by the 3-prepolygraph qS , defined by $qS_i = \mathcal{AC}_i$ for $0 \leq i \leq 2$, and $qS_3 := \mathcal{AC}_3 qS'_3$, where qS'_3 consists of the following 3-generators:

$$\gamma_{a,b,c,d} : \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ d \\ a \\ b \end{array} \rightarrow \sum_{s=\max(0, c-b)}^{\min(c, d)} \begin{bmatrix} a-b+c-d \\ s \end{bmatrix}_q \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d-s \\ c-s \\ a \\ b \end{array}$$

for all $a, b, c, d \geq 0$ with $d \leq a$ and $c \leq b + d$.

5.1.2. Stratification of qS_3 . We stratify the extension qS'_3 according to the values of the parameters a, b, c, d of the rule $\gamma_{a,b,c,d}$. Several special choices of parameters yield degenerate forms of $\gamma_{a,b,c,d}$:

i) If $c = 0$ or $d = 0$, the rule reduces to the identity:

$$\gamma_{a,b,0,d} : \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d \\ d \\ a \\ b \end{array} \rightarrow \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d \\ d \\ a \\ b \end{array} \quad \text{or} \quad \gamma_{a,b,c,0} : \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ c \\ a \\ b \end{array} \rightarrow \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ c \\ a \\ b \end{array}.$$

ii) When $b = 0$, the rule takes the form

$$\gamma_{a,0,c,d} : \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} c \\ d-c \\ d \\ a \end{array} \rightarrow \begin{bmatrix} a+c-d \\ c \end{bmatrix}_q \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d-c \\ c \\ a \\ a \end{array}. \quad (5.1.3)$$

If moreover $c = d$, the rule further simplifies to

$$\gamma_{a,0,c,d} : \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d \\ d \\ a \\ a \end{array} \rightarrow \begin{bmatrix} a \\ d \end{bmatrix}_q \begin{array}{c} \text{Green triangle up} \\ \text{Orange triangle down} \\ \text{Green triangle up} \\ \text{Orange triangle down} \end{array} \begin{array}{c} d \\ d \\ a \\ a \end{array}. \quad (5.1.4)$$

iii) When $b + d - c = 0$, we obtain

$$\gamma_{a,b,c,d} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{bmatrix} a \\ d \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array}. \quad (5.1.5)$$

If moreover $b = 0$, this further reduces to

$$\gamma_{a,0,c,d} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{bmatrix} a \\ d \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array}.$$

iv) When $a - d = 0$, we have

$$s\gamma_{a,b,c,d} = \begin{array}{c} \text{diagram} \end{array} \quad \text{and} \quad t\gamma_{a,b,c,d} = \sum_{s=\max(0,c-b)}^{\min(c,a)} \begin{bmatrix} c-b \\ s \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array}.$$

If $c \geq b$, then $\begin{bmatrix} c-b \\ s \end{bmatrix}_q = 0$ for all $s > c - b$. Since $c - b \leq s \leq \min(c, a)$, it follows that $t\gamma_{a,b,c,d}$ has only one non-zero term with $s = c - b$. In this case, $\gamma_{a,b,c,d}$ is the identity. If $c < b$, we rewrite $\gamma_{a,b,c,d}$ in the oriented form

$$\gamma_{a,b,c,d} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{array}{c} \text{diagram} \end{array} - \sum_{s=1}^{\min(c,a)} \begin{bmatrix} c-b \\ s \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array}. \quad (5.1.6)$$

The orientation is induced by the monomial order defined in §5.2.1, which ensures termination of qS.

Therefore, we construct a 5-stratification $F^*\text{qS}$ of the prepolygraph qS as follows. We set $F_s^1\text{qS} = F_s^1\text{AC}$ and $F_s^2\text{qS} = F_s^2\text{AC}$ from Example 4.3.1. The extension $F_s^3\text{qS}$ consists of the following rules:

$$\gamma_{a,b,c}^{\Delta} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{bmatrix} a \\ c \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array}, \quad \gamma_{a,b,c}^{\nabla} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{bmatrix} a \\ c \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array}, \quad \gamma_{a,c}^{\diamond} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{bmatrix} a \\ c \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array},$$

for all $a, b, c > 0$, which correspond respectively to the degenerate rules in (5.1.3), (5.1.5), and (5.1.4). The extension $F_s^4\text{qS}$ consists of the following non-degenerate rules:

$$\gamma_{a,b,c,d} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \sum_{s=\max(0,c-b)}^{\min(c,d)} \begin{bmatrix} a-b+c-d \\ s \end{bmatrix}_q \begin{array}{c} \text{diagram} \end{array},$$

for all $a, b, c, d > 0$, with $a > d$ and $b + d > c$. Finally, the extension $F_s^5 \text{qS}$ consists of the following degenerate rules:

$$\eta_{a,b,c} : \begin{array}{c} \text{Diagram 1} \\ \text{Diagram 2} \end{array} \rightarrow \begin{array}{c} \text{Diagram 3} \\ \text{Diagram 4} \end{array} - \sum_{s=1}^{\min(c,a)} \begin{bmatrix} c-b \\ s \end{bmatrix}_q \begin{array}{c} \text{Diagram 5} \end{array},$$

for all $a, b, c > 0$, with $b > c$, which correspond to the degenerate rules in (5.1.6).

Hence, the 5-stratified 3-prepolygraph $F^* \text{qS}$ presents the q -Schur category $q\mathcal{S}$. Its rules will be written simply by omitting subscripts whenever no confusion arises.

5.2. Termination by stratification

5.2.1. A total preorder on qS_2° . We define a total preorder $\succsim_q$ on qS_2° as follows.

i) For any two parallel 2-cells f, g in qS_2° of size 1, written as

$$f = a_n \cdots a_1 \circ_0 \varphi \circ_0 b_1 \cdots b_m \quad \text{and} \quad g = a'_{n'} \cdots a'_1 \circ_0 \varphi' \circ_0 b'_1 \cdots b'_{m'},$$

where $a_i, a'_i, b_i, b'_i \in \text{qS}_1$ and $\varphi, \varphi' \in \text{qS}_2$. Since $sf = sg$, we have

$$\sum_{i=1}^n a_i + \sum_{i=1}^m b_i + \deg(\varphi) = \sum_{i=1}^{n'} a'_i + \sum_{i=1}^{m'} b'_i + \deg(\varphi'),$$

and denote this common number by r , where $\deg(\varphi) := c$ if $\varphi = \begin{array}{c} \text{Diagram 6} \end{array}$ or $\varphi = \begin{array}{c} \text{Diagram 7} \end{array}$. We set $j := \min(n + m + 1, n' + m' + 1)$, and define $f \succsim_q g$ if

$$(a_n, \dots, a_1, \varphi, b_1, \dots, b_m)_j \succ_{[q, \dots, q]} (a'_{n'}, \dots, a'_1, \varphi', b'_1, \dots, b'_{m'})_j,$$

where $(-)_j$ denotes the truncation to the first j components from the left and the order $\succsim_q$ on $\text{qS}_1 \sqcup \text{qS}_2$ is defined by

$$\cdots \succ_q \begin{array}{c} \text{Diagram 8} \end{array} \succ_q \begin{array}{c} \text{Diagram 9} \end{array} \succ_q \begin{array}{c} \text{Diagram 10} \end{array} \succ_q \begin{array}{c} \text{Diagram 11} \end{array} \succ_q \begin{array}{c} \text{Diagram 12} \end{array} \succ_q \begin{array}{c} \text{Diagram 13} \end{array} \succ_q \begin{array}{c} \text{Diagram 14} \end{array} \succ_q \cdots \succ_q \begin{array}{c} \text{Diagram 15} \end{array} \succ_q \begin{array}{c} \text{Diagram 16} \end{array}. \quad (5.2.2)$$

ii) More generally, for any two parallel 2-cells $f, g \in \text{qS}_2^\circ$, write their unique $\circ_1$ -decompositions as

$$f = C_1[\varphi_1] \circ_1 C_2[\varphi_2] \circ_1 \cdots \circ_1 C_n[\varphi_n] \quad \text{and} \quad g = D_1[\psi_1] \circ_1 D_2[\psi_2] \circ_1 \cdots \circ_1 D_m[\psi_m].$$

We define $f \succsim_q g$ if $n > m$, or $n = m$ and

$$(C_1[\varphi_1], \dots, C_n[\varphi_n]) \succ_{[q, \dots, q]} (D_1[\psi_1], \dots, D_m[\psi_m]). \quad (5.2.3)$$

5.2.4. Lemma. *The total preorder $\preccurlyeq_q$ is monotone and well-founded.*

Proof. As in the case of the lexicographic order on free words, it is straightforward that the order $\preccurlyeq_q$ on qS_2° is monotone. By the definition of $\preccurlyeq_q$ on parallel 2-cells of arbitrary size given in (5.2.3), it remains to prove that $\preccurlyeq_q$ is well-founded on parallel 2-cells of size 1, following [28, Lem. 1.15].

From §5.2.1-i), the set of 2-cells of size 1 is decomposed into subsets $(Z_i)_{i \geq 1}$, where each Z_i consists of 2-cells of the form

$$f = a_n \cdots a_1 \circ_1 \varphi \circ_1 b_1 \cdots b_m \quad \text{with} \quad i = \sum_{j=1}^n a_j + \sum_{j=1}^m b_j + s\varphi.$$

For each Z_i , (5.2.2) shows that the order $\leq_q$ is well-founded on $qS_1 \sqcup qS_2$ with minimal element $\begin{smallmatrix} | \\ i \end{smallmatrix}$. It follows that $\leq_q$ is well-founded on Z_i . Therefore, the order $\leq_q$ is well-founded on the set of parallel 2-cells of size 1. $\square$

We define a well-founded total order $\leq$ on the set of generating 2-cells of qS by setting

$$\begin{smallmatrix} \blacktriangle \\ a \ b \end{smallmatrix} > \begin{smallmatrix} \blacktriangle \\ c \ d \end{smallmatrix} > \begin{smallmatrix} a \ b \\ \blacktriangledown \end{smallmatrix} > \begin{smallmatrix} c \ d \\ \blacktriangledown \end{smallmatrix} > \begin{smallmatrix} | \\ m \end{smallmatrix} > \begin{smallmatrix} | \\ n \end{smallmatrix},$$

if $(a, b) > (c, d)$ with respect to the lexicographic order and $m > n$. By §2.3.3, this order induces a lexicographic order $\leq_{\text{lex}}$ on qS_2° . By Lemma 5.2.4, the order $\leq_{q, \text{lex}}$ is a monomial order on qS_2° , and we denote it by $\leq_{\text{Sch}} := \leq_{q, \text{lex}}$.

5.2.5. Lemma. *The order $\leq_{\text{Sch}}$ is a qS -compatible monomial order on qS_2° . Hence, the prepolygraph qS is terminating.*

Proof. We compare the sources and targets of all rules in mqS_3 with respect to $\leq_q$.

i) For the extension $F_s^1 qS$, since $\begin{smallmatrix} a+b \\ \blacktriangle \\ a \ b \end{smallmatrix} >_q \begin{smallmatrix} | \\ a \end{smallmatrix}$, it follows that

$$\begin{smallmatrix} \blacktriangle \\ \dots \\ a \ b \end{smallmatrix} \begin{smallmatrix} \blacktriangle \\ \dots \\ a \ b \end{smallmatrix} >_q \begin{smallmatrix} \blacktriangle \\ a \ b \end{smallmatrix} \begin{smallmatrix} \blacktriangle \\ \dots \\ a \ b \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} \blacktriangle \\ \dots \\ a \ b \end{smallmatrix} \begin{smallmatrix} \blacktriangledown \\ \dots \\ a \ b \end{smallmatrix} >_q \begin{smallmatrix} \blacktriangle \\ a \ b \end{smallmatrix} \begin{smallmatrix} \blacktriangledown \\ \dots \\ a \ b \end{smallmatrix}.$$

Similarly, since $\begin{smallmatrix} \blacktriangledown \\ a \end{smallmatrix} >_q \begin{smallmatrix} | \\ a \end{smallmatrix}$, we have

$$\begin{smallmatrix} \blacktriangledown \\ \dots \\ a \end{smallmatrix} \begin{smallmatrix} \blacktriangledown \\ \dots \\ a \end{smallmatrix} >_q \begin{smallmatrix} \blacktriangledown \\ a \end{smallmatrix} \begin{smallmatrix} \blacktriangledown \\ \dots \\ a \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} \blacktriangledown \\ a \end{smallmatrix} \begin{smallmatrix} \blacktriangle \\ \dots \\ a \ b \end{smallmatrix} >_q \begin{smallmatrix} \blacktriangledown \\ a \end{smallmatrix} \begin{smallmatrix} \blacktriangledown \\ \dots \\ a \ b \end{smallmatrix}.$$

ii) For the extension $F_s^2 qS$, since $\begin{smallmatrix} \blacktriangle \\ a \ b \end{smallmatrix} >_q \begin{smallmatrix} | \\ a \end{smallmatrix}$, $\begin{smallmatrix} a \ b+c \\ \blacktriangledown \\ a+b+c \end{smallmatrix} = \begin{smallmatrix} a+b \ c \\ \blacktriangledown \\ a+b+c \end{smallmatrix}$ and $\begin{smallmatrix} | \\ a \end{smallmatrix} >_q \begin{smallmatrix} a \ b \\ \blacktriangledown \\ a+b \end{smallmatrix}$, we obtain

$$\begin{smallmatrix} \blacktriangle \\ \blacktriangle \\ a \ b \ c \end{smallmatrix} >_q \begin{smallmatrix} \blacktriangle \\ a \ b \ c \end{smallmatrix} \quad \text{and} \quad \begin{smallmatrix} a \ b \ c \\ \blacktriangledown \end{smallmatrix} >_q \begin{smallmatrix} a \ b \\ \blacktriangledown \end{smallmatrix} \begin{smallmatrix} c \\ \blacktriangledown \end{smallmatrix}.$$

iii) For the extension $F_s^3 qS$, comparing the sizes of 2-cells yields

$$\begin{smallmatrix} a \ b \\ \blacktriangle \\ c \end{smallmatrix} >_q \begin{smallmatrix} a \ b \\ \blacktriangledown \end{smallmatrix}, \quad \begin{smallmatrix} \blacktriangle \\ \blacktriangle \\ a \ b \end{smallmatrix} >_q \begin{smallmatrix} \blacktriangle \\ a \ b \end{smallmatrix}, \quad \text{and} \quad \begin{smallmatrix} c \\ \blacktriangledown \\ a \end{smallmatrix} >_q \begin{smallmatrix} | \\ a \end{smallmatrix}.$$

iv) For the extension $F_s^4 \mathbf{qS}$, since $\begin{array}{c} \blacktriangledown \\ a \end{array} >_q \begin{array}{c} | \\ a \end{array}$ for every $s \geq 0$, we have

$$\begin{array}{c} \begin{array}{c} \blacktriangle \\ c \\ \blacktriangledown \\ d \\ \blacktriangledown \\ a \end{array} \begin{array}{c} | \\ b \end{array} >_q \begin{array}{c} \begin{array}{c} \blacktriangle \\ d-s \\ \blacktriangledown \\ c-s \\ \blacktriangledown \\ a \end{array} \begin{array}{c} | \\ b \end{array} \end{array}.$$

v) For the extension $F_s^5 \mathbf{qS}$, since $\begin{array}{c} \blacktriangle \\ a \ c \end{array} >_q \begin{array}{c} \blacktriangle \\ a \ c-s \end{array}$ for every $s > 0$, and by comparing the sizes, we obtain

$$\begin{array}{c} \begin{array}{c} \blacktriangle \\ a \\ \blacktriangledown \\ c \\ \blacktriangledown \\ a \end{array} \begin{array}{c} | \\ b \end{array} >_q \begin{array}{c} \begin{array}{c} \blacktriangle \\ a-s \\ \blacktriangledown \\ c-s \\ \blacktriangledown \\ a \end{array} \begin{array}{c} | \\ b \end{array} \end{array} \quad \text{and} \quad \begin{array}{c} \begin{array}{c} \blacktriangle \\ a \\ \blacktriangledown \\ c \\ \blacktriangledown \\ a \end{array} \begin{array}{c} | \\ b \end{array} >_q \begin{array}{c} \begin{array}{c} \blacktriangle \\ c \\ \blacktriangledown \\ a \end{array} \begin{array}{c} | \\ b \end{array} \end{array}.$$

Therefore, the order $\leq_{\text{Sch}} = \leq_{q, \text{lex}}$ is also compatible. $\square$

5.3. Stratified normalization and normal form basis

In this subsection, we denote by $|n|$ the n vertical strands, and by $|\leq n|$ any i vertical strands $|i|$ with $1 \leq i \leq n$. The notation $\begin{array}{c} \blacktriangle \\ |1| \end{array}$ (resp. $\begin{array}{c} |1| \\ \blacktriangledown \end{array}$) is defined to be the identity. For $n \geq 2$, we set

$$\begin{array}{c} \blacktriangle \\ |2| \end{array} := \begin{array}{c} \blacktriangle \\ \blacktriangledown \end{array} \quad (\text{resp. } \begin{array}{c} |2| \\ \blacktriangledown \end{array} := \begin{array}{c} \blacktriangledown \\ \blacktriangledown \end{array}),$$

and define inductively

$$\begin{array}{c} \blacktriangle \\ |n+1| \end{array} := \begin{array}{c} \begin{array}{c} \blacktriangle \\ |n| \end{array} \\ \blacktriangledown \end{array} \quad (\text{resp. } \begin{array}{c} |n+1| \\ \blacktriangledown \end{array} := \begin{array}{c} |n| \\ \begin{array}{c} \blacktriangledown \\ \blacktriangledown \end{array} \end{array}).$$

We also introduce the following notation

$$\begin{array}{c} \boxed{f} \quad \boxed{g} \quad \boxed{h} := \begin{array}{c} \boxed{f} \quad \dots \quad \dots \\ \dots \quad \boxed{g} \quad \dots \\ \dots \quad \dots \quad \boxed{h} \end{array}.$$

A 2-cell f in $\mathbf{qS}_2^\circ$ is *minimal* if $f \downarrow$ cannot be written as $\boxed{f_1} \mid \dots \mid \boxed{f_i}$, for every $i \geq 2$.

We now compute the $\leq_{\text{Sch}}$ -normalization $F^* \widehat{\mathbf{qS}}$ of the 5-stratified prepolygraph $F^* \mathbf{qS}$. We set $F_s^1 \widehat{\mathbf{qS}} := F_s^1 \mathbf{qS}$. Thanks to Example 4.3.1, we have $F_s^2 \widehat{\mathbf{qS}} := F_s^2 \mathbf{qS}$ and $\text{Nf}(F_s^2 \widehat{\mathbf{qS}}) = \text{Nf}_m(\widehat{\mathbf{AC}})$.

5.3.1. $F^2 \widehat{\mathbf{qS}}$ -normal reductions. For each $\gamma^\diamond \in F_s^3 \mathbf{qS}$, there exist two contexts

$$C_1[\gamma^\diamond] : \begin{array}{c} \blacktriangle \\ \begin{array}{c} \blacktriangle \\ \blacktriangledown \end{array} \end{array} \rightarrow \begin{array}{c} \blacktriangle \\ \blacktriangledown \end{array} \quad \text{and} \quad C_2[\gamma^\diamond] : \begin{array}{c} \begin{array}{c} \blacktriangle \\ \blacktriangledown \end{array} \\ \blacktriangledown \end{array} \rightarrow \begin{array}{c} \blacktriangledown \\ \blacktriangledown \end{array}$$

such that $\gamma^\Delta = C_1[\gamma^\diamond] \downarrow_2 \leq$ and $\gamma^\nabla = C_2[\gamma^\diamond] \downarrow_2 \leq$. Consequently, we obtain

$$\llbracket \gamma^\Delta \rrbracket \downarrow_2 \subseteq \llbracket \gamma^\diamond \rrbracket \downarrow_2 \quad \text{and} \quad \llbracket \gamma^\nabla \rrbracket \downarrow_2 \subseteq \llbracket \gamma^\diamond \rrbracket \downarrow_2.$$

It suffices to compute $\llbracket \gamma^\diamond \rrbracket_{\downarrow_2}$ in order to obtain $F_s^3 \widehat{\mathbf{qS}}$.

Since $s\gamma^\diamond \in \text{Nf}(F^2 \widehat{\mathbf{qS}})$, we have the stratum $L^0 \min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2} = \{\gamma^\diamond\}$. A straightforward verification shows that only C_1 and C_2 yield new rules to $\min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2}$ among contexts of size 1. Hence

$$L^1 \min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2} = \{\gamma^\diamond, \gamma^\Delta, \gamma^\nabla\}.$$

Moreover, among contexts of size 2, none adds new rules to $\min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2}$ except the following one:

$$C_3[\gamma_{a,c}^\diamond] : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{array}{c} \text{diagram} \end{array} \quad \rightsquigarrow \quad C_3[\gamma_{a,c}^\diamond]_{\downarrow_2} : \begin{array}{c} \text{diagram} \end{array} \rightarrow \begin{array}{c} \text{diagram} \end{array}.$$

Therefore, we obtain

$$L^2 \min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2} = \{\gamma^\diamond, \gamma^\Delta, \gamma^\nabla, C_3[\gamma_{a,c}^\diamond]_{\downarrow_2}\}.$$

Finally, one checks that $L^2 \min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2} = L^3 \min_{\sqsubseteq} \llbracket \gamma^\diamond \rrbracket_{\downarrow_2}$. By Proposition 4.2.3, it follows that

$$F_s^3 \widehat{\mathbf{qS}} = \{\gamma^\diamond, \gamma^\Delta, \gamma^\nabla, C_3[\gamma_{a,c}^\diamond]_{\downarrow_2}\}.$$

By induction, there are two families of 3-cells in $(F_s^3 \widehat{\mathbf{qS}})^{\circ \ell}$:

$$\begin{array}{c} \text{diagram} \end{array} \rightarrow \prod_{i=2}^m \begin{array}{c} \text{diagram} \end{array} \quad \text{or} \quad \begin{array}{c} \text{diagram} \end{array} \rightarrow \prod_{i=2}^m \begin{array}{c} \text{diagram} \end{array},$$

where $b = a_1 + \dots + a_m$. This shows that any $\begin{array}{c} \text{diagram} \end{array}$ and $\begin{array}{c} \text{diagram} \end{array}$ share at most one common strand. Hence $F_s^3 \widehat{\mathbf{qS}}$ is confluent, since every source of critical branchings in $F_s^3 \widehat{\mathbf{qS}}$ admits a unique normal form [1, Thm. 4.3.3]. In fact, the minimal 2-cell in $\text{Nf}(F^3 \widehat{\mathbf{qS}})$ can be described as follows:

$$\begin{array}{c} \text{diagram} \end{array}, \quad (5.3.2)$$

where $2 \leq m \leq 3$ and $n \geq 2$, with each occurrence of m and n taking independent values.

5.3.3. $F_s^3 \widehat{\mathbf{qS}}$ -normal reductions. For simplicity, we write

$$\sum_s := \sum_{s=\max(0, c-b)}^{\min(c, d)} \quad \text{and} \quad \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} := \begin{bmatrix} a - b + c - d \\ s \end{bmatrix}_q.$$

For $\gamma \in F_s^3 \widehat{\mathbf{qS}}$, among contexts of size 1, namely C_1 and C_2 , and of size 2, namely C_3 ,

$$C_1[s\gamma] : \begin{array}{c} \text{diagram} \end{array} \quad C_2[s\gamma] : \begin{array}{c} \text{diagram} \end{array} \quad C_3[s\gamma] : \begin{array}{c} \text{diagram} \end{array},$$

we have $C_1[\partial\gamma]\downarrow_3 = 0$ and $C_2[\partial\gamma]\downarrow_3 = 0$, and the rule $C_3[\gamma]\downarrow_3^{\leq}$ is trivial (see Appendix A.1). Therefore, none of these contexts yields a new rule in $\min_{\sqsubseteq} \llbracket \gamma \rrbracket \downarrow_3$.

Note that the extension $\min_{\sqsubseteq} \llbracket \gamma \rrbracket \downarrow_3$ is not bounded. It suffices to apply active contexts on γ along the two directions indicated by the arrows in the following diagram:

$$\gamma : \begin{array}{c} \text{Diagram with two green triangles and two orange triangles} \end{array} \longrightarrow \sum \begin{array}{c} \text{Diagram with two green triangles and two orange triangles} \end{array}$$

and to compute the corresponding $F^3\widehat{\text{qS}}$ -normal reductions. Consequently, $F_s^4\widehat{\text{qS}} = \sqcup \min_{\sqsubseteq} \llbracket \gamma_{a,b,c,d} \rrbracket \downarrow_3$ consists of

$$\begin{array}{c} \text{Diagram with a box labeled } f \text{ and labels } e, c, d, a+e, b \end{array} \longrightarrow \sum_s \Psi_s \begin{array}{c} \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \text{ and } \begin{array}{c} \text{Diagram with a box labeled } f \text{ and labels } e, c, d, a+e, b \end{array} \end{array},$$

where $f \in \text{qS}_2^{\circ}$ such that the 2-cells of the form of the box of the source of the above 3-cell belong to $\text{Nf}(F^3\widehat{\text{qS}})$. We also write the above rule as $\gamma_{a,b,c,d}$ when no confusion arises.

5.3.4. Lemma. *The extension $F_s^4\widehat{\text{qS}}$ is confluent.*

Proof. We list below the three families of generic critical branchings:

$$\left(\begin{array}{c} \text{Diagram with labels } g, c, d, a, b, f \end{array} \right) \gamma \begin{bmatrix} b+d \\ f \\ b+d-c+f-g \\ b+d-c \end{bmatrix} \gamma \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad \left(\begin{array}{c} \text{Diagram with labels } g, c, d, a, b \end{array} \right) \gamma \begin{bmatrix} f+d \\ b \\ c \\ d \end{bmatrix} \gamma \begin{bmatrix} a-d \\ c \\ g \\ f \end{bmatrix}, \quad \left(\begin{array}{c} \text{Diagram with labels } c, d, a, b \end{array} \right) \gamma \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \gamma' \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix},$$

where $\gamma' : \boxed{f} \rightarrow \sum_i \lambda_i \boxed{f_i}$ is a $F^4\widehat{\text{qS}}$ -rewriting step. Here we prove confluence for the last family, while

that of the other two is proved in Appendix A.2.

$$\sum_s \Psi_s \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} \longrightarrow \sum_{s,i} \lambda_i \Psi_s \begin{array}{c} \text{Diagram with box } f_i \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array}.$$

□

By Theorem 4.1.9, the extension $F^4\widehat{\mathbf{qS}}$ is confluent. The shapes of the minimal 2-cells in $\text{Nf}(F_s^4\widehat{\mathbf{qS}})$ are described in pictorial language in §6.1.2. We now compute the final extension $F_s^5\widehat{\mathbf{qS}}$. Without loss of generality, we restrict to active contexts C such that $C[s\eta]$ is not in $\text{Nf}(F_s^4\widehat{\mathbf{qS}})$ in order to compute the corresponding $F^4\widehat{\mathbf{qS}}$ -normal reductions. There are three cases.

5.3.5. Active context of type I. For $\eta_{a,b,c} \in F_s^5\widehat{\mathbf{qS}}$, we consider the $F^4\widehat{\mathbf{qS}}$ -active context $D_1 = \text{Diagram with box } \square \text{ and input } d$. Its application to $\eta_{a,b,c}$ gives

$$D_1[\eta_{a,b,c}] : \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} \longrightarrow \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} - \sum_{s=1}^{\min(c,a)} \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array}.$$

Under the order $\leq_{\text{Sch}}$, its $F^4\widehat{\mathbf{qS}}$ -normal reduction is

$$D_1[\eta_{a,b,c}] \downarrow_4^{\leq} : \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} \longrightarrow \sum_{s=1}^{\min(c,a)} \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} + \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array},$$

since $\begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} >_q \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array}$ and $\begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array} >_q \begin{array}{c} \text{Diagram with box } f \text{ and inputs } a, b \text{ at bottom, } c, d \text{ at top} \end{array}$ for $b > c$. The target of $D_1[\eta_{a,b,c}] \downarrow_4^{\leq}$ is not in $\text{Nf}(F_s^5\widehat{\mathbf{qS}})$, since we can further apply $\eta_{a,b+d,c}$ to the second term on the right-hand side above, which yields a new rule

in $F_s^5 \widehat{\mathbf{qS}}$, denoted by $\zeta_{a,b,c,d}$:

$$\begin{array}{c} \text{Diagram: A vertical line with a green triangle at the top, a red triangle at the bottom, and a green triangle in the middle. The top green triangle is labeled 'c', the middle green triangle is labeled 'b', and the bottom red triangle is labeled 'a'. The line is labeled 'b+d' at the bottom.} \end{array} \rightarrow \begin{bmatrix} b-c+d & \\ & d \end{bmatrix}_q + \sum_{s=1}^{\min(c,a)} \left(\begin{bmatrix} c-b & \\ & s \end{bmatrix}_q \begin{bmatrix} b-c+s+d & \\ & d \end{bmatrix}_q - \begin{bmatrix} b-c+d & \\ & d \end{bmatrix}_q \begin{bmatrix} c-b-d & \\ & s \end{bmatrix}_q \right) \begin{array}{c} \text{Diagram: A vertical line with a green triangle at the top, a red triangle at the bottom, and a green triangle in the middle. The top green triangle is labeled 'c', the middle green triangle is labeled 'a-s', and the bottom red triangle is labeled 'a'. The line is labeled 'b+d' at the bottom.} \end{array}.$$

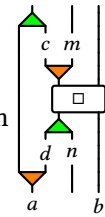
In fact, when $b = c$, we have $\zeta_{a,b,c,d} = \eta_{a,b+d,c}$, which shows that the rule η can be replaced by ζ in $F_s^5 \widehat{\mathbf{qS}}$. For simplicity, for $b \geq c$, we write

$$\zeta_{a,b,c,d} : \begin{array}{c} \text{Diagram: A vertical line with a green triangle at the top, a red triangle at the bottom, and a green triangle in the middle. The top green triangle is labeled 'c', the middle green triangle is labeled 'b', and the bottom red triangle is labeled 'a'. The line is labeled 'b+d' at the bottom.} \end{array} \rightarrow \sum_s \Phi_s \begin{bmatrix} c & \\ b & \\ & d \end{bmatrix} \begin{array}{c} \text{Diagram: A vertical line with a green triangle at the top, a red triangle at the bottom, and a green triangle in the middle. The top green triangle is labeled 'c', the middle green triangle is labeled 'a-s', and the bottom red triangle is labeled 'a'. The line is labeled 'b+d' at the bottom.} \end{array},$$

where

$$\sum_s := \sum_{s=1}^{\min(c,a)} \quad \text{and} \quad \Phi_s \begin{bmatrix} c & \\ b & \\ & d \end{bmatrix} := \left(\begin{bmatrix} c-b & \\ & s \end{bmatrix}_q \begin{bmatrix} b-c+s+d & \\ & d \end{bmatrix}_q - \begin{bmatrix} b-c+d & \\ & d \end{bmatrix}_q \begin{bmatrix} c-b-d & \\ & s \end{bmatrix}_q \right).$$

5.3.6. Active context of type II. We consider the $F^4 \widehat{\mathbf{qS}}$ -active context D_2 of the form



then compute $sD_2[\zeta_{d+n,y,c+m,b-y}]\downarrow_4^{\leq}$ and $tD_2[\zeta_{d+n,y,c+m,b-y}]\downarrow_4^{\leq}$, which are respectively given by

$$\sum_s \Psi_s \begin{bmatrix} a & \\ n+y & \\ c & \\ d & \end{bmatrix} \begin{array}{c} \text{Diagram: A vertical line with a green triangle at the top labeled 'm', a red triangle at the bottom labeled 'b', and a green triangle in the middle labeled 'x'. The line is labeled 'b' at the bottom.} \end{array} \quad \text{and} \quad \sum_{s,t} \Phi_t \begin{bmatrix} c+m & \\ y & \\ b-y & \end{bmatrix} \Psi_s \begin{bmatrix} a & \\ n+c+m-t & \\ c & \\ d & \end{bmatrix} \begin{array}{c} \text{Diagram: A vertical line with a green triangle at the top labeled 'm', a red triangle at the bottom labeled 'b', and a green triangle in the middle labeled 'x'. The line is labeled 'b' at the bottom.} \end{array}.$$

where $c + m \leq y$ and $x = y + d + n - c - m$. Under the order $\leq_{\text{Sch}}$, the corresponding $F^4\widehat{\text{qS}}$ -normal reduction is given by

$$D_2[\zeta]_{\downarrow_4} : \begin{array}{c} \text{Diagram 1} \end{array} \rightarrow - \sum_{s=1} \Psi_s \begin{bmatrix} a \\ n+y \\ c \\ d \end{bmatrix} \begin{array}{c} \text{Diagram 2} \end{array} + \sum_{s,t} \Phi_t \begin{bmatrix} c+m \\ y \\ b-y \end{bmatrix} \Psi_s \begin{bmatrix} a \\ n+c+m-t \\ c \\ d \end{bmatrix} \begin{array}{c} \text{Diagram 3} \end{array}.$$

The target of $D_2[\zeta]_{\downarrow_4}^{\leq}$ is not in $\text{Nf}(F_s^5\widehat{\text{qS}})$, since we can further apply $D_2[\zeta]_{\downarrow_4}^{\leq}$ to the first term on the right-hand side above, which yields a new rule in $F_s^5\widehat{\text{qS}}$, denoted by θ_1 :

$$\begin{array}{c} \text{Diagram 1} \end{array} \rightarrow \sum_t \Phi_t \begin{bmatrix} c+m \\ y \\ b-y \end{bmatrix} \begin{array}{c} \text{Diagram 2} \end{array},$$

where $a > d$ and $c + m \leq y$.

In fact, by repeating the above construction, we can apply the context D_2 to the rule θ_{k-1} inductively, thereby obtaining further rules in $F_s^5\widehat{\text{qS}}$, which we denote by θ_k :

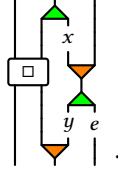
$$\begin{array}{c} \text{Diagram 1} \end{array} \rightarrow \begin{array}{c} \text{Diagram 2} \end{array},$$

where $c + m \leq y$,

$$f := \begin{array}{c} \text{Diagram 1} \end{array} \quad \text{and} \quad g := \begin{array}{c} \text{Diagram 2} \end{array},$$

and the 2-cells in the above boxes cannot be reduced by θ_i for $i < k$.

5.3.7. Active context of type III. We consider the $F^4\widehat{\mathbf{qS}}$ -active context D_3 of the form



We then compute $sD_3[\zeta_{a,b,c,d}]\downarrow_4^{\leq}$ and $tD_3[\zeta_{a,b,c,d}]\downarrow_4^{\leq}$, which are respectively given by

$$\sum_t \Psi_t \begin{bmatrix} d+y \\ e \\ x \\ y \end{bmatrix} \begin{array}{c} \text{diagram with labels } c, x-t, a, b, y-t, b+d+y, e \end{array} \quad \text{and} \quad \sum_{s,t} \Psi_t \begin{bmatrix} b+d+y-c+s \\ e \\ x \\ y \end{bmatrix} \Phi_s \begin{bmatrix} c \\ b \\ d \end{bmatrix} \begin{array}{c} \text{diagram with labels } c, x-t, a, c-s, y-t, b+d+y, e \end{array},$$

where $c \leq b$. Under the order $\leq_{\text{Sch}}$, the corresponding $F^4\widehat{\mathbf{qS}}$ -normal reduction is given by

$$D_3[\zeta]\downarrow_4^{\leq} : \begin{array}{c} \text{diagram with labels } c, x, a, b, y, b+d+y, e \end{array} \rightarrow - \sum_{t=1} \Psi_t \begin{bmatrix} d+y \\ e \\ x \\ y \end{bmatrix} \begin{array}{c} \text{diagram with labels } c, x-t, a, b, y-t, b+d+y, e \end{array} + \sum_{s,t} \Psi_t \begin{bmatrix} b+d+y-c+s \\ e \\ x \\ y \end{bmatrix} \Phi_s \begin{bmatrix} c \\ b \\ d \end{bmatrix} \begin{array}{c} \text{diagram with labels } c, x-t, a, c-s, y-t, b+d+y, e \end{array}.$$

The target of $D_3[\zeta]\downarrow_4^{\leq}$ is not in $\text{Nf}(F_s^5\widehat{\mathbf{qS}})$, since we can further apply $D_3[\zeta]\downarrow_4^{\leq}$ to the first term on the right-hand side above, which yields a new rule in $F_s^5\widehat{\mathbf{qS}}$, denoted by ϑ_1 :

$$\begin{array}{c} \text{diagram with labels } c, x, a, b, y, b+d+y, e \end{array} \rightarrow \sum_s \Phi_s \begin{bmatrix} c \\ b \\ d \end{bmatrix} \begin{array}{c} \text{diagram with labels } c, x, a, c-s, y, b+d+y, e \end{array},$$

where $c \leq b$ and $d+y > x$.

By repeating the above construction, we can apply the context D_3 to the rule ϑ_{k-1} inductively, thereby obtaining further rules in $F_s^5\widehat{\mathbf{qS}}$, which we denote by ϑ_k :

$$\begin{array}{c} \text{diagram with labels } c, x, a, b, y, b+d+y, e \end{array} \dots k \dots \begin{array}{c} \text{diagram with labels } c, x, a, b, y, b+d+y, e \end{array} \rightarrow \sum_s \Phi_s \begin{bmatrix} c \\ b \\ d \end{bmatrix} \begin{array}{c} \text{diagram with labels } c, x, a, c-s, y, b+d+y, e \end{array} \dots k \dots \begin{array}{c} \text{diagram with labels } c, x, a, c-s, y, b+d+y, e \end{array},$$

where $c \leq b$ and the 2-cells in the above boxes cannot be reduced by ϑ_i for $i < k$.

As a result, $F_s^5\widehat{\mathbf{qS}}$ consists of the rules $\zeta_{a,b,c,d}$, θ_k , and ϑ_k , together with their closure under $F_s^4\widehat{\mathbf{qS}}$ -stable contexts. Note that the rules in $F_s^4\widehat{\mathbf{qS}}$ and $F_s^5\widehat{\mathbf{qS}}$ change the shapes and the labels of 2-cells in $\mathbf{qS}^\circ$, respectively. Hence, we call the rules in $F_s^4\widehat{\mathbf{qS}}$ the *shaping rules* and those in $F_s^5\widehat{\mathbf{qS}}$ the *labeling rules*. In Section 6, we give a pictorial description of the 2-cells in $\text{Nf}(F_s^5\widehat{\mathbf{qS}})$.

5.3.8. Lemma. *The extension $F_s^5\widehat{\mathbf{qS}}$ is confluent.*

Proof. Let (σ, τ) be a $F_s^5 \widehat{qS}$ -critical branching. Since $F_s^5 \widehat{qS}$ consists only of labeling rules, the normal forms $t\sigma \downarrow_s$ and $t\tau \downarrow_s$ have the same underlying diagrammatic shapes. It remains to compare their labels and scalar coefficients. For each 2-monomial appearing in either normal form, the labels outside the rewritten region remain unchanged, while those inside are determined by the original labels and the summation indices. For instance, a summand indexed by s in the target of $\zeta_{a,b,c,d}$ carries the labels $a-s$ and $c-s$. The rules θ_k and ϑ_k extend this local relabeling through the iterated contexts described above. We therefore collect terms with the same fully labeled diagram and compare their coefficients, assigning coefficient zero to a diagram absent from either expression. Substituting the explicit rule coefficients reduces these comparisons to scalar identities that can be checked symbolically using SageMath [44]. Hence $t\sigma \downarrow_s = t\tau \downarrow_s$, so every critical branching is confluent. $\square$

Consequently, each $F_s^i \widehat{qS}$ is left-monomial and confluent for $1 \leq i \leq 5$. By Theorem 4.1.9, we obtain the main result of this section.

5.3.9. Theorem. *The set of normal forms $\text{Nf}(\widehat{qS})$ is a hom-basis of the q -Schur category.*

The hom-basis $\text{Nf}(\widehat{qS})$ is called the *normal-form basis*, and in Procedure 6.1.6, we give an algorithm for computing it in each hom-space using picture calculus.

6. A PICTURE CALCULUS FOR THE NORMAL FORM BASIS

Given any two 1-cells $(\mathbf{a}, \mathbf{b})$ of the q -Schur category qS , as pointed out by Brundan [6], the set indexing the hom-basis of $qS(\mathbf{a}, \mathbf{b})$ admits several descriptions, from double coset diagrams to matrices of non-negative integers of whose row and column sums are the entries of $\mathbf{a}$ and of $\mathbf{b}$. Such a set of matrices will be denoted by $\text{Mat}(\mathbf{a}, \mathbf{b})$. For example, take $\mathbf{a} = (2, 5)$ and $\mathbf{b} = (3, 4)$ then there are 3 matrices in $\text{Mat}((2, 5), (3, 4))$:

$$A = \begin{bmatrix} 0 & 3 \\ 2 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 3 \end{bmatrix}, \quad C = \begin{bmatrix} 2 & 1 \\ 0 & 4 \end{bmatrix}.$$

These sets are also in bijection with the set of double cosets $S_{\mathbf{a}} \setminus S_N / S_{\mathbf{b}}$, where N is the sum of entries in $\mathbf{a}$ and $\mathbf{b}$ and $S_{\mathbf{a}}$ is the subgroup of the symmetric group S_n given by blocks: $S_{\mathbf{a}} = S_{a_1} \times \cdots \times S_{a_n}$, for $\mathbf{a} = (a_1, \dots, a_n)$.

In Subsection 6.1, we introduce diagrams that we call *pictures* and that slightly generalise double coset diagrams. Pictures allow to represent 2-cells in the q -Schur category, in a somehow easier way. We perform a rewriting procedure on the set of pictures. Then, in Subsection 6.2, we compare the normal form basis with three other bases of the q -Schur category.

6.1. Linear picture rewriting systems

For the normalization $F^* \widehat{qS}$ introduced in Section 5, its set of normal forms satisfies

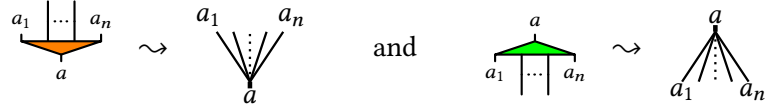
$$\text{Nf}(\widehat{qS}) = \text{Nf}(F^4 \widehat{qS}) \cap \text{Nf}(F_s^5 \widehat{qS}) = \text{Nf}(F^4 \widehat{qS}) \cap (qS_2^\circ \setminus [F_s^5 \widehat{qS}]) = \text{Nf}(F^4 \widehat{qS}) \setminus [F_s^5 \widehat{qS}],$$

where $[F_s^5 \widehat{qS}] := \{ C[s\alpha] \mid C \in \text{Cont}(qS_2^\circ) \text{ and } \alpha \in F_s^5 \widehat{qS} \}$. Therefore, computing the normal form basis of the q -Schur category reduces to two tasks:

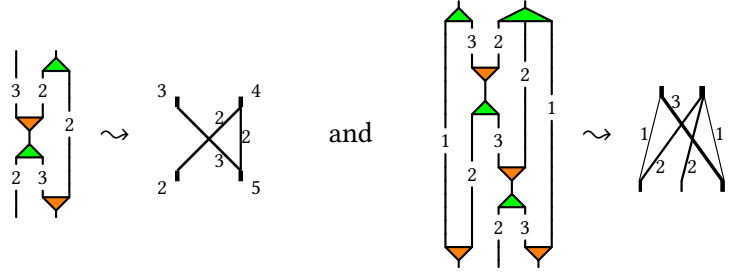
- i) computing all diagrams in $\text{Nf}(F^4\widehat{\text{qS}})$ with fixed source and target;
- ii) excluding diagrams that contain a subdiagram of the form $s\alpha$ for some $\alpha \in F_s^5\widehat{\text{qS}}$.

We now describe the normal form basis using pictures, giving a pictorial procedure for the normal forms.

6.1.1. Linear picture rewriting. We represent the 2-cells in qS_2° by using the following pictures respectively



From these two forms, every 2-cell of $\text{Nf}(F^3\widehat{\text{qS}})$ in §5.3.2 admits a unique picture representation. For instance,



The pictures corresponding to $\text{Nf}(F^4\widehat{\text{qS}})$ are called F_4 -pictures. They are those in $\text{Nf}(F^3\widehat{\text{qS}})$ that do not contain subpictures of the following form:



We denote by $P(\mathbf{a}; \mathbf{b})$ the set of all F^4 -pictures with source $\mathbf{a}$ and target $\mathbf{b}$, and by $P^\ell(\mathbf{a}; \mathbf{b})$ the set obtained by linearizing the pictures of $P(\mathbf{a}; \mathbf{b})$. Given a set R of oriented binary relations on $P^\ell(\mathbf{a}; \mathbf{b})$, we call a pair $(P^\ell(\mathbf{a}; \mathbf{b}), R)$ a *linear picture rewriting system*.

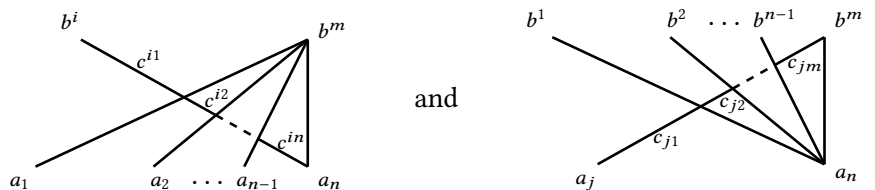
6.1.2. F^4 -picture procedure. We now present a procedure to compute all F^4 -pictures with source $\mathbf{a} = (a_1, \dots, a_n)$ and target $\mathbf{b} = (b^1, \dots, b^m)$. We define two families of *oriented segments*

$$(c^i : b^i \rightarrow a_n)_{1 \leq i \leq m} \quad \text{and} \quad (c_j : a_j \rightarrow b^m)_{1 \leq j \leq n},$$

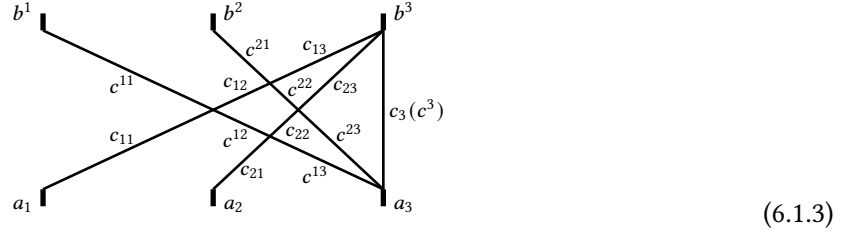
that decompose into two families oriented subsegments

$$c^i = (c^{ik})_{1 \leq k \leq n}, \quad \text{and} \quad c_j = (c_{jk})_{1 \leq k \leq m},$$

where $c^{i1}, c^{i2}, \dots, c^{in}$ (resp. $c_{j1}, c_{j2}, \dots, c_{jm}$) denote the oriented subsegments resulting by the crossing with the n (resp. m) oriented segments from a_k to b^m for $1 \leq k \leq n$ (resp. b^k to a_n for $1 \leq k \leq m$), as drawn on the following pictures



For instance, when $n = m = 3$, we have



We regard all labels c^i and c_j as natural numbers, and erase the corresponding segment if the label is 0. In order for them to define a pictures, the labels are required to satisfy the following relations:

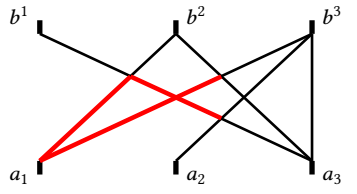
$$\begin{cases} c^m = c_n, & c^{i1} = b^i \ (1 \leq i \leq m-1), & c_{j1} = a_j \ (1 \leq j \leq n-1), \\ c_n + \sum_{i=1}^{n-1} c_{im} = b^m, & c^m + \sum_{i=1}^{m-1} c^{in} = a_n, \\ c^{ij} + c_{ji} = c^{i,j+1} + c_{j,i+1}, & (1 \leq i \leq m-1, 1 \leq j \leq n-1). \end{cases}$$

6.1.4. Lemma. *There is a one-to-one correspondence between the set of pictures defined in §6.1.2 and $P(\mathbf{a}; \mathbf{b})$.*

Proof. Set $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b^1, \dots, b^m)$. It is straightforward that all pictures defined in §6.1.2 are F^4 -pictures. Moreover, any additional connecting segment between b^i and a_j with $i < m$ and $j < n$ yields a subpicture of the form



and hence does not correspond to any picture in $P(\mathbf{a}; \mathbf{b})$. For example, if we connect the points a_1 and b_2 in (6.1.3), the resulting picture contains the subpicture highlighted in red below.



Therefore, the result follows. □

6.1.5. Collapsing rules. We depict the source and target of rules in $F_s^5 \widehat{\mathbf{qS}}$ as F^4 -pictures, and define the associated rewriting rules on P^ℓ :

$$\Gamma_1 : \begin{array}{c} \boxed{c} \\ \diagdown \quad \diagup \\ a \quad b+d \end{array} \rightarrow \sum_s \Phi_s \begin{array}{c} \left[\begin{array}{c} c \\ b \\ d \end{array} \right] \\ \diagdown \quad \diagup \\ a \quad b+d \end{array} \quad \text{where } c \leq b,$$

$$\begin{array}{c}
\Gamma^k : \text{Diagram 1} \rightarrow \sum_t \Phi_t \begin{bmatrix} c+m \\ y \\ b-y \end{bmatrix} \text{Diagram 2} \quad \text{where } c+m \leq y, \\
\Gamma_k : \text{Diagram 3} \rightarrow \sum_s \Phi_s \begin{bmatrix} c \\ b \\ d \end{bmatrix} \text{Diagram 4} \quad \text{where } c \leq b.
\end{array}$$

The diagrams are as follows:

- Diagram 1:** A sequence of crossings. The first crossing has a vertical line labeled $\boxed{m}$ and a diagonal line labeled x . The second crossing has a vertical line labeled d and a diagonal line labeled c . The third crossing has a vertical line labeled y and a diagonal line labeled n .
- Diagram 2:** A sequence of crossings. The first crossing has a vertical line labeled m and a diagonal line labeled $d+n-l$. The second crossing has a vertical line labeled d and a diagonal line labeled c . The third crossing has a vertical line labeled $c+m-l$ and a diagonal line labeled n .
- Diagram 3:** A sequence of crossings. The first crossing has a vertical line labeled $\boxed{c}$ and a diagonal line labeled a . The second crossing has a vertical line labeled $\boxed{b}$ and a diagonal line labeled a . The third crossing has a vertical line labeled a and a diagonal line labeled a .
- Diagram 4:** A sequence of crossings. The first crossing has a vertical line labeled c and a diagonal line labeled $a-s$. The second crossing has a vertical line labeled $c-s$ and a diagonal line labeled $a-s$. The third crossing has a vertical line labeled a and a diagonal line labeled a .

where k is the number of iterations. We define the set

$$\mathbf{Col} = \{C_1[\Gamma_1], C^k[\Gamma^k], C_k[\Gamma_k]\}$$

for all contexts C_1, C^k, C_k such that $C_1[s\Gamma_1], C^k[s\Gamma^k], C_k[s\Gamma_k]$ are F^4 -pictures. Since Γ_1, Γ^k , and Γ_k correspond to labeling rules in $F_s^5\widehat{qS}$, the pictures $C_1[t\Gamma_1], C^k[t\Gamma^k]$, and $C_k[t\Gamma_k]$ are also F^4 -pictures. Therefore, every rule in $\mathbf{Col}$ is a rewriting rule in P^ℓ .

We call the rules in $\mathbf{Col}$ the *collapsing rules*. It follows from Lemma 5.3.8 that the linear picture rewriting system $(P^\ell(a; b), \mathbf{Col})$ is convergent.

6.1.6. Normal form basis procedure. The following procedure computes the normal form basis of the q -Schur category in terms of pictures.

Input: Two tuples $\mathbf{a} = (a_1, \dots, a_n)$ and $\mathbf{b} = (b^1, \dots, b^m)$ with $a_1 + \dots + a_n = b^1 + \dots + b^m$.

Output: The normal form basis pictures with source $\mathbf{a}$ and target $\mathbf{b}$.

$T \leftarrow \emptyset$

foreach $p \in P(\mathbf{a}; \mathbf{b})$ **do**

if $p \in \mathbf{Nf}(\mathbf{Col})$ **then**

$T \leftarrow T \cup \{p\}$

else

$T \leftarrow T \cup \text{supp}(p \downarrow_{\mathbf{Col}})$

end

end

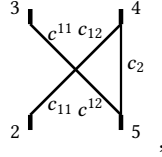
return T

When the tuples $\mathbf{a}$ and $\mathbf{b}$ are fixed, the set $P(\mathbf{a}; \mathbf{b})$ is finite by the F^4 -picture procedure in §6.1.2. Hence, Procedure (6.1.6) always terminates. By Theorem 5.3.9, we obtain the following result.

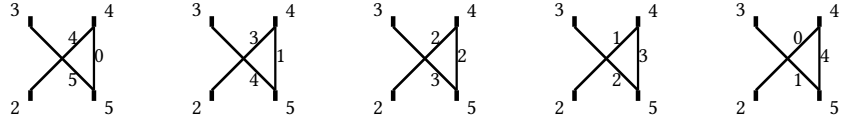
6.1.7. Proposition. For any tuples $\mathbf{a}$ and $\mathbf{b}$, Procedure (6.1.6) computes a linear basis of the hom-space $qS_2(\mathbf{a}, \mathbf{b})$.

6.2. Comparison of hom-bases for the q -Schur category

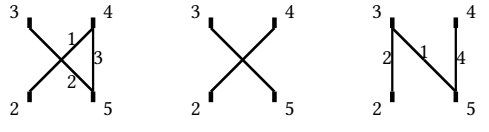
In this subsection, we compute four types of linear bases in hom-space $q\mathcal{S}((2, 5); (3, 4))$ and determine the transition matrices between them. Following F^4 -picture procedure, we list below the F_4 -pictures in $P((2, 5); (3, 4))$ together with their labels



where $c^{11} = 3$, $c_{11} = 2$, $c^{11} + c_{12} = c_{11} + c^{12}$, $c_{12} + c_2 = 4$, and $c^{12} + c_2 = 5$. By computation, we obtain the following solutions:



Among them, the following pictures belong to $\text{Nf}(\text{Col})$



which is the normal form basis in $q\mathcal{S}((2, 5); (3, 4))$.

These pictures correspond to diagrams in $\text{Nf}(F^3 q\mathcal{S})$. They satisfy the *collapsing relations* induced by the collapsing rules of §6.1.5, together with the following two relations:

$$\begin{array}{c} \text{diagram with strands } a, b \text{ and crossings } c, d \end{array} = \sum_s \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{array}{c} \text{diagram with strands } a, b \text{ and crossings } c, d \end{array} \quad \text{and} \quad \begin{array}{c} \text{diagram with strands } a_1, \dots, a_m \end{array} = \prod_{i=2}^m \begin{bmatrix} a_1 + \dots + a_i \\ a_i \end{bmatrix}_q \begin{array}{c} \text{diagram with strands } a_1 + \dots + a_m \end{array}.$$

These two relations correspond to the rules γ and $\gamma^\diamond$ in $F_s^4 q\mathcal{S}$ and $F_s^3 q\mathcal{S}$, respectively. Therefore, we can use collapsing relations to compute the transition matrices between the normal form basis and other bases of the q -Schur category.

6.2.1. Standard basis. In the very first definition of the q -Schur category $q\mathcal{S}$ given by Brundan [6, Theorem 5.1], the hom-spaces are equipped with a *standard basis*, in the spirit of Iwahori-Hecke algebras. More precisely, given $\mathbf{a}$ and $\mathbf{b}$ two 1-cells, $q\mathcal{S}(\mathbf{a}, \mathbf{b})$ is the free $\mathbb{Z}[q, q^{-1}]$ -module with basis (ξ_A) , where A runs over the set $\text{Mat}(\mathbf{a}, \mathbf{b})$.

In the example $(\mathbf{a}, \mathbf{b}) = ((2, 5); (3, 4))$, for the standard basis, we get the three pictures

$$\xi_A = \begin{array}{c} \text{diagram with strands } 3, 4, 2, 5 \end{array}, \quad \xi_B = \begin{array}{c} \text{diagram with strands } 3, 4, 2, 5 \end{array}, \quad \xi_C = \begin{array}{c} \text{diagram with strands } 3, 4, 2, 5 \end{array}.$$

The transition matrix is given by:

$$\left(\begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array} , \begin{array}{c} \text{3} \quad \text{4} \\ \diagup \quad \diagdown \\ \text{2} \quad \text{5} \end{array} , \begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array} \right) = \left(\begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array} , \begin{array}{c} \text{3} \quad \text{4} \\ \diagup \quad \diagdown \\ \text{2} \quad \text{5} \end{array} , \begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array} \right) \begin{bmatrix} \Phi_1 u & 1 & 0 \\ \Phi_{-2} u & 0 & 0 \\ \Phi_2 u & \begin{bmatrix} 4 \\ 3 \end{bmatrix}_q \Psi_1 v & 1 \end{bmatrix},$$

where $u = [3 \ 3 \ 2]^\top$ and $v = [2 \ 2 \ 2 \ 1]^\top$.

6.2.2. Canonical basis. Given two 1-cells (a, b) in the q -Schur category, there is only one element of maximal length d_A^+ in the double coset associated to A in $S_a \setminus S_N / S_b$ (here the length ℓ is in terms of the Coxeter generators of the symmetric group).

The *canonical basis* of the hom-space $q\mathcal{S}(a, b)$ is given as a linear combination in the standard basis as such

$$\Theta_A = \sum_{B \in \text{Mat}(a, b)} P_{A, B}(q) \xi_B$$

with $P_{A, B}$ is the polynomial defined by

$$P_{A, B} = q^{\ell(d_A^+) - \ell(d_B^+)} P_{d_A^+, d_B^+}(q^{-2}),$$

where $P_{d_A^+, d_B^+}$ is the Kazhdan-Lusztig polynomial associated to the two permutations d_A^+ and d_B^+ .

In the case $((2, 5), (3, 4))$, the computations of the 3 maximal length elements in the 3 double cosets associated to A, B and C give

$$\begin{aligned} d_A^+ &= s_1 s_2 s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_1 s_5 s_4 s_3 s_2 s_1 s_6 s_5 s_4 s_3 s_2 s_1, \\ d_B^+ &= s_1 s_2 s_1 s_4 s_3 s_6 s_5 s_4 s_3 s_5 s_6 s_5 s_2 s_3 s_1 s_4 s_3 s_5, \\ d_C^+ &= s_1 s_2 s_1 s_4 s_5 s_4 s_6 s_3 s_4 s_5 s_4 s_3 s_6. \end{aligned}$$

of respective lengths $\ell(d_A^+) = 21$ (the maximal one), $\ell(d_B^+) = 18$ and $\ell(d_C^+) = 13$. Then we get the Kazhdan-Lusztig polynomials,

$$P_{A, B} = q^3(1), \quad P_{A, C} = q^8(1), \quad P_{B, C} = q^5(1 + q^{-2}),$$

using the computer algebraic system SageMath [44] for the computation of the last one. Therefore, the canonical basis reads as follows

$$\begin{aligned} \Theta_A &= \xi_A + q^3 \xi_B + q^8 \xi_C, \\ \Theta_B &= \xi_B + (q^5 + q^3) \xi_C, \\ \Theta_C &= \xi_C \end{aligned}$$

and the transition matrix with the standard basis is upper triangular.

6.2.3. Codeterminant basis. In [6], Brundan also introduces a third basis, called the *codeterminant basis*. Its definition uses yet another description of the double cosets in $S_a \setminus S_N / S_b$ that we do not recall here. But in the case $((2, 5), (3, 4))$, the computations of pictures associated to the codeterminant basis give

$$\begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array} \quad \begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array} \quad \begin{array}{c} \text{3} \quad \text{4} \\ \diagdown \quad \diagup \\ \text{2} \quad \text{5} \end{array}.$$

And the transition matrix is given by

$$\left(\begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \begin{array}{c} \text{1} \\ \diagup \quad \diagdown \\ \text{2} \end{array} \begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{4} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \right), \left(\begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{4} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \right), \left(\begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{4} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \right) = \left(\begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \begin{array}{c} \text{1} \\ \diagup \quad \diagdown \\ \text{2} \end{array} \begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{4} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \right), \left(\begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{4} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \right), \left(\begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{2} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \begin{array}{c} \text{1} \\ \diagup \quad \diagdown \\ \text{2} \end{array} \begin{array}{c} \text{2} \\ \diagup \quad \diagdown \\ \text{3} \end{array} \begin{array}{c} \text{3} \\ \diagdown \quad \diagup \\ \text{4} \end{array} \begin{array}{c} \text{4} \\ \diagup \quad \diagdown \\ \text{5} \end{array} \right) \begin{bmatrix} \Phi_1 u & 0 & \Phi_1 v \\ \Phi_{-2} u & 1 & \Phi_{-2} v \\ \Phi_2 u & 0 & \Phi_2 v \end{bmatrix},$$

where $u = [3 \ 4 \ 1]^\top$ and $v = [3 \ 3 \ 2]^\top$.

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Use of computer systems. We used the computer algebra system SageMath [44] to compute q -binomial coefficients arising in the confluence proofs. Artificial intelligence tools were used for proofreading, literature searches, and assistance with the preparation of formal code and diagrams. They were not used to produce or establish any mathematical results. The authors adhere to the *Leiden Declaration on Artificial Intelligence and Mathematics*. Diagrams were typeset using Catex [19].

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A. APPENDICES

A.1. Normalization of the contexts C_1 , C_2 , and C_3

We provide a detailed verification of $C_1[\partial\gamma]\downarrow_3 = 0$ and $C_2[\partial\gamma]\downarrow_3 = 0$, and show that the rule $C_3[\gamma]\downarrow_3 \Leftarrow$ is trivial in §5.3.3.

i) We have

$$C_1[\gamma_{a,b,c,d}] : \begin{array}{c} \text{diagram with two vertical lines, top nodes } c, d, \text{ bottom nodes } a, b, \text{ with a crossing} \end{array} \rightarrow \sum_s \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{array}{c} \text{diagram with two vertical lines, top nodes } d-s, c-s, \text{ bottom nodes } a, b, \text{ with a crossing} \end{array},$$

where $C_1[s\gamma_{a,b,c,d}]\downarrow_3 = \begin{bmatrix} b+d \\ b+d-c \end{bmatrix}_q \begin{bmatrix} a \\ d \end{bmatrix}_q \begin{array}{c} \text{diagram with two vertical lines, top nodes } a, b \end{array}$ and

$$C_1[t\gamma_{a,b,c,d}]\downarrow_3 = \sum_s \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} a+c-s \\ d-s \end{bmatrix}_q \begin{bmatrix} b \\ b-c+s \end{bmatrix}_q \begin{array}{c} \text{diagram with two vertical lines, top nodes } a, b \end{array}.$$

A direct computation using the computer algebraic system SageMath [44] yields

$$\begin{bmatrix} b+d \\ b+d-c \end{bmatrix}_q \begin{bmatrix} a \\ d \end{bmatrix}_q = \sum_s \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{bmatrix} a+c-s \\ d-s \end{bmatrix}_q \begin{bmatrix} b \\ b-c+s \end{bmatrix}_q.$$

Consequently, one has $C_1[\partial\gamma_{a,b,c,d}]\downarrow_3 = 0$. Alternatively, if they were not equal, then in the q -Schur category we would have the 1-generators $\begin{array}{c} \text{diagram with two vertical lines, top nodes } a, b \end{array} = 0$ for all $a, b > 0$, which is a clear contradiction. Similarly, one proves that $C_2[\partial\gamma_{a,b,c,d}]\downarrow_3 = 0$.

ii) Applying the context $C_3 = \begin{array}{c} \text{diagram with a box containing a square, top node } e, \text{ bottom node } \end{array}$ to $\gamma_{a,b,c,d}$, we obtain

$$\begin{array}{c} \text{diagram with two vertical lines, top nodes } c, d, \text{ bottom nodes } a+e, b, \text{ with a crossing} \end{array} \xrightarrow{C_3[\gamma_{a,b,c,d}]\downarrow_3 \Leftarrow} \sum_s \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \begin{array}{c} \text{diagram with two vertical lines, top nodes } d-s, c-s, \text{ bottom nodes } e, b, \text{ with a crossing} \end{array} \rightarrow \sum_{s,t} \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Psi_t \begin{bmatrix} a+e \\ c-s \\ a+c-d \\ a \end{bmatrix} \begin{array}{c} \text{diagram with two vertical lines, top nodes } a-t, c-s, \text{ bottom nodes } a+e, b+c-s, \text{ with a crossing} \end{array}$$

$$\rightarrow \sum_{s,t} \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Psi_t \begin{bmatrix} a+e \\ c-s \\ a+c-d \\ a \end{bmatrix} \begin{bmatrix} b+d-a-c+t \\ b-c+s \end{bmatrix}_q \begin{array}{c} \text{diagram with two strands } a+e \text{ and } b, \text{ with crossings and labels } a-t, a+c-d-t \end{array}.$$

Using the computer algebraic system SageMath [44], one can check

$$\Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \Psi_t \begin{bmatrix} a+e \\ c-s \\ a+c-d \\ a \end{bmatrix} \begin{bmatrix} b+d-a-c+t \\ b-c+s \end{bmatrix}_q = \Psi_r \begin{bmatrix} a+e \\ b \\ c \\ d \end{bmatrix}$$

for all $r = d + t - a$. Since

$$\gamma_{a+e,b,c,d} : \begin{array}{c} \text{diagram with strands } a+e \text{ and } b, \text{ with crossings and labels } c, d \end{array} \rightarrow \sum_r \Psi_r \begin{bmatrix} a+e \\ b \\ c \\ d \end{bmatrix} \begin{array}{c} \text{diagram with strands } a+e \text{ and } b, \text{ with crossings and labels } d-r, c-r \end{array}.$$

belongs to $F_s^4 \mathbf{qS}$, it follows that $C_3[\gamma_{a,b,c,d}] \downarrow_3 \leq$ is trivial.

A.2. Confluence of the extension $F_s^4 \widehat{\mathbf{qS}}$

In this subsection, we prove the confluence diagrams for two families of critical branchings listed in Lemma 5.3.4.

i) The confluent diagram of critical branching $\left(\gamma \begin{bmatrix} b+d \\ f \\ b+d-c+f-g \\ b+d-c \end{bmatrix}, \gamma \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \right) :$

$$\begin{array}{c} \begin{array}{c} \text{diagram with strands } a, b, f, \text{ and } g, \text{ with crossings and labels } c, d \end{array} \longrightarrow \sum_s \Psi_s \begin{array}{c} \text{diagram with strands } a, b, f, \text{ and } g, \text{ with crossings and labels } d, \text{ and } g-b-d+c+s \end{array} \longrightarrow \sum_{s,t} \Psi_t \Psi_s \begin{array}{c} \text{diagram with strands } a, b, f, \text{ and } g, \text{ with crossings and labels } d-t, \text{ and } g-b-d+c+s \end{array} \\ \downarrow \\ \sum_n \Psi_n \begin{array}{c} \text{diagram with strands } a, b, f, \text{ and } g, \text{ with crossings and labels } d-n, \text{ and } c-n \end{array} \longrightarrow \sum_{n,m} \Psi_m \Psi_n \begin{array}{c} \text{diagram with strands } a, b, f, \text{ and } g, \text{ with crossings and labels } d-n-m, \text{ and } c-n \end{array} \longrightarrow \sum_{n,m,l} \Psi_l \Psi_m \Psi_n \begin{array}{c} \text{diagram with strands } a, b, f, \text{ and } g, \text{ with crossings and labels } d-n-m, \text{ and } g-b-d+c+m+l \end{array} \end{array}$$

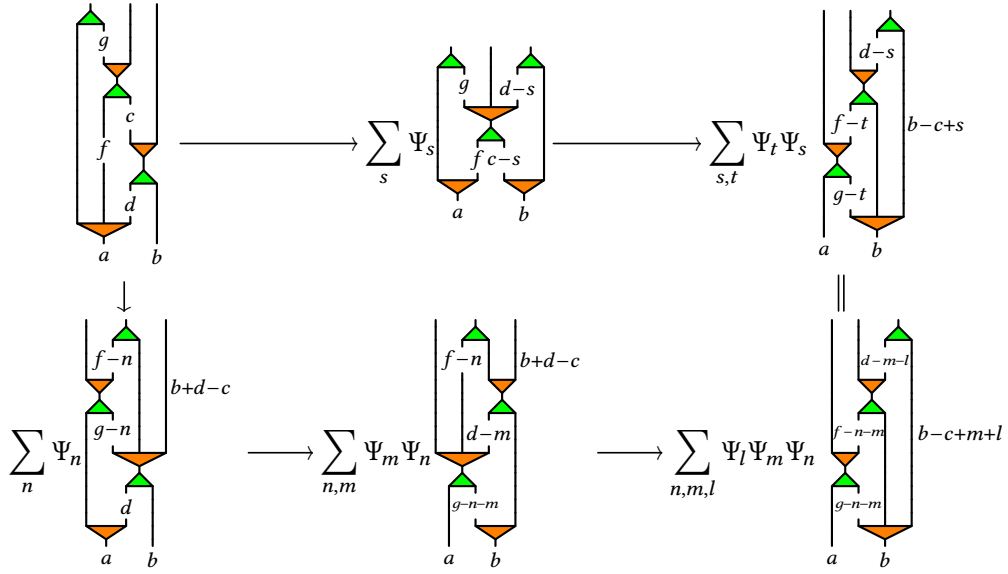
where

$$\Psi_s := \Psi_s \begin{bmatrix} b+d \\ f \\ b+d-c+f-g \\ b+d-c \end{bmatrix}, \quad \Psi_t := \Psi_t \begin{bmatrix} a \\ f-g+2b+d-c-s \\ f-g+b+d \\ d \end{bmatrix},$$

$$\Psi_n := \Psi_n \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad \Psi_m := \Psi_m \begin{bmatrix} a+c-n \\ b-c+n+f \\ b-c+f+d-g \\ d-n \end{bmatrix}, \quad \Psi_l := \Psi_l \begin{bmatrix} b \\ f \\ f+b+d-c-g-m \\ b-c+n \end{bmatrix}.$$

Set $t = n + m$ and $s = m + l$. With this choice, the last two diagrams above have the same shape. Using the computer algebraic system SageMath [44], one can verify that $\Psi_t \Psi_s = \Psi_l \Psi_m \Psi_n$, for all $t = n + m$ and $s = m + l$. Therefore, this family of critical branchings is confluent.

ii) The confluent diagram of critical branching $\left(\gamma \begin{bmatrix} f+d \\ b \\ c \\ d \end{bmatrix}, \gamma \begin{bmatrix} a-d \\ c \\ g \\ f \end{bmatrix} \right)$:



where

$$\Psi_s := \Psi_s \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}, \quad \Psi_t := \Psi_t \begin{bmatrix} a \\ c-s \\ g \\ f \end{bmatrix},$$

$$\Psi_n := \Psi_n \begin{bmatrix} a-d \\ c \\ g \\ f \end{bmatrix}, \quad \Psi_m := \Psi_m \begin{bmatrix} a \\ b \\ g-n \\ d \end{bmatrix}, \quad \Psi_l := \Psi_l \begin{bmatrix} a+g-n-m \\ b-g+n+m \\ c+n-g \\ d-m \end{bmatrix}.$$

Set $t = n + m$ and $s = m + l$. With this choice, the last two diagrams above have the same shape. Using the computer algebraic system SageMath [44], we prove that $\Psi_t \Psi_s = \Psi_l \Psi_m \Psi_n$, for all $t = n + m$ and $s = m + l$. Therefore, this family of critical branchings is confluent.