Lectures on
Algebraic Rewriting

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Abstract Rewriting

The principle of rewriting comes from combinatorial algebra. It was introduced by Thue when he considered systems of transformation rules on combinatorial objects such as strings, trees or graphs in order to solve the word problem, [Thu14]. Given a collection of objects and a system of transformation rules on these objects, the word problem is

INSTANCE: given two objects,

QUESTION: can one of these objects be transformed to the other by means of a finite number of applications of the transformation rules?

Dehn described the word problem for finitely presented groups, [Deh10] and Thue studied this problems for strings, which correspond to the word problem for finitely presented monoids, [Thu14]. Note that it was only much later, that the problem was shown to be undecidable, independently by Post [Pos47] and Markov [Mar47a, Mar47b]. Afterwards, the word problem have been considered in many contexts in algebra and in computer science.

Far beyond the precursor works on this decidability problem on strings, rewriting theory has been mainly developed in theoretical computer science, producing numerous variants corresponding to different syntaxes of the formulas being transformed: strings in a monoid, [BO93, GM18], paths in a graph,
1.1. Abstract Rewriting Systems

terms in an algebraic theory, \[\text{[BN98, Klo92, Ter03]}\], terms modulo, \(\lambda\)-terms, trees, Boolean circuits, \[\text{[Laf03]}\], graph grammars, etc. Rewriting appears also on various forms in algebra, for commutative algebras, \[\text{[Buc65, Buc87]}\], Lie algebras, \[\text{[Shi62]}\], with the notion of Gröbner-Shirshov bases, or associative algebras, \[\text{[Bok76, Ber78, Mor94, Ufn95, GHM19]}\] and operads, \[\text{[DK10]}\], as well as on topological objects, such as Reidemeister moves, knots or braids, \[\text{[Bur01]}\], or in higher-dimensional categories, \[\text{[GM09, GMT2a, Mim10, Mim14]}\].

Many of the basic definitions and fundamental properties of these forms of rewriting can be stated on the most abstract version of rewriting that is given by a binary relation on set. In this chapter, we present the notion of abstract rewriting system and the main abstract rewriting properties used in these lectures. We refer the reader to \[\text{[BN98, Klo92, Ter03]}\] for a complete account on the abstract rewriting theory.

1.1. ABSTRACT REWRITING SYSTEMS

1.1.1. Abstract Rewriting Systems. An abstract rewriting system, ARS for short, is a data \((A, \rightarrow_1)\) made of a set \(A\) and a sequence \(\rightarrow_1\) of binary relations on \(A\) indexed by a set \(I\), that is,

\[\rightarrow_1 = (\rightarrow_\alpha)_{\alpha \in I}, \quad \text{and} \quad \rightarrow_\alpha \subseteq A \times A.\]

The relation is called reduction or rewrite relation on \(A\). An element \((a, b)\) in the relation \(\rightarrow\) will be denoted by \(a \rightarrow b\), and we said that \(b\) is a one-step reduct of \(a\), and that \(a\) is a one-step expansion of \(b\). An element of \(\rightarrow\) is called a reduction step. In most cases the elements of \(A\) have a syntactic or graphical nature (string, term, tree, graph, polynomial...). We will denoted by \(\equiv\) the syntactical or graphical identity.

1.1.2. Reduction sequence. A reduction sequence, or rewriting sequence, with respect to a reduction relation \(\rightarrow\) is a finite or infinite sequence of reduction steps

\[a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \ldots\]

If we have a reduction sequence

\[a \equiv a_0 \rightarrow a_1 \rightarrow a_2 \rightarrow \ldots \rightarrow a_{n-1} \rightarrow a_n \equiv b\]

we say that \(a\) reduces to \(b\). The length of a finite reduction sequence is the number of its reduction steps.

1.1.3. Composition. Given two reduction relations \(\rightarrow_1\) and \(\rightarrow_2\) on \(A\), their composition is denoted by \(\rightarrow_1 \cdot \rightarrow_2\) and defined by

\[a \rightarrow_1 \cdot \rightarrow_2 b \quad \text{if} \quad a \rightarrow_1 c \rightarrow_2 b, \quad \text{for some} \ c \in A.\]

1.1.4. Notations. The identity relation is denoted by

\[\overset{0}{\rightarrow} = \{(a, a) \mid a \in A\}.\]
1.1.5. Branchings and confluence pairs

The inverse relation of \( \rightarrow \) is denoted by \( \leftarrow \), or by \( \rightarrow^{-1} \), and defined by:

\[
\leftarrow = \{(b, a) \mid a \rightarrow b\}.
\]

A relation is reflexive if \( \rightarrow \subseteq \rightarrow \) and transitive if \( \rightarrow \cdot \rightarrow \subseteq \rightarrow \). The reflexive closure of \( \rightarrow \) is denoted by \( \equiv \rightarrow \) and defined by

\[
\equiv = \rightarrow \cup \rightarrow^{-1}.
\]

The symmetric closure of \( \rightarrow \) is denoted by \( \leftrightarrow \) and defined by

\[
\leftrightarrow = \rightarrow \cup \leftarrow.
\]

The transitive closure of \( \rightarrow \) is denoted by \( \rightarrow^{+} \) and defined by

\[
\rightarrow^{+} \subseteq \bigcup_{i>0} \rightarrow^{i},
\]

where \( \rightarrow^{i} = \rightarrow^{i-1} \cdot \rightarrow \) for all \( i > 0 \). The reflexive and transitive closure of \( \rightarrow \) is denoted by \( \rightarrow^{\ast} \), or by \( \rightarrow^{\ast} \), and defined by

\[
\rightarrow^{\ast} = \rightarrow^{+} \cup \rightarrow^{-1}.
\]

The reflexive, transitive and symmetric closure of \( \rightarrow \) is denoted by \( \leftrightarrow^{\ast} \) and defined by

\[
\leftrightarrow^{\ast} = \left(\rightarrow^{\ast}\right)^{\ast}.
\]

In particular, we have \( a \rightarrow b \) is there is a rewriting sequence from \( a \) to \( b \) and we have \( a \leftrightarrow^{\ast} b \) if and only if there is a zig-zag of rewriting sequence from \( a \) to \( b \):

\[
a \equiv a_0 \leftrightarrow a_1 \leftrightarrow a_2 \leftrightarrow \ldots \leftrightarrow a_{n-1} \leftrightarrow a_n \equiv b.
\]

The relation \( \leftrightarrow^{\ast} \) is equal to the equivalence relation generated by \( \rightarrow \).

1.1.5. Branchings and confluence pairs. A branching (resp. local branching) of the relation \( \rightarrow \) is an element of the composition \( \rightarrow \cdot \rightarrow \) (resp. \( \leftarrow \cdot \rightarrow \)). It is defined by a triple \( a \leftarrow c \rightarrow b \) (resp. \( a \leftarrow c \rightarrow b \)) as pictured by the following diagram:

\[
\begin{array}{c}
\text{(resp.} \quad \begin{array}{c}
\text{c} \\
\text{c}
\end{array} \\
\text{b} \\
\text{b}
\end{array}
\end{array}
\]

A confluence pair (resp. local confluence pair) of the relation \( \rightarrow \) is an element of the composition \( \rightarrow \cdot \leftarrow \) (resp. \( \leftarrow \cdot \rightarrow \)). It is defined by a triple \( a \rightarrow d \leftarrow b \) as pictured by the following diagram:

\[
\begin{array}{c}
\text{(resp.} \quad \begin{array}{c}
\text{a} \\
\text{a}
\end{array} \\
\text{a} \\
\text{b}
\end{array}
\end{array}
\]

Note that the relations \( \leftarrow \cdot \rightarrow \) and \( \equiv \cdot \rightarrow \) are symmetric.
1.2. Confluence

1.1.6. Commutation. Two relations $\rightarrow_1$ and $\rightarrow_2$ on $A$ commute if

$$1 \closehat{\rightarrow_1 \subseteq \rightarrow_2 \cdot \rightarrow_2, \cdot \rightarrow_2 \subseteq \rightarrow_1 \cdot \rightarrow_1}.$$ 

1.2. CONFLUENCE

1.2.1. Diamond property. A relation $\rightarrow$ has the diamond property if it commutes with itself. This means that for any local branching $a \leftarrow c \rightarrow b$ there exists a local confluence:

```
      c
      ↙  ↙
    a   b
  d ↘  ↘
```

This property is hard to obtain in general. Let us give the main confluence patterns used in rewriting.

1.2.2. Confluence patterns. A reduction relation $\rightarrow$ is called

i) Church-Rosser if $\leftrightarrow \subseteq \rightarrow \cdot \leftarrow$.

ii) confluent if the relation $\rightarrow$ commutes, that is $\leftrightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$.

iii) semi-confluent if $\leftrightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$.

iv) strongly-confluent if $\leftrightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \equiv$.

v) locally confluent if $\leftrightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$.

vi) has the diamond property if the relation $\rightarrow$ commutes, that is $\leftrightarrow \cdot \rightarrow \subseteq \rightarrow \cdot \leftarrow$. 

```
      *  ↘
    d \ b  ↘
```

i) 

```
      ↘  ↘
    a   b
  d ↘  ↘
```

ii) 

```
      ↘  ↘
    a   b
  d ↘  ↘
```

iii) 

```
      ↘  ↘
    a   b
  d ↘  ↘
```

iv) 

```
      ↘  ↘
    a   b
  d ↘  ↘
```

v) 

```
      ↘  ↘
    a   b
  d ↘  ↘
```

vi)
1.2.3. Remark. The diamond property implies the Church-Rosser property, \([\text{New}42, \text{Theorem 1}]\). Note that in \([\text{New}42]\) Newman called confluence the Church-Rosser property defined above. He showed that these properties coincide. Obviously, any Church-Rosser property is confluent, and the reverse implication is shown by the following diagram:

![Diagram](image-url)

1.2.4. Proposition. For an abstract rewriting system \((A, \rightarrow)\) the following conditions are equivalent

i) \(\rightarrow\) is confluent,

ii) \(\rightarrow\) is semi-confluent,

iii) \(\rightarrow\) has the Church-Rosser property.

Proof. Prove that iii) implies i). Suppose that \(\rightarrow\) is Church-Rosser. Given a branching \(a \leftrightarrow c \rightarrow b\), we have \(a \leftrightarrow b\). Hence by the Church-Rosser property, there is a confluence pair \(a \rightarrow d \leftarrow b\), hence \(\rightarrow\) is confluent. Obviously i) implies ii). Prove that ii) implies iii). Suppose that \(\rightarrow\) is semi-confluent and let \(a \rightarrow b\). Prove by induction on the length of the sequence of reductions between \(a\) and \(b\), that there is a confluence pair \(a \rightarrow d \leftarrow b\). This is obvious when the sequence is of length 0, that is \(a \equiv b\), or when the sequence is of length 1, that is \(a \rightarrow b\) or \(a \leftarrow b\). Let consider a sequence of reductions \(a \rightarrow^{n-1} b' \leftarrow^1 b\). By induction hypothesis, there is a confluence pair \(a \rightarrow d \leftarrow b'\). If \(b \rightarrow b'\), that is

\[
\begin{array}{cccccc}
a & \rightarrow^{n-1} & b' & \leftarrow^1 & b \\
\text{Induction} & & & & \\
& d &
\end{array}
\]

by induction, this gives a confluence pair \(a \rightarrow d \leftarrow b\). In the other case, if \(b' \rightarrow b\), by semi-confluence,
1.3. Normalisation

there is a confluence pair \( d \rightarrow d' \leftarrow b \):

\[
\begin{array}{c}
\text{a} \\
\downarrow n-1 \\
\downarrow \text{Induction} \\
\downarrow d \\
\downarrow d' \\
\uparrow \text{Semi-Confluence} \\
\uparrow b' \\
\uparrow b \\
\end{array}
\]

hence, by induction we have a confluence pair \( a \rightarrow d' \leftarrow b \). It follows that the relation \( \rightarrow \) is Church-Rosser.

\( \square \)

1.2.5. Exercise. Prove that strong confluence implies confluence.

1.2.6. Exercise. Let \( A \) be a set and let \( \rightarrow_1 \) and \( \rightarrow_2 \) be two reduction relations on \( A \).
1. Prove that the confluence of \( \rightarrow_1 \) and \( \rightarrow_2 \) does not imply the confluence of \( \rightarrow_1 \cup \rightarrow_2 \).
2. Prove that
\[
\rightarrow_1 \subseteq \rightarrow_2 \subseteq \rightarrow_1 \quad \text{implies} \quad \rightarrow_1 = \rightarrow_2 .
\]
3. Prove that if \( \rightarrow_1 \subseteq \rightarrow_2 \subseteq \rightarrow_1 \) and \( \rightarrow_2 \) is strongly confluent, then \( \rightarrow_1 \) is confluent.
4. Prove that if \( \rightarrow_1 \) and \( \rightarrow_2 \) are confluent and commute, then the relation \( \rightarrow_1 \cup \rightarrow_2 \) is also confluent.

1.3. NORMALISATION

Let \( (A, \rightarrow) \) be an abstract rewriting system.

1.3.1. Normal form. An element \( a \) in \( A \) is in normal form, or irreducible, with respect to \( \rightarrow \) if there is no \( b \) in \( A \) such that \( a \rightarrow b \). It is reducible if it is no irreducible. We denote by \( \text{NF}(\rightarrow) \) the set of normal forms in \( A \) with respect to \( \rightarrow \).

1.3.2. Normalizing. An element \( a \) in \( A \) is (weakly) normalizing if \( a \rightarrow b \) for some \( b \) in \( \text{NF}(\rightarrow) \). Then we say that \( a \) has a normal form \( b \) and \( b \) is called a normal form of \( a \). The relation \( \rightarrow \) is (weakly) normalizing if every element \( a \) in \( A \) is normalizing.

1.3.3. Termination. An element \( a \) in \( A \) is strongly normalizing if every reduction sequence starting from \( a \) is finite. The relation \( \rightarrow \) is strongly normalizing, or terminating, or noetherian if every \( a \) in \( A \) is
1.3.4. Convergence. We say that $\rightarrow$ is convergent, or complete, canonical, uniquely terminating, if $\rightarrow$ is confluent and terminating.

1.3.5. Normal form property. The relation $\rightarrow$ has the normal form property if for any $a$ in $A$ and any normal form $b$ in $A$

$$a \leftrightarrow b \quad \text{implies} \quad a \rightarrow b.$$ 

The relation $\rightarrow$ has the unique normal form property if for all normal forms $a$ and $b$ in $A$

$$a \leftrightarrow b \quad \text{implies} \quad a \equiv b.$$ 

1.3.6. Semi-convergence. We say that $\rightarrow$ is semi-convergent, or semi-complete, if $\rightarrow$ has the unique normal form property and is normalizing. If $(A, \rightarrow)$ is semi-convergent, then every element $a$ in $A$ reduces to a unique normal form denoted by $\hat{a}$.

1.3.7. Confluence and unicity of the normal form. If $\rightarrow$ is confluent, every element has at most one normal form. As an immediate consequence of the equivalence of the Church-Rosser property and the confluence property, Proposition 1.2.4, we have

1.3.8. Theorem. For an abstract rewriting system $(A, \rightarrow)$ the following implications hold:

i) The normal form property implies the unique normal form property.

ii) If $\rightarrow$ is confluent then $\rightarrow$ has the normal form property.

iii) If $\rightarrow$ is semi-convergent then it is confluent.

For a confluent abstract rewriting system $(A, \rightarrow)$, two elements $a$ and $b$ in $A$ are equivalent if there are joignable: $a \leftarrow \cdots \rightarrow b$. The test of joignability may be not possible when the relation is not terminating. For example, how to test the joignability of $-n$ and $n$ in the following example:

$$\cdots \leftarrow -2 \leftarrow -1 \leftarrow 0 \rightarrow 1 \rightarrow 2 \rightarrow \cdots$$

Let us show that normalisation suffices to determine joignability. If $\rightarrow$ is normalizing and confluent, every element $a$ in $A$ has a unique normal form denoted by $\hat{a}$.

1.3.9. Theorem. If $\rightarrow$ is normalizing and confluent then we have

$$a \overset{*}{\rightarrow} b \quad \text{if and only if} \quad \hat{a} \equiv \hat{b}.$$ 

As a consequence of the previous result, for a normalizing and confluent abstract rewriting system $(A, \rightarrow)$ an equivalence test of two elements $a$ and $b$ in $A$ is to check the syntactical equality of their normal forms $\hat{a}$ and $\hat{b}$. If the normal forms are computable and the syntactic identity is decidable then the equivalence is decidable.
1.3. Normalisation

1.3.10. Exercise. Prove Theorem 1.3.8 and Theorem 1.3.9.

1.3.11. Examples. The abstract rewriting system

\[
\begin{array}{ccc}
  a & \rightarrow & a' \\
  b & \leftarrow & \leftarrow
\end{array}
\]

is confluent, not terminating and admits a unique normal form. The abstract rewriting system

\[
\begin{array}{ccc}
  c & \leftarrow & b \\
  & \rightarrow & a \\
  & \rightarrow & a' \\
  \end{array}
\]

is not confluent, not terminating and admits a unique normal form.

1.3.12. Example. Let \( A = \mathbb{N} - \{0, 1\} \). Consider the relation on \( A \) defined by the

\[
\{(m, n) \mid m > n \text{ and } n \text{ divides } m \}.
\]

Then \( m \) is in \( \text{NF}(\rightarrow) \) if and only if \( m \) is prime. An element \( p \) is a normal form of \( m \) if and only if \( p \) is a prime factor of \( m \). We have \( m \rightarrow \cdot \leftarrow n \) if and only if \( m \) and \( n \) are not relatively prime. The transitive closure of \( \rightarrow \) coincide with \( \rightarrow \) because \( > \) and divide relations are already transitive. We have \( \leftrightarrow = A \times A \) and \( \rightarrow \) terminates and it is not confluent.

1.3.13. Example. Let \( A = \{a, b\}^* \) be the free monoid on \( \{a, b\} \). We consider the relation \( \rightarrow \) defined by the set

\[
\{(uabv, uabv) \mid u, v \in A\}.
\]

Then an element of \( A \) is in normal form if and only if it is of the form \( a^n b^m \) for \( n, m \in \mathbb{N} \). Not that the relation \( \rightarrow \) terminates and its confluent. Thus every element of \( A \) has a unique normal form and we have \( w \rightarrow \cdot \leftarrow w' \) if and only if \( w \leftrightarrow w' \) if and only if \( w \) and \( w' \) contain the same number of as and bs. We will see in the next chapter on string rewriting that such an abstract rewriting system can be specified by only one rewriting rule \( ba \rightarrow ab \).

1.3.14. Exercise, [Jan88]. Consider the set \( \mathbb{N} \times \mathbb{N} \) with the reduction relation \( \rightarrow_1 \) defined by \( (x, y) \rightarrow_1 (x', y') \) if

\[
\left( (x' = x - 2 \text{ and } y' = y + 1) \right) \text{ or } \left( (x' = x + 2 \text{ and } y' = y - 1) \right).
\]

1. Show that \( \rightarrow_1 \) is terminating.
2. Show that \( \rightarrow_1 \) is not confluent.
3. Define a reduction relation \( \rightarrow_2 \) on \( \mathbb{N} \times \mathbb{N} \) that is terminating, confluent and equivalent to \( \rightarrow_1 \), that is the relations \( \leftrightarrow \leftrightarrow_1 \) and \( \leftrightarrow \leftrightarrow_2 \) are equal.
1.3.15. Well-founded induction. The principle of induction for natural numbers ensures that a property \( P(n) \) holds for all natural numbers \( n \) if we can show that \( P(n) \) holds under the hypothesis that \( P(m) \) holds for all \( m < n \). The principle is a consequence of the fact that there is no infinitely descending chain of natural numbers.

The \textit{well-founded induction} principle for an abstract rewriting system \((A, \rightarrow)\) can be stated as follows. Given a property \( P \) on elements of \( A \), then

\[
\forall a \in A, \ ( \forall b \in A, \ a \rightarrow b \ \text{implies} \ P(b) ) \ \text{implies} \ P(a)
\]

implies

\[
\forall a \in A, \ P(a).
\]

With this principle, the property \( P(a) \) is proved for all elements \( a \) in \( A \) by proving that the property \( P(b) \) holds for any element \( b \) in \( A \) such that there is a rewriting sequence \( a \rightarrow b \).

1.3.16. Theorem. If \( \rightarrow \) terminates then the principle of noetherian induction holds

\[\text{Proof.} \\] Suppose that the principle of induction does not hold, that is

\[
\forall a \in A, \ ( \forall b \in A, \ a \rightarrow b \ \text{implies} \ P(b) ) \ \text{implies} \ P(a)
\]

holds and that there exist an element \( c \) in \( A \) such that \( P(c) \) does not hold. Then there exists \( c' \) such that \( c \rightarrow c' \) and \( P(c') \) does not hold. In this way, we construct an infinite reduction sequence starting on \( c \). Hence, the reduction relation \( \rightarrow \) does not terminate.

Conversely, if the noetherian induction principle holds for an abstract rewriting system \((A, \rightarrow)\), then it terminates. It suffices to apply the induction principle to the property:

\[
P(a) \equiv (\text{there is no infinite reduction sequence starting on } a).
\]

1.3.17. Exercise. Let \((A, \rightarrow)\) be an abstract rewriting system. The relation \( \rightarrow \) is called \textit{finitely branching} if each element \( a \) of \( A \) has only finitely many direct successors, that is elements \( b \) such that \( a \rightarrow b \). The relation is called \textit{globally finite} if the relation \( \rightarrow \) is finitely branching, that is each element \( a \) in \( A \) has only finitely many successors.

1. Suppose that the relation \( \rightarrow \) is terminating and finitely branching. Prove that it is globally finite.

2. Show that it is not true that a finitely branching relation is terminating if it is globally finite. A relation is \textit{acyclic} if there is no element \( a \) in \( A \) such that \( a \rightarrow a \).

3. Show that any acyclic relation is terminating if it is globally finite.

4. Show that a finitely branching and acyclic relation is terminating if and only if it is globally finite.

1.3.18. Exercise. Let \((A, \rightarrow)\) be an abstract rewriting system such that every element \( a \) in \( A \) has a unique irreducible descendant. Prove that the relation \( \rightarrow \) is confluent.
1.4. FROM LOCAL TO GLOBAL CONFLUENCE

The local confluence does not generally imply confluence, however these properties are equivalent for terminating rewriting systems. This result is also due to Newman.

1.4.1. Theorem (Newman’s lemma, [New42 Theorem 3]). A terminating relation is confluent if it is locally confluent.

A short proof by Noetherian induction is given by Huet in [Hue80]. Due to this proof, Newman’s Lemma is also called the diamond lemma.

Proof. Suppose that $\rightarrow$ is locally confluent and terminating. We prove its confluence by Noetherian induction. Given $a_0$ in $A$, we suppose that for all $a$ with $a_0 \rightarrow a$ and for all branching

\[
\begin{array}{c}
  a_0 \\
  \Downarrow \\
  a_1 \\
  \Downarrow \\
  a_2 \\
\end{array}
\]

there exists a confluence

\[
\begin{array}{c}
  a_1 \\
  \Downarrow \\
  t \\
  \Downarrow \\
  a_2 \\
\end{array}
\]

Let us consider a branching

\[
\begin{array}{c}
  a_0 \\
  \Downarrow \\
  a' \\
  \Downarrow \\
  a'' \\
\end{array}
\]

The cases $a' \equiv a_0$ or $a'' \equiv a_0$ are obvious. In the other case, the length of the reductions $a_0 \rightarrow a'$ and $a_0 \rightarrow a''$ are greater than 1:

\[
\begin{array}{c}
  a_0 \\
  \Downarrow \\
  a_1 \\
  \Downarrow \\
  a'' \\
\end{array}\quad \text{loc. confl.} \quad \begin{array}{c}
  a_0 \\
  \Downarrow \\
  a_1 \\
  \Downarrow \\
  a'' \\
\end{array}
\]

\[
\begin{array}{c}
  a_1 \\
  \Downarrow \\
  \text{ind. hyp.} \\
  \Downarrow \\
  d' \\
\end{array}\quad \begin{array}{c}
  a_2 \\
  \Downarrow \\
  \text{ind. hyp.} \\
  \Downarrow \\
  d \\
\end{array}
\]

We conclude using the induction hypothesis and local confluence.
1.4.2. 

1.4.2. Example, [Hue80]. The following examples illustrate that the requirement of noetherianity is necessary to prove confluence from local confluence. The following abstract rewriting system is locally confluent but not confluent.

\[
\begin{array}{ccc}
  & b & \\
  & a & \swarrow \\
 a & \searrow & a' \\
 & b' & \\
\end{array}
\]

The following abstract rewriting system with \(2n \rightarrow a\), \(2n + 1 \rightarrow b\) and \(n \rightarrow n + 1\) for all \(n\) in \(\mathbb{N}\) without cycle is local confluent but not confluent:

\[
\begin{array}{ccccccccc}
  & b & \downarrow & 3 & 5 & 7 & 9 & 11 \\
  & 1 & \downarrow & 4 & 6 & 8 & 10 & \ldots \\
  & 0 & \downarrow & 2 & 4 & 6 & 8 & 10 \\
 a & \swarrow & \searrow & \swarrow & \searrow & \swarrow & \searrow & \\
\end{array}
\]

It is locally confluent but not confluent.
1.4. From local to global confluence
A string rewriting system, SRS for short, historically called a semi-Thue system, is a rewriting system over a set of strings on an alphabet. String rewriting systems are Turing complete in the sense that they give a calculus that is equivalent to that of the Turing machine. String rewriting system appear in the formal language theory. They are also used in combinatorial algebra as a tool for presentation of semigroups, groups or monoids. For a fuller treatment on string rewriting systems we refer the reader to [BO93] and [Jan88].

In this chapter, string rewriting system will be describe in the categorical language of 2-polygraphs as in [GM18] and [GM12b, Section 4]. A 2-polygraph is a rewriting system over a set of paths of a given directed graph. String rewriting system is the particular case when the directed graph has only one vertex.

2.1. PRELIMINARIES: ONE AND TWO-DIMENSIONAL CATEGORIES
2.1. Preliminaries: one and two-dimensional categories

2.1.1. Categories. A (small) category (or 1-category) is a data \( C \) made of

i) a set \( C_0 \), whose elements are called the \( 0 \)-cells of \( C \),

ii) for every \( 0 \)-cells \( x \) and \( y \) of \( C \), a set \( C(x, y) \), whose elements are called the \( 1 \)-cells from \( x \) to \( y \) of \( C \),

iii) for every \( 0 \)-cells \( x, y \) and \( z \) of \( C \), a map

\[
\star_{x,y,z}^0 : C(x, y) \times C(y, z) \to C(x, z),
\]

called the composition (or \( 0 \)-composition) of \( C \),

iv) for every \( 0 \)-cell \( x \), a specified element \( 1_x \) of \( C(x, x) \), called the identity of \( x \).

The following relations are required to hold

v) the composition is associative, \( \text{i.e.} \), for every \( 0 \)-cells \( x, y, z \) and \( t \) and for every \( 1 \)-cells \( u \in C(x, y) \), \( v \in C(y, z) \) and \( w \in C(z, t) \),

\[
\star_{x,z,t}^0 (\star_{x,y,z}^0 (u, v), w) = \star_{x,y,t}^0 (u, \star_{y,z,t}^0 (v, w)),
\]

vi) the identities are local units for the composition, \( \text{i.e.} \), for every \( 0 \)-cells \( x \) and \( y \) and for every \( 1 \)-cell \( u \in C(x, y) \),

\[
\star_{x,y}^0 (1_x, u) = u = \star_{x,y}^0 (u, 1_y).
\]

We write \( u : x \to y \) to mean that \( u \) is in \( C(x, y) \). The \( 0 \)-cell \( x \) is the source of \( u \) denoted by \( s_0(u) \) and the \( 0 \)-cell \( y \) is the target of \( u \) denoted by \( t_0(u) \). The composition \( \star_{x,y}^0 (u, v) \) will be denoted by \( u \star_0 v \), or simply by juxtaposition \( uv \).

2.1.2. Monoids. A monoid \( M \) with product \( \cdot \) and identity element \( 1_M \) corresponds to a category \( M \) with only one \( 0 \)-cell, denoted by \( \bullet \), and the \( 1 \)-cells of \( M(\bullet, \bullet) \) are the elements of the monoid \( M \). The identity arrow \( 1_\bullet \) of \( M \) is the identity element \( 1_M \) and the composition of \( u \star_0 v \) of \( 1 \)-cells in \( M(\bullet, \bullet) \) if the product \( u \cdot v \) in the monoid \( M \). The associativity and unitary properties of the composition, making \( M \) into a category, are induced by the corresponding properties of the product \( \cdot \). In this way, any monoid can be thought of as a one-0-cell category and a category can be thought of as a "monoid with several \( 0 \)-cells".

2.1.3. Internal definition. A category \( C \) can also be defined as an internal category in the category \( \text{Set} \) of sets. Explicitly, it is defined by a diagram in \( \text{Set} \):

\[
\begin{array}{c}
\text{C}_0 \xleftarrow{t_0} \text{C}_1 \xrightarrow{\star_0} \text{C}_1 \times \text{C}_0
\end{array}
\]
2.1.4. Product of categories

where \( C_1 \times_{C_0} C_1 \) is defined by the following pullback diagram in the category \( \text{Set} \):

\[
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_1 & \rightarrow & C_0 \\
\end{array}
\]

Elements of \( C_1 \times_{C_0} C_1 \) are pairs \((u,v)\) of 0-composable 1-cells \( u \) and \( v \), that is satisfying \( t_0(u) = s_0(v) \).

The maps \( s_0 \), \( t_0 \) and \( \star_0 \) satisfy the axioms in such a way that the diagram above defines a category.

Explicitly, the following diagrams commute in the category \( \text{Set} \):

\[
\begin{array}{ccc}
C_0 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_0 & \rightarrow & C_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \rightarrow & C_1 \\
\downarrow & & \downarrow \\
C_1 & \rightarrow & C_0 \\
\end{array}
\]

\[
\begin{array}{ccc}
C_1 \times_{C_0} C_1 & \rightarrow & C_1 \\
\downarrow \star_0 & & \downarrow \\
C_0 \times_{C_0} C_1 & \rightarrow & C_0 \\
\end{array}
\]

where \( \pi_1 \) and \( \pi_2 \) denote respectively first and second projection.

2.1.4. Product of categories. Given two categories \( C \) and \( D \), the product category \( C \times D \) is defined as follows

i) the 0-cells are the pairs \((x,y)\), where \( x \) is a 0-cell of \( C \) and \( y \) is a 0-cell of \( D \),

ii) the 1-cells are the pairs \((u,v)\) where \( u \) is a 1-cell of \( C \) and \( v \) is a 1-cell of \( D \),

iii) the composition is component-wise: \((u,v)(u',v') = (uu', vv')\),

iv) the identities are the pairs of identities: \( 1_{(x,y)} = (1_x, 1_y) \).

2.1.5. Functors. Let \( C \) and \( D \) be categories. A functor \( F : C \rightarrow D \) is a data made of

i) a map \( F_0 : C_0 \rightarrow D_0 \),

ii) for every 0-cells \( x \) and \( y \) of \( C \), a map

\[
F_{x,y} : C(x, y) \rightarrow D(F(x), F(y)),
\]

such that the following relations are satisfied:
2.1. Preliminaries: one and two-dimensional categories

iii) for every 0-cells \( x, y \) and \( z \) and every 1-cells \( u : x \to y \) and \( v : y \to z \) of \( C \),

\[
F_{x,z}(u \ast_0 v) = F_{x,y}(u) \ast F_{y,z}(v),
\]

iv) for every 0-cell \( x \) of \( C \),

\[
F_{x,x}(1_x) = 1_{F(x)}.
\]

We will write \( F(x) \) for \( F_0(x) \) and \( F(u) \) for \( F_{x,y}(u) \). A functor \( F \) is a monomorphism (resp. an epimorphism, resp. an isomorphism) if the map \( F_0 \) and each map \( F_{x,y} \) is an injection (resp. a surjection, resp. a bijection).

2.1.6. Functors as morphisms of graphs. A functor \( F : C \to D \) can be seen as a morphism of graphs

\[
\begin{array}{c}
\begin{array}{c}
C_0 \xrightarrow{s_0} C_1 \\
\text{F}_0 \downarrow \quad \downarrow \text{F}_1
\end{array} \\
\begin{array}{c}
D_0 \xrightarrow{s_0} D_1 \\
\text{t}_0 \quad \text{t}_0
\end{array}
\end{array}
\]

where, for every 1-cell \( u : x \to y \) of \( C \), the 1-cell \( F_1(u) \) is defined as \( F_{x,y}(u) \).

2.1.7. One-dimensional polygraphs. A 1-polygraph is a directed graph \( \Sigma \), i.e., a diagram of sets and maps

\[
\begin{array}{c}
\Sigma_0 \xrightarrow{s_0} \Sigma_1 \\
\text{t}_0 \quad \text{t}_0
\end{array}
\]

The elements of \( \Sigma_0 \) and \( \Sigma_1 \) are called the 0-cells and the 1-cells of \( \Sigma \), respectively. If there is no confusion, we just write \( \Sigma = (\Sigma_0, \Sigma_1) \). Note that the notion of 1-polygraph is equivalent to the notion of abstract rewriting system given in (1.1.1). A 1-polygraph is finite if it has finitely many 0-cells and 1-cells.

2.1.8. Free categories. If \( \Sigma \) is a 1-polygraph, the free category over \( \Sigma \) is the category denoted by \( \Sigma^* \) and defined as follows:

i) the 0-cells of \( \Sigma^* \) are the ones of \( \Sigma \),

ii) the 1-cells of \( \Sigma^* \) from \( x \) to \( y \) are the finite paths of \( \Sigma \), i.e., the finite sequences

\[
x \xrightarrow{u_1} x_1 \xrightarrow{u_2} x_2 \xrightarrow{u_3} \cdots \xrightarrow{u_{n-1}} x_{n-1} \xrightarrow{u_n} y
\]

of 1-cells of \( \Sigma \),

iii) the composition is given by concatenation,

iv) the identities are the empty paths.

If \( \Sigma \) has only one 0-cell, then the 1-cells of the free category \( \Sigma^* \) form the free monoid over the set \( \Sigma_1 \).
2.1.9. Generating 1-polygraph. Let $\mathbf{C}$ be a category. A 1-polygraph $\Sigma$ generates $\mathbf{C}$ if there exists an epimorphism

$$\pi : \Sigma \rightarrow \mathbf{C}$$

that is the identity on 0-cells. In that case, the 1-polygraph $\Sigma$ has the same 0-cells as $\mathbf{C}$ and, for every 0-cells $x$ and $y$ of $\mathbf{C}$, the map

$$\pi : \Sigma(x, y) \rightarrow \mathbf{C}(x, y)$$

is surjective. A category is finitely generated if it admits a finite generating 1-polygraph.

2.1.10. Spheres and cellular extensions of categories. A sphere of a category $\mathbf{C}$ is a pair $\gamma = (u, v)$ of parallel 1-cells of $\mathbf{C}$, that is, with the same source, $s_0(u) = s_0(v)$, and the same target, $t_0(u) = t_0(v)$. The 1-cell $u$ is the source of $\gamma$ and $v$ is its target. A cellular extension of $\mathbf{C}$ is a set $\Gamma$ equipped with a map from $\Gamma$ to the set of spheres of $\mathbf{C}$. It is equivalent to the data of a set $\Gamma$ with two maps

$$\mathbf{C} \xrightarrow{s_1} \xrightarrow{t_1} \Gamma,$$

satisfying the following gobular relations:

$$s_0s_1 = s_0t_1, \quad t_0s_1 = t_0t_1.$$

An element of $\Gamma$ will be graphically represented by a 2-cell with the following globular shape

$\alpha$

that relates parallel 1-cells $u$ and $v$ in $\mathbf{C}$, also denoted by $u \xrightarrow{\alpha} v$ or by $\alpha : u \Rightarrow v$.

2.1.11. Congruences. A congruence on a category $\mathbf{C}$ is an equivalence relation $\equiv$ on the parallel 1-cells of $\mathbf{C}$ that is compatible with the composition of $\mathbf{C}$, that is, for every 1-cells

$$x \xrightarrow{w} y \xleftarrow{\mu} z \xrightarrow{w'} t$$

of $\mathbf{C}$ such that $u \equiv v$, we have $wuw' \equiv wvw'$. If $\Gamma$ is a cellular extension of $\mathbf{C}$, the (Thue) congruence generated by $\Gamma$ is denoted by $\equiv_\Gamma$ and defined as the smallest congruence relation such that, if $\gamma$ is in $\Gamma$, then $s_1(\gamma) \equiv_\Gamma t_1(\gamma)$.
2.1. Preliminaries: one and two-dimensional categories

2.1.12. Quotient categories. If $C$ is a category and $\Gamma$ is a cellular extension of $C$, the *quotient of $C$ by $\Gamma$* is the category denoted by $C/\Gamma$ and defined as follows:

i) the 0-cells of $C/\Gamma$ are the ones of $C$,

ii) for every 0-cells $x$ and $y$ of $C$, the set $C/\Gamma(x,y)$ of 1-cell with source $x$ and target $y$ is the quotient of $C(x,y)$ by the restriction of $\equiv_\Gamma$.

We will denote by $\pi_\Gamma : C \rightarrow C/\Gamma$ the canonical projection. We will denote by $u_\Gamma$ for the image through $\pi_\Gamma$ of a 1-cell $u$ in $C$. The superscript $\Gamma$ in $u_\Gamma$ will be omitted whenever ambiguity is not introduced.

2.1.13. Two-dimensional categories. A (strict) 2-category is a category enriched over the cartesian monoidal category $\text{Cat}$ of categories. Explicitly, is a data $C$ made of a set $C_0$, whose elements are called the 0-cells of $C$, and, for every 0-cells $x$ and $y$ of $C$, a category $C(x,y)$, whose 0-cells and 1-cells are respectively called the 1-cells and the 2-cells from $x$ to $y$ of $C$. This data is equipped with the following algebraic structure:

i) for every 0-cells $x$, $y$, and $z$ of $C$, a functor $\ast_0^{x,y,z} : C(x,y) \times C(y,z) \rightarrow C(x,z)$,

ii) for every 0-cell $x$, a specified 0-cell $1_x$ of the category $C(x,x)$.

The following relations are required to hold:

iii) the composition is associative, i.e., for every 0-cells $x$, $y$, $z$ and $t$,

\[
\ast_0^{x,z,t} \circ (\ast_0^{x,y,z} \times \text{Id}_{C(z,t)}) = \ast_0^{x,y,t} \circ (\text{Id}_{C(x,y)} \times \ast_0^{y,z,t}),
\]

iv) the identities are local units for the composition, i.e., for every 0-cells $x$ and $y$,

\[
\ast_0^{x,y} \circ (1_x \times \text{Id}_{C(x,y)}) = \text{Id}_{C(x,y)} = \ast_0^{x,y} \circ (\text{Id}_{C(x,y)}, 1_y).
\]

2.1.14. Globular definition. A 2-category can, equivalently, be defined as a 2-graph

\[
\begin{array}{c}
C_0 \xrightarrow{s_0} C_1 \xrightarrow{s_1} C_2 \\
\downarrow t_0 \quad \quad \quad \downarrow t_1
\end{array}
\]

equipped with an additional algebraic structure. The definition of 2-graph requires that the source and target maps satisfy the globular relations:

\[s_0 \circ s_1 = s_0 \circ t_1 \quad \text{and} \quad t_0 \circ s_1 = t_0 \circ t_1.\]

The 2-graph is equipped with two compositions, the 0-*composition* $\ast_0$ and the 1-*composition* $\ast_1$, respectively defined on 0-composable 1-cells and 2-cells, and on 1-composable 2-cells. We also have an inclusion of $C_0$ into $C_1$ given by the identities of the 2-category, and an inclusion of $C_1$ into $C_2$ induced by the identities of the hom-categories. Explicitly, we have the following operations:
2.1.14. Globular definition

i) for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, a 0-composite 1-cell

$$u *_0 v : x \rightarrow z,$$

ii) for every 2-cells $x \xrightarrow{f} y \xrightarrow{g} z$, a 0-composite 2-cell

$$u *_0 v : x \xrightarrow{f *_0 g} z,$$

iii) for every 2-cells $x \xrightarrow{f} y \xrightarrow{g} z$, a 1-composite 2-cell

$$u *_0 v : x \xrightarrow{f \star_1 g} y,$$

iv) for every 0-cell $x$, an identity 1-cell

$$1_x : x \rightarrow x,$$

v) for every 1-cell $x \xrightarrow{u} y$, an identity 2-cell

$$1_u : u \rightarrow u.$$

The 0-composition and the 1-composition satisfy the following relations:

- for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z \xrightarrow{w} t$, $(u *_0 v) *_0 w = u *_0 (v *_0 w)$,
- for every 1-cell $x \xrightarrow{u} y$, $1_x *_0 u = u = u *_0 1_y$,
- for every 1-cells $x \xrightarrow{u} y \xrightarrow{v} z$, $1_{u*0v} = 1_u *_0 1_v$,
- for every 2-cells $u \xrightarrow{f} v \xrightarrow{g} w \xrightarrow{h} x$, $(f \star_1 g) \star_1 h = f \star_1 (g \star_1 h)$. 

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- for every 2-cells 

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{u \ar@{=>}[rr]^f \\
\downarrow & 
\downarrow & 
\downarrow \\
v & z \ar@{=>}[rr]^g & w \\
u' \ar@{=>}[rr]^h & v' \ar@{=>}[rr]^t & w'}
\end{array}
\end{array} \]

, \quad (f \ast_0 g) \ast_0 h = f \ast_0 (g \ast_0 h),

- for every 2-cell 

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{f \ar@{=>}[rr] & \\
v & y \ar@{=>}[rr] & w}
\end{array}
\end{array} \]

, \quad 1_x \ast_0 f = f = f \ast_0 1_y,

- for every 2-cell 

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{u \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

, \quad 1_u \ast_1 f = f = f \ast_1 1_v,

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{u \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

, \quad (f \ast_1 f') \ast_0 (g \ast_1 g') = (f \ast_0 g) \ast_1 (f' \ast_0 g').

The last relation is usually called the exchange relation or the interchange law for the compositions \( \ast_0 \) and \( \ast_1 \). This globular definition of 2-categories is equivalent to the enriched one.

The 0-composition of 2-cells with identity 1-cells defines the whiskering operations:

- for every 

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{w \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

, \quad the left whiskering is

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{w \ast_0 u \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

,

- for every 

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{w \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

, \quad the right whiskering is

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{u \ast_0 w \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

.

The left and right whiskering operations satisfy the following relations:

- for every 

\[ \begin{array}{c}
\begin{array}{c}
\xymatrix{u \ar@{=>}[rr]^f & \\
v \ar@{=>}[rr]^g & z}
\end{array}
\end{array} \]

, \quad u \ast_0 (f \ast_1 f') = (u \ast_0 f) \ast_1 (u \ast_0 f'),
2.2. String rewriting systems

2.2.1. Two-dimensional polygraphs. A 2-polygraph \( \Sigma = (\Sigma_0, \Sigma_1, \Sigma_2) \) made of a 1-polygraph \( (\Sigma_0, \Sigma_1) \), often simply denoted by \( \Sigma_1 \), and a cellular extension \( \Sigma_2 \) of the free category \( \Sigma_1^* \). In other terms, a 2-polygraph \( \Sigma \) is a 2-graph

\[
\begin{array}{cccc}
\Sigma_0 & \xleftarrow{s_0} & \Sigma_1 & \xrightarrow{s_1} \\
\xrightarrow{t_0} & \Sigma_2 & \xleftarrow{t_1}
\end{array}
\]

whose 0-cells and 1-cells form a free category. The elements of \( \Sigma_k \) are called the \( k \)-cells of \( \Sigma \) and \( \Sigma \) is finite if it has finitely many cells in every dimension.

2.2.2. Example. The string rewriting system on the alphabet \( \{a\} \) with only one rewriting rule \( aa \rightarrow \alpha \) is described by the 2-polygraph \( \Sigma \), where

\[
\Sigma_0 = \{\bullet\}, \quad \Sigma_1 = \{a\}, \quad \Sigma_2 = \{aa \Rightarrow \alpha\}.
\]

The rule \( aa \rightarrow \alpha \) corresponds to the following globular 2-cell

\[
\begin{array}{c}
\alpha \rightarrow \alpha \\
\xrightarrow{\alpha} \bullet
\end{array}
\]
2.2. String rewriting systems

2.2.3. Presentations of categories. If \( \Sigma \) is a 2-polygraph, the category presented by \( \Sigma \) is the category denoted by \( \Sigma \) and defined by

\[
\Sigma = \Sigma_1 / \Sigma_2.
\]

If \( C \) is a category, a presentation of \( C \) is a 2-polygraph \( \Sigma \) such that \( C \) is isomorphic to \( \Sigma \). In that case, the 1-cells of \( \Sigma \) are the generators of \( C \), and the 2-cells of \( \Sigma \) are the relations of \( C \).

Two 2-polygraphs \( \Sigma \) and \( \Upsilon \) are said to be Tietze-equivalent if they present isomorphic categories, that is there exists an isomorphism of categories \( \Sigma \simeq \Upsilon \).

2.2.4. Free 2-categories. Let \( \Sigma \) be a 2-polygraph. The free 2-category over \( \Sigma \) is the 2-category denoted by \( \Sigma^* \) and defined as follows:

\begin{itemize}
  \item[i)] the 0-cells of \( \Sigma^* \) are the ones of \( \Sigma \),
  \item[ii)] for every 0-cells \( x, y \) of \( \Sigma \), the category \( \Sigma^*_2(x, y) \) is defined as
    \begin{itemize}
      \item the free category over the 1-polygraph
        \begin{itemize}
          \item whose 0-cells are the 1-cells in \( \Sigma_1^*(x, y) \),
          \item whose 1-cells are the
        \end{itemize}
        \begin{equation}
        \begin{array}{c}
        x \xrightarrow{w} y \\
        \overset{\alpha}{\Downarrow} z \xrightarrow{w'} t
        \end{array}
        \end{equation}
      \item quotented by the congruence generated by the cellular extension made of all the possible
        \[
        \alpha wv \equiv u\nu\beta \equiv uw\beta \equiv \alpha wv',
        \]
    \end{itemize}
  \item[iii)] for every 0-cells \( x, y \) and \( z \) of \( \Sigma \) the composition functor is given by the concatenation on 1-cells and, on 2-cells, as follows:
    \begin{itemize}
      \item \( u_1 \alpha_1 \cdots \alpha_m u_m \beta_1 \cdots \beta_n \)
      \item \( v_1 \Gamma_1 \cdots \Gamma_n \)
      \item \( u \Gamma_1 \cdots \Gamma_n u' \)
      \item \( v \Gamma_1 \cdots \Gamma_n v' \)
    \end{itemize}
\end{itemize}
2.2.5. Rewriting sequences

A rewriting step of a 2-polygraph $\Sigma$ is a 2-cell of the free 2-category $\Sigma^*$ with shape

\[
\begin{array}{c}
x \\ \Downarrow \phi \\ \downarrow \quad \Downarrow \psi \\ z \quad v
\end{array}
\Rightarrow
\begin{array}{c}
y \\ \\ \Downarrow \phi \\ \downarrow \psi \\ u \\ \Downarrow \phi \\ \downarrow \psi \\ v
\end{array}
\]

where $\phi : l \Rightarrow r$ is a generating 2-cell in $\Sigma$ and $u$ and $v$ are 1-cells of $\Sigma^*$. Such a rewriting step will be denoted by $u \phi v \Rightarrow_{\Sigma^*} \Sigma^*$. The subscript $\Sigma$ will be omitted whenever ambiguity is not introduced.

A rewriting sequence of $\Sigma$ is a finite or infinite sequence

\[
\sigma_1 \Rightarrow_{\Sigma^*} \sigma_2 \Rightarrow_{\Sigma^*} \cdots \Rightarrow_{\Sigma^*} \sigma_n \Rightarrow_{\Sigma^*} \cdots
\]

of rewriting steps. If $\Sigma$ has a non-empty rewriting sequence from $w$ to $w'$, we say that $w$ rewrites into $w'$. Let us note that every 2-cell $f$ of the 2-category $\Sigma^*$ decomposes into a finite rewriting sequence of $\Sigma$, this decomposition being unique up to exchange relations.
2.2. String rewriting systems

2.2.6. Leftmost reduction. Let \( \Sigma \) be a 2-polygraph. A reduction step \( w \Rightarrow w' \) is leftmost, and we denote \( w \Rightarrow^l w' \), if the two following conditions are satisfied

i) if \( w = ulv \) and \( w' =urv \) for some \( l \Rightarrow r \) in \( \Sigma_2 \) with \( u \) and \( v \) in \( \Sigma_1^* \),

ii) for any factorisation \( w = ul'l'r' \) for some \( l' \Rightarrow r' \) in \( \Sigma_2 \), then \( ul \) is a proper prefix of \( ul' \) or \( ul = ul' \) and \( u \) is a prefix of \( ul' \).

2.2.7. Rewriting properties of 2-polygraphs. To any 2-polygraph \( \Sigma \), we associate an abstract rewriting system whose elements are 1-cells in \( \Sigma_1^* \) and the reduction relations is the relation \( \Rightarrow_{\Sigma_2} \). We say that a 2-polygraph has a rewriting property \( P \), such as normalisation, termination or confluence, if the associated abstract rewriting system \((\Sigma_1^* \Rightarrow_{\Sigma_2})\) has the property \( P \). In particular, a 2-polygraph is confluent if and only if it is Church-Rosser and by Newman’s lemma, Theorem 5.5.12, for a terminating 2-polygraph, local confluence and confluence are equivalent properties.

We will denote by \( \Sigma_1^{nf} \) the set of 1-cells of \( \Sigma_1^* \) in normal form with respect to \( \Sigma_2 \).

2.2.8. Theorem. Let \( \Sigma \) be a terminating 2-polygraph. Then \( \Sigma \) is confluent if and only if the restriction of the canonical projection

\[ \pi : \Sigma_1^* \rightarrow \Sigma \]

to the irreducible 1-cells induces a bijection for any 0-cells \( x \) and \( y \):

\[ \tilde{\pi}_{x,y} : \Sigma_1^{nf}(x,y) \rightarrow \Sigma(x,y). \]

2.2.9. Exercise. Prove Theorem 2.2.8

2.2.10. Termination order. A termination order on \( \Sigma \) is an order relation \( \prec \) on parallel 1-cells of \( \Sigma_1^* \) such that the following three conditions are satisfied:

i) the composition of 1-cells of \( \Sigma_1^* \) is strictly monotone in both arguments, i.e., \( u' \prec u \) implies \( vu'w \prec vuw \) for all composable 1-cells \( u, u', v \) and \( w \) in \( \Sigma_1^* \),

ii) the relation is wellfounded, i.e., every decreasing family \((u_n)_{n \in \mathbb{N}}\) of parallel 1-cells of \( \Sigma_1^* \) is stationary,

iii) for every 2-cell \( \alpha \) of \( \Sigma_2 \), the strict inequality \( t(\alpha) \prec s(\alpha) \) holds.

As a direct consequence of the definition, if a 2-polygraph admits a termination order, then it terminates.

2.2.11. Lexicographic order. A useful example of termination order is the left degree-wise lexicographic order (or deglex for short) generated by a given order on the 1-cells of \( \Sigma \). It is defined by the following strict inequalities, where each \( x_i \) and \( y_j \) is a 1-cell of \( \Sigma \):

\[ x_1 \cdots x_p < y_1 \cdots y_q, \quad \text{if } p < q, \]

\[ x_1 \cdots x_{k-1}x_k \cdots x_p < x_1 \cdots x_{k-1}y_k \cdots y_p, \quad \text{if } x_k < y_k. \]

The deglex order is total if and only if the original order on the set \( \Sigma_1 \) is total.
2.2.12. **Reduced 2-polygraph.** A 2-polygraph $\Sigma$ is

i) *left-reduced* if for any $l \Rightarrow r$ in $\Sigma_2$ then $l$ is irreducible with respect to $\Sigma_2 \setminus \{l \Rightarrow r\}$,

ii) *right-reduced* if for any $l \Rightarrow r$ in $\Sigma_2$ then $r$ is irreducible with respect to $\Sigma_2$,

iii) *reduced* if it is left-reduced and right-reduced.

2.2.13. **Exercise [Mét83], [Squ87, Theorem 2.4].** Show that every finite convergent 2-polygraph is Tietze equivalent to a finite reduced convergent 2-polygraph.

### 2.3. The Word Problem

2.3.1. **The word problem.** The *word problem* for a 2-polygraph $\Sigma$ is the following decision problem

**INSTANCE:** two 1-cells $u$ and $v$ in $\Sigma_1^*$.  

**QUESTION:** Does $u \leftarrow \Sigma_2 \rightarrow v$?

There are finite string rewriting systems for which the word problem is algorithmically unsolvable. Hence, the word problem for finite string rewriting systems is undecidable in general. When a string rewriting system is finite and convergent, then its word problem is decidable by the normal form procedure.

2.3.2. **Normal form procedure.** Given a convergent 2-polygraph $\Sigma$, every 1-cell $u$ of $\Sigma_1^*$ has a unique normal form, denoted by $\hat{u}$, so that we have $\overline{u} = \overline{v}$ in $\Sigma$ if, and only if, $\hat{u} = \hat{v}$ holds in $\Sigma_1^*$. This defines a section

$$\Sigma \twoheadrightarrow \Sigma_1^*$$

of the canonical projection $\Sigma_1^* \rightarrow \Sigma$, mapping a 1-cell $u$ of $\Sigma$ to the unique normal form of its representative 1-cells in $\Sigma_1^*$, still denoted by $\hat{u}$. As a consequence, a finite and convergent 2-polygraph $\Sigma$ yields a decision procedure for the word problem of the category $\Sigma$ it presents: the *normal-form procedure*:

**Input:** $u$, $v$ two 1-cells of $\Sigma_1^*$.  

**begin**

| reduce $u$ to its normal form $\hat{u}$ with respect to $\Sigma_2$ ; |
| reduce $v$ to its normal form $\hat{v}$ with respect to $\Sigma_2$ ; |
| if $\hat{u} = \hat{v}$ then |
| | Accept |
| else |
| | Reject |

**end**

**Algorithm 1:** Normal form procedure

Note that finiteness is used to test whether a given 1-cell $u$ is a normal form or not, by examination of all the relations and their possible applications on $u$. Then, the equality $\overline{u} = \overline{v}$ holds in $\Sigma$ if, and only if, the equality $\hat{u} = \hat{v}$ holds in $\Sigma_1^*$.  

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2.4. Branchings

2.3.3. Complexity of the word problem for a finite 2-polygraph. For a finite convergent 2-polygraph $\Sigma$, consider a function $f_\Sigma : \mathbb{N} \to \mathbb{N}$ such that for any 1-cell $u$ in $\Sigma_1^+$, the leftmost reduction sequence from $u$ to its normal form contains at most $f_\Sigma(\ell(u))$ many steps. In [Boo82], Book proves that for a finite convergent and reduced 2-polygraph $\Sigma$, the normal form for a 1-cell $u$ in $\Sigma_1^+$ can be computed in time $O(\ell(u) + f_\Sigma(\ell(u)))$. As a consequence, if a 2-polygraph $\Sigma$ is length-reducing and confluent, then its word problem is decidable in linear time.

2.3.4. Decidability of the word problem and Tietze invariance. It is well-known that the decidability of the word problem is an invariant property of finite presentations of monoids:

2.3.5. Proposition. Let $\Sigma$ and $\Upsilon$ two finite Tietze-equivalent 2-polygraphs. Then the word problem for $\Sigma$ is decidable if and only if the word problem for $\Upsilon$ is decidable.

We can thus talk about the decidability of the word problem in a finitely generated monoid. Finally, let us mention the following result obtained by Avenhaus and Madlener in [AM78a, AM78b] for presentations of groups, but the proof can be applied to presentation of monoids.

2.3.6. Theorem. Let $\Sigma$ and $\Upsilon$ be two Tietze-equivalent finite 2-polygraphs. If the word problem can be decided for $\Sigma$ in time $O(f(n))$, then the word problem for $\Upsilon$ can be solved in time $O(f(c.n))$ for some constant natural number $c > 0$.

2.4. BRANCHINGS

2.4.1. Branchings. Recall from (1.1.5), that a branching of $\Sigma$ is a pair $(f, g)$ of 2-cells of $\Sigma_2^+$ with a common source, as in the following diagram

\[
\begin{array}{c}
\text{v} \\
\downarrow \\
\text{u} \\
\downarrow \\
\text{w} \\
\uparrow \\
\text{g} \\
\downarrow \\
\text{f} \\
\downarrow \\
\text{v} \\
\end{array}
\]

The 1-cell $u$ is the source of this branching and the pair $(v, w)$ is its target. A branching $(f, g)$ is local if $f$ and $g$ are rewriting steps. A branching

\[
\begin{array}{c}
\text{v} \\
\downarrow \\
\text{u} \\
\downarrow \\
\text{w} \\
\uparrow \\
\text{g} \\
\downarrow \\
\text{f} \\
\downarrow \\
\text{v} \\
\end{array}
\]

is confluent if there exist 2-cells $f'$ and $g'$ in $\Sigma_2^+$, as in the following diagram:

\[
\begin{array}{c}
\text{v} \\
\downarrow \\
\text{u} \\
\downarrow \\
\text{w} \\
\uparrow \\
\text{g} \\
\downarrow \\
\text{f} \\
\downarrow \\
\text{v} \\
\end{array}
\]
2.4.2. **Local branchings.** Local branchings belong to one of the three following families. The *aspherical* branchings have shape

\[ u \xleftarrow{f} v \]

where \( f \) is a rewriting step. The *orthogonal* branchings, also called *Peiffer* branchings, have shape

\[ u \xrightarrow{v} u'v \]
\[ uv \xrightarrow{g} uv' \]

where \( f : u \Rightarrow u' \) and \( g : v \Rightarrow v' \) are rewriting steps. The *overlapping* branchings are the remaining local branchings.

2.4.3. **Critical branchings.** Local branchings are compared by the order \( \sqsubseteq \) generated by the relations

\[(f, g) \sqsubseteq (uv, ugv)\]

given for any local branching \((f, g)\) and any possible 1-cells \( u \) and \( v \) of \( \Sigma_1^* \). An overlapping local branching that is minimal for the order \( \sqsubseteq \) is called a *critical branching*, or a *critical pair*. Note that a 2-polygraph has two kinds of critical branchings, namely *inclusion* ones and *overlapping* ones, respectively corresponding to the two situations pictured on Figure 2.4.3:

![Figure 2.4.3: Critical branchings by inclusion and overlapping](image)

2.4.4. **Theorem (Critical pair theorem).** A 2-polygraph is locally confluent if, and only if, all its critical branchings are confluent.

*Proof.* Every aspherical branching is confluent:
2.4. Branchings

We also have confluence of every Peiffer local branching:

\[ f\nu \rightarrow \ u'\nu = \ u'\nu' \]
\[ \u \rightarrow \ u'\nu \]
\[ g \rightarrow \ u'\nu' \]

Finally, in the case of an overlapping but not minimal local branching \((f, g)\), there exist factorisations \(f = uhv\) and \(g = ukv\) with

\[ h \rightarrow w_1 \]
\[ w \]
\[ k \rightarrow w_2 \]

a critical branching of \(\Sigma\). By hypothesis, the branching \((h, k)\) is confluent:

\[ h \rightarrow w_1 \rightarrow h' \]
\[ w \]
\[ k \rightarrow w_2 \rightarrow k' \]

then so is \((f, g)\):

\[ f \rightarrow uw_1v = uh'v \]
\[ uwv \]
\[ g \rightarrow uw_2v = uk'v \]

2.4.5. Example. Consider the 2-polygraph \(\Sigma\), with \(\Sigma_1 = \{a, b\}\) and \(\Sigma_2 = \{\alpha : aba \Rightarrow 1\}\). The 2-polygraph \(\Sigma\) is terminating since \(\ell(u) > \ell(v)\) whenever \(u \Rightarrow v\). The polygraphs admits one critical branching and this branching is not confluent:

\[ aba \rightarrow ab \]
\[ ababa \]
\[ \alpha \rightarrow ba \]

It follows that the 2-polygraph \(\Sigma\) is not confluent.
2.4.6. Example. Consider the 2-polygraph $\Sigma$, with $\Sigma_1 = \{a, b, c\}$ and

$$\Sigma_2 = \{\alpha : ab \Rightarrow ca, \beta : bc \Rightarrow ab, \gamma : ca \Rightarrow bc\}.$$ 

The 2-polygraph $\Sigma$ is not terminating and local confluent with three confluent critical branchings:

\[
\begin{array}{c}
\alpha \beta \rightarrow aab \xrightarrow{\alpha \gamma} aca \xrightarrow{\alpha \gamma} abc \\
\alpha c \rightarrow cac \xrightarrow{\gamma c} bcc \xrightarrow{\beta c} abc \\
\end{array}
\quad
\begin{array}{c}
\gamma b \rightarrow bcb \xrightarrow{\beta b} abb \xrightarrow{\alpha b} cab \\
\beta a \rightarrow aba \xrightarrow{\alpha a} cca \xrightarrow{\gamma a} abc \\
\end{array}
\]

2.4.7. Example: reduced standard presentation. Given a category $C$, we call reduced standard polygraphic presentation of $C$, the 2-polygraph $\Sigma$ defined as follows:

i) it has one 0-cell for each 0-cell of $C$ and one 1-cell $\hat{u} : x \rightarrow y$ for every non-identity 1-cell $u : x \rightarrow y$ of $C$,

ii) it has one 2-cell

\[
\begin{array}{c}
\hat{u} \xrightarrow{\mu_{\hat{u},\hat{v}}} y \\
\widehat{\mu_{\hat{u},\hat{v}}} \\
x \xrightarrow{uv} z
\end{array}
\]

for every non-identity 1-cells $u : x \rightarrow y$ and $v : y \rightarrow z$ of $C$ such that $uv$ is not an identity,

iii) it has one 2-cell

\[
\begin{array}{c}
\hat{u} \xrightarrow{\mu_{\hat{u},\hat{v}}} y \\
\widehat{\mu_{\hat{u},\hat{v}}} \\
x \xrightarrow{1_x} x
\end{array}
\]

for every non-identity 1-cells $u : x \rightarrow y$ and $v : y \rightarrow x$ of $C$ such that $uv = 1_x$.

The 2-polygraph $\Sigma$ is reduced and convergent. It has one critical branching $(\mu_{\hat{u},\hat{v}}, \widehat{\mu_{\hat{u},\hat{v}}})$ for every triple $(u, v, w)$ of non-identity composable 1-cells in $C$. Each of these critical branchings is confluent,
with four possible cases, depending on whether $uv$ or $vw$ is an identity or not:

Following Theorem 2.4.4, one can decide whether a finite string rewriting system is convergent by checking confluence of critical branchings. If the set of rules is finite, there are only finitely many critical branchings. It thus can be tested whether every such branching is confluent. The result follows because, the rewriting system is locally confluent if and only if every critical branching is confluent.

2.5. Completion

2.5.1. Knuth-Bendix’s completion procedure. Let $\Sigma$ be a terminating 2-polygraph, equipped with a total termination order $\prec$. A Knuth-Bendix’s completion of $\Sigma$ is a 2-polygraph $KB(\Sigma)$ obtained by the following procedure.

2.4.8. Theorem ([Niv73]). Let $\Sigma$ be a finite terminating string rewriting system. Then, whether or not $\Sigma$ is locally confluent, is decidable. Hence, it is decidable whether or not $\Sigma$ is confluent.
**2.5.1. Knuth-Bendix’s completion procedure**

**Input:** \( \Sigma \) be a terminating 2-polygraph with a total termination order \( \prec \).

\[ \mathcal{KB}(\Sigma) \leftarrow \Sigma \]

\( \mathcal{Cb} \leftarrow \{ \text{critical branchings of } \Sigma \} \)

**while** \( \mathcal{Cb} \neq \emptyset \) **do**

1. Picks a branching in \( \mathcal{Cb} \):
   
   \[ f \rightarrow v \]
   
   \[ g \rightarrow w \]

   \( \mathcal{Cb} \leftarrow \mathcal{Cb} \setminus \{(f, g)\} \)

2. Reduce \( v \) to a normal form \( \hat{v} \) with respect to \( \mathcal{KB}(\Sigma)_2 \)

3. Reduce \( w \) to a normal form \( \hat{w} \) with respect to \( \mathcal{KB}(\Sigma)_2 \)

4. If \( \hat{v} \neq \hat{w} \) then
   - If \( \hat{v} > \hat{w} \) then
     
     \[ \mathcal{KB}(\Sigma)_2 \leftarrow \mathcal{KB}(\Sigma)_2 \cup \{ \alpha : \hat{v} \Rightarrow \hat{w} \} \]

   - If \( \hat{w} > \hat{v} \) then
     
     \[ \mathcal{KB}(\Sigma)_2 \leftarrow \mathcal{KB}(\Sigma)_2 \cup \{ \alpha : \hat{w} \Rightarrow \hat{v} \} \]

5. \( \mathcal{Cb} \leftarrow \mathcal{Cb} \cup \{ \text{critical branching created by } \alpha \} \)

**end**

**Algorithm 2: Knuth-Bendix completion procedure**

If the procedure stops, it returns the 2-polygraph \( \mathcal{KB}(\Sigma) \). Otherwise, it builds an increasing sequence of 2-polygraphs, whose limit is denoted by \( \mathcal{KB}(\Sigma) \). Note that, if the starting 2-polygraph \( \Sigma \) is already convergent, then the Knuth-Bendix’s completion of \( \Sigma \) is \( \Sigma \).
2.6. Existence of finite convergent presentations

2.5.2. Theorem ([KB70]). The Knuth-Bendix’s completion $KB(\Sigma)$ of a 2-polygraph $\Sigma$ is a convergent presentation of the category $\Sigma$. Moreover, the 2-polygraph $KB(\Sigma)$ is finite if, and only if, the 2-polygraph $\Sigma$ is finite and if the Knuth-Bendix’s completion procedure halts.

2.5.3. Exercise. Find a finite convergent presentation of the monoid generated by two generators $a$ and $b$ and submitted to the relation $aba = 1$.

2.6. EXISTENCE OF FINITE CONVERGENT PRESENTATIONS

When a string rewriting system is not convergent, one wishes to determine whether there exists a Tietze equivalent convergent string rewriting system. We can formulate the two following problems of existence of finite convergent presentations.

2.6.1. Problem.

INSTANCE: A finite string rewriting system $(\Sigma_1, \Sigma_2)$.

QUESTION: Does $(\Sigma_1, \Sigma_2)$ is Tietze equivalent to a finite convergent string rewriting system $(\Sigma_1, \Upsilon_2)$?

2.6.2. Problem.

INSTANCE: A finite string rewriting system $(\Sigma_1, \Sigma_2)$.

QUESTION: Does $(\Sigma_1, \Sigma_2)$ is Tietze equivalent to a finite convergent string rewriting system?

2.6.3. Theorem ([BO84]). The problems 2.6.1 and 2.6.2 are undecidable.

2.6.4. Existence of finite convergent presentations. The normal form procedure proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem. The converse implication was still an open problem in the middle of the eighties. Jantzen in [Jan82, Jan85] asked the following question.

2.6.5. Question. Does every finitely presented monoid with a decidable word problem admit a finite convergent presentation?

2.6.6. Example. In [KN85], Kapur and Narendran consider Artin’s presentation of the monoid $B_3^+$ of positive braids on three strands:

$$\Sigma = \langle s, t \mid sts \Rightarrow tst \rangle.$$ 

The generators $s$ and $t$ correspond to the following braids

$$s = \begin{array}{c} \scriptsize \bigcirc \\ \scriptsize \bigcirc \end{array} \quad \text{and} \quad t = \begin{array}{c} \scriptsize \bigcirc \\ \scriptsize \bigcirc \end{array},$$

and the rule $sts \Rightarrow tst$ corresponds to the Yang-Baxter relation:

$$\begin{array}{c} \scriptsize \bigcirc \\ \scriptsize \bigcirc \end{array} = \begin{array}{c} \scriptsize \bigcirc \\ \scriptsize \bigcirc \end{array} \begin{array}{c} \scriptsize \bigcirc \\ \scriptsize \bigcirc \end{array}.$$
2.6.9. Question

They proved that the word problem for $B_3^+$ is decidable and that this monoid admits no finite convergent presentation on the two generators $s$ and $t$. However, Bauer and Otto, [BO84], have found a finite convergent presentation of the monoid $B_3^+$ by adjunction of a new generator $a$ standing for the product $st$:

$$\Gamma = \langle s, t, a \mid \alpha : ta \Rightarrow as, \; \beta : st \Rightarrow a \rangle.$$  

Indeed, this rewriting system can be completed by applying the Knuth-Bendix completion procedure, [KB70], into the following convergent presentation

$$KB(\Gamma) = \langle s, t, a \mid \alpha : ta \Rightarrow as, \; \beta : st \Rightarrow a, \; \gamma : sas \Rightarrow aa, \; \delta : saa \Rightarrow aat \rangle \quad (2.6.7)$$

with the following four critical branchings:

\begin{itemize}
  \item $\beta a \Rightarrow aa$
  \item $\gamma t \Rightarrow aat$
  \item $\gamma as \Rightarrow aas \Leftarrow \alpha a$
  \item $\gamma aa \Rightarrow aaaa \Leftarrow \alpha aat$
\end{itemize}

As a consequence, the word problem for $B_3^+$ is solvable by the normal form algorithm. The result of Kapur and Narendran shows that the existence of a finite convergent presentation depends on the specific presentation of the monoid, in particular on the chosen generators. In their example, by adding new letters in the alphabet it is possible to obtain a finite convergent string rewriting system. However, is it always possible to obtain such Tietze equivalent system by adding a finite set of letters? Thus, to provide the awaited negative answer to the open question, one would have to exhibit a monoid with a decidable word problem but with no finite convergent presentation on any possible set of generators.

2.6.9. Question. Which condition a monoid need to satisfy to admit a presentation by a finite convergent rewriting system?

Diekert solved the problem for the case of abelian groups. He derived a whole class of finite string rewriting systems presenting abelian groups with a decidable word problem which are not Tietze equivalent to finite convergent string rewriting systems on the same alphabet, [Die86]. Moreover, he constructed a necessary and sufficient conditions for the existence of convergent presentation for finitely generated abelian groups. However, the problem for general monoids was still open. At this point, new methods had to be introduced for a problem which seems to concern intrinsic properties of the presented monoid.

2.6.10. Exercise. Compute a convergent presentation of the monoid $B_3^+$ with two generating 1-cells.

2.6.11. Exercise, [KN85]. Consider the monoid $B_3^+$ of positive braids on three strands and the Artin’s presentation $\Sigma = \langle s, t \mid \gamma : sts \Rightarrow tst \rangle$.

1. Show that the word problem is decidable for $B_3^+$.

2. Show that for any $i \geqslant 0$ and any $j \geqslant 0$, the words $s^{i+1}t^{j+2}st$ and $tst^{i+2}s^{j+1}$ are equals in $B_3^+$. 

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2.6. Existence of finite convergent presentations

3. Denote by \([w]\) the equivalence class modulo the relation \(\gamma\) containing the word \(w\). Prove that for any \(n > 0\) the two following equalities hold
\[
[t^n_{st}] = \{t^{n-i}sts^i \mid 0 \leq i \leq n\},
\]
\[
[tst^n] = \{s^jts^{n-j} \mid 0 \leq j \leq n\}.
\]

4. Show that there does not exist any finite convergent presentation of the monoid \(B^+_3\) with two generators \(s\) and \(t\).

2.6.12. Example: plactic monoid. The structure of plactic monoids appeared in the combinatorial study of Young tableaux by Schensted \([\text{Sch61}]\) and Knuth \([\text{Knu70}]\). The plactic monoid of rank \(n > 0\), denoted by \(P_n\), is generated by the set \(\{1, \ldots, n\}\) and subject to the Knuth relations:
\[
zxy = xzy \quad \text{for} \quad 1 \leq x \leq y < z \leq n, \quad yzx = yxz \quad \text{for} \quad 1 \leq x < y \leq z \leq n.
\]
For instance, the monoid \(P_2\) is generated by \(\{1, 2\}\) and submitted to the relations \(211 = 121\) and \(221 = 212\). These relations can be oriented with respect to the lexicographic order as follows
\[
\eta_{1,1,2} : 211 \Rightarrow 121 \quad \epsilon_{1,2,2} : 221 \Rightarrow 212.
\]
In this way, the Knuth presentation of the monoid \(P_2\) is convergent with a unique critical branching:

With respect to the lexicographic order, the Knuth presentation of the monoid \(P_3\) is not convergent, but it can be completed by adding 3 relations to get a convergent presentation with 27 critical branchings. For the monoid \(P_d\) we have 4 generators and 20 relations, and its completion is infinite. More generally, Kubat and Okniński showed in \([\text{KO14}]\) that for rank \(n > 3\), a finite convergent presentation of the monoid \(P_n\) cannot be obtained by completion of the Knuth presentation with the degree lexicographic order. Bokut, Chen, Chen and Li in \([\text{BCCL15}]\), Cain, Gray and Malheiro in \([\text{CGM15}]\), and Hage in \([\text{Hag15}]\) for type C, constructed with independent methods a finite convergent presentation by adding column generators to the Knuth presentation.
Coherent presentations and syzygies

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The notion of coherent presentation extends those of presentation of a category by globular homotopy generators taking into account the relations amongst the relations. In this chapter we show how to compute a coherent presentation for a category using the completion procedure introduced in the previous chapter. The method follows a construction introduced by Squier in [SOK94] in his homotopical and homological study of finiteness conditions for finite convergence of finitely presented monoids. Many constructions presented in this chapter come from [GM18].

3.1. INTRODUCTION

3.1.1. Syzygies of Knuth’s relations. Recall from (2.6.12), that for \( n > 0 \), the plactic monoid \( P_n \) is generated by the set \( \{1, \ldots, n\} \) and subject to the Knuth relations:

\[
    zxy = xzy \quad \text{for} \quad 1 \leq x < y < z \leq n, \quad yzx = yxz \quad \text{for} \quad 1 \leq x < y < z \leq n.
\]

We consider the problem of finding all independent irreducible algebraic relations amongst these relations. Such a relation is called a 2-syzygy, and we aim to to give an algorithmic method that computes all
2-syzgies of the presentation, and in particular a family of generators for these syzygies. For instance, the monoid $P_2$ is generated by $\{1, 2\}$ and submitted to the relations

$$\eta_{1,1,2} : 211 \Rightarrow 121, \quad \varepsilon_{1,2,2} : 221 \Rightarrow 212.$$ 

There are two ways to prove the equality $2211 = 2121$ in the monoid $P_2$, either by applying the first relation or the second relation. This two equalities are related by a syzygy:

We will prove that this syzygy generates all the syzygies of the above presentation. The proof is based on a categorical description of syzygies of such a presentation, and an extension of the Knuth-Bendix completion procedure given in (2.5.1), by keeping track of syzygies created when adding rules during the completion. The correctness of the procedure follows the coherent Squier theorem, [SOK94], which states that a convergent presentation of a monoid extended by the homotopy generators defined by the confluence diagrams induced by critical branchings forms a coherent convergent presentation.

3.1.2. Positive braid monoids. Let us illustrate the notion of syzygy on the presentation of the braid monoid $B_3^+$ studied in (2.6.6):

$$\langle s, t \mid \alpha : sts \Rightarrow tst \rangle.$$ 

One proves that there is no nontrivial syzygy amongst the relations induce by the rule $\alpha$. Now consider the braid monoid $B_4^+$ on four strands with the following presentation:

$$\langle r, s, t \mid rsr = srs, \, sts = tst, \, rt = tr \rangle.$$ 

The generators corresponds to the following generating braids on four strands:

$$r = \begin{array}{c|c|c|c} & & & \\ \times & | & \end{array}, \quad s = \begin{array}{c|c|c|c} & & & \\ \times & \times & \end{array}, \quad t = \begin{array}{c|c|c|c} & & & \\ \times & | & \end{array},$$

so that the relations read as follows:

$$\begin{array}{c|c|c|c} & & & \\ x & x & x & x \end{array} = \begin{array}{c|c|c|c} & & & \\ x & x & x & x \end{array}, \quad \begin{array}{c|c|c|c} & & & \\ x & x & x & x \end{array} = \begin{array}{c|c|c|c} & & & \\ x & x & x & x \end{array}, \quad \begin{array}{c|c|c|c} & & & \\ x & x & x & x \end{array} = \begin{array}{c|c|c|c} & & & \\ x & x & x & x \end{array}.$$

In that case, one proves [Del97, GGM15], that all the syzygies are generated by the following
3.2. Categorical preliminaries

Zamolodchikov relation:

3.2. CATEGORICAL PRELIMINARIES

In this section, we recall the notion of 2-functor, (2, 1)-category, and 3-category used in this chapter.

3.2.1. Two-dimensional functors. In (2.1.13), we have introduced the notion of 2-category as a category enriched over the cartesian monoidal category \( \text{Cat} \) of categories. A (strict) 2-functor between 2-categories is a functor enriched in categories. Explicitly, given two 2-categories \( C \) and \( D \), a 2-functor \( F : C \to D \) is a data made of

i) a map \( F_0 : C_0 \to D_0 \),

ii) for every 0-cells \( x \) and \( y \) of \( C_0 \), a functor

\[
F_{x,y} : C(x, y) \to D(F_0(x), F_0(y)),
\]

such that the following diagrams commute in the category \( \text{Cat} \), for every 0-cells \( x, y, z \) in \( C_0 \)

\[
\begin{array}{ccc}
C(x, y) \times C(y, z) & \xrightarrow{\pi_0^{x,y,z}} & C(x, z) \\
F_{x,y} \times F_{y,z} & & F_{x,z}
\end{array}
\]

\[
\begin{array}{ccc}
D(F_0(x), F_0(y)) \times D(F_0(y), F_0(z)) & \xrightarrow{\pi_0^{F_0(x), F_0(y), F_0(z)}} & D(F_0(x), F_0(z)) \\
F_{F_0(x), F_0(y), F_0(z)} & & F_{F_0(x), F_0(y), F_0(z)}
\end{array}
\]
3.2. Categorical preliminaries

where $1$ denotes the terminal category:

$I$

with a single $0$-cell and a single $1$-cell, and the downward arrows map the single $0$-cell on identities $1$-cells.

If there is no possible confusion, we will write $F(x)$ for $F_0(x)$ and $F(u)$ (resp. $F(\alpha)$) for $F_{x,y}(u)$ (resp. $F_{x,y}(\alpha)$), where $u$ and $\alpha$ are $1$-cells and $2$-cells of $\mathbf{C}$ respectively.

The $2$-categories and their $2$-functors form a category that we will denote by $2\mathbf{Cat}$.

3.2.2. (2,1)-categories. A (small) groupoid, or $(1,0)$-category, is a $1$-category $\mathbf{C}$ in which all $1$-cells are isomorphisms, that is there is an inverse map $(-)^{-1} : \mathbf{C}_1 \to \mathbf{C}_1$ such that for any $1$-cell $u$ in $\mathbf{C}_1$, the following conditions hold:

$$uu^{-1} = 1_{s_0(u)}, \quad u^{-1}u = 1_{t_0(u)}.$$  \hspace{1cm}

A (2,1)-category is a category enriched over the cartesian monoidal category $\mathbf{Gpd}$ of groupoids. That is, it is a $2$-category $\mathbf{C}_2$, whose $2$-cells are invertible for the $1$-composition: for any $2$-cell $f : u \Rightarrow v$, there exists a $2$-cell $f^{-1} : v \Rightarrow u$, such that

$$f \circ f^{-1} = 1_u, \quad f^{-1} \circ f = 1_v.$$  \hspace{1cm}

3.2.3. Free (2,1)-category. Given a $2$-polygraph $\Sigma$, the free (2,1)-category over $\Sigma$ is denoted by $\Sigma_2^\top$ and defined as the free $2$-category generated by $\Sigma$, and whose every $2$-cell is invertible. Explicitly, its set of $0$-cells is $\Sigma_0$ and, for all $0$-cells $x$ and $y$, the groupoid $\Sigma_2^\top(x,y)$ is given as the quotient

$$\Sigma_2^\top(x,y) = (\Sigma \amalg \Sigma^\top)^\ast(x,y)/\text{Inv}(\Sigma_2),$$  \hspace{1cm}

where:

i) the $2$-polygraph $\Sigma^\top$ is defined from $\Sigma$ by reversing its $2$-cells, that is

$$\Sigma^\top_2 = \{ t_1(\alpha) \Rightarrow s_1(\alpha) \mid \alpha \in \Sigma_2 \},$$  \hspace{1cm}

ii) the cellular extension $\text{Inv}(\Sigma_2)$ contains the following two relations for every $2$-cell $\alpha$ of $\Sigma$ and all possible $1$-cells $u$ and $v$ of $\Sigma^\top_1$ such that $s(u) = x$ and $t(v) = y$:

$$u\alpha v \ast_1 u\alpha^\top v \equiv 1_{u\alpha(\alpha)\nu} \quad \text{and} \quad u\alpha^\top v *_1 u\alpha v \equiv 1_{u\alpha(\alpha)\nu}.$$  \hspace{1cm}

By definition of the $(2,1)$-category $\Sigma^\top_2$, for all $1$-cells $u$ and $\nu$ of $\Sigma^\top_1$, we have $\overline{u} = \overline{\nu}$ in the quotient category $\overline{\Sigma}$ if, and only if, there exists a $2$-cell $f : u \Rightarrow \nu$ in the $(2,1)$-category $\Sigma^\top_2$.

3.2.4. Lemma. Let $\mathbf{C}$ be a category and let $\Sigma$ and $\Upsilon$ be two $2$-polygraphs that present $\mathbf{C}$. There exist two $2$-functors

$$F : \Sigma_2^\top \to \Upsilon_2^\top \quad \text{and} \quad G : \Upsilon_2^\top \to \Sigma_2^\top$$  \hspace{1cm}

and, there exist two families of $2$-cells

$$(\sigma_u : GF(u) \Rightarrow u)_{u \in \Sigma_1^\top} \quad \text{and} \quad (\tau_\nu : FG(\nu) \Rightarrow \nu)_{\nu \in \Upsilon_1^\top}$$  \hspace{1cm}

in $\Sigma_2^\top$ and $\Upsilon_2^\top$ respectively, such that the following conditions are satisfied:
i) the 2-functors $F$ and $G$ induce the identity through the canonical projections onto $C$, that is the two following diagrams commute

\[
\begin{array}{ccc}
\Sigma_2 \overset{\pi_\Sigma}{\longrightarrow} C & \quad & \Sigma_2 \overset{\pi_\Sigma}{\longrightarrow} C \\
F \downarrow & & \downarrow G \\
\Upsilon_2 \overset{\pi_\Upsilon}{\longrightarrow} C & = & \Upsilon_2 \overset{\pi_\Upsilon}{\longrightarrow} C
\end{array}
\]

ii) the 2-cells $\sigma_u$ and $\tau_v$ are functorial in $u$ and $v$, that is

\[
\sigma_{uu'} = \sigma_u \sigma_{u'}, \quad \sigma_{1_x} = 1_{1_x},
\]

for any 1-cells $u$ and $u'$ and 0-cell $x$ and

\[
\tau_{vv'} = \tau_v \tau_{v'}, \quad \tau_{1_y} = 1_{1_y},
\]

for any 1-cells $v$ and $v'$ and 0-cell $y$.

**Proof.** We prove the existence of the functor $F$. The proof of the existence of the functor $G$ is similar. For a 0-cell $x$, we set $F(x) = x$. If $\alpha : x \to y$ is a 1-cell of $\Sigma$, we choose, in an arbitrary way, a 1-cell $F(\alpha) : x \to y$ in $\Upsilon_2$ such that $\pi_\Upsilon F(\alpha) = \pi_\Sigma(\alpha)$. Then, we extend $F$ to every 1-cell of $\Sigma_2$ by functoriality. Let $\alpha : u \Rightarrow u'$ be a 2-cell of $\Sigma_2$. Since $\Sigma$ is a presentation of the category $C$, we have $\pi_\Sigma(u) = \pi_\Sigma(u')$, so that $\pi_\Upsilon F(u) = \pi_\Upsilon F(u')$ holds. Using the fact that $\Upsilon$ is a presentation of the category $C$, we arbitrarily choose a 2-cell $F(\alpha) : F(u) \Rightarrow F(u')$ in the $(2, 1)$-category $\Upsilon_2$. Then, we extend $F$ to every 2-cell of $\Sigma_2$ by functoriality.

Now, let us define $\sigma$, the case of $\tau$ being symmetric. Let $\alpha$ be a 1-cell of $\Sigma$. By construction of $F$ and $G$, we have:

\[
\pi_\Sigma GF(\alpha) = \pi_\Upsilon F(\alpha) = \pi_\Sigma(\alpha).
\]

Since $\Sigma$ is a presentation of $C$, there exists a 2-cell $\sigma_\alpha : GF(\alpha) \Rightarrow \alpha$ in $\Sigma_2$. We extend $\sigma$ to every 1-cell $u$ of $\Sigma_2$ by functoriality.

3.2.5. 3-categories. The notion of 3-category is defined as the one of 2-category but by replacement of the hom-categories and composition functors by hom-2-categories and composition 2-functors. A (strict) 3-category is a category enriched in the category $\text{2Cat}$ of 2-categories. In particular, in a 3-category, the 3-cells can be composed in three different ways:

i) by $\star_0$, along their 0-dimensional boundary:
3.3. Coherent presentation of categories

ii) by $\star_1$, along their 1-dimensional boundary:

\[
\begin{array}{c}
\text{u} \\
\text{x} \quad \text{f} \Downarrow \text{A} \Rightarrow \text{f}' \Downarrow \text{v} \\
\text{g} \Rightarrow \text{B} \Rightarrow \text{g}' \\
\text{w} \quad \text{y}
\end{array}
\quad \mapsto
\begin{array}{c}
\text{u} \\
\text{x} \quad \text{f}_1 \Downarrow \text{A} \Rightarrow \text{f}'_1 \Downarrow \text{v} \\
\text{g} \Rightarrow \text{B} \Rightarrow \text{g}' \\
\text{w} \quad \text{y}
\end{array}
\]

iii) by $\star_2$, along their 2-dimensional boundary:

\[
\begin{array}{c}
\text{u} \\
\text{x} \quad \text{f} \Downarrow \text{A} \Rightarrow \text{g} \Rightarrow \text{B} \Rightarrow \text{h} \\
\text{v} \quad \text{y}
\end{array}
\quad \mapsto
\begin{array}{c}
\text{u} \\
\text{x} \quad \text{f} \Downarrow \text{A} \Rightarrow \text{h} \\
\text{g} \Rightarrow \text{B} \Rightarrow \text{h} \\
\text{v} \quad \text{y}
\end{array}
\]

The compositions in a 3-category satisfy the exchange relation, for every $0 \leq i < j \leq 2$:

\[(A \star_1 B) \star_1 (A' \star_1 B') = (A \star_1 A') \star_1 (B \star_1 B').\]

3.2.6. (3, 1)-categories. A (3, 1)-category is a 3-category whose 2-cells are invertible for the composition $\star_1$ and whose 3-cells are invertible for the composition $\star_2$.

3.2.7. Exercise. Show that in a (3, 1)-category, all the 3-cells are invertible for the composition $\star_1$.

3.3. COHERENT PRESENTATION OF CATEGORIES

3.3.1. Cellular extension of 2-categories. Let $\mathbf{C}$ be a 2-category. A 2-sphere of $\mathbf{C}$ is a pair $(f, g)$ of parallel 2-cells of $\mathbf{C}$, that is such that $s_1(f) = s_1(g)$ and $t_1(f) = t_1(g)$:

\[
\begin{array}{c}
\text{f} \\
\text{g}
\end{array}
\]

A cellular extension of the 2-category $\mathbf{C}$ is a set $\Gamma$ equipped with a map from $\Gamma$ to the set of 2-spheres of $\mathbf{C}$. It is equivalent to the data of a set $\Gamma$ with two maps

\[
\begin{array}{c}
\text{C}_2 \\
\text{s}_2 \\
\text{t}_2
\end{array}
\quad \Gamma
\]
satisfying the globular relations $s_1s_2 = s_1t_2$ and $t_1s_2 = t_1t_2$. A congruence on the 2-category $C$ is an equivalence relation $\equiv$ on the parallel 2-cells of $C$ such that, for every cells

$$
\begin{array}{c}
\text{x} \xrightarrow{w} \text{y} \\
\Downarrow f \\
\Downarrow g
\end{array}
\quad
\begin{array}{c}
\text{z} \xrightarrow{w'} \text{t}
\Downarrow k
\Downarrow h
\end{array}
$$

of $C$, if $f \equiv g$, then

$$w \ast_0 (h \ast_1 f \ast_1 k) \ast_0 w' \equiv w \ast_0 (h \ast_1 g \ast_1 k) \ast_0 w'.$$

If $\Gamma$ is a cellular extension of $C$, the congruence generated by $\Gamma$ is denoted by $\equiv_\Gamma$ and defined as the smallest congruence such that, if $\Gamma$ contains a 3-cell $\gamma : f \Rightarrow g$, then $f \equiv_\Gamma g$. The quotient 2-category of a 2-category $C$ by a congruence relation $\equiv$ is the 2-category, denoted by $C/ \equiv$, whose 0-cells and 1-cells are those of $C$ and the 2-cells are the equivalence classes of 2-cells of $C$ modulo the congruence $\equiv$.

### 3.3.2. Acyclicity

A cellular extension $\Gamma$ of a 2-category $C$ is called **acyclic** if for every parallel 2-cells $f$ and $g$ of $C$, we have $f \equiv_\Gamma g$, that is, the equality $\tilde{f} = \tilde{g}$ holds in the quotient 2-category $C/ \equiv_\Gamma$. For instance, the set of 2-spheres of $C$ forms an acyclic extension of $C$. In the literature, an acyclic extension of $C$ is also called an **homotopy basis** of $C$.

### 3.3.3. $(3,1)$-polygraphs

A $(3,1)$-polygraph is a data $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ made of a 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ and a cellular extension $\Sigma_3$ of the free $(2,1)$-category $\Sigma_2^\top$ over $\Sigma_2$, as summarised in the following diagram:

$$
\begin{array}{c}
\Sigma_0 \xrightarrow{s_0} \Sigma_1 \xrightarrow{s_1} \Sigma_2 \xrightarrow{s_2} \Sigma_3 \\
\xrightarrow{t_0} \xrightarrow{t_1} \xrightarrow{t_2} \Gamma_3
\end{array}
$$

### 3.3.4. Coherent presentations

A **coherent presentation** of a 1-category $C$ is a $(3,1)$-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2, \Sigma_3)$ such that the 2-polygraph $(\Sigma_0, \Sigma_1, \Sigma_2)$ is a presentation of $C$ and $\Sigma_3$ is an acyclic cellular extension of the $(2,1)$-category $\Sigma_2^\top$.

### 3.3.5. Free $(3,1)$-categories

Given a $(3,1)$-polygraph $\Sigma$, the **free $(3,1)$-category over $\Sigma$** is denoted by $\Sigma_2^\top$ and defined as follows:

i) its underlying 2-category is the free $(2,1)$-category $\Sigma_2^\top$,

ii) its 3-cells are all the formal compositions by $\ast_0$, $\ast_1$ and $\ast_2$ of 3-cells of $\Sigma_3$, of their inverses and of identities of 2-cells, up to associativity, identity, exchange and inverse relations.

In particular, we get that $\Sigma_3$ is an acyclic extension of $\Sigma_2^\top$ if, and only if, for every pair $(f, g)$ of parallel 2-cells of $\Sigma_2^\top$, there exists a 3-cell $A : f \Rightarrow g$ in $\Sigma_3$.
3.4. Finite derivation type

3.4.1. 2-polygraphs of finite derivation type. A 2-polygraph \( \Sigma \) is of finite derivation type, FDT for short, if it is finite and if the free \((2,1)\)-category \(\Sigma_2^T\) admits a finite acyclic cellular extension. A category \( \mathbf{C} \) is said to be of finite derivation type if it admits a finite coherent presentation. Let us prove now that this property does not depend on this presentation provide is finite. The proof is based on the following theorem, that allows transfers of acyclic cellular extensions of two \((2,1)\)-categories that present the same category.

3.4.2. Homotopy bases transfer theorem. Given a category \( \mathbf{C} \) a category, we consider two presentations \( \Sigma \) and \( \Upsilon \) of \( \mathbf{C} \). By Lemma 3.2.4 there exist 2-functors

\[
F : \Sigma_2^T \to \Upsilon_2^T \quad \text{and} \quad G : \Upsilon_2^T \to \Sigma_2^T
\]

and for every 1-cell \( v \) of \( \Upsilon_2^T \), there exists a 2-cell \( \tau_v : FG(v) \Rightarrow v \) in \( \Upsilon_2^T \) that satisfy the conditions given in Lemma 3.2.4.

Let define the cellular extension \( \tau_\Gamma \) of the \((2,1)\)-category \( \Upsilon_2^T \) that contains one 3-cell

\[
\begin{array}{cccc}
FG(\alpha) & FG(v) & \tau_v \\
FG(u) & \tau_u & \alpha & v \\
\end{array}
\]

for every 2-cell \( \alpha : u \Rightarrow v \) of \( \Upsilon_2 \).

Given a cellular extension \( \Gamma \) of the \((2,1)\)-category \( \Sigma_2^T \), we denote by \( F(\Gamma) \) the cellular extension of \( \Sigma_2^T \) that contains one 3-cell

\[
\begin{array}{cccc}
F(f) & F(g) & \tau_\gamma \\
F(u) & F(\gamma) & F(v) \\
\end{array}
\]

for every 3-cell \( \gamma : f \Rightarrow g \) of \( \Gamma \).

Using these notations, we can formulate the following result, called the acyclicity transfer theorem in [GM18].

3.4.3. Theorem. If \( \Gamma \) is an acyclic cellular extension of the \((2,1)\)-category \( \Sigma_2^T \), then the cellular extension

\[
\Delta = F(\Gamma) \sqcup \tau_\Gamma
\]

is an acyclic cellular extension of the \((2,1)\)-category \( \Upsilon_2^T \).
3.4.2. Homotopy bases transfer theorem

Proof. Let us define, for every 2-cell \( f : u \Rightarrow v \) of \( \Upsilon^2 \), a 3-cell \( \tau_f \) of the free \((3, 1)\)-category \( \Delta^3 \) with the following shape:

![Diagram](image)

We extend the notation \( \tau_\alpha \), where \( \alpha \) is a 2-cell of \( \Upsilon^2 \) in a functorial way, according to the following formulas:

\[
\tau_{1u} = 1_{\tau_u}, \quad \tau_{fg} = \tau_f \tau_g, \quad \tau_f^* = FG(f)^* \tau_f^* 1^*, \quad \tau_{f*}^* = (FG(f)^* \tau_g^*) \tau_{f*}^* (FG(f)^* \tau_g^*).
\]

One checks that the 3-cells \( \tau_f \) are well-defined, i.e., that their definition is compatible with the relations on 2-cells, such as the exchange relation:

\[
\tau_{fg*}^* 1^* 1^* = \tau_{(f^* g)^*}^* 1^* 1^*.
\]

Now, let us consider parallel 2-cells \( f, g : u \Rightarrow v \) of \( \Upsilon^2 \). The 2-cells \( G(f) \) and \( G(g) \) are parallel in \( \Sigma^2 \) so that, since \( \Gamma \) is an acyclic cellular extension of \( \Sigma^2 \), there exists a 3-cell

![Diagram](image)

in \( \Gamma^3 \). An application of \( F \) to \( A \) gives the 3-cell

![Diagram](image)

which, by definition of the cellular extension \( \Delta \) and functoriality of \( F \), is in \( \Delta^3 \). Using the 3-cells \( F(A) \), \( \tau_f \) and \( \tau_g \), we get the following 3-cell from \( f \) to \( g \) in \( \Delta^3 \):

![Diagram](image)
3.5. Coherence from convergence

This concludes the proof that $\Delta = F(\Gamma) \amalg \tau_\Upsilon$ is an acyclic cellular extension of the $(2, 1)$-category $\Upsilon_2^\top$.

We deduce from Theorem 3.4.3 that the finite derivation type property is Tietze invariant for finite 2-polygraphs:

3.4.4. Theorem ([SOK94, Theorem 4.3]). Let $\Sigma$ and $\Upsilon$ be two Tietze-equivalent finite 2-polygraphs. Then $\Sigma$ is of finite derivation type if and only if $\Upsilon$ is of finite derivation type.

The result of the following exercise is useful to prove that a presentation admits no finite acyclic cellular extensions.

3.4.5. Exercise. Let $\Sigma$ be a 2-polygraph and let $\Gamma$ be an acyclic cellular extension of the free $(2, 1)$-category $\Sigma_2^\top$. Show that if $\Sigma_2^\top$ admits a finite acyclic cellular extension, then there exists a finite subset of $\Gamma$ that is an acyclic cellular extension of $\Sigma_2^\top$.

3.5. Coherence from convergence

3.5.1. Generating confluences. Squier’s completion procedure provides a way to extend a convergent presentation of a 1-category $C$ into a coherent presentation of $C$. We fix a convergent 2-polygraph $\Sigma$. A family of generating confluences of $\Sigma$ is a cellular extension of the free $(2, 1)$-category $\Sigma_2^\top$ that contains exactly one 3-cell for every critical branching $(f, g)$ of $\Sigma$.

Note that, if $\Sigma$ is confluent, it always admits a family of generating confluences. However, such a family is not necessarily unique, since the 3-cell $A_{f,g}$ can be directed in the reverse way and, for a given branching $(f, g)$, we can have several possible 2-cells $f'$ and $g'$ with the required shape. Later, we will define the notion of normalisation strategies that provide a deterministic way to construct a family of generating confluences.

3.5.2. Squier’s completion for convergent presentations. A Squier’s completion of a convergent 2-polygraph $\Sigma$ is the $(3, 1)$-polygraph denoted by $S(\Sigma)$ and defined by $S(\Sigma) = (\Sigma, \Gamma)$, where $\Gamma$ is a chosen family of generating confluences of $\Sigma$. The first proof of the following result is due to Squier, [SOK94], in the case where the category $C$ is a monoid. We present the proof given in [GM18] in the language of polygraphs.

3.5.3. Theorem ([SOK94, Theorem 5.2]). For every convergent presentation $\Sigma$ of a category $C$, Squier’s completion of $\Sigma$ is a coherent presentation of $C$.

Proof. We proceed in three steps.
3.5.4. Step 1. We prove that, for every local branching \((f, g) : u \Rightarrow (v, w)\) of \(\Sigma\), there exist 2-cells \(f' : v \Rightarrow u'\) and \(g' : w \Rightarrow u'\) in \(\Sigma_2\) and a 3-cell \(A : f \ast_1 f' \Rightarrow g \ast_1 g'\) in \(\delta(\Sigma)_3\), as in the following diagram:

![Diagram](image)

As we have seen in the study of confluence of local branchings, in the case of an aspherical or Peiffer branching, we can choose \(f'\) and \(g'\) such that \(f \ast_1 f' = g \ast_1 g'\): an identity 3-cell is enough to link them. Moreover, if we have an overlapping branching \((f, g)\) that is not critical, we have \((f, g) = (uhv, ukv)\) with \((h, k)\) critical. We consider the 3-cell \(\alpha : h \ast_1 h' \Rightarrow k \ast_1 k'\) of \(\delta(\Sigma)\) corresponding to the critical branching \((h, k)\) and we conclude that the following 2-cells \(f'\) and \(g'\) and 3-cell \(A\) satisfy the required conditions:

\[
f' = uh'v \quad g' = uk'v \quad A = u\alpha v.
\]

3.5.5. Step 2. We prove that, for every parallel 2-cells \(f\) and \(g\) of \(\Sigma_2\) whose common target is a normal form, there exists a 3-cell from \(f\) to \(g\) in \(\delta(\Sigma)_3\). We proceed by noetherian induction on the common source \(u\) of \(f\) and \(g\), using the termination of \(\Sigma\). Let us assume that \(u\) is a normal form: then, by definition, both 2-cells \(f\) and \(g\) must be equal to the identity of \(u\), so that \(1_u : 1_u \Rightarrow 1_u\) is a 3-cell of \(\delta(\Sigma)_3\) from \(f\) to \(g\).

Now, let us fix a 1-cell \(u\) with the following property: for any 1-cell \(v\) such that \(u\) rewrites into \(v\) and for any parallel 2-cells \(f, g : v \Rightarrow \tilde{v}\) of \(\Sigma_2\), there exists a 3-cell from \(f\) to \(g\) in \(\delta(\Sigma)_3\). Let us consider parallel 2-cells \(f, g : u \Rightarrow \tilde{u}\) and let us prove the result by progressively constructing the following composite 3-cell from \(f\) to \(g\) in \(\delta(\Sigma)_3\):

![Diagram](image)

Since \(u\) is not a normal form, we can decompose \(f = f_1 \ast_1 f_2\) and \(g = g_1 \ast_1 g_2\) so that \(f_1\) and \(g_1\) are rewriting steps. They form a local branching \((f_1, g_1)\) and we build the 2-cells \(f'_1\) and \(g'_1\), together with the 3-cell \(A\) as in the first part of the proof. Then, we consider a 2-cell \(h\) from \(u'\) to \(\tilde{u}\) in \(\Sigma_2\), that must exist by confluence of \(\Sigma\) and since \(\tilde{u}\) is a normal form. We apply the induction hypothesis to the parallel 2-cells \(f_2\) and \(f_1 \ast_1 h\) in order to get \(B\) and, symmetrically, to the parallel 2-cells \(g_2\) and \(g_1 \ast_1 h\) to get \(C\).
3.5. Coherence from convergence

3.5.6. Step 3. We prove that every $2$-sphere of $\Sigma^T_3$ is the boundary of a $3$-cell of $\mathcal{S}(\Sigma)_3^T$. First, let us consider a $2$-cell $f : u \Rightarrow v$ in $\Sigma^T_2$. Using the confluence of $\Sigma$, we choose $2$-cells

$$\sigma_u : u \Rightarrow \hat{u} \quad \text{and} \quad \sigma_v : v \Rightarrow \hat{v} = \hat{u}$$

in $\Sigma^T_2$. By construction, the $2$-cells $f \ast_1 \sigma_v$ and $\sigma_u$ are parallel and their common target $\hat{u}$ is a normal form. Thus by Step 2, there exists a $3$-cell $f \ast_1 \sigma_v$ from $f \ast_1 \sigma_v$ to $\sigma_u$, or, equivalently, a $3$-cell $\sigma_f$ from $f$ to $\sigma_u \ast_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)_3^T$, as in the following diagram:

![Diagram](attachment:image.png)

Moreover, the free $(3,1)$-category $\mathcal{S}(\Sigma)_3^T$ contains a $3$-cell $\sigma_f$ from $f^-$ to $\sigma_v \ast_1 \sigma_u^-$, given as the following composite:

![Diagram](attachment:image.png)

Now, let us consider a general $2$-cell $f : u \Rightarrow v$ of $\Sigma^T_2$. By construction of the free $(2,1)$-category $\Sigma^T_2$, the $2$-cell $f$ can be decomposed into a “zig-zag”, that is non-unique in general,

![Diagram](attachment:image.png)

where each $f_i$ and $g_i$ is a $2$-cell of $\Sigma^T_2$. We define $\sigma_f$ as the following composite $3$-cell of $\mathcal{S}(\Sigma)_3^T$, with source $f$ and target $\sigma_u \ast_1 \sigma_v$:

![Diagram](attachment:image.png)

We proceed similarly for any other $2$-cell $g : u \Rightarrow v$ of $\Sigma^T_2$, to get a $3$-cell $\sigma_g$ from $g$ to $\sigma_u \ast_1 \sigma_v^-$ in $\mathcal{S}(\Sigma)_3^T$. Thus, the composite $\sigma_f \ast_2 \sigma_g^-$ is a $3$-cell of the free $(3,1)$-category $\mathcal{S}(\Sigma)_3^T$ from $f$ to $g$, concluding the proof.

Theorem 3.5.3 is extended to higher-dimensional polygraphs in [GM09] Proposition 4.3.4.

3.5.7. Theorem ([SOK94 Theorem 5.3]). If a monoid admits a finite convergent presentation, then it is of finite derivation type.
3.5.8. Example. Consider the convergent presentation $KB(\Gamma)$ of the braid monoid $B_3^+$ given in (2.6.7). It has four critical branchings given in (2.6.8). We deduce an acyclic extension of the $(2,1)$-category $KB(\Gamma)\uparrow$, with the following 3-cells:

\[
\begin{align*}
\beta a & \rightarrow aa \\
\gamma t & \rightarrow aat \\
\gamma s & \rightarrow aas \\
\gamma a & \rightarrow aaaa
\end{align*}
\]

3.5.10. Example [LP91]. Consider the following 2-polygraph:

\[
\Sigma = \langle a, b, c, d \mid \alpha : ab \Rightarrow a, \beta : da \Rightarrow ac \rangle.
\]

The 2-polygraph $\Sigma$ is not convergent and can be completed into the following infinite but convergent polygraph

\[
KB(\Sigma) = \langle a, b, c, d \mid \alpha_n : ac^n b \Rightarrow ac^n, \ n \in \mathbb{N}, \beta : da \Rightarrow ac \rangle,
\]

with an infinity of confluent critical branchings:

\[
\begin{align*}
d\alpha_n & \rightarrow dac^n \\
dac^n b & \rightarrow \beta e^n \\
\beta e^n b & \rightarrow ac^{n+1} b \\
ac^{n+1} b & \rightarrow A_n
\end{align*}
\]

By Theorem 3.5.3, the 2-polygraph $KB(\Sigma)$ can be extended into a coherent presentation of the monoid $\Sigma$ presented by $\Sigma$ with infinitely many $3$-cells $A_n$, for $n$ in $\mathbb{N}$.

Now, consider the following 2-polygraph

\[
\Gamma = \langle a, b, c, d \mid \alpha : ab \Rightarrow a, \gamma : ac \Rightarrow da \rangle.
\]

It presents the monoid $\Sigma$ and it is convergent with no critical branching. It follows that it forms a coherent presentation of the monoid $\Sigma$ with no $3$-cell.

3.5.11. Exercise. The *standard presentation* of a category $C$ is the 2-polygraph $Std_2(C)$ defined as follows. The 0-cells and 1-cells of $Std_2(C)$ are the ones of $C$, with $\hat{u}$ denoting a 1-cell $u$ of $C$ when seen as a 1-cell of $Std_2(C)$. The 2-polygraph $Std_2(C)$ contains a 2-cell

\[
\begin{align*}
\hat{u} & \rightarrow y \\
y & \rightarrow y_{u,v} \\
\hat{v} & \rightarrow z
\end{align*}
\]
3.5. Coherence from convergence

for all 1-cells \( u : x \to y \) and \( v : y \to z \) of \( C \), and a 2-cell

\[
\begin{array}{c}
1_x \\
\circlearrowleft_{l_x} \\
1_x
\end{array}
\]

for every 0-cell \( x \) of \( C \).

Extend this 2-polygraph into a coherent presentation of the category \( C \).

3.5.12. Exercise. Let us consider the monoid \( M \) presented by the 2-polygraph

\[
\Sigma = \langle x, y \mid \alpha : xyx \Rightarrow yy \rangle.
\]

1. Prove that \( \Sigma \) terminates.

2. Complete \( \Sigma \) into a coherent presentation of the monoid \( M \).

3.5.13. Squier’s example. In \([\text{Squ87}]\), Squier defines, for every \( k \geq 1 \), the monoid \( S_k \) presented by the 2-polygraph

\[
\langle a, b, t, x_1, \ldots, x_k, y_1, \ldots, y_k \mid (\alpha_n)_{n \in \mathbb{N}}, (\beta_i)_{1 \leq i \leq k}, (\gamma_i)_{1 \leq i \leq k}, (\delta_i)_{1 \leq i \leq k}, (\epsilon_i)_{1 \leq i \leq k} \rangle
\]

with

\[
\alpha_n \at^{n} b \Rightarrow 1, \quad x_i a \Rightarrow a t x_i, \quad x_i t \Rightarrow t x_i, \quad x_i b \Rightarrow b x_i, \quad x_i y_i \Rightarrow 1.
\]

In \([\text{SOK94}]\), Squier proves the following finiteness properties for the monoid \( S_1 \). With similar arguments, the result extends to every monoid \( S_k \), for \( k \geq 1 \).

3.5.14. Theorem (SOK94, Theorem 6.7, Corollary 6.8). For every \( k \geq 1 \), the monoid \( S_k \) satisfies the following properties:

i) it is finitely presented,

ii) it has a decidable word problem,

iii) it is not of finite derivation type,

iv) it admits no finite convergent presentation.

This result shows in particular, that the property of being decidable is not sufficient for finitely presented monoids to have a finite convergent presentation or to have finite derivation type. Let us prove the result in the case of the monoid \( S_1 \), with the following infinite presentation:

\[
\Sigma^{\text{Sq}_1} = \langle a, b, t, x, y \mid (\alpha_n)_{n \in \mathbb{N}}, \beta, \gamma, \delta, \epsilon \rangle
\]

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whose rules are defined by

\[
\begin{align*}
\alpha_n &: at^n b \implies 1, \\
\beta &: x a \implies at x, \\
\gamma &: x t \implies tx, \\
\delta &: x b \implies bx, \\
\epsilon &: x y \implies 1.
\end{align*}
\]

We will denote by \( \gamma_n : xt^n \implies t^n x \) the 2-cell of \((\Sigma^{Sq_1})^*_2\) defined by induction on \( n \) as follows:

\[
\gamma_0 = 1_x \quad \text{and} \quad \gamma_{n+1} = \gamma t^n \star t \gamma_n.
\]

For every \( n \), we write \( f_n : x at^n b \implies at^{n+1} bx \) the 2-cell of \((\Sigma^{Sq_1})^*_2\) defined as the following composite:

\[
x at^n b \xrightarrow{\beta t^n b} atxt^nb \xrightarrow{at\gamma_n b} at^{n+1}xb \xrightarrow{at^{n+1}\delta} at^{n+1}bx.
\]

3.5.15. Exercise.

1. Show that the monoid \( S_1 \) admits the following finite presentation:

\[
\langle a, b, t, x, y \mid \alpha_0, \beta, \gamma, \delta, \epsilon \rangle.
\]

2. Show that the monoid \( S_1 \) has a decidable word problem.

3.5.16. Exercise, \cite{GM18}.

1. Show that the 2-polygraph \( \Sigma^{Sq_1} \) is convergent and Squier’s completion of \( \Sigma^{Sq_1} \) contains a 3-cell \( A_n \) for every natural number \( n \) with the following shape:

\[
\begin{array}{c}
\xymatrix{ & at^{n+1}bx & \\
\alpha_{n+1}x & & \\
\ar@{=>}[ur]^{f_n} & & \\
\ar@{=>}[urr] & A_n & \\
\ar@{=>}[rr] & & x \\
xat^n b & & \\
\ar@{=>}[urr] & & \alpha_n \\
\ar@{=>}[ur] & \}
\end{array}
\]

2. Show that the monoid \( S_1 \) is not of finite derivation type.

3.5.17. Exercise, \cite{LP91, Laf95}. Consider the monoid \( M \) presented by the following 2-polygraph:

\[
\langle a, b, c, d, d' \mid \alpha_0 : ab \Rightarrow a, \beta : da \Rightarrow ac, \gamma : d'a \Rightarrow ac \rangle.
\]

Show that the monoid \( M \) admits a finite presentation, it has a decidable word problem, yet it is not of finite derivation type and, as a consequence, it does not admit a finite convergent presentation.
3.5. Coherence from convergence
In this chapter we present the result obtained by Squier relating the finite-convergence of a string rewriting system with the homotopical type left-FP3. \cite{Squ87}. The constructions developed in this chapter come from \cite{GM18}.

4.1. Preliminaries on modules

In this section, we fix a ring \( R \). We will say “R-module” of “module” for “left R-module”. All the notions presented are defined in the same manner for right R-modules since every right R-module is a left \( R^{\text{op}} \)-module, where \( R^{\text{op}} \) is the opposite ring. We will say “homomorphism” for a homomorphism of left R-modules. We refer the reader to \cite{Lan02} or to \cite{Rot09} for a deeper presentation and the proofs of the results given in this preliminary part on modules.

4.1.1. Exact sequences. Two homomorphisms of modules

\[ M' \xrightarrow{f} M \xrightarrow{g} M'' \]
4.1. Preliminaries on modules

are exact at $M$ if $\text{Im} \ f = \ker g$. A sequence of homomorphisms

$$\cdots \rightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \rightarrow \cdots$$

is exact if each adjacent pair of homomorphisms is exact.

4.1.2. Examples. If $0 \rightarrow M \xrightarrow{f} M' \rightarrow 0$ is exact, then the map $f$ is injective. If $M \xrightarrow{f} M' \rightarrow 0$ is exact, then the map $f$ is surjective. If the sequence $0 \rightarrow M \xrightarrow{f} M' \rightarrow 0$ is exact, then the map $f$ is an isomorphism.

4.1.3. Free modules. A $R$-module $M$ is free if it is a direct sum of copies of $R$. If $M = \bigoplus_{i \in I} Rx_i$, with $R \xrightarrow{\sim} Rx_i$, the the set $\{x_i | i \in I\}$ is called a basis of $M$. It follows that each element $x$ in $M$ has a unique decomposition

$$x = \sum_{i \in I} \lambda_i x_i,$$

where $\lambda_i \in R$ and almost all $\lambda_i$ are zero.

4.1.4. Proposition. Let $X = \{x_i | i \in I\}$ be a basis of a free module $M$. For any module $N$ and any map $f : X \rightarrow N$, there is a unique map $\tilde{f} : M \rightarrow N$ extending $f$, i.e., such that the following diagram commutes:

\[\begin{array}{ccc}
N & \xrightarrow{f} & X \\
\downarrow & & \downarrow \\
M & \xrightarrow{\tilde{f}} & \text{graph of } f
\end{array}\]

4.1.5. Proposition. Let $X$ be a set. There exists a free $R$-module having $X$ as a basis.

4.1.6. Proposition. Every $R$-module is a quotient of a free $R$-module.

4.1.7. Finitely generated modules. A $R$-module $M$ is finitely generated if there is a finite subset $\{x_1, x_2, \ldots, x_n\}$ of $M$ such that for all $x$ in $M$, there exist $r_1, r_2, \ldots, r_n$ in $R$ with $x = r_1x_1 + r_2x_2 + \ldots + r_nx_n$. Then the set $\{x_1, x_2, \ldots, x_n\}$ is referred to as a generating set for $M$. The finite generators need not be a basis, since they need not be linearly independent over $R$. A $R$-module $M$ is finitely generated if and only if there is a surjective homomorphism:

$$R^n \rightarrow M$$

for some $n$. That is, $M$ is a quotient of a free module of finite rank.
4.1.8. Proposition. Let $F$, $M$ and $N$ be left $R$-modules. If $F$ is free, $\varepsilon : M \rightarrow N$ is a surjective homomorphism and $f : F \rightarrow N$ is any homomorphism, then there exists a homomorphism $\tilde{f} : F \rightarrow M$ such that following diagram commutes

\[
\begin{array}{c}
F \\
\downarrow f \\
M \xrightarrow{\varepsilon} N \xrightarrow{f} 0
\end{array}
\]

As a consequence of Proposition 4.1.8, for any free $R$-module, the functor $\text{Hom}_R(F, -)$ is exact.

4.1.9. Projective modules. A projective module is a module which behaves as the free module $F$ in Proposition 4.1.8. More explicitly, a $R$-module $P$ is projective if whenever $\varepsilon : M \rightarrow N$ is a surjective homomorphism and $f : P \rightarrow N$ is any homomorphism, there exists a homomorphism $\tilde{f} : P \rightarrow M$ making the following diagram commutative:

\[
\begin{array}{c}
P \\
\downarrow f \\
M \xrightarrow{\varepsilon} N \xrightarrow{f} 0
\end{array}
\]

In particular, any free module is projective.

The following result gives several ways to characterise projective modules.

4.1.10. Proposition. The following conditions are equivalent for a $R$-module $P$:

i) $P$ is projective,

ii) if $f : M \rightarrow P$ is a surjective homomorphism, then there exists $h : P \rightarrow M$ such that $fh = \text{Id}_P$,

iii) if $f : M \rightarrow P$ is a surjective homomorphism, then $M \simeq P \oplus \ker f$,

iv) the functor $\text{Hom}_R(P, -)$ is exact, that is for any exact sequence $0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$, the induced sequence $0 \rightarrow \text{Hom}_R(P, M') \rightarrow \text{Hom}_R(P, M) \rightarrow \text{Hom}_R(P, M'') \rightarrow 0$ is exact,

v) $P$ is a summand of a free module, that is there exists a free $R$-module $F$ such that $F \simeq P \oplus Q$, for some $R$-module $Q$.

4.1.11. Proposition (Schanuel’s Lemma). Given exact sequences of $R$-modules

\[
0 \rightarrow K_1 \rightarrow P_1 \rightarrow M \rightarrow 0,
\]

\[
0 \rightarrow K_2 \rightarrow P_2 \rightarrow M \rightarrow 0,
\]

where $P_1$ and $P_2$ are projective. Then $K_1 \oplus P_2 \simeq K_2 \oplus P_1$.

\[\text{by the equivalence, the } R\text{-module } Q \text{ is necessarily projective.}\]
4.1. Preliminaries on modules

4.1.12. Exercise. Prove Proposition 4.1.11

4.1.13. Proposition (Generalised Schanuel’s Lemma). Given exact sequences of $R$-modules

\[ 0 \rightarrow K \rightarrow P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0, \]
\[ 0 \rightarrow L \rightarrow Q_k \rightarrow Q_{k-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0, \]

where all the $P_i$ and $Q_i$ are projective. Let

\[ P_{\text{odd}} = \bigoplus_{i \text{ odd}} P_i, \quad P_{\text{even}} = \bigoplus_{i \text{ even}} P_i, \]

and

\[ Q_{\text{odd}} = \bigoplus_{i \text{ odd}} Q_i, \quad Q_{\text{even}} = \bigoplus_{i \text{ even}} Q_i. \]

Then the following properties hold

i) If $k$ is even, then $K \oplus Q_{\text{even}} \oplus P_{\text{odd}} \simeq L \oplus Q_{\text{odd}} \oplus P_{\text{even}}$.

ii) If $k$ is odd then $K \oplus Q_{\text{odd}} \oplus P_{\text{even}} \simeq L \oplus Q_{\text{even}} \oplus P_{\text{odd}}$.

Let us mention a consequence of the Proposition 4.1.13


\[ 0 \rightarrow K \rightarrow P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M \rightarrow 0, \]
\[ 0 \rightarrow L \rightarrow Q_k \rightarrow Q_{k-1} \rightarrow \cdots \rightarrow Q_1 \rightarrow Q_0 \rightarrow M \rightarrow 0, \]

where all the $P_i$ and $Q_i$ are finitely generated and projective, then the $R$-module $K$ is finitely generated if and only if $L$ is finitely generated.

4.1.15. Exercise. Prove Proposition 4.1.13

4.1.16. Chain’s complex. A (chain) complex of $R$-modules is a sequence $(M_n)_{n \in \mathbb{N}}$ of $R$-modules, together with a sequence $(d_n)_{n \in \mathbb{N}}$ of homomorphisms

\[ \cdots \rightarrow M_n \xrightarrow{d_n} M_{n-1} \rightarrow \cdots \rightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \]

such that we have the inclusion

\[ \text{Im } d_{n+1} \subseteq \ker d_n \]

for all $n$, or equivalently, the relation $d_n d_{n+1} = 0$ holds for all $n$. The map $d_n$ are called boundary maps.
4.1.17. Resolutions. A resolution of a R-module M is an exact sequence of R-modules

\[ \cdots \rightarrow M_n \xrightarrow{d_n} M_{n-1} \rightarrow \cdots \rightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \rightarrow 0 \]

From the definition, the homomorphism \( \varepsilon \) is surjective and

\[ \text{Im } d_1 = \text{ker } \varepsilon, \quad \text{and } \text{Im } d_{n+1} = \text{ker } d_n, \text{ for all } n. \]

Such a resolution is called projective (resp. free) if all the modules \( M_n \) are projective (resp. free).

Given a natural number \( n \), a partial resolution of length \( n \) of \( M \) is defined in a similar way but with a bounded sequence \( (M_k)_{0 \leq k \leq n} \) of R-modules:

\[ \cdots \rightarrow M_n \xrightarrow{d_n} M_{n-1} \rightarrow \cdots \rightarrow M_2 \xrightarrow{d_2} M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \rightarrow 0 \]

4.1.18. Proposition. Every R-module M has a free resolution.


4.1.20. Contracting homotopies. Given a complex of R-modules

\[ \cdots \rightarrow M_{n+1} \xrightarrow{d_{n+1}} M_n \xrightarrow{d_n} M_{n-1} \rightarrow \cdots \rightarrow M_1 \xrightarrow{d_1} M_0 \xrightarrow{\varepsilon} M \rightarrow 0 \]

(4.1.21)
a method to prove that such a complex is a resolution of M is to construct a contracting homotopy, that is a sequence of homomorphisms of \( \mathbb{Z} \)-modules

\[ \cdots \xleftarrow{i_{n+1}} M_{n+1} \xleftarrow{i_{n}} M_n \xleftarrow{i_{n-1}} M_{n-1} \xleftarrow{\cdots} M_1 \xleftarrow{i_1} M_0 \xleftarrow{i_0} M \]

satisfying the following equalities

\[ \varepsilon i_0 = \text{Id}_M, \]
\[ d_1 i_1 + i_0 \varepsilon = \text{Id}_M, \]
\[ d_{n+1} i_{n+1} + i_n d_n = \text{Id}_{M_n}, \text{ for all } n \in \mathbb{N}. \]

Indeed, in that case, the first equality proves that the homomorphism \( \varepsilon \) is surjective. Moreover, for every natural number \( n \) and every \( x \) in \( \text{ker } d_n \), the equality \( d_{n+1} i_{n+1}(x) = x \) holds, proving that \( x \) is in \( \text{Im } d_{n+1} \), so that \( \text{ker } d_n \subseteq \text{Im } d_{n+1} \) holds. As a consequence, the complex (4.1.21) is a resolution of the R-module M.

4.2. MONOIDS OF FINITE HOMOLOGICAL TYPE

Let \( M \) be a monoid. We denote by \( \mathbb{Z}M \) the ring generated by \( M \), that is, the free abelian group over \( M \), equipped with the canonical extension of the product of \( M \):

\[ \left( \sum_{u \in M} \lambda_u u \right) \left( \sum_{v \in M} \lambda_v v \right) = \sum_{u, v \in M} \lambda_u \lambda_v uv = \sum_{w \in M} \sum_{uv = w} \lambda_u \lambda_v w, \]

with \( \lambda_u, \lambda_v \) in \( \mathbb{Z} \). The trivial \( \mathbb{Z}M \)-module is the abelian group \( \mathbb{Z} \) equipped with the trivial action \( \mathbb{Z}n = n \), for every \( u \) in \( M \) and \( n \) in \( \mathbb{Z} \).
4.2. Monoids of finite homological type

4.2.1. Homological type left-\(\text{FP}_n\). A monoid \(M\) is of homological type left-\(\text{FP}_n\), for a natural number \(n\), if there exists a partial resolution of length \(n\) of the trivial \(\mathbb{Z}M\)-module \(\mathbb{Z}\) by projective, finitely generated \(\mathbb{Z}M\)-modules:

\[
P_n \xrightarrow{d_n} P_{n-1} \xrightarrow{d_{n-1}} \cdots \xrightarrow{d_2} P_1 \xrightarrow{d_1} P_0 \xrightarrow{d_0} \mathbb{Z} \xrightarrow{} 0.
\]

A monoid \(M\) is of homological type left-\(\text{FP}_\infty\) if there exists a resolution of \(\mathbb{Z}\) by projective, finitely generated \(\mathbb{Z}M\)-modules.

4.2.2. Proposition. Let \(M\) be a monoid and let \(n\) be a natural number. The following assertions are equivalent:

i) The monoid \(M\) is of homological type left-\(\text{FP}_n\).

ii) There exists a free, finitely generated partial resolution of the trivial \(\mathbb{Z}M\)-module \(\mathbb{Z}\) of length \(n\)

\[
F_n \xrightarrow{} F_{n-1} \xrightarrow{} \cdots \xrightarrow{} F_0 \xrightarrow{} \mathbb{Z} \xrightarrow{} 0.
\]

iii) For every \(0 \leq k < n\) and every projective, finitely generated partial resolution of the trivial \(\mathbb{Z}M\)-module \(\mathbb{Z}\) of length \(k\)

\[
P_k \xrightarrow{d_k} P_{k-1} \xrightarrow{d_{k-1}} \cdots \xrightarrow{} P_0 \xrightarrow{d_0} \mathbb{Z} \xrightarrow{} 0,
\]

the \(\mathbb{Z}M\)-module \(\ker d_k\) is finitely generated.

4.2.3. Exercise. Show Proposition 4.2.2 using Proposition 4.1.14.

4.2.4. Homological type \(\text{FP}_0\). The augmentation map of a monoid \(M\) is the ring homomorphism

\[
\varepsilon : \mathbb{Z}M \rightarrow \mathbb{Z}
\]

defined by

\[
\varepsilon \left( \sum_{u \in M} \lambda_u u \right) = \sum_{u \in M} \lambda_u.
\]

The ring homomorphism \(\varepsilon\) is extended to a homomorphism of \(\mathbb{Z}M\)-modules in the obvious way. If we consider the homomorphism of \(\mathbb{Z}\)-modules \(i_0 : \mathbb{Z} \rightarrow \mathbb{Z}M\) defined by \(i_0(1) = 1\), we have \(\varepsilon i_0 = \text{Id}_{\mathbb{Z}}\). Hence the homomorphism \(\varepsilon\) is surjective. It follows that

4.2.5. Proposition. Every monoid \(M\) is of homological type \(\text{FP}_0\).
4.2.6. Remark. Every \( R \)-module admits a free resolution. In particular, given a monoid \( M \), there exists a resolution
\[
\cdots \longrightarrow F_{n+1} \overset{d_{n+1}}{\longrightarrow} F_n \overset{d_n}{\longrightarrow} F_{n-1} \longrightarrow \cdots \longrightarrow F_1 \overset{d_1}{\longrightarrow} F_0 \overset{d_0}{\longrightarrow} Z \longrightarrow 0
\]
of the trivial \( \mathbb{Z}M \)-module \( Z \) by free \( \mathbb{Z}M \)-modules. We can build such a resolution by setting \( F_0 = \mathbb{Z}M \) and \( d_0 = \varepsilon \). Let \( F_1 \) be the free \( \mathbb{Z}M \)-module generated by \( \ker \varepsilon \), and let \( d_1 : F_1 \to \mathbb{Z}M \) be the canonical homomorphism induced by the homomorphism \( \ker \varepsilon \to F_0 \). Then, for any \( n \geq 2 \), \( F_n \) is the free \( \mathbb{Z}M \)-module generated by \( \ker d_{n-1} \), and the homomorphism \( d_n : F_n \to F_{n-1} \) is induced by the homomorphism \( \ker d_{n-1} \to F_{n-1} \).

Note that in this way, the obtained resolution is too big in general. In the rest of this section, we show how to construct a partial resolution which is more “economic” in the sense that the free modules are generated by a reduced number of generators.

4.2.7. Normalisation strategies. Given a monoid \( M \), we consider a presentation of \( M \) by a 2-polygraph \( \Sigma \) with a single 0-cell \( • \). Let \( \pi : \Sigma \to M \) be the canonical projection. We will write \( u \) instead of \( \pi(u) \).

We consider a section \( M \to \Sigma \) of \( \pi \), i.e., we choose, for every 1-cell \( u \) of \( M \), a 1-cell \( \hat{u} \) of \( \Sigma \) such that \( \pi(\hat{u}) = u \). In general, we cannot assume that the chosen section is functorial, that is \( \hat{uv} = \hat{u} \hat{v} \) holds in \( \Sigma \). However, we will assume that \( \hat{1} = 1 \) holds. Given a 1-cell \( u \) of \( \Sigma \), we simply write \( \hat{u} \) for \( \pi(u) \).

Such a section being fixed, a normalisation strategy for \( \Sigma \) is a map
\[
\sigma : \Sigma^+ \to \Sigma^2
\]
that sends every 1-cell \( u \) of \( \Sigma^+ \) to a 2-cell
\[
\sigma_u : u \Rightarrow \hat{u}
\]
of the free \((2,1)\)-category \( \Sigma^2 \), such that \( \sigma_{uv} = 1_{\hat{uv}} \) holds for every 1-cell \( u \) of \( \Sigma^+ \).

4.2.8. Left and right normalisation strategies. Let \( \Sigma \) be a 2-polygraph, with a chosen section. A normalisation strategy \( \sigma \) for \( \Sigma \) is a left one (resp. a right one) if it satisfies
\[
\sigma_{uv} = (\sigma_u \circ_0 v ) \star_1 \sigma_{\hat{uv}}, \quad \text{resp.} \quad \sigma_{uv} = (u \circ_0 \sigma_v ) \star_1 \sigma_{\hat{uv}} \).
\]
That is
\[
\sigma_{uv} = \begin{array}{c}
\sigma_u
\end{array} \hat{u} \Rightarrow \begin{array}{c}
v
\end{array} \quad \text{resp.} \quad \sigma_{uv} = \begin{array}{c}
u
\end{array} \Rightarrow \begin{array}{c}
\sigma_v
\end{array} \hat{uv}.
\]

4.2.10. Proposition. Any 2-polygraph admits a left (resp. right) normalisation strategy.
4.2. Monoids of finite homological type

Proof. Let $\Sigma$ be a 2-polygraph with a chosen section. Prove that $\Sigma$ admits a left normalisation strategy $\sigma : \Sigma^* \rightarrow \Sigma^\bot_2$. The proof of the existence of a right normalisation strategies is similar.

Let us arbitrarily choose a 2-cell $\sigma_{\hat{u}a} : \hat{u}a \Rightarrow \hat{u}a$ in $\Sigma^\bot_2$, for every 1-cell $u$ of $\Sigma^*$ and every 1-cell $a$ of $\Sigma_1$, such that $\hat{u}a \neq \hat{u}a$. Then, we extend $\sigma$ into a left normalisation strategy for $\Sigma$ by setting $\sigma_{\hat{u}a} = 1_{\hat{u}}$, for any $u$ in $\Sigma^*$, and for $u \neq \hat{u}$ by setting

$$\sigma_u = \sigma_v \ast_1 \sigma_{va}$$

if $u = va$ with $v$ in $\Sigma^*$ and $a$ in $\Sigma_1$:

The relations $\sigma_{1_*} = 1_{1_*}$ and [4.2.9] are immediate consequences of the definition of the map $\sigma$.

4.2.11. Leftmost and rightmost normalisation strategies. If $\Sigma$ is a reduced 2-polygraph, then, for every 1-cell $u$ of $\Sigma^*$, the set of rewriting steps with source $u$ can be ordered from left to right: for two rewriting steps $f = v\alpha v'$ and $g = w\beta w'$ with source $u$, we have $f \prec g$ if the length of $v$ is strictly smaller than the length of $w$. If $\Sigma$ is finite, then the order $\prec$ is total and the set of rewriting steps of source $u$ is finite. Hence, this set contains a smallest element $\lambda_u$ and a greatest element $\rho_u$, respectively called the leftmost and the rightmost rewriting steps on $u$. If, moreover, the 2-polygraph $\Sigma$ terminates, the iteration of $\lambda_u$ (resp. $\rho_u$) yields a normalisation strategy $\sigma$ called the leftmost (resp. rightmost) normalisation strategy of $\Sigma$:

$$\sigma_u = \lambda_u \ast_1 \sigma_{t(\lambda_u)} \quad \text{(resp. } \sigma_u = \rho_u \ast_1 \sigma_{t(\rho_u)}).$$

The leftmost and rightmost normalisation strategies give a way to make constructive some of the results we present here. For example, when $\Sigma$ is convergent they provide a deterministic choice of a confluence diagram

for every branching $(f,g)$ of $\Sigma$.

4.2.12. Exercice. Prove (by noetherian induction) that the leftmost (resp. rightmost) normalisation strategy of $\Sigma$ is a left (resp. right) normalisation strategy.

4.2.13. Presentations and partial resolutions of length 2. Let $M$ be a monoid and let $\Sigma$ be a presentation of $M$. Let us define a partial resolution of length 2 of $\mathbb{Z}$ by free $\mathbb{Z}M$-modules

$$\mathbb{Z}M[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}M[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}M \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0.$$
4.2.13. Presentations and partial resolutions of length 2

The \( \mathbb{Z}M \)-modules \( \mathbb{Z}M[\Sigma_1] \) and \( \mathbb{Z}M[\Sigma_2] \) are the free \( \mathbb{Z}M \)-modules over \( \Sigma_1 \) and \( \Sigma_2 \), respectively: they contain the formal sums of elements denoted by \( u[x] \), where \( u \) is an element of \( M \) and \( x \) is a 1-cell of \( \Sigma \). Let us note that \( \mathbb{Z}M \) is isomorphic to the free \( \mathbb{Z}M \)-module over the singleton \( \Sigma_0 \). The map \( \varepsilon \) is the augmentation map defined in (4.2.4) and the boundary maps are defined, on generators, by

\[
d_1([x]) = x - 1 \quad d_2([\alpha]) = [s_1(\alpha)] - [t_1(\alpha)].
\]

The map \( d_2 \) is called the Reidemeister-Fox Jacobian of \( \Sigma \). In the definition of \( d_2 \), the bracket \([\cdot]\) is extended to the 1-cells of \( \Sigma_1 \) thanks to the relation

\[
[1] = 0 \quad \text{and} \quad [uv] = [u] + \tau [v],
\]

for all 1-cells \( u \) and \( v \) of \( \Sigma_1 \).

4.2.15. Lemma. For any \( u \) in \( \Sigma_1 \), we have \( d_1(u) = \varepsilon u - 1 \).

Proof. We prove the relation by induction on the length of \( u \). For the unit, we have \( d_1[1] = d_1(0) = 0 \) and \( \varepsilon - 1 = 0 \). Then, for a composite 1-cell \( uv \), such that the result holds for both \( u \) and \( v \), we get

\[
d_1[uv] = d_1[u] + \tau d_1[v] = \varepsilon u - 1 + \varepsilon v - u = \varepsilon uv - 1.
\]

4.2.16. Proposition. Let \( M \) be a monoid and let \( \Sigma \) be a presentation of \( M \). The sequence of \( \mathbb{Z}M \)-modules

\[
\mathbb{Z}M[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}M[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}M \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{\eta} 0
\]

is a partial free resolution of length 2 of \( \mathbb{Z} \).

Proof. We first note that the sequence is a chain complex. Indeed, the augmentation map is surjective by definition. Moreover, we have

\[
\varepsilon d_1[x] = \varepsilon(x) - \varepsilon(1) = 1 - 1 = 0,
\]

for every 1-cell \( x \) of \( \Sigma_1 \). The relation \( d_1 d_2 = 0 \) is consequence of Lemma 4.2.15. Indeed, we have

\[
d_1 d_2[\alpha] = d_1[s(\alpha)] - d_1[t(\alpha)] = s(\alpha) - t(\alpha) = 0,
\]

for every 2-cell \( \alpha \) of \( \Sigma_2 \), where the last equality comes from \( s(\alpha) = \varepsilon t(\alpha) \), that holds since \( \Sigma \) is a presentation of the monoid \( M \).

The rest of the proof consists in defining contracting homotopies \( i_0, i_1, i_2 \):

\[
\mathbb{Z}M[\Sigma_2] \xrightarrow{i_2} \mathbb{Z}M[\Sigma_1] \xrightarrow{i_1} \mathbb{Z}M \xrightarrow{i_0} \mathbb{Z}
\]

We choose a representative \( \hat{u} \) in \( \Sigma_1 \) for every element \( u \) of \( M \), with \( \hat{1} = 1_x \) for every 0-cell \( x \) of \( \Sigma \), and we fix a normalisation strategy \( \sigma \) for \( \Sigma \). Then we define the homomorphisms of \( \mathbb{Z} \)-modules \( i_0, i_1 \) and \( i_2 \) by setting

\[
i_0(1) = 1, \quad i_1(u) = [\hat{u}], \quad i_2(u[x]) = [\sigma_{\hat{u}x}],
\]

(4.2.18)
4.2. Monoids of finite homological type

for any \( u \in M \) and \( x \in \Sigma_1 \).

In \((4.2.18)\), the element \([\hat{\sigma}ux]\) is defined using an extension of the bracket notation \([\cdot]\) on 2-cells of \( \Sigma_2 \)

\[
[\cdot] : \Sigma_2^\top \to \mathbb{Z}M[\Sigma_2]
\]

thanks to the relations

\[
[1_u] = 0, \quad [uvf] = [u][f] \quad \text{and} \quad [f \ast g] = [f] + [g],
\]

for all 1-cells \( u \) and \( v \) and 2-cells \( f \) and \( g \) of \( \Sigma_2^\top \) such that the composites \( ufv \) and \( f \ast g \) are defined.

First, we have \( \varepsilon i_0 = \text{Id}_Z \). Next, for every \( u \in M \), we have \( i_0 \varepsilon (u) = 1 \) and

\[
d_1 i_1 (u) = d_1 [\hat{u}] = u - 1.
\]

Thus \( d_1 i_1 + i_0 \varepsilon = \text{Id}_{ZM} \). Finally, we have, on the one hand,

\[
i_1 d_1 (u[x]) = i_1 (u[x] - u) = [\hat{u}x] - [\hat{u}]
\]

and, on the other hand,

\[
d_2 i_2 (u[x]) = d_2 [\hat{\alpha}ux] = [\hat{u}x] - [\hat{u}x] = u[x] + [\hat{u}] - [\hat{u}x].
\]

For this equality, we check that \( d_2 [f] = [s(f)] - [t(f)] \) holds for every 2-cell \( f \) of \( \Sigma_2^\top \) by induction on the size of \( f \). Hence we have \( d_2 i_2 + i_1 d_1 = \text{Id}_{ZM[\Sigma_1]} \), thus concluding the proof.

From Proposition \(4.2.16\) we deduce the following result:

4.2.19. Theorem. The following properties hold.

i) Every monoid is of homological type left-FP\(_0\).

ii) Every finitely generated monoid is of homological type left-FP\(_1\).

iii) Every finitely presented monoid is of homological type left-FP\(_2\).

4.2.20. Examples. Let us consider the monoid \( M \) presented by the 2-polygraph

\[
\Sigma = \langle a, c, t \mid \alpha_n : at^{n+1} \Rightarrow ct^n, \ n \in \mathbb{N} \rangle.
\]

The monoid \( M \) is finitely generated and, thus, it is of homological type left-FP\(_1\). However, for every natural number \( n \), we have

\[
d_2 [\alpha_{n+1}] = [at^{n+2}] - [ct^{n+1}],
\]

\[
= [at^{n+1} + at^{n+1}[t] - [ct^n] - ct^n[t],
\]

\[
= d_2 [\alpha_n] + (at^{n+1} - ct^n)[t].
\]

The equality \( at^{n+1} = ct^n \) holds in \( M \) by definition, yielding \( d_2 [\alpha_{n+1}] = d_2 [\alpha_n] \). As a consequence, the \( ZM \)-module \( \ker d_2 \) is generated by the elements \([\alpha_n] - [\alpha_0]\). Since the \( ZM \)-module \( \ker d_1 \) is equal

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to \( \text{Im} \ d_2 \), hence isomorphic to \( \mathbb{Z}M[\Sigma_2]/\ker d_2 \), it follows that \( \ker d_1 \) is generated by \( [\alpha_0] \) only, so that, by Lemma 3.2.2, the monoid \( M \) is of homological type \( \text{left-FP}_2 \). This can also be obtained by simply observing that \( M \) admits the finite presentation \( \langle a, c, t \mid \alpha_0 \rangle \).

Now, let us consider the monoid \( M \) presented by the 2-polygraph

\[ \Sigma = \langle a, b, t \mid \alpha_n : at^n b \Rightarrow 1, n \in \mathbb{N} \rangle. \]

The monoid \( M \) is of homological type \( \text{left-FP}_1 \), but not \( \text{left-FP}_2 \). This is proved by showing that \( \ker d_1 \) is not finitely generated as a \( \mathbb{Z}M \)-module, which is tedious by direct computation in this case. Another way to conclude is to extend the partial resolution of Proposition 4.2.16 by one dimension: it will then be sufficient to compute \( \text{Im} d_3 \), which is trivial in this case because \( \Sigma \) has no critical branching, so that \( \ker d_2 = 0 \) and, as a consequence, \( \ker d_1 \) is isomorphic to \( \mathbb{Z}M[\Sigma_2] \). Convergent presentations provide a method to obtain such a length-three partial resolution.

### 4.3. Squier’s homological theorem

#### 4.3.1. Coherent presentations and partial resolutions of length 3.

Let \( M \) be a monoid and let \( \Sigma \) be a coherent presentation of \( M \). Let us extend the partial resolution of length 3

\[
\mathbb{Z}M[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}M[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}M[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}M \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{} 0,
\]

where the \( \mathbb{Z}M \)-module \( \mathbb{Z}M[\Sigma_3] \) is the free \( \mathbb{Z}M \)-module over \( \Sigma_3 \), formed by the linear combination of elements \( u[\gamma] \), with \( u \) in \( M \) and \( \gamma \) a 3-cell of \( \Sigma_3 \). The boundary map \( d_3 \) is defined, for every 3-cell \( \gamma \) of \( \Sigma_3 \), by

\[ d_3[\gamma] = [s_2(\gamma)] - [t_2(\gamma)]. \]

The bracket notation \( [\cdot] \) defined on 3-cells of \( \Sigma_3 \) can be extended into a map

\[ [\cdot] : \Sigma_3^3 \rightarrow \mathbb{Z}M[\Sigma_3] \]

thanks to the relations

\[ [uAv] = u[A], \quad [A *_1 B] = [A] + [B], \quad [A *_2 B] = [A] + [B], \]

for all 1-cells \( u \) and \( v \) and 3-cells \( A \) and \( B \) of \( \Sigma_3^3 \) such that the composites are defined. In particular, the latter relation implies \( [1_f] = 0 \) for every 2-cell \( f \) of \( \Sigma_3^2 \). We check, by induction on the size, that \( d_3[A] = [s_2(A)] - [t_2(A)] \) holds for every 3-cell \( A \) of \( \Sigma_3^3 \).

#### 4.3.2. Proposition.

Let \( M \) be a monoid and let \( \Sigma \) be a coherent presentation of \( M \). The sequence of \( \mathbb{Z}M \)-modules

\[
\mathbb{Z}M[\Sigma_3] \xrightarrow{d_3} \mathbb{Z}M[\Sigma_2] \xrightarrow{d_2} \mathbb{Z}M[\Sigma_1] \xrightarrow{d_1} \mathbb{Z}M \xrightarrow{\varepsilon} \mathbb{Z} \xrightarrow{} 0
\]

is a partial free resolution of length 3 of \( \mathbb{Z} \).
4.3. Squier’s homological theorem

**Proof.** We proceed with the same notations as the ones of the proof of Proposition 4.2.16 with the extra hypothesis that $\sigma$ is a left normalisation strategy for $\Sigma$. This implies that $i_2(u[v]) = [\sigma_{uv}]$ holds for all $u$ in $M$ and $v$ in $\Sigma_1^\top$, by induction on the length of $v$.

We have $d_2d_3 = 0$ because $s_1s_2 = s_1t_2$ and $t_1s_2 = t_1t_2$. Then, we define the following homomorphism of $\mathbb{Z}$-modules

$$
\mathbb{Z}M[\Sigma_2] \xrightarrow{i_3} \mathbb{Z}M[\Sigma_3]
$$

where $\sigma_{u\alpha}$ is a 3-cell of $\Sigma_3^\top$ with the following shape, with $v = s(\alpha)$ and $w = t(\alpha)$:

Let us note that such a 3-cell necessarily exists in $\Sigma_3^\top$ because $\Sigma_3$ is an acyclic extension of the free $(2, 1)$-category $\Sigma_2^\top$. Then we have, on the one hand,

$$i_2d_2(u[\alpha]) = i_2(u[v] - u[w]) = [\sigma_{uv}] - [\sigma_{uw}]$$

and, on the other hand,

$$d_3i_3(u[\alpha]) = [\hat{u}\alpha] + [\sigma_{uw}] - [\sigma_{uv}],
= u[\alpha] + [\sigma_{uw}] - [\sigma_{uv}].$$

Hence $d_3i_3 + i_2d_2 = \text{Id}_{\mathbb{Z}M[\Sigma_2]}$, concluding the proof. \qed

4.3.3. **Remark.** The proof of Proposition 4.3.2 uses the fact that $\Sigma_3$ is an acyclic extension to produce, for every 2-cell $\alpha$ of $\Sigma_2$ and every $u$ in $M$, a 3-cell $\sigma_{u\alpha}$ with the required shape. The hypothesis on $\Sigma_3$ could thus be modified to only require the existence of such a 3-cell in $\Sigma_3^\top$. It is proved in [GM12b] that this implies that $\Sigma_3$ is an acyclic extension of the free $(2, 1)$-category $\Sigma_2^\top$.

From Proposition 4.3.2, we deduce

4.3.4. **Theorem ([CO94, Theorem 3.2], [Laf95, Theorem 3], [Pri95]).** Let $M$ be a finitely presented monoid. If $M$ is of finite derivation type, then it is of homological type left-FP$_3$.

By Theorem 3.5.7 this implies

4.3.5. **Theorem ([Squ87, Theorem 4.1]).** If a monoid admits a finite convergent presentation, then it is of homological type left-FP$_3$. 

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4.3.6. Example. Let us consider the monoid $M$ with the convergent presentation

$$\langle a \mid \mu : aa \Rightarrow a \rangle.$$

With the leftmost normalisation strategy $\sigma$, we get, writing the 2-cell $\mu$ as a string diagram $\Downarrow$:

$$\sigma_a = 1_a \quad \sigma_{aa} = \Downarrow \quad \sigma_{aaa} = \mu a \ast 1 \mu = \Downarrow \Downarrow.$$

The presentation has exactly one critical branching, whose corresponding generating confluence can be written in the two equivalent ways

![Diagram](image)

The $\mathbb{Z}M$-module $\ker d_2$ is generated by

$$d_3[\Downarrow] = [1\Downarrow] - [\Downarrow\Downarrow]$$
$$= [\Downarrow\Downarrow] + [\Downarrow\Downarrow] - [\Downarrow\Downarrow] - [\Downarrow]$$

$$= a[\Downarrow\Downarrow] - [\Downarrow].$$

4.4. HOMOLOGY OF MONOIDS WITH INTEGRAL COEFFICIENTS

4.4.1. Morphism of resolutions. Let $M$ be a monoid. Consider two free resolutions of the trivial $\mathbb{Z}M$-module $\mathbb{Z}$ by $\mathbb{Z}M$-modules

$$\mathcal{F} : \cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{d_n} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{\varepsilon} \mathbb{Z} \rightarrow 0$$

$$\mathcal{F}' : \cdots \rightarrow F_{n+1} \xrightarrow{d'_{n+1}} F_n' \xrightarrow{d_n'} F_{n-1} \rightarrow \cdots \rightarrow F_1' \xrightarrow{d_1'} F_0' \xrightarrow{\varepsilon'} \mathbb{Z} \rightarrow 0$$

A homomorphism of resolutions $f : \mathcal{F} \rightarrow \mathcal{F}'$ is a family of homomorphisms $f = (f_n : F_n \rightarrow F'_n)_{n \in \mathbb{N}}$ making the following diagrams commutative

$$\cdots \rightarrow F_{n+1} \xrightarrow{d_{n+1}} F_n \xrightarrow{f_{n+1}} F'_n \xrightarrow{d_n'} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{d_1} F_0 \xrightarrow{f_0} \mathbb{Z} \rightarrow 0$$

$$\cdots \rightarrow F_{n+1} \xrightarrow{d'_{n+1}} F_n' \xrightarrow{d_n'} F_{n-1} \rightarrow \cdots \rightarrow F_1' \xrightarrow{d_1'} F_0' \xrightarrow{\varepsilon'} \mathbb{Z} \rightarrow 0$$

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4.4. Homology of monoids with integral coefficients

4.4.2. Homotopy of resolutions. Given two homomorphisms of resolutions $f, g : \mathcal{F} \to \mathcal{F}'$ given by

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & F_{n+1} & \overset{d_{n+1}}{\longrightarrow} & F_n & \overset{d_n}{\longrightarrow} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \overset{d_1}{\longrightarrow} & F_0 & \overset{\varepsilon}{\longrightarrow} & Z & \longrightarrow & 0 \\
\cdots & \longrightarrow & F'_{n+1} & \overset{d'_{n+1}}{\longrightarrow} & F'_n & \overset{d'_n}{\longrightarrow} & F'_{n-1} & \longrightarrow & \cdots & \longrightarrow & F'_1 & \overset{d'_1}{\longrightarrow} & F'_0 & \overset{\varepsilon'}{\longrightarrow} & Z & \longrightarrow & 0 \\
\end{array}
$$

We say that $f$ is homotopic to $g$ if there exists a family of homomorphisms $h = (h_n : F_n \to F'_n)_{n \in \mathbb{Z}}$ such that

$$
f_0 - g_0 = d'_1 h_0,
$$

$$
f_n - g_n = d'_{n+1} h_n + h_{n-1} d_n,
$$

for all $n \geq 1$. It is easy to see that homotopy is an equivalence relation on the set of homomorphisms of resolutions from $\mathcal{F}$ to $\mathcal{F}'$.

4.4.3. Proposition. Between two free resolutions, there exists a homomorphism. Moreover, two such homomorphisms are homotopic.

4.4.4. Exercise. Prove Proposition 4.4.3.

4.4.5. Homology with integral coefficients. Let $\mathbf{M}$ be a monoid. To a free resolution of the trivial $\mathbb{Z}\mathbf{M}$-module $Z$ by left $\mathbb{Z}\mathbf{M}$-modules

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & F_{n+1} & \overset{d_{n+1}}{\longrightarrow} & F_n & \overset{d_n}{\longrightarrow} & F_{n-1} & \longrightarrow & \cdots & \longrightarrow & F_1 & \overset{d_1}{\longrightarrow} & F_0 & \overset{\varepsilon}{\longrightarrow} & Z & \longrightarrow & 0 \\
\end{array}
$$

we associate the following complex of $\mathbb{Z}$-modules

$$
\begin{array}{ccccccc}
\cdots & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_{n+1} & \overset{\tilde{d}_{n+1}}{\longrightarrow} & \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_n & \overset{\tilde{d}_n}{\longrightarrow} & \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_{n-1} & \longrightarrow & \cdots & \longrightarrow & \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_1 & \overset{\tilde{d}_1}{\longrightarrow} & \mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_0 \\
\end{array}
$$

where $\tilde{d}_n = \text{Id} \otimes d_n$. Note that the $\mathbb{Z}$-module $\mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_n$ is obtained from $F_n$ by trivialising the action of $\mathbf{M}$, that is $F_n$ quotiented by all relations $ux = x$ for $u \in \mathbf{M}$ and $x \in F_n$. In particular, if $F_n = \mathbb{Z}[\mathbf{X}]$, then $\mathbb{Z} \otimes_{\mathbb{Z}\mathbf{M}} F_n \cong \mathbb{Z}[\mathbf{X}]$ is the free $\mathbb{Z}$-module on $\mathbf{X}$. We obtain a chain complex, because $d_n \circ d_{n+1} = 0$ induces that $\text{Id} \otimes d_n \circ (\text{Id} \otimes d_{n+1}) = 0$.

We define the $n$-th homology group of $\mathbf{M}$ with integral coefficient $\mathbb{Z}$ as the quotient $\mathbb{Z}$-module:

$$
H_n(\mathbf{M}, \mathbb{Z}) = \ker(\tilde{d}_n) / \text{Im}(\tilde{d}_{n+1}),
$$

with the convention that $d_0 = 0$. For any monoid $\mathbf{M}$, we have $H_0(\mathbf{M}, \mathbb{Z}) \cong \mathbb{Z}$.

4.4.6. Proposition. For $n \geq 0$, the group $H_n(\mathbf{M}, \mathbb{Z})$ does not depend on a particular choice of a free resolution, but only on the monoid $\mathbf{M}$ itself.
4.4.7. **Exercise.** Prove the Proposition 4.4.6.

4.4.8. **Proposition.** If a monoid $M$ is of homological type left-FP$_n$ for all $n \geq 0$, then the groups $H_n(M, \mathbb{Z})$ are all finitely generated.

In particular, we have the following consequence that gives a necessary condition for a monoid to have a finite convergent presentation.

4.4.9. **Corollary.** If a monoid admits a finite convergent presentation, then the group $H_3(M, \mathbb{Z})$ is finitely generated.

4.4.10. **Exercise.** Consider the monoid $M$ presented by the following 2-polygraph:

\[ \langle a, b, c \mid \alpha_n : a c^n b \Rightarrow 1, \ n \in \mathbb{N} \rangle. \]

1. Compute homology groups $H_1(M, \mathbb{Z})$ and $H_2(M, \mathbb{Z})$.
2. Show that $M$ is a finitely generated monoid which cannot be finitely presented.

4.4.11. **Exercise.** Consider the monoid $M$ presented by the following 2-polygraph:

\[ \langle a, b, c, d \mid a b \Rightarrow a, \ da \Rightarrow ac \rangle. \]

Compute the homology groups $H_n(M, \mathbb{Z})$, for $n = 1, 2, 3$.

4.4.12. **Exercise.** Consider the monoid $M$ presented by the following 2-polygraph:

\[ \langle a, b, c, d, d' \mid a b \Rightarrow a, \ da \Rightarrow ac, \ d'a \Rightarrow ac \rangle. \]

Compute the homology groups $H_n(M, \mathbb{Z})$, for $n = 1, 2, 3$.

4.5. **HISTORICAL NOTES**

4.5.1. **Homological finiteness condition.** Jantzen in [Jan82, Jan85] asked the following question: does every finitely presented monoid with a decidable word problem admit a finite convergent presentation? At the end of the eighties, using a homological argument Squier answered the Jantzen question negatively by showing that there are finitely presented monoids with a decidable word problem which do not have a finite convergent presentation, [SO87, Squ87]. He linked the existence of a finite convergent presentation for a finitely presented monoid to the homological type left-FP$_3$ property, Theorem 4.3.5. He showed that a monoid needs to satisfy this invariant to have a finite convergent presentation. Giving examples, recalled in Example 3.5.13 of finitely presented monoids that have a decidable word problem and that do not have homological type left-FP$_3$, he proved that there are finitely presented monoids with a decidable word problem that cannot be presented by a finite convergent string rewriting system. However, it still remains open to characterize the class of monoids with a decidable word problem and having a finite convergent presentation. Squier result leads to the following question: is the homological finiteness condition left-FP$_3$ sufficient for a finitely presented monoid with a decidable word problem to admit a finite convergent presentation?
4.5. Historical notes

4.5.2. Homotopical finiteness condition. Squier answered this question negatively in another article. In [SOK94], he related the existence of a finite convergent presentation to a new finiteness condition of finitely presented monoids, called finite derivation type, Definition 3.4.1. This property is a natural extension of the properties of being finitely generated and finitely presented. Squier defined the finite derivation type for a monoid as a finiteness property on a 2-dimensional combinatorial complex associated to a presentation of the monoid. Note that this complex was defined independently by Kilibarda, [Kil97], and Pride, [Pri95]. Squier proved that the finite derivation type property is an invariant property for finitely presented monoids, Theorem 3.4.4. As a consequence, the property finite derivation type can be defined for monoids independently of a considered presentation: a monoid is of finite derivation type if its finite presentations are of finite derivation type. The proof given by Squier is based on Tietze transformations. Finally, Squier proved that, if a monoid admits a finite convergent presentation, then it is of finite derivation type, Theorem 3.5.7. This result corresponds to a “homotopical” version of Newman’s Lemma 5.5.12 for string rewriting systems. Squier used this result to give another proof that there exist finitely presented monoids with a decidable word problem that do not admit a finite convergent presentation. Moreover, he showed that the homological finiteness condition left-FP\(3\) is not sufficient for a finitely presented monoid with a decidable word problem to admit a finite convergent presentation. Indeed, he showed that the finitely presented monoid \(S_1\) given in Example 3.5.13 has a decidable word problem and is of homological type left-FP\(3\), but it is not of finite derivation type, and, thus, it does not admit a finite convergent presentation.

The article [SOK94] concludes with the following question: for finitely presented monoids does the property of having finite derivation type implies the existence of a finite convergent presentation? The answer is negative, indeed there exist finitely presented groups of homological type left-FP\(3\) that have undecidable word problems, [Mil92]. Since for finitely presented groups the property of having finite derivation type is equivalent to the homological type left-FP\(3\), [CO96], it follows that a finitely presented group can have an undecidable word problem even if it has finite derivation type. Hence in general the finite derivation property is not sufficient for the existence of a finite convergent presentation.

4.5.3. Extensions of Squier’s finiteness conditions. By his results, Squier has opened a homological direction and a homotopical one, in the quest for a complete characterisation of the existence of finite convergent presentations of monoids. In the homological direction, it has been shown that a finitely presented monoid admitting a finite convergent presentation satisfies the more restrictive condition homological type left-FP\(\infty\), Definition 4.2.1. Further proofs of the following result can be found in the literature.

4.5.4. Theorem ([Ani86, Kob90, Gro90, Bro92]). If a monoid admits a finite convergent presentation, then it is of homological type left-FP\(\infty\).

The proofs are based on distinct ways to describe the \(n\)-fold critical branchings of a convergent rewriting system. Note that the converse implication of this result is false in general. By this fact, there were numerous finiteness conditions introduced with the goal to have a sufficient condition for the finite-convergence, [WP00, KO01, KO02, KO03, PO04, MPP05, GM13]. However, all these conditions were necessarily but not sufficient. The characterization of the class of finitely presented monoids having a presentation by a finite convergent rewriting system is still an open problem.

Beyond this problem, the methods initiated by Squier have opened the way to homotopical and homological analysis of rewriting systems. Moreover, it was shown in [Ani86, Kob90, Gro90, Bro92]...
4.5.5. Question ([LM09, LMW10])

Mal03, GHM19 that this methods highlight the way to compute “effectively” free resolutions for groups, monoids, associative algebras or small categories using rewriting.

Finally, the question of putting all this work in a higher-categorical framework was posed by Lafont and Métayer. [Laf95, Met03, LM09]. In particular, is it possible to describe in the higher-categorical framework the constructions developed in [Ani86, Kob90, Gro90, Bro92]:

4.5.5. Question ([LM09, LMW10]). Is it true that a monoid presented by a finite convergent rewriting system always has a finite cofibrant approximation in the folk model structure on \( \infty \)-categories?

We will see that in fact the higher-dimensional strict categories constitute a natural setting for the analysis of rewriting systems.
We must be careful when we rewrite in a linear structure defined over a field. For example, consider a rewriting system over a ring or an algebra. We expect that the rewriting rules are compatible with the linear structure in the following way. For a rewriting rule

\[ f \rightarrow g \]

relating two elements of an algebra on a ground field \( K \), then for any scalar \( \lambda \) in \( K \) we would like the reduction:

\[ \lambda f \rightarrow \lambda g, \]

and for any other element \( h \) of the algebra, we would like the following reduction:

\[ f + h \rightarrow g + h. \]
5.1. Linear 2-polygraphs

Taken together, these two reductions lead to losing termination of rewriting. Indeed, in that case from the rule \( f \rightarrow g \), we deduce the reductions \( -f \rightarrow -g \) and \( -f + (f + g) \rightarrow -g + (f + g) \). Finally, we deduce the following reduction

\[ g \rightarrow f. \]

As a consequence, the system will never terminate. Further to this remark, it is necessary to adapt the notion of rewriting system to linear situations. In the example presented above the reduction \( -f + (f + g) \rightarrow -g + (f + g) \) appears as the source of the nontermination problem.

There are two ways to solve this problem. The most well-known method is to choose an orientation of the rules induced by a monomial order, which is well-founded by definition, see [5.4.1]. This approach is used in various paradigms of linear rewriting as recalled in Chapter 6. In this chapter, we present the categorical description of linear rewriting that extends to associative algebras the notion of 2-polygraph, with an appropriated notion of reduction. The constructions given in this chapter come from [GHM19].

The ground field will be denoted by \( \mathbb{K} \). We denote by \( \text{Vect} \) the category of vector spaces over \( \mathbb{K} \) and linear maps. This category is a monoidal category with the tensor product over \( \mathbb{K} \), denoted by \( \otimes \). We will denote by \( \text{Alg} \) the category of (unital associative) algebras over \( \mathbb{K} \).

5.1. LINEAR 2-POLYPGRAPHS

We have seen in (2.1.2) that a category can be thought of as a "monoid with several 0-cells". Similarly, the notion of 1-algebroid describes the concept of associative algebra with several 0-cells.

5.1.1. Algebroids. A 1-algebroid over a ground field \( \mathbb{K} \) is a category enriched over the monoidal category \( \text{Vect} \). Explicitly, a 1-algebroid \( A \) is specified by the following data:

i) a set \( A_0 \) of 0-cells, that we will denote by \( p, q \),...

ii) for every 0-cells \( p \) and \( q \), a vector space \( A(p, q) \), whose elements are the 1-cells of \( A \), with source \( p \) and target \( q \), that we will denote by \( f, g \),...

iii) for every 0-cells \( p, q \) and \( r \), a linear map

\[ *_0 : A(p, q) \otimes A(q, r) \rightarrow A(p, r) \]

called the 0-composition of \( A \) and whose image on \( f \otimes g \) is denoted by \( f *_0 g \) or \( fg \). This composition is associative, that is the relation:

\[ (f *_0 g) *_0 h = f *_0 (g *_0 h), \]

holds for any 0-composable 1-cells \( f, g \) and \( h \), and unitary, that is, for any 0-cell \( p \), there is a 1-cell \( 1_p \) such that for any 1-cell \( f \) in \( A(p, q) \), the following relation holds

\[ 1_p *_0 f = f *_0 1_q = f. \]

A 1-cell \( f \) with source \( p \) and target \( q \) will be graphically represented by

\[ p \xrightarrow{f} q \]
5.1.2. Remarks. A 1-algebra is a 1-algebroid with a single one 0-cell, that can be identified to an algebras over \( \mathbb{K} \). The notion of 1-algebroid was first introduced by Mitchell as ring with several objects called \( \mathbb{K} \)-category in [Mit72], terminology linear category appear also in the literature. A small \( \mathbb{Z} \)-category is called a ringoid and a one-0-cell ringoid is a ring.

5.1.3. Free 1-algebroid. The free 1-algebroid on a 1-polygraph \( \Lambda = (\Lambda_0, \Lambda_1) \) is the 1-algebroid, denoted by \( \Lambda_1^f \), whose set of 0-cells is \( \Lambda_0 \), and for any 0-cells \( p, q \), \( \Lambda_1^f(p, q) \) is the free vector space on \( \Lambda_1^f(p, q) \). In other words, any 1-cell in the space \( \Lambda_1^f(p, q) \) is a linear combination of paths from \( p \) to \( q \) generated by the 1-polygraph \( \Lambda \). If \( \Lambda_0 \) has only one 0-cell, \( \Lambda_1^f \) is the free algebra with basis \( \Lambda_1 \). The source and target maps \( s_0 \) and \( t_0 \) of the 1-polygraph \( \Lambda \) are extended into maps on \( \Lambda_1^f \), denoted by \( s_0^f \) and \( t_0^f \), in a natural way making the following two diagrams commutative:

\[ \begin{array}{ccc}
\Lambda_0 & \overset{s_0}{\leftarrow} & \Lambda_1^f \\
\Lambda_0 & \overset{s_0}{\leftarrow} & \Lambda_1^f \\
\Lambda_1 & \overset{t_0}{\leftarrow} & \Lambda_1 \\
\end{array} \]

where \( \iota_1 \) denotes the inclusion of 1-cells of \( \Lambda_1 \) in the free algebroid \( \Lambda_1^f \).

5.1.4. Two-dimensional linear polygraphs. A cellular extension of the 1-algebroid \( \Lambda_1^f \) is a set \( \Lambda_2 \) equipped with two maps

\[ \Lambda_1^f \overset{s_1}{\underset{t_1}{\leftarrow}} \Lambda_2 \]

such that, for every \( \alpha \) in \( \Lambda_2 \), the pair \( (s_1(\alpha), t_1(\alpha)) \) is a 1-sphere in \( \Lambda_1^f \), that is, the following globular relations hold \( s_0s_1(\alpha) = s_0t_1(\alpha) \) and \( t_0s_1(\alpha) = t_0t_1(\alpha) \). As in the non linear situation of 2.1.10, an element of the cellular extension \( \Lambda_2 \) will be graphically represented by a 2-cell with the following globular shape

\[ \begin{array}{ccc}
f & \overset{\alpha}{\underset{\alpha}{\simeq}} & g \\
p & \overset{\alpha}{\underset{\alpha}{\simeq}} & q \\
\end{array} \]

that relates parallel 1-cells \( f \) and \( g \) in \( \Lambda_1^f \), also denoted by \( f \overset{\alpha}{\Rightarrow} g \) or by \( \alpha : f \Rightarrow g \).

We define a linear 2-polygraph as a triple \( (\Lambda_0, \Lambda_1, \Lambda_2) \), where \( (\Lambda_0, \Lambda_1) \) is a 1-polygraph and \( \Lambda_2 \) is a cellular extension of the free 1-algebroid \( \Lambda_1^f \):

\[ \begin{array}{ccc}
\Lambda_0 & \overset{s_0}{\leftarrow} & \Lambda_1^f \\
\Lambda_0 & \overset{t_0}{\leftarrow} & \Lambda_1^f \\
\Lambda_1 & \overset{s_1}{\leftarrow} & \Lambda_2 \\
\end{array} \]

The elements of \( \Lambda_2 \) are called the 2-cells of \( \Lambda \), or the rewriting rules of \( \Lambda \).
5.1. Linear 2-polygraphs

In the sequel, we will consider polygraphs with one 0-cell denoted •.

5.1.5. The ideal of a linear 2-polygraph. Given a linear 2-polygraph \( \Lambda \). We denote by \( I(\Lambda) \) the two-sided ideal of the free algebra \( \Lambda_1^f \) generated by the following set of 1-cells

\[ \{ s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2 \}. \]

The ideal \( I(\Lambda) \) is made of the linear combinations

\[ \sum_{i=1}^{p} \lambda_i u_i (s_1(\alpha_i) - t_1(\alpha_i))v_i, \]

for pairwise distinct 2-monomials \( u_1 \alpha_1 v_1, \ldots, u_p \alpha_p v_p \) of \( \Lambda_1^f \), and nonzero scalars \( \lambda_1, \ldots, \lambda_p \).

5.1.6. Presentations of algebras. The algebra presented by a linear 2-polygraph \( \Lambda \), and denoted by \( \overline{\Lambda} \), is the quotient of the free algebra \( \Lambda_1^f \) by the two-sided ideal \( I(\Lambda) \). We denote by \( \overline{f} \) the image of a 1-cell \( f \) of \( \Lambda_1^f \) through the canonical projection \( \pi : \Lambda_1^f \to A \).

We say that a linear 2-polygraph \( \Lambda \) is a presentation of an algebra \( A \) if the algebra presented by \( \Lambda \) is isomorphic to \( A \). Two linear 2-polygraphs are said to be Tietze equivalent if they present isomorphic algebras.

5.1.7. First toy example. Here our first toy example that we will use through this chapter:

\[ \Lambda = \langle x, y, z \mid xyz \Rightarrow x^3 + y^3 + z^3 \rangle. \]

The free 1-algebroid generated by \( \Lambda^f_1 = \langle x, y, z \rangle \) is the free algebra \( \mathbb{K}\langle x, y, z \rangle \). The algebra presented by the linear 2-polygraph \( \Lambda \) is the quotient of the algebra \( \mathbb{K}\langle x, y, z \rangle \) by the two-sided ideal generated by the 1-cell \( xyz - x^3 - y^3 - z^3 \).

5.1.8. Other toy examples. We will consider the two following Tietze equivalent linear 2-polygraphs:

\[ \langle x, y \mid x^2 \Rightarrow yx \rangle, \quad \langle x, y \mid yx \Rightarrow x^2 \rangle. \]

5.1.9. 2-algebras. We define a 2-algebra \( A \) as an internal 1-category in the category \( \text{Alg} \). Explicitly, it is defined by a diagram

\[
\begin{array}{ccc}
A_1 & \xrightarrow{s_1} & A_2 \\
\downarrow t_1 & \downarrow *_1 & \downarrow \times_{A_1} \\
A_2 \times_{A_2} A_1 & \xrightarrow{i_2} & A_2
\end{array}
\]

(5.1)

where \( A_2 \times_{A_1} A_2 \) is the algebra defined by the following pullback diagram in the category \( \text{Alg} \):

\[
\begin{array}{ccc}
A_2 \times_{A_1} A_2 & \xrightarrow{j} & A_2 \\
\downarrow & \downarrow & \downarrow s_1 \\
A_2 & \xrightarrow{t_1} & A_1
\end{array}
\]

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Elements of the algebra $A_2 \times A_1$, called $1$-cells of $A$, are graphically pictured as follows

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \]

or

\[ \bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \]

The elements of $A_2$, called $2$-cells of $A$, are graphically represented by

\[ \bullet \xrightarrow{s_1(a)} \bullet \xrightarrow{t_1(a)} \]

5.1.10. Notations. For a $1$-cell $f$, the identity $2$-cell $i_2(f)$ is denoted by $1_f$, or $f$ if there is no possible confusion. The $1$-composite $\star_1(a, a')$ of $1$-composable $2$-cells $a$ and $a'$, will be denoted by $a \star_1 a'$. Elements of the algebra $A_1$, called $1$-cells of $A$, are graphically pictured as follows

\[ \bullet \xrightarrow{g} \bullet \xrightarrow{h} \]

The elements of $A_2$, called $2$-cells of $A$, are graphically represented by

\[ \bullet \xrightarrow{s_1(a)} \bullet \xrightarrow{t_1(a)} \]
5.1. Linear 2-polygraphs

Given 2-cells

\[
\begin{array}{c}
\bullet \downarrow a \quad \bullet \\
\downarrow f' \quad \downarrow g'
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\bullet \downarrow b \\
\downarrow f' \quad \downarrow g'
\end{array},
\]

the source and target maps \( s_1 \) and \( t_1 \) being morphisms of algebras, we have

\[ s_1(ab) = s_1(a)s_1(b), \quad \text{and} \quad t_1(ab) = t_1(a)t_1(b), \]

and for any scalars \( \lambda \) and \( \mu \) in \( \mathbb{K} \), we have

\[ s_1(\lambda a + \mu b) = \lambda s_1(a) + \mu s_1(b), \quad \text{and} \quad t_1(\lambda a + \mu b) = \lambda t_1(a) + \mu t_1(b). \]

Hence

\[
\begin{array}{c}
\bullet \downarrow \lambda a + \mu b \\
\downarrow \lambda f' + \mu g'
\end{array}
\quad \text{and} \quad
\begin{array}{c}
\bullet \downarrow \lambda a + \mu b \\
\downarrow \lambda f' + \mu g'
\end{array}
\]

Given 1-cells \( h, f, f' \) and \( k \) in \( A_1 \) and a 2-cell \( \alpha \) in \( A_2 \) such that

\[
\begin{array}{c}
\bullet \\
\downarrow f' \quad \downarrow f
\end{array}
\end{array}
\begin{array}{c}
\bullet \\
\downarrow f' \quad \downarrow f
\end{array}
\]

we will denote by \( h\alpha k : hfk \Rightarrow hf'k \) the 0-composite \( 1_h \ast 0 \alpha \ast 0 1_k \).

5.1.11. Properties of 1-composition. Given 1-composable 2-cells:

\[
\begin{array}{c}
\bullet \\
\downarrow f' \quad \downarrow f
\end{array}
\end{array}
\begin{array}{c}
\bullet \\
\downarrow f' \quad \downarrow f
\end{array}
\]

in \( A_2 \ast A_1, A_2 \), the 1-composition \( \ast_1 \) being linear, \( \alpha \ast_1 \alpha' + b \ast_1 b' \) is a 2-cell from \( f + g \) to \( f' + g' \) and we have

\[
(a + b) \ast_1 (a' + b') = \ast_1(a + b, a' + b'),
\]

\[
= \ast_1(a, a') + \ast_1(b, b'),
\]

\[
= a \ast_1 a' + b \ast_1 b'.
\]

Furthermore, for any scalar \( \lambda \) in \( \mathbb{K} \), \( \lambda(\alpha \ast_1 \alpha') \) is a 2-cell from \( \lambda f \) to \( \lambda f'' \) and we have

\[
(\lambda a) \ast_1 (\lambda a') = \lambda(a \ast_1 a').
\]
Finally, the compatibility with the product induces that $\star_1((a, a')(b, b')) = \star_1(ab, a'b')$. Hence, we have

$$\star_1(ab, a'b') = (a \star_1 a')(b \star_1 b'). \quad (5.2)$$

Relation (5.2) corresponds to the exchange law in the 2-algebra $A$ between the 1-composition and the product.

### 5.1.12. Remarkable identities in a 2-algebra

The following properties hold in a 2-algebra $A$

i) for any 1-composable 2-cells $a$ and $a'$ in $A$, we have

$$a \star_1 a' = a + a' - t_1(a), \quad (5.3)$$

ii) any 2-cell $a$ in $A$ is invertible for the $\star_1$-composition, and its inverse is given by

$$a^{-1} = -a + s_1(a) + t_1(a). \quad (5.4)$$

iii) for any 2-cells $a$ and $b$ in $A$, we have

$$ab = as_1(b) + t_1(a)b - t_1(a)s_1(b) = s_1(a)b + at_1(b) - s_1(a)t_1(b). \quad (5.5)$$

Relation (5.3) is a consequence of the linearity of the 1-composition $\star_1$. Indeed, for any $(a, a')$ in $A_2 \times_{A_1} A_2$, we have

$$a \star_1 a' = (a - s_1(a') + s_1(a')) \star_1 (t_1(a) - t_1(a) + a'),
= a \star_1 t_1(a) - s_1(a') \star_1 t_1(a) + s_1(a') \star_1 a',
= a - t_1(a) + a'.$$

### 5.1.13. Exercise

Show identities (5.4) and (5.5).

### 5.1.14. The free 2-algebra on a linear 2-polygraph

The free 2-algebra over a linear 2-polygraph $\Lambda$ is the 2-algebra, denoted by $\Lambda^f_1$, defined as follows. In dimension 1, it is the free 1-algebra $\Lambda^f_1$ over $\Lambda_1$. For dimension 2, we consider the following diagram in the category of $\Lambda^f_1$-bimodule

$$\Lambda^f_1 \xrightarrow{t_1} \Lambda^M_2 \xleftarrow{s_1} \Lambda^M_2,$$

where $\Lambda^f_1$ is seen as $\Lambda^f_1$-bimodule, $\Lambda^M_2$ is the $\Lambda^f_1$-bimodule $(\Lambda^f_1 \otimes \mathbb{K} \Lambda_2 \otimes \Lambda^f_1) \oplus \Lambda^f_1$ and where the linear maps $s_1$, $t_1$ and $t_2$ are defined by:

$$s_1(f \alpha g) = fs_1(\alpha)g, \quad t_1(f \alpha g) = ft_1(\alpha)g \quad \text{and} \quad s_1(h) = t_1(h) = t_2(h) = h,$$

for all 2-cell $\alpha$ in $\Lambda_2$, and 1-cells $f, g, h$ in $\Lambda_1$. The quotient of the $\Lambda^f_1$-bimodule $\Lambda^M_2$ by the equivalence relation generated by

$$as_1(b) + t_1(a)b - t_1(a)s_1(b) \sim s_1(a)b + at_1(b) - s_1(a)t_1(b),$$
for all $a$ and $b$ in $\Lambda^1_1 \otimes K \Lambda_2 \otimes \Lambda^1_1$, has a structure of algebra, denoted by $\Lambda^2_1$, and whose product is given by
\[
a b = a s_1(b) + t_1(a) b - t_1(a) s_1(b).
\]
One proves that the source and target maps are compatible with this quotient, so giving a structure of 2-algebra:
\[
\Lambda^1_1 \xrightarrow{\text{t}_i} \Lambda^1_2 \xrightarrow{\text{s}_i} \Lambda^1_1.
\]

### 5.1.15. Exercise.
Let $\Lambda$ be a linear 2-polygraph. Given 1-cells $f$ and $g$ in $\Lambda^1_1$, show that the 1-cell $f-g$ belongs to $1(\Lambda)$ if and only if there exists a 2-cell $\alpha : f \Rightarrow g$ in $\Lambda^2_1$. As a consequence, the algebra presented by $\Lambda$ is obtained by identifying in $\Lambda^1_1$ all the 1-cells $s_1(a)$ and $t_1(a)$, for every 2-cell $\alpha$ in $\Lambda^2_1$.

### 5.1.16. Monomials.
A monomial in the free 2-algebra $\Lambda^2_1$ is a 1-cell of the free monoid $\Lambda^1_1$ over $\Lambda_1$. The set monomials of $\Lambda^2_1$, also denoted by $\Lambda^1_1$, forms a linear basis of the free algebra $\Lambda^1_1$. As a consequence, every nonzero 1-cell $f$ of $\Lambda^1_1$ can be uniquely written as a linear combination of pairwise distinct monomials $u_1, \ldots, u_p$:
\[
f = \lambda_1 u_1 + \ldots + \lambda_p u_p
\]
with $\lambda_i \in K \setminus \{0\}$, for all $i = 1, \ldots, p$. The set of monomials $\{u_1, \ldots, u_p\}$ will be called the support of $f$ and denoted by $\text{Supp}(f)$.

### 5.1.17. 2-monomials.
A 2-monomial of a free 2-algebra $\Lambda^2_2$ is a 2-cell of $\Lambda^1_2$ with shape $u \alpha v$, where $\alpha$ is a 2-cell in $\Lambda_2$, and $u$ and $v$ are monomials in $\Lambda^1_1$:
\[
\bullet \xrightarrow{s_1(\alpha)} \bullet \xrightarrow{\text{t}_1(\alpha)} \bullet
\]

By construction of the free 2-algebra $\Lambda^2_2$, and by freeness of $\Lambda^1_1$, every non-identity 2-cell $a$ of $\Lambda^2_1$ can be written as a linear combination of pairwise distinct 2-monomials $a_1, \ldots, a_p$ and of an 1-cell $h$ of $\Lambda^1_1$:
\[
a = \lambda_1 a_1 + \ldots + \lambda_p a_p + h.
\]

### 5.1.18. Exercise.
Prove that the decomposition in (5.6) is unique up to the following relations
\[
as_1(b) + t_1(a) b - t_1(a) s_1(b) = s_1(a) b + a t_1(b) - s_1(a) t_1(b),
\]
for all 2-monomials $a$ and $b$ in $\Lambda^2_2$. 

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5.1.19. Monomial linear 2-polygraphs. A linear 2-polygraph $\Lambda$ is left-monomial if, for every 2-cell $\alpha$ of $\Lambda_2$, the source $s_1(\alpha)$ is a monomial in $\Lambda_1^* \setminus \operatorname{Supp}(t_1(\alpha))$. Note that a non-left monomial linear 2-polygraph would produce useless ambiguity only due to the linear structure.

A linear 2-polygraph $\Lambda$ is monomial if it is left-monomial and for every 2-cell $\alpha$ of $\Lambda_2$, $t_1(\alpha) = 0$ holds. A monomial algebra is an algebra admitting a presentation by a monomial linear 2-polygraph.

5.1.20. Exercise. Show that any linear 2-polygraph is Tietze equivalent to a left-monomial linear 2-polygraph.

5.1.21. Examples. The linear 2-polygraph $\Lambda$ given in Example 5.1.7 is left-monomial. The linear 2-polygraph $\langle x, y \mid x^2 + y^2 \Rightarrow 2xy \rangle$ is not left-monomial, but it is Tietze equivalent to the following left-monomial 2-polygraph:

$$\Lambda' = \langle x, y \mid xy \Rightarrow \frac{1}{2}(x^2 + y^2) \rangle.$$  

The linear 2-polygraphs $\langle x \mid x^2 \Rightarrow 0 \rangle$ and $\langle x, y \mid xy \Rightarrow 0 \rangle$ are monomials.

5.1.22. Degrees and length. For monomials $u$ and $v$ in $\Lambda_1^*$, we denote by $\deg_v u$ the number of different occurrences of the monomial $v$ in the monomial $u$. For instance $\deg_x x^2 x^4 = 3$ and $\deg_y x^3 = 0$. For a subset $M$ of monomials in $\Lambda_1^*$, we denote

$$\deg_M u = \sum_{v \in M} \deg_v u.$$  

The length of a monomial $u$ in $\Lambda_1^*$, denoted by $\ell(u)$, is equal to $\deg_{\Lambda_1} u$.

5.2. Linear rewriting steps

5.2.1. Elementary 2-cells. Let $\Lambda$ be a linear 2-polygraph. An elementary 2-cell of the free 2-algebra $\Lambda_2^*$ is a 2-cell of $\Lambda_2^*$ with shape

$$\begin{array}{c}
\lambda \\
\text{\downarrow}
\end{array} \begin{array}{c}
a \\
\text{\downarrow}
\end{array} + \begin{array}{c}
g \\
\text{\downarrow}
\end{array}$$

where $a$ is a 2-monomial, $g$ is a 1-cell of $\Lambda_1^*$ and $\lambda$ is a nonzero scalar in $\mathbb{K}$.

5.2.2. Example. With the polygraph $\Lambda'$ of Example 5.1.21, the 2-cell

$$2x\alpha'y + y^3 : 2x^2y^2 \Rightarrow x^3y + xy^3 - y^3$$

is elementary and the 2-cell

$$x\alpha' + \alpha'y : x^2y + xy^2 \Rightarrow \frac{1}{2}(x^3 + xy^2 + x^2y + y^3)$$

is not elementary.
5.2. Linear rewriting steps

5.2.3. Exercise. Show that any 2-cell in a free 2-algebra \( \Lambda^2_\ell \) can be decomposed into a 1-composition of elementary 2-cells of \( \Lambda^2_\ell \).

5.2.4. Rewriting steps. Let \( \Lambda \) be a left-monomial linear 2-polygraph. A rewriting step of \( \Lambda \) is an elementary 2-cell

\[
\begin{array}{c}
\lambda \\
\downarrow \alpha \\
\downarrow f
\end{array} + \begin{array}{c}
g
\end{array}
\]

of \( \Lambda^2_\ell \) such that \( \lambda \) is a nonzero scalar and \( u \) is not in the support of \( g \).

5.2.5. Examples. For the linear 2-polygraph given in Example 5.1.7, the 2-cell

\[
3xy - 3xz^3 : 3x^2yz - 3xz^3 \implies 3x^4 + 3xy^3
\]

is a rewriting step. For a linear 2-polygraph having a rule \( \alpha : u \Rightarrow f \), the 2-cell

\[
-\alpha + (u + f) : -u + (u + f) \implies -f + (u + f)
\]

is not a rewriting step because the monomial \( u \) appears in the context \( u + f \).

5.2.6. Exercise. Let \( \Lambda \) be a left-monomial linear 2-polygraph and let \( \alpha \) be an elementary 2-cell of the 2-algebra \( \Lambda^2_\ell \). Show that \( \alpha \) can be factorised in the 2-algebra \( \Lambda^2_\ell \) into

\[
\begin{array}{c}
\alpha \\
b
\end{array} = \begin{array}{c}
c
\end{array}
\]

where \( b \) and \( c \) are either identities of rewriting steps.

5.2.7. Example. Let \( \Lambda \) be a linear 2-polygraph and let \( \alpha : u \Rightarrow v \) be a 2-cell of \( \Lambda_2 \). The 2-cell

\[
-\alpha + (u + v) \quad \text{and} \quad \alpha + (5u + 4v)
\]

are not rewriting steps of \( \Lambda \). They can be decomposed respectively as follows:

\[
\begin{array}{c}
-\alpha + (u + v) \\
v
\end{array} = \begin{array}{c}
-\alpha + (u + v) \\
(1 - 1)u + v
\end{array} \quad \alpha + (5u + 4v) = \begin{array}{c}
\alpha + (5u + 4v) \\
6\alpha + 4v
\end{array} \quad 10v \leq \begin{array}{c}
5\alpha + 5v
\end{array}
\]

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5.2.8. Rewriting sequences. A 2-cell \( \alpha \) of \( \Lambda_2^f \) is positive, or a rewriting sequence, if it is an identity or a 1-composite
\[
f_0 \xrightarrow{\alpha_1} f_1 \Rightarrow \cdots \Rightarrow f_{k-1} \xrightarrow{\alpha_k} f_k
\]
of rewriting steps of \( \Lambda \).

5.2.9. Reduced cells. A 1-cell \( f \) of \( \Lambda_1^f \) is called reduced, or irreducible, with respect to \( \Lambda_2 \), if there is no rewriting step of \( \Lambda \) with source \( f \). As a consequence, a 1-cell is reduced if and only if it is the zero 1-cell of \( \Lambda_1^f \), or a linear combination of reduced monomials in \( \Lambda_1^f \). The reduced 1-cells of \( \Lambda_1^f \) form a vector subspace of \( \Lambda_1^f \), denoted by \( \Lambda_1^{nf} \). Since \( \Lambda \) is left-monomial, the set of reduced monomials of \( \Lambda_1^f \), denoted by \( \Lambda_1^{irm} \), forms a basis of the vector space \( \Lambda_1^{nf} \).

We denote by \( s_1(\Lambda) \) the set of redex of a reduced left-monomial linear 2-polygraph \( \Lambda \) defined by
\[
s_1(\Lambda) = \{ s_1(\alpha) \mid \alpha \text{ in } \Lambda_2 \}.
\]
In [Am86], a redex is called an obstruction. The number of possible application of rules of \( \Lambda_2 \) to a monomial \( u \) is \( \deg_{s_1(\Lambda)} u \).

5.2.10. Reduced linear 2-polygraphs. We say that a linear 2-polygraph \( \Lambda \) is left-reduced if, for every 2-cell \( \alpha \) in \( \Lambda_2 \), the 1-cell \( s_1(\alpha) \) is reduced with respect to \( \Lambda_2 \setminus \{ \alpha \} \). We say that \( \Lambda \) is right-reduced if, for every 2-cell \( \alpha \) of \( \Lambda \), the 1-cell \( t_1(\alpha) \) is reduced. The linear polygraph \( \Lambda \) is reduced if it is both left-reduced and right-reduced.

5.2.11. Exercise. Show that any left-monomial linear 2-polygraph is Tietze equivalent to a reduced left-monomial linear 2-polygraph.

5.2.12. Normal forms. If \( f \) is a 1-cell of \( \Lambda_1^f \), a normal form for \( f \) with respect to \( \Lambda_2 \) is a reduced 1-cell \( g \) of \( \Lambda_1^f \) such that there exists a positive 2-cell \( \alpha : f \Rightarrow g \) in \( \Lambda_2^f \).

5.3. Termination for linear 2-polygraphs

We recall the notion of rewrite relation for linear 2-polygraphs from [GHM19]. Let us fix a left-monomial linear 2-polygraph \( \Lambda \).

5.3.1. Termination. The rewrite relation of \( \Lambda \) is the binary relation, denoted by \( \prec_\Lambda \) on the set of monomial \( \Lambda_1^* \) defined by
\[
\text{i) } w \prec_\Lambda u \text{ for every 2-cell } \alpha : u \Rightarrow f \text{ of } \Lambda_2 \text{ and every monomial } w \text{ in } \text{Supp}(f),
\]
\[
\text{ii) } u' \prec_\Lambda u \text{ implies } \nu u' \prec_\Lambda \nu u w \text{ for all monomials } u, u', \nu \text{ and } w \text{ of } \Lambda_1^*.
\]
We say that \( \Lambda \) terminates if its rewrite relation \( \prec_\Lambda \) is wellfounded, that is, there is no infinite descending chains in \( \Lambda_1^* \):
\[
u_1 \succ_\Lambda \nu_2 \succ_\Lambda \nu_3 \succ_\Lambda \cdots \succ_\Lambda \nu_n \succ_\Lambda \nu_{n+1} \succ_\Lambda \cdots
\]
5.4. Monomial orders

5.3.2. Example. Consider the linear 2-polygraph $\Lambda = \langle x, y \mid xy \xrightarrow{3} x^2 + y^2 \rangle$. We have $x^2 \prec_\Lambda xy$ and $y^2 \prec_\Lambda xy$. It follows that $x^2y \succ_\Lambda x^2 + y^2$. Hence the relation $\prec_\Lambda$ is not a wellfounded and the polygraph is not terminating. Note that, we have an infinite sequence of rewriting steps:

\[
x^3 + xy \\
x^2y \xrightarrow{3} x^3 + xy^2 \implies x^3 + y^3 + x^2y \implies \ldots
\]

5.3.3. The rewrite relation on 1-cells. The rewrite relation $\prec_\Lambda$ is extended to the 1-cells of $\Lambda^1$ by setting, for any 1-cells $f$ and $g$, $g \prec_\Lambda f$ if the following two conditions hold

i) there exists a monomial $v$ in $\text{Supp}(f)$ which is not in $\text{Supp}(g)$,

ii) for any monomial $v$ in $\text{Supp}(g) \setminus \text{Supp}(f)$, there exists a monomial $u$ in $\text{Supp}(f) \setminus \text{Supp}(g)$, such that $v \prec_\Lambda u$

5.3.4. Proposition. The rewrite relation $\prec_\Lambda$ is wellfounded on 1-cells if and only if it is wellfounded on monomials.

If $\Lambda$ terminates, then for every rewriting step $a$ of $\Lambda$, we have $t_1(a) \prec_\Lambda s_1(a)$. This implies that the 2-algebra $\Lambda^2$ contains no infinite sequence of pairwise 1-composable rewriting steps

\[
f_0 \xrightarrow{a_1} f_1 \Rightarrow \ldots \Rightarrow f_{k-1} \xrightarrow{a_k} f_k \Rightarrow \ldots
\]

so that every 1-cell of $\Lambda^1$ admits at least one normal form with respect to $\Lambda^2$.

5.4. MONOMIAL ORDERS

5.4.1. Monomial orders. A total order $\prec$ on the set of monomials $\Lambda^1_*$ is a monomial order if the following conditions are satisfied

i) $\prec$ is a well-order, that is, there is no infinite descending chains in $\Lambda^1_*$.

\[
u_1 > u_2 > u_3 > \ldots > u_n > u_{n+1} > \ldots
\]

ii) $\prec$ is compatible with the multiplicative structure on monomials, that is

\[
u \prec u' \text{ implies } vuw \prec vu'w,
\]

for all monomials $u$, $u'$, $v$, and $w$ in $\Lambda^1_*$.

5.4.2. Example. Given a total order relation $\prec$ on $\Lambda_1$, we define the left degree-wise lexicographic order generated by $\prec$, or deglex order generated by $\prec$, as the order $\prec_{\text{deglex}}$ on $\Lambda^1_*$ that compare two monomials first by degree and then lexicographically. It is defined by

i) $y_1 \ldots y_p \prec_{\text{deglex}} x_1 \ldots x_q$, if $p < q$.

ii) $y_1 \ldots y_j \ldots y_p \prec_{\text{deglex}} y_1 \ldots y_j \ldots x_p$, if $y_j \prec x_j$. 

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5.4.3. Exercise. Show that the order \( \prec_{\text{deglex}} \) is a monomial order.

5.4.4. Exercise. Explain why the pure lexicographic order is not a monomial order. Show that it is neither a well-order nor compatible with the product of monomials.

5.4.5. Polygraph compatible with a monomial order. A linear 2-polygraph \( \Lambda \) is say to be compatible with a monomial order \( \prec \) if for every 2-cell \( \alpha : u \Rightarrow f \) of \( \Lambda_2 \), then \( w \prec u \) for any monomial \( w \) in the support of \( f \). The monomial order \( \prec \) is thus a well-founded rewrite relation for \( \Lambda \). It follows that any linear 2-polygraph compatible with a monomial order is terminating. The converse is false in general as we will see in Exercise 5.4.7.

5.4.6. Example. Consider the linear 2-polygraph \( \Lambda = \langle x, y \mid x^2 \triangleright x,y - y^2 \rangle \). It is Tietze equivalent to the linear 2-polygraph of Example 5.3.2 but it is terminating. Indeed, having \( x,y \prec x^2 \) and \( y^2 \prec x^2 \), the linear 2-polygraph \( \Lambda \) is compatible with the deglex order \( \prec_{\text{deglex}} \) induced by \( y \prec x \), hence it is terminating. An other way to prove that \( \Lambda \) is terminating, is to count the number of occurrence of \( x \) in monomials. For any \( u \) in \( \Lambda_1 \), let denote by \( A(u) \) the number of occurrence of \( x \) in \( u \). To prove that the linear 2-polygraph \( \Lambda \) terminates, it is sufficient to check that, for every rewriting step \( \alpha : s_1(\alpha) \Rightarrow f \), we have \( A(s_1(\alpha)) > A(v) \), for any monomial \( v \) in \( \text{Supp}(f) \).

5.4.7. Exercise. Show that the linear 2-polygraph \( \Lambda \) given in Example 5.1.7 is terminating. Show that \( \Lambda \) is not compatible with a monomial order.

5.4.8. Exercise, [Ber78, Exercise 5.2.1.]. Examine termination of the linear 2-polygraph \( \langle x, y \mid \alpha \rangle \) in each of the following situations

\[
x^2 y \triangleright yx, \quad yx \triangleright x^2 y, \quad x^2 y^2 \triangleright yx, \quad yx \triangleright x^2 y^2.
\]

5.4.9. Noetherian induction. Let us recall the principle of noetherian induction for terminating rewriting systems, see [Hue80] for more details. Let \( \Lambda \) be a left-monomial terminating linear 2-polygraph. Given a property \( P(f) \) of the 1-cells \( f \) of \( \Lambda_1 \). In order to show that \( P(f) \) holds for any 1-cell \( f \) of \( \Lambda_1 \), it suffices to show that

i) \( P(f) \) holds for \( f \) reduced with respect to \( \Lambda_2 \),

ii) \( P(f) \) holds under the assumption that \( P(g) \) is hold for every \( g \prec_{\Lambda} f \).

5.4.10. Leading terms. Let \( \Lambda_1 \) be a free algebra over a set \( \Lambda_1 \) and let \( \prec \) be a monomial order on \( \Lambda_1 \). For a nonzero 1-cell \( f \) of \( \Lambda_1 \), the leading monomial of \( f \) with respect to \( \prec \) is the monomial of \( f \), denoted by \( \text{lm}(f) \), such that \( w \prec \text{lm}(f) \), for any monomial \( w \) in the support of \( f \). The leading coefficient of \( f \) is the coefficient \( \text{lc}(f) \) of \( \text{lm}(f) \) in \( f \), and the leading term of \( f \) is the 1-cell \( \text{lt}(f) = \text{lc}(f) \text{lm}(f) \) of \( \Lambda_1 \). We also define \( \text{lt}(0) = \text{lc}(0) = \text{lm}(0) = 0 \).

Note that for any 1-cells \( f \) and \( g \) in \( \Lambda_1 \), we have \( f \prec g \) if and only if either \( \text{lm}(f) \prec \text{lm}(g) \) or \( (\text{lm}(f) = \text{lm}(g) \) and \( f - \text{lt}(f) \prec g - \text{lt}(g) \). The following property

\[ \text{lt}(fg) = \text{lt}(f) \text{lt}(g), \]
5.5. Confluence and convergence

for any 1-cells \( f \) and \( g \) is also useful.

5.4.11. Leading polygraph. Given a monomial order \(<\) on \( \Lambda_1^\ell \) and a nonzero 1-cell \( g \) in \( \Lambda_1^\ell \), we define the 2-cell:

\[
\alpha_{g,<} : \text{lm}(g) \Rightarrow \text{lm}(g) - \frac{1}{\text{lc}(g)} g.
\]

For any set \( \mathcal{G} \) of nonzero 1-cells in \( \Lambda_1^\ell \), the leading 2-polygraph associated to \( \mathcal{G} \) with respect to \(<\) is the linear 2-polygraph \( \Lambda(\mathcal{G},<) \) whose set of 1-cells is \( \Lambda_1 \) and

\[
\Lambda(\mathcal{G},<)_2 = \{ \alpha_{g,<} \mid g \in \mathcal{G} \}.
\]

By definition, the leading polygraph \( \Lambda(\mathcal{G},<) \) is compatible with the monomial order \(<\).

A monomial \( w \) in \( \Lambda_1^\ast \) is \( \mathcal{G} \)-reduced with respect to the monomial order \(<\) if it reduced with respect to \( \Lambda(\mathcal{G},<)_2 \), that is, there is no factorisation \( w = uv \) with \( u \) and \( v \) monomials in \( \Lambda_1^\ast \) and \( g \) in \( \mathcal{G} \). A set \( \mathcal{G} \) of 1-cells is reduced with respect to the monomial order \(<\) if for any 1-cell \( g \) in \( \mathcal{G} \), any monomial in the support of \( g \) is \( (\mathcal{G} \setminus \{g\}) \)-reduced.

5.5. CONFLUENCE AND CONVERGENCE

5.5.1. Suppose that \( \Lambda \) is a terminating left-monomial linear 2-polygraph. Every 1-cell \( f \) of \( \Lambda_1^\ell \) admits at least a normal form \( \tilde{f} \). That is, \( \tilde{f} \) is reduced and there exists a positive 2-cell \( \alpha : f \Rightarrow \tilde{f} \) in \( \Lambda_2^\ell \). As a consequence, we have a decomposition

\[
f = \tilde{f} + (f - \tilde{f}),
\]

with \( \tilde{f} \) in \( \Lambda_1^{nf} \) and \( f - \tilde{f} \) in \( \text{I}(\Lambda) \) by Exercise 5.1.15. It follows that the vector space \( \Lambda_1^\ell \) admits the following decomposition

\[
\Lambda_1^\ell = \Lambda_1^{nf} + \text{I}(\Lambda).
\]

In this section we show that the decomposition (5.8) is direct if and only if the polygraph \( \Lambda \) is confluent.

5.5.2. Example. Note that the decomposition (5.8) is not direct in general. Indeed, consider the linear 2-polygraph \( \Lambda = \langle x, y \mid x^2 \Rightarrow \beta y \rangle \). It is terminating thanks to the deglex order generated by \( x > y \). Consider the two following reduction sequences reducing the 1-cell \( x^3 \):

\[
\beta x \Rightarrow xyx
\]

\[
\begin{align*}
x^3 & \Rightarrow \beta y \Rightarrow x^2y = xy^2 \\
x\beta & \Rightarrow x^2y
\end{align*}
\]

Thus the 1-cell

\[
xyx - xy^2 = -(x^2 - xy)x + x(x^2 - xy) + (x^2 - xy)y
\]

is both in \( \Lambda_1^{nf} \) and \( \text{I}(\Lambda) \). It follows that the sum \( \Lambda_1^{nf} + \text{I}(\Lambda) \) is not direct.
5.5.3. Branchnings and confluence. Let $\Lambda$ be a left-monomial linear 2-polygraph. A branching of $\Lambda$ is a non-ordered pair $(a, b)$ of positive 2-cells of $\Lambda_2^I$ with a common source $s_1(a) = s_1(b)$. A branching $(a, b)$ is local if both $a$ and $b$ are rewriting steps of $\Lambda$. A branching $(a, b)$ of $\Lambda$ is confluent if there exist positive 2-cells $a'$ and $b'$ of $\Lambda$ as in the following diagram

![Diagram of branching](image)

We say that $\Lambda$ is confluent (resp. locally confluent) if every branching (resp. local branching) of $\Lambda$ is confluent. An immediate consequence of the confluence property is that every 1-cell of $\Lambda_1^I$ admits at most one normal form.

5.5.4. Proposition. Let $\Lambda$ be a terminating left-monomial linear 2-polygraph. The following conditions are equivalent.

i) $\Lambda$ is confluent.

ii) Every 1-cell of $I(\Lambda)$ admits $\emptyset$ as a normal form with respect to $\Lambda_2$.

iii) The vector space $\Lambda_1^I$ admits the direct decomposition $\Lambda_1^I = \Lambda_{nf}^I \oplus I(\Lambda)$.

Proof. i) $\Rightarrow$ ii). Let $f$ be a 1-cell in the ideal $I(\Lambda)$, then there exists a 2-cell $a : f \Rightarrow \emptyset$ in $\Lambda_2^I$. The polygraph $\Lambda$ being confluent, the 1-cells $f$ and $\emptyset$ have the same normal form. Finally, $\emptyset$ being reduced, this implies that $\emptyset$ is a normal form for $f$.

ii) $\Rightarrow$ iii). Prove that $\Lambda_{nf}^I \cap I(\Lambda) = \emptyset$. If $f$ is in $\Lambda_{nf}^I$, then $f$ is reduced and, thus, admits itself as normal form. If $f$ is in $I(\Lambda)$, then $f$ admits $\emptyset$ as a normal form by ii). Hence $\Lambda_{nf}^I \cap I(\Lambda) = \emptyset$.

iii) $\Rightarrow$ i). Consider a branching $(a, b)$ of $\Lambda$ with $a : f \Rightarrow g$ and $b : f \Rightarrow h$. Since $\Lambda$ terminates, each of $g$ and $h$ admits at least one normal form. Hence, there exist positive 2-cells $a_1$ and $b_1$ in $\Lambda_2^I$:

![Diagram of branching](image)

with $g_1$ and $h_1$ reduced. It follows that $g_1 - h_1$ is also reduced. Moreover, the 2-cell $(a \ast_1 a_1) \ast_1 (b \ast b_1)$ has $g_1$ as source and $h_1$ as target. This implies that $g_1 - h_1$ is also in $I(\Lambda)$. As $\Lambda_{nf}^I \cap I(\Lambda) = \emptyset$, we have $g_1 - h_1 = \emptyset$, hence the branching $(a, b)$ is confluent.

5.5.5. Convergence. We say that a left-monomial linear 2-polygraph $\Lambda$ is convergent if it terminates and it is confluent. In that case, every 1-cell $f$ of $\Lambda_1^I$ has a unique normal form, denoted by $\widehat{f}$, such that $\widehat{f} = \overline{g}$ holds in $\Lambda$ if and only if $\widehat{f} = \overline{g}$ holds in $\Lambda_1^I$.

As a consequence, if $\Lambda$ is a convergent presentation of an algebra $A$, the assignment of every 1-cell $f$ of $\Lambda$ to the normal form $\widehat{f}$, defines a section $\iota : A \rightarrow \Lambda_1^I$ of the canonical projection $\pi : \Lambda^I \rightarrow A$. The section $\iota$ is a linear map, i.e., it satisfies $\lambda \widehat{f} + \mu \overline{g} = \lambda \overline{f} + \mu \overline{g}$, and it preserves the identities because $\Lambda$ terminates.
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5.5.6. Exercise. Show that the section $\iota$ is not a morphism of algebras in general.

5.5.7. Theorem. Let $A$ be an algebra and $\Lambda$ be a convergent presentation of $A$. The set $\Lambda_{irr}^f$ of reduced monomials is a linear basis of $A$. Moreover, the vector space $\Lambda_{nf}^f$ equipped with the product defined by $f \cdot g = \hat{f}g$, for any 1-cells $f$ and $g$ in $\Lambda_{nf}^f$, is an algebra isomorphic to $A$.

Proof. Suppose that $\Lambda$ is a convergent linear 2-polygraph. By Proposition 5.5.4 the following sequence of vector spaces is exact:

$$0 \longrightarrow I(\Lambda) \longrightarrow \Lambda_{irr}^f \longrightarrow \Lambda_{nf}^f \longrightarrow 0$$

The vector space $\Lambda_{nf}^f$ admits $\Lambda_{irr}^f$ as a basis, hence $\Lambda_{irr}^f$ forms a basis of the vector space underlying the quotient algebra $\Lambda_{irr}^f / I(\Lambda)$, that is the algebra $A$. The polygraph $\Lambda$ being convergent, any 1-cell of $\Lambda_{irr}^f$ has a unique normal form, hence the product defined by $f \cdot g = \hat{f}g$ is associative. Indeed, for any 1-cells $f, g$ and $h$, we have

$$(f \cdot g) \cdot h = \hat{f}g \cdot h = \hat{f} \hat{g}h = \hat{f}g \cdot h = f \cdot (g \cdot h).$$

It follows that this product equips $\Lambda_{nf}^f$ with a structure of algebra in such a way that $\Lambda_{nf}^f$ is isomorphic to the algebra $A$. \qed

5.5.8. Exercise. Compute a linear basis of the algebra presented by \langle $x, y \mid xy = x^2$\rangle.

5.5.9. Exercise. Compute a linear basis for the symmetric algebra on $k$ variables presented by

\langle $x_1, \ldots, x_k \mid x_i x_j \mapsto x_j x_i \mid 1 \leq i < j \leq k$\rangle

and for the skew-polynomial algebra on $k$ variables presented by

\langle $x_1, \ldots, x_k \mid x_i x_j \mapsto q_i^j x_j x_i \mid 1 \leq i < j \leq k$\rangle,

where the $q_i^j$ are scalars in $K$.

5.5.10. Exercise: Poincaré-Birkhoff-Witt theorem, [Bok76, §1.], [Ber78, Theorem 3.1]. Consider an ordered bases $x_1 \prec x_2 \prec \ldots \prec x_k$ of a Lie algebra $\mathfrak{g}$. Consider the following ideals of the free tensor algebra $T(\mathfrak{g})$ over $\mathfrak{g}$:

$$I = \langle x_i x_j - x_j x_i \mid 1 \leq i < j \leq k \rangle,$$

$$J = \langle x_i x_j - x_j x_i + [x_i, x_j] \mid 1 \leq i < j \leq k \rangle.$$

Show that the symmetric algebra $S_\mathfrak{g} = T(\mathfrak{g}) / I$ and the enveloping algebra $U_\mathfrak{g} = T(\mathfrak{g}) / J$ are isomorphic as vector spaces.
5.5.11. From local to global confluence. The Newman lemma, also called the diamond lemma, states that for terminating rewriting systems local confluence and confluence are equivalent properties. This result was proved by Newman in [New42] for abstract rewriting systems. A short and simple proof of this result was given by Huet in [Hue80] using the principle of noetherian induction. Let us recall the arguments of this proof for linear 2-polygraphs.

5.5.12. Theorem (Newman’s Lemma). Let $\Lambda$ be a terminating left-monomial linear 2-polygraph. Then $\Lambda$ is confluent if and only if it is locally confluent.

Proof. One implication is trivial. Suppose $\Lambda$ locally confluent and prove that it is confluent at every 1-cell $f$ of $\Lambda^1_1$. We proceed by noetherian induction on $f$. If $f$ is reduced, the only branching with source $f$ is $(1_f, 1_f)$ which is confluent.

Suppose that $f$ is a nonreduced 1-cell of $\Lambda^1_1$ and such that $\Lambda$ is confluent at every 1-cell $g \prec f$. Consider a branching $(a, b)$ of $\Lambda$ with source $f$. If $a$ or $b$ is an identity, then $(a, b)$ is confluent. Otherwise, we prove that the branching $(a, b)$ is confluent by induction. Since $a$ and $b$ are not identities, they admit decompositions $a = a_1 \star_1 a_2$ and $b = b_1 \star_1 b_2$ where $a_1$ and $b_1$ are rewriting steps, and $a_2$ and $b_2$ are positive 2-cells. By local confluence, the local branching $(a_1, b_1)$ is confluent. Hence there exist positive 2-cells $a'_1$ and $b'_1$ as indicated in the following diagram

We have $g_1 \prec_A f$ and $h_1 \prec_A f$. Then we apply the induction hypothesis on the branching $(a_2, a'_1)$ to get positive 2-cells $a'_1$ and $c$, and, then, to the branching $(b'_1 \star_1 c, b_2)$ to get positive 2-cells $d$ and $b'_2$, which complete the proof.

5.6. The Critical Branchings Theorem

5.6.1. Local branchings. A case analysis leads to a partition of the local branchings of a left-monomial linear 2-polygraph $\Lambda$ into the following four families:

i) Aspherical branchings, for all 2-monomial $\alpha : u \Rightarrow f$ of $\Lambda^2_2$, nonzero scalar $\lambda$, and 1-cell $h$ of $\Lambda^1_1$
such that the monomial $u$ is not in the support of $h$:

$$
\begin{align*}
\lambda a + h \\
\lambda u + h &\rightarrow \lambda f + h \\
\lambda a + h
\end{align*}
$$

ii) Additive branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : v \Rightarrow g$ of $\Lambda^2_\ell$, nonzero scalars $\lambda$ and $\mu$, and 1-cell $h$ of $\Lambda^1_\ell$ such that the monomials $u$ and $v$ are not in the support of $h$:

$$
\begin{align*}
\lambda a + \mu v + h &\rightarrow \lambda f + \mu v + h \\
\lambda u + \mu v + h \\
\lambda u + \mu b + h &\rightarrow \lambda u + \mu g + h
\end{align*}
$$

iii) Peiffer branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : v \Rightarrow g$ of $\Lambda^2_\ell$, nonzero scalar $\lambda$, and 1-cell $h$ of $\Lambda^1_\ell$ such that the monomial $uv$ is not in the support of $h$:

$$
\begin{align*}
\lambda \alpha v + h &\rightarrow \lambda f v + h \\
\lambda u v + h \\
\lambda u b + h &\rightarrow \lambda u g + h
\end{align*}
$$

iv) Overlapping branchings, for all 2-monomials $a : u \Rightarrow f$ and $b : u \Rightarrow g$ of $\Lambda^2_\ell$ such that the branching $(a, b)$ is neither aspherical nor Peiffer, and all nonzero scalar $\lambda$ and 1-cell $h$ of $\Lambda^1_\ell$ such that the monomial $u$ is not in the support of $h$:

$$
\begin{align*}
\lambda a + h &\rightarrow \lambda f + h \\
\lambda u + h \\
\lambda b + h &\rightarrow \lambda g + h
\end{align*}
$$

5.6.2. Critical branchings. A critical branching of a left-monomial linear 2-polygraph $\Lambda$ is an overlapping branchings, as defined in 5.6.1 with $\lambda = 1$ and $h = 0$, and that is minimal for the relation on branchings defined by

$$(a, b) \sqsubseteq (waw', wbw') \quad \text{for any } w \text{ and } w' \text{ in } \Lambda^*_\ell.$$
By case analysis on the source of critical branchings, they must have one of the following two shapes

\[
\begin{array}{ccc}
\alpha & \parallel & \beta \\
\downarrow & & \downarrow \\
\beta & \parallel & \beta \\
\end{array}
\]

with \( \alpha, \beta \) in \( \Lambda_2 \). When the linear 2-polygraph \( \Lambda \) is reduced, the first case cannot occur since, otherwise, the monomial \( s_1(\alpha) \) would be reducible by \( \beta \).

**5.6.3. Exercise.** Let \( \Lambda \) be a reduced linear 2-polygraph. Show that for any critical branching

\[
\begin{array}{ccc}
\alpha & \parallel & \beta \\
\downarrow & & \downarrow \\
\beta & \parallel & \beta \\
\end{array}
\]

the monomial \( u, v \) and \( w \) are reduced and cannot be identities or null.

**5.6.4. Critical branching lemma.** By the Newman lemma [5.5.12] for terminating rewriting systems, local confluence and confluence are equivalent properties. It turns out that one can decide whether a rewriting system is convergent by checking local confluence. For string rewriting systems, that is 2-polygraphs, the critical branching lemma states that local confluence is equivalent to the confluence of all critical branching, see [GM18, 3.1.5] for details. For linear 2-polygraphs the critical branching lemma given in [GHM19] differs from the case of 2-polygraphs. Indeed, in the linear setting the termination hypothesis is required. Moreover, nonoverlapping branchings may be non confluent as illustrated by the following example in which an additive branching is nonconfluent.

**5.6.5. Example.** Some local branchings can be nonconfluent without termination, even if critical confluence holds. Indeed, consider for instance the following linear 2-polygraph

\[
\langle x, y, z, t \mid xy \xrightarrow{\alpha} xz, zt \xrightarrow{\beta} 2yt \rangle
\]

has no critical branching, but it has a nonconfluent additive branching:

\[
\begin{align*}
2x\beta & \rightarrow 2xtz \\
2xtz & \rightarrow xzt + x\beta \\
xzt + x\beta & \rightarrow xzt + 2xtz \\
xzt + 2xtz & \rightarrow \cdots
\end{align*}
\]

\[
\begin{align*}
xyt + xzt & \rightarrow xzt + 2xyt \\
xzt + 2xyt & \rightarrow \cdots
\end{align*}
\]

\[
\begin{align*}
xyt + x\beta & \rightarrow 3xyt \\
3xyt & \rightarrow \alpha + 2xyt \\
\alpha + 2xyt & \rightarrow \cdots
\end{align*}
\]

\[
\begin{align*}
\alpha t + xzt & \rightarrow 2xzt \\
2xzt & \rightarrow 4xt + 4\alpha t \\
4xt + 4\alpha t & \rightarrow 4x\beta \\
4x\beta & \rightarrow \cdots
\end{align*}
\]

\[
\begin{align*}
3\alpha t & \rightarrow 3xzt \\
3xzt & \rightarrow 6xyt \\
6xyt & \rightarrow \cdots
\end{align*}
\]
5.6. The Critical Branchings Theorem

5.6.6. If a linear 2-polygraph $\Lambda$ is terminating and with any critical branching confluent, we can show that such an additive branching is confluent by noetherian induction on the sources of the branchings. Let consider an additive branching $(\lambda u + \mu v + h, \lambda u + \mu g + h)$ as in (5.6.1) and suppose that $\Lambda$ is locally confluent at every $g \prec_{\Lambda} \lambda u + \mu v + h$. By linearity of the 1-composition, the following equation

$$(\lambda a + \mu v + h) \circ \lambda f + \mu b + h = (\lambda u + \mu b + h) \circ (\lambda a + \mu g + h)$$

holds in the free 2-algebra $\Lambda^2$.

Note that the dotted 2-cells $\lambda a + \mu g + h$ and $\lambda f + \mu b + h$ may be not positive in general. Indeed, the monomial $u$ can be in the support of $g$ or the monomial $v$ can be in the support of $f$, as illustrated in Example 5.6.5. However, those 2-cells are elementary, hence there exist, see Exercise 5.2.6, positive 2-cells $a_1'$, $b_1'$, $c$ and $d$ that satisfy

$$a_1' = (\lambda f + \mu b + h) \circ c \quad \text{and} \quad b_1' = (\lambda a + \mu g + h) \circ d.$$  

We have $f \prec_{\Lambda} u$ and $g \prec_{\Lambda} v$, hence $\lambda f + \mu g + h \prec_{\Lambda} \lambda u + \mu v + h$. Thus, the branching $(c, d)$ is confluent by induction hypothesis, yielding the positive 2-cells $a_1'$ and $b_1'$.

In this way, one shows that under terminating hypothesis, all local branching given in (5.6.1) are confluent if all critical branching are confluent.

5.6.7. Theorem (Critical branching lemma). A terminating left-monomial linear 2-polygraph is locally confluent if and only if all its critical branchings are confluent.

As consequence of the critical branching lemma and of the Newman lemma 5.5.12, a terminating left-monomial linear 2-polygraph is confluent if all its critical branchings are confluent. In particular a terminating left-monomial 2-polygraph with no critical branching is convergent.

5.6.8. Example. The linear 2-polygraph given in Example 5.1.7 is terminating, see Exercise 5.4.7. Moreover, it does not have critical branching, hence it is convergent.

5.6.9. The Knuth-Bendix completion procedure. The completion procedure for terminating 2-polygraphs given in (2.5.1) can be adapted to linear 2-polygraphs as follows. Let $\Lambda$ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on $\Lambda^1$. A Knuth-Bendix completion of $\Lambda$ is a linear
2-polygraph $KB(\Lambda)$ obtained by the following procedure that examines the confluence of the set of critical branchings.

**Input:** $\Lambda$ be a left-monomial linear 2-polygraph compatible with a monomial order $\prec$ on $\Lambda_1^*$.  
$KB(\Lambda) := \Lambda$  
$Cb := \{ \text{critical branchings with respect to } \Lambda_2 \}$  
while $Cb \neq \emptyset$ do  
Picks a branching in $Cb$:  
\[
\begin{array}{c}
 f \rightrightarrows v \\
 u \rightarrowright g \rightarrowright w \\
 g \rightarrowright w \rightarrowright v \\
 f \rightarrowright v \\
 u \rightarrowright g \rightarrowright w \\
 g = \hat{v} - \hat{w}
\end{array}
\]
if $g \neq 0$ then  
\[
KB(\Lambda)_2 := KB(\Lambda)_2 \cup \{ \alpha_{g,\prec} : \text{lm}(g) \Rightarrow \text{lm}(g) - \frac{1}{\text{lc}(g)}g \} \\
Cb := Cb \cup \{ \text{critical branching created by } \alpha_{g,\prec} \}
\]  
end

If the procedure stops, it returns a finite convergent left-monomial linear 2-polygraph $KB(\Lambda)$. Otherwise, it builds an increasing sequence of left-monomial linear 2-polygraphs, whose limit is also denoted by $KB(\Lambda)$. Note that, if the starting linear 2-polygraph $\Lambda$ is convergent, then the Knuth-Bendix completion of $\Lambda$ is $\Lambda$ itself. The linear 2-polygraph $KB(\Lambda)$ obtained by this procedure depends on the order of examination of the critical branchings. Finally, since all the operations of adding new rules performed by the procedure are Tietze transformations, the linear 2-polygraph $KB(\Lambda)$ is Tietze-equivalent to $\Lambda$.

**5.6.10. Exercice.** Prove that the following linear 2-polygraph has a nonconfluent Peiffer branching

\[
\langle x, y, z \mid xy \xrightarrow{\alpha} 2x, yz \xrightarrow{\beta} z \rangle.
\]

**5.6.11. Weyl algebras.** Let $\mathbb{K}$ be a field of characteristic zero. The *Weyl algebra* of dimension $n$ over $\mathbb{K}$ is the algebra presented by the linear 2-polygraph whose 1-cells are

\[
x_1, \ldots, x_n, \partial_1, \ldots, \partial_n
\]

and with the following 2-cells:

\[
x_i x_j \Rightarrow x_j x_i, \quad \partial_1 \partial_j \Rightarrow \partial_1 \partial_i, \quad \partial_i x_j \Rightarrow x_j \partial_i, \quad \text{for any } 1 \leq i < j \leq n,
\]

\[
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\]
5.7. Coherent presentations of algebras

\[ \partial_i x_i \Rightarrow x_i \partial_i + 1, \quad \text{for any } 1 \leq i \leq n. \]

This polygraph is convergent with the following six families of confluent critical branchings:

where \( 1 \leq i < j \leq n \).

5.6.12. Exercice. In his seminal paper on the diamond lemma, Bergman point out that he was first led to the ideas of his paper with the following American Mathematical Monthly Advanced Problem 5082, [Ber78, 2.1.].

Let \( R \) be a ring in which, if either \( x + x = 0 \) or \( x + x + x = 0 \), it follows that \( x = 0 \). Suppose that \( a, b, c \) and \( a + b + c \) are all idempotents in \( R \). Does it follows that \( ab = 0 \)?

Solve this problem. [Hints. Consider the following linear 2-polygraph:

\[ \Lambda = \langle a, b, c \mid a^2 \Rightarrow a, \ b^2 \Rightarrow b, \ c^2 \Rightarrow c, \ ba \Rightarrow -ab - bc - cb - ac - ca \rangle. \]

1/ List all critical branchings of \( \Lambda \). 2/ Compute a convergent left-monomial linear 2-polygraph \( KB(\Lambda) \) by applying the Knuth-Bendix completion procedure to \( \Lambda \). 3/ List all irreducible monomials with respect to \( KB(\Lambda)_2 \). 4/ Conclude that \( ab \neq 0 \).]

5.7. COHERENT PRESENTATIONS OF ALGEBRAS

In this last section, we recall from [GHM19] the notion of coherent presentation for an algebra as a presentation of the algebra extended by a family of generating syzygies. We explain how to generate syzygies when the presentation is convergent.
5.7.1. **Linear 3-polygraph.** Let $\Lambda$ be a linear 2-polygraph. A **cellular extension** of the free 2-algebroid $\Lambda^\ell_2$ is a set $\Lambda_3$ equipped with maps

$$\Lambda^\ell_2 \xleftarrow{s_2} t_2 \Lambda_3$$

such that, for every $F$ in $\Lambda_3$, the pair $(s_2(F), t_2(F))$ is a 2-sphere in $\Lambda^\ell_2$, that is, $s_1 s_2(F) = s_1 t_2(F)$ and $t_1 s_2(F) = t_1 t_2(F)$ hold in $\Lambda^\ell_2$. The elements of $\Lambda_3$ are the **3-cells** of the cellular extension and graphically represented by

$$f \downarrow \downarrow F \xrightarrow{g}$$

A **linear 3-polygraph** is a data $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$, where $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a linear 2-polygraph and $\Lambda_3$ is a cellular extension of the free 2-algebroid $\Lambda^\ell_2$:

$$\begin{array}{cccc}
\Lambda_0 & & \Lambda^\ell_2 & & \Lambda_3 \\
\downarrow s_0 & & \downarrow \downarrow F & & \downarrow \downarrow g \\
\Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow t_0 & & \downarrow \downarrow s_1 & & \downarrow \downarrow t_1 \\
\Lambda_0 & & \Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow s_0 & & \downarrow \downarrow t_1 & & \downarrow \downarrow s_1 & & \downarrow \downarrow t_1 \\
\Lambda_0 & & \Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow s_0 & & \downarrow \downarrow t_1 & & \downarrow \downarrow s_1 & & \downarrow \downarrow t_1 \\
\Lambda_0 & & \Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\end{array}$$

5.7.2. **Three-dimensional algebras.** We define a **3-algebra** as an internal 2-category in the category $\text{Alg}$:

$$\begin{array}{cccc}
\Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow s_1 & & \downarrow \downarrow F & & \downarrow \downarrow G \\
\Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow t_1 & & \downarrow \downarrow s_2 & & \downarrow \downarrow t_2 \\
\Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow s_1 & & \downarrow \downarrow t_2 & & \downarrow \downarrow s_2 & & \downarrow \downarrow t_2 \\
\Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\downarrow s_1 & & \downarrow \downarrow t_2 & & \downarrow \downarrow s_2 & & \downarrow \downarrow t_2 \\
\Lambda_1 & & \Lambda_2 & & \Lambda_3 \\
\end{array}$$

In particular, the algebras $\Lambda_1$ and $\Lambda_2$ with composition $*_{1} : \Lambda_2 \times_{\Lambda_1} \Lambda_2 \to \Lambda_2$ form a 2-algebra. The 3-cells can be composed in two different ways:

$$*_{1} : \Lambda_3 \times_{\Lambda_1} \Lambda_3 \to \Lambda_3 \quad *_{2} : \Lambda_3 \times_{\Lambda_2} \Lambda_3 \to \Lambda_3$$

by $*_{1}$ along their 1-dimensional boundary, and by $*_{2}$ along their 2-dimensional boundary as pictured in (3.2.5). The source and target maps $s_1$, $s_2$ and $t_1$, $t_2$ being morphisms of algebras, the product of 3-cells $F$ and $G$ satisfies:

$$f g$$

These compositions and the product satisfy remarkable properties similar to those given in (5.1.12) for 2-algebras.
5.7. Coherent presentations of algebras

5.7.3. Free 3-algebras. The free 3-algebra over a linear 3-polygraph $\Lambda$ is constructed similarly to the free 2-algebra given in (5.1.14). It is the 3-algebra, denoted by $\Lambda^3$, whose underlying 2-algebra is the free 2-algebra $\Lambda^2$, and its 3-cells are all the formal 1-composition, 2-composition and product of 3-cells of $\Lambda^3$, of identities of 2-cells, up to associativity, identity, exchange and inverse relations, see [GHM19] for more details.

5.7.4. Coherent presentations of algebras. A coherent presentation of an algebra $A$ is a linear 3-polygraph $\Lambda$ such that

i) the linear 2-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2)$ is a presentation of $A$,

ii) $\Lambda_3$ is a homotopy basis of the free 2-algebra $\Lambda^2$, that is, a cellular extension

$$\Lambda^2 \xleftarrow{s_2} t_2 \Lambda^3$$

such that for every 2-sphere $(a, b)$ of the free 2-algebra $\Lambda^2$, there exists a 3-cell $A$ in the free 3-algebra $\Lambda^3$ such that $s_2(A) = a$ and $t_2(A) = b$.

5.7.5. Squier’s completion. Let $\Lambda$ be a left-monomial linear 2-polygraph. Suppose that all critical branching of $\Lambda$ are confluent. For every critical branching $(a, b)$ in $\Lambda$, we choose two positive 2-cells $a'$ and $b'$ making the branching confluent:

$$\begin{array}{c}
a \xrightarrow{g} a' \\
b \xrightarrow{f} b' \\
\end{array}$$

(5.7.6)

For any such a confluent branching, we consider a 3-cell $F_{(a, b)} : a \ast_1 a' \Rightarrow b \ast_1 b'$. The set of such 3-cells

$$\Lambda_3 = \{ F_{(a, b)} \mid (a, b) \text{ is a critical branching} \}$$

forms a cellular extension of the free 2-algebra $\Lambda^2$. The linear 3-polygraph $(\Lambda_0, \Lambda_1, \Lambda_2, \Lambda_3)$ is a Squier’s completion of $\Lambda$. When the polygraph is confluent, there exists such a Squier’s completion. However, the cellular extension $\Lambda_3$ is not unique in general. Indeed, the 3-cells can be directed in the reverse way and a branching $(a, b)$ can have several possible positive 2-cells $a'$ and $b'$ making the branching confluent.

The following result is a formulation of the Squier Lemma, [SOK94], in the setting of linear 2-polygraphs.

5.7.7. Theorem ([GHM19, Thm. 4.3.2]). Let $A$ be an algebra and let $\Lambda$ be a convergent left-monomial presentation of $A$. Any Squier’s completion of $\Lambda$ is a coherent presentation of $A$. 

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5.7.8. Linear oriented syzygies. Let $\Lambda$ be presentation of an algebra $A$. Any nontrivial 2-sphere $(a, b)$ in the free 2-algebra $\Lambda_2^f$ is called a linear oriented 3-syzygy of the presentation $\Lambda$. If $\Lambda$ is extended into a coherent presentation $(\Lambda, \Lambda_3)$ of the algebra $A$, the quotient 2-algebra $\Lambda_2^f/\Lambda_3$ is aspherical, that is, for any 2-sphere $(a, b)$ in $\Lambda_2^f/\Lambda_3$, we have $a = b$. In other words, the cellular extension $\Lambda_3$ forms a generating set of linear 3-syzygies of the presentation $\Lambda$. Theorem 5.7.7 say that, when the presentation $\Lambda$ is convergent the 3-cells defined by confluence diagrams of the critical branchings, as in (5.7.6), form a family of generator for 3-syzygies.

5.7.9. Exercise. Let $\{F_1, \ldots, F_k\}$ be a generating set for linear 3-syzygies of a linear 2-polygraph $\Lambda$. Prove that $\{F_1^-, \ldots, F_k^-\}$ is also a generating set for linear 3-syzygies of $\Lambda$.

5.7.10. Example. The linear 2-polygraph $\langle x \mid x^3 \Rightarrow 0 \rangle$ has one critical branching

\[
\begin{array}{c}
\xymatrix{ x^3 \ar@{-->}[r]^\alpha \ar@{->}[d] \ar@{->}[ld] & \alpha x \ar@{->}[d] \\
\ar@{->}[r]^F & 0 }
\end{array}
\]

which is confluent. The polygraph being convergent the 3-cell $F : \alpha x \Rightarrow x\alpha$ generates all linear 3-syzygies of this presentation.

5.7.11. Example. Consider the algebra $A$ presented by the linear 2-polygraph $\Lambda = \langle x, y, z \mid xyz \Rightarrow x^3 + y^3 + z^3 \rangle$ given in Example 5.1.7. It does not have critical branching, hence any Squier’s completion of $\Lambda$ is empty. As a consequence, $\Lambda$ can be extended into a coherent presentation with an empty homotopy basis. That is, there is no 3-syzygy for this presentation.

The linear 2-polygraph $\langle x, y, z \mid \alpha, \beta \rangle$ considered in Example 6.3.7 is Tietze equivalent to $\Lambda$, convergent and compatible with a monomial order. It has three critical branchings, as shown in Example 6.3.7. It can be extended into a coherent presentation of $A$ with three generating 3-syzygies.

5.7.12. Exercise. Give an explicit description of the 3-cells of a coherent presentation on the linear 2-polygraph $\Lambda'$ of Example 5.7.11.

5.7.13. Exercise. Compute a coherent presentation for the algebras presented by the following linear 2-polygraphs

1) $\langle x, y \mid xyx \Rightarrow y^2 \rangle$.

2) $\langle x, y, z \mid yz \Rightarrow -x^2, zy \Rightarrow -\lambda^{-1}x^2 \rangle$, where $\lambda \in K \setminus \{0, 1\}$, see [PP05, 4.3].

5.7.14. Exercise. Compute a minimal coherent presentation for the algebra presented by the linear 2-polygraph $\langle x \mid x^3 = 0 \rangle$. 

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5.7. Coherent presentations of algebras
Paradigms of linear rewriting

In this chapter, we survey several approaches in linear rewriting. The most well-known is given by the Gröbner basis theory for ideals in commutative polynomial rings introduced by Buchberger in [Buc65]. A subset $G$ of an ideal $I$ in the polynomial ring $\mathbb{K}[x]$ of commutative polynomials is a Gröbner basis of $I$ with respect to a given monomial order $\prec$, if the leading term ideal of $I$ is generated by the set of leading monomials of $G$, that is

$$\langle \text{lt}_\prec(I) \rangle = \langle \text{lt}_\prec(G) \rangle.$$  

Buchberger introduced the notion of S-polynomial to describe the obstructions to local confluence and gave an algorithm for computation of Gröbner bases, [Buc65, Buc06], see also [Buc87] for an historical account. Any ideal $I$ of a commutative polynomial ring $\mathbb{K}[x]$ has a finite Gröbner basis. Indeed, the Buchberger algorithm on a finite family of generators of an ideal $I$ always terminates and returns a Gröbner basis of the ideal $I$.

Shirshov introduced in [Shi62] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of composition of elements in a free Lie algebra, that corresponds to the notion of S-polynomial in the work of Buchberger. He gave an algorithm to compute bases in free algebras having the computational properties of the Gröbner bases. He proved that irreducible elements for such a basis forms a linear basis of the Lie algebra. This result is called now the Composition Lemma for Lie algebras.
6.1. Composition Lemma

Subsequently, the Gröbner basis theory has been developed for other types of algebras, such as associative algebras by Bokut in [Bok76] and by Bergman in [Ber78]. They prove Newman’s Lemma for rewriting systems in free associative algebras compatible with a monomial order stating that local confluence and confluence are equivalent properties. This result was called Composition Lemma by Bokut and Diamond Lemma for ring theory by Bergman, see also [Mor94, Ufn95]. In general, the Buchberger algorithm does not terminate for ideals in a noncommutative polynomial ring $\mathbb{K}\langle x \rangle$. Indeed, its termination would give a decision procedure of the undecidable word problem. Even if the ideal is finitely generated it may not have a finite Gröbner basis. However, when $\mathbb{K}$ is a field an infinite Gröbner basis can be computed, [Mor94, Ufn98].

Note that ideas in the spirit of the Gröbner basis approach appear in several others works. Let us mention works by Hironaka in [Hir64] and Grauert in [Gra72] that compute bases of ideals in rings of power series having analogous properties to Gröbner bases but without a constructive method for computing such bases. In [Coh65], Cohn gave a method to decide the word problem by a normal form algorithm based on a confluence property. Finally, Janet [Jan20], Thomas [Tho37] and Pommaret [Pom78] developed the notion of involutive bases that are particular cases of Gröbner bases in the context of partial differential algebra. We refer the reader to [LM19] for an historical account on involutives bases and their applications to algebraic analysis of linear partial differential systems. Much more recently, Gröbner basis theory was developed in various noncommutative contexts such as Weyl algebras, see [SST00], or operads [DK10].

6.1. Composition Lemma

6.1.1. Compositions in free Lie algebras. Shirshov introduced in [Shi62] an algorithm to compute a linear basis of a Lie algebra defined by generators and relations. He used the notion of composition of elements in a free Lie algebra, that corresponds to the notion of $S$-polynomial in the work of Buchberger, [Buc65]. This work remained unknown outside the USSR and the two theories were developed in parallel. The algorithm completes a given set of elements in a free algebra by adding all nontrivial compositions. This algorithm corresponds to the completion algorithm given by Knuth-Bendix for term rewriting systems, [KB70], and by Buchberger for commutative polynomials, [Buc65]. The Shirshov completion constructs a set, that may be infinite, such that every composition of its elements is trivial. Such a subset is called a Lie Gröbner-Shirshov basis. The key result in [Shi62] states that the set of irreducible elements for a Gröbner-Shirshov basis $S$ forms a linear basis of the Lie algebra with defining relations $S$. This result is called now the Composition-Diamond Lemma for Lie algebras. For a recent account of the theory of Gröbner-Shirshov we refer the reader to [BC14].

In this subsection we summarize without proofs an analogue of Shirshov’s composition-diamond lemma for associative algebras given by Bokut in [Bok76].

6.1.2. Compositions. Let $\Lambda^\ell_1$ be a free algebra over a set $\Lambda_1$ and let $\prec$ be a monomial order on $\Lambda^\ell_1$. Bokut introduced in [Bok76] the notion of composition of elements of a free associative algebra as follows. Given two 1-cells $f$ and $g$ in $\Lambda^\ell_1$ and a monomial $w$ in $\Lambda^*_1$. There are two kinds of compositions:

i) if $w = \text{lm}(f)v = u \text{lm}(g)$ with $\ell(\text{im}(f)) + \ell(\text{lm}(g)) > \ell(w)$, for some monomials $u$ and $v$ in $\Lambda^*_1$, 


Let \( 6.1.3. \) Gröbner-Shirshov’s bases.

Combinations creating several compositions.

\[ \text{gs-basis} G \text{ to the monomial ordering } \prec \text{.} \]

6.1.5. Theorem (The Composition Lemma, \[Bok76\], Proposition 1 & Corollary 1).

Let \( \text{gs-basis} G \text{ to the monomial order } \prec \text{.} \)

6.1.4. Exercise. Let \( G \) be a minimal Gröbner-Shirshov basis in a free algebra \( \Lambda_1 \). Suppose that there exists a decomposition

\[ w = u_1 \text{Im}(g_1)v_1 = u_2 \text{Im}(g_2)v_2, \]

with \( u_1, v_1, u_2, v_2 \in \Lambda_1 \) and \( g_1, g_2 \in G \). Show that \( u_1g_1v_1 - u_2g_2v_2 \) is trivial modulo \( (G, w) \).

6.1.5. Theorem (The Composition Lemma, \[Bok76\], Proposition 1 & Corollary 1). Let \( \Lambda_1 \) be a free algebra and let \( \prec \) be a monomial order on \( \Lambda_1 \). Let \( G \) be a set of 1-cells in \( \Lambda_1 \) and let \( I \) be the ideal generated by \( G \). Denote by \( A \) the algebra given by the quotient of the free algebra \( \Lambda_1 \) by the ideal \( I \). The following conditions are equivalent.

i) \( G \) is a GS-basis.

ii) For any \( f \) in \( I \), there exists a decomposition \( \text{Im}(f) = u \text{Im}(g)v \) for some \( u, v \) in \( \Lambda_1 \) and \( g \) in \( G \).

iii) The set of \( G \)-reduced monomial forms a linear basis of the algebra \( A \).
6.2. Reduction operators

Yet another approach of rewriting in associative algebras were developed by Bergman in [Ber78]. With a functional description of linear rewriting reductions he obtained an equivalent result of the composition lemma 6.1.5.

6.2.1. Reduction operators. Given $\Lambda_1^t$ a free algebra over a set $\Lambda_1$, he defines a reduction system as a set $S$ of pairs $\sigma^t = (w_\sigma, f_\sigma)$, where $w_\sigma$ is a monomial of $\Lambda_1^t$ and $f_\sigma$ is a 1-cell of $\Lambda_1^t$. Given $\sigma$ in $S$ and two monomials $u, v$ in $\Lambda_1^t$, he considers the linear map $r_{u\sigma v} : \Lambda_1^t \rightarrow \Lambda_1^t$ defined by

$$r_{u\sigma v}(w) = \begin{cases} uf_\sigma v & \text{if } w = uw_\sigma v, \\ w & \text{otherwise.} \end{cases}$$

The endomorphism $r_{u\sigma v}$ is called reduction by $\sigma$. Note that this notion of reduction corresponds to the notion of rewriting step given in (5.2.4).

A 1-cell $f$ in $\Sigma_1^t$ is irreducible under $S$ if every reduction by elements of $S$ acts trivially on $f$, that is $uf_\sigma v$ is not in the support of $f$, for any $\sigma$ in $S$ and monomials $u, v$ in $\Sigma_1^t$. As in the case of linear 2-polygraphs, we denote by $\Lambda_1^t_{lu}$ the vector subspace of $\Lambda_1^t$ of all irreducible 1-cells of $\Lambda_1^t$.

6.2.2. Reduction-unique. Bergman introduced the notion of confluence for reduction systems as follows. A finite sequence of reductions $r_1, \ldots, r_n$ is final on a 1-cell $f$, if the 1-cell $r_n \ldots r_1 (f)$ is irreducible. A 1-cell $f$ of $\Lambda_1^t$ is reduction-finite if for any infinite sequence $(r_n)_{n \geq 1}$ of reductions, $r_l$ acts trivially on $r_{l-1} \ldots r_1 (f)$ for a sufficiently large $l$. A 1-cell $f$ is reduction-unique if it is reduction-finite and if its images under all final sequences of reduction are the same. This common image is denoted by $r_S(f)$. A reduction system $S$ is reduction-unique if all 1-cells of $\Lambda_1^t$ are reduction-unique under $S$. 

6.2.3. Exercise, [Ber78, Lemma 1.1.].

1) Show that the set of reduction-unique 1-cells of $\Lambda_1^t$ forms a subspace of $\Lambda_1^t$ denoted by $\Lambda_1^t_{lu}$ and that $r_S : \Lambda_1^t_{lu} \rightarrow \Lambda_1^t_{lu}$ defines a linear map.

2) Given monomials $w_f, w_g$ and $w_h$ in the support of the 1-cells $f, g$ and $h$ respectively, such that the product $w_f w_g w_h$ is in $\Lambda_1^t_{lu}$. Show that for any finite composition of reductions $r$, then $r(f)g$ is in $\Lambda_1^t_{lu}$ and that $r_S (r(f)g) h) = r_S (fg)h)$ holds.

6.2.4. Ambiguities. A 5-tuple $(\sigma, \tau, u, v, w)$ with $\sigma, \tau$ in $S$ and $u, v, w$ monomials in $\Lambda_1^t$, such that $w_\sigma = uv$ and $w_\tau = vv$ (resp. $\sigma \neq \tau$, $w_\sigma = v$ and $w_\tau = uvw$) is an overlap ambiguity (resp. inclusion ambiguity) of $S$. Such an ambiguity is resolvable if there exist compositions of reductions $r$ and $r'$ that satisfy the confluence condition:

$$r(f_\sigma w) = r'(uf_\sigma) \quad \text{resp.} \quad r(u f_\sigma w) = r'(f_\sigma).$$
6.2.5. Reduction system compatible with a monomial order

The diamond lemma obtained by Bergman concern reduction systems compatible with a monomial order. A reduction system $S$ is compatible with a monomial order $\prec$, if for any $\sigma = (w_\sigma, f_\sigma)$ in $S$, we have $w \prec w_\sigma$ for any monomial $w$ in the support of $f_\sigma$.

Given a reduction system compatible with a monomial order $\prec$. For a monomial $w$ in $\Sigma_1^*$, we denote by $I_{w \prec}$ the subspace of $\Lambda_{1}^{\ell}$ defined by

$$I_{w \prec} = \text{Span}_{K} \left( u(w, f - f_\sigma) v \mid (w_\sigma, f_\sigma) \in S \text{ and } uw_\sigma v \prec w \right).$$

An overlap ambiguity (resp. inclusion ambiguity) $(\sigma, \tau, u, v, w)$ is resolvable relative to $\prec$ if

$$f_\sigma w - uf_\tau \in I_{w \prec}, \quad (\text{resp. } uf_\sigma w - f_\tau \in I_{w \prec}).$$

Let $G$ be a subset of 1-cells of $\Lambda_{1}^{\ell}$ and let $\prec$ be a monomial order on $\Lambda_{1}^{\ell}$. We denote by $S(G, \prec)$ the reduction system generated by $G$ with respect to $\prec$ defined by

$$S(G, \prec) = \left\{ \left( \text{lcm}(f), \text{lcm}(f) - \frac{1}{\text{lcm}(f)} f \right) \mid f \in G \right\}.$$

6.2.6. Theorem (The Diamond Lemma, [Ber78, Theorem 1.2]). Let $S$ be a reduction system compatible with a monomial order $\prec$. The following conditions are equivalent.

i) All the ambiguities of $S$ are resolvable.

ii) All the ambiguities of $S$ are resolvable relative to $\prec$.

iii) $S$ is reduction-unique.

A fourth equivalent condition is given in [Ber78, Theorem 1.2] as follows. Consider the algebra $A$ given as the quotient of the free algebra $\Lambda_{1}^{\ell}$ by the two-side ideal

$$I(S) = \{ w_\sigma - f_\sigma \mid \sigma \in S \}.$$ 

If the reduction system $S$ is compatible with a monomial order $\prec$, the confluence conditions i) - iii) above hold if and only if the set $\Lambda_{1}^{\ell}_{\text{irr}}$ of irreducible monomial under $S$ is a linear basis of the algebra $A$. In this case, the $K$-algebra $A$ is isomorphic to the $K$-algebra $\Lambda_{1}^{n}$, whose product is given by $f \cdot g = r_{S}(fg)$, for any 1-cells $f$ and $g$ in $\Lambda_{1}^{n}$.

6.3. NONCOMMUTATIVE GRÖBNER BASES

6.3.1. Noncommutative Gröbner bases. Let $\Lambda_{1}^{\ell}$ be a free algebra over a set $\Lambda_{1}$ and let $\prec$ be a monomial order on $\Lambda_{1}^{\ell}$. A (noncommutative) Gröbner basis of an ideal $I$ of $\Lambda_{1}^{\ell}$ with respect to the monomial order $\prec$ is a subset $G$ of $I$ such that the ideal generated by the leading monomials of the 1-cells of $I$ coincides with the ideal generated by the leading monomials of the 1-cells of $G$:

$$\langle \text{lm}(I) \rangle = \langle \text{lm}(G) \rangle.$$ 

Equivalently, for every 1-cell $f$ in $I$, there exists $g$ in $G$ with $\text{lm}(f) = u \text{lm}(g)v$, where $u$ and $v$ are monomials of $\Lambda_{1}^{\ell}$.

The two following results show that the notion of noncommutative Gröbner basis corresponds to the notion of left-monomial convergent linear 2-polygraph compatible with a monomial order.
6.3. Noncommutative Gröbner bases

6.3.2. Proposition. Let $\Lambda$ be a convergent left-monomial linear 2-polygraph, compatible with a monomial order $\prec$ on $\Lambda_1$. The set of 1-cells $\{s_1(\alpha) - t_1(\alpha) \mid \alpha \in \Lambda_2\}$ is a Gröbner basis of the ideal $I(\Lambda)$ for the monomial order $\prec$.

6.3.3. Exercise. Prove Proposition 6.3.2.

6.3.4. Proposition. Let $I$ be an ideal of a free 1-algebra $\Lambda_1$. Let $G$ be a Gröbner basis for $I$ with respect to a monomial order $\prec$. Then the leading 2-polygraph $\Lambda(G, \prec)$ is convergent and $I(\Lambda(G, \prec)) = I$ holds.

Proof. Suppose that $G$ is a Gröbner basis of the ideal $I$ with respect to $\prec$. By definition, the ideal $I(\Lambda(G, \prec))$ is equal to the ideal $I$ generated by $G$. Prove that the linear 2-polygraph $\Lambda(G, \prec)$ is convergent. Its termination is a consequence of its compatibility with the monomial order $\prec$. The monomials in $\Lambda_1^*$ reduced with respect to $\Lambda(G, \prec)$ are the monomials that cannot be decomposed as $u \cdot \text{lm}(g) \cdot v$ with $g \in G$ and $u$ and $v$ monomials in $\Lambda_1^*$. As a consequence, if a reduced 1-cell $f$ of $\Lambda_1^*$ is contained in the ideal $I$, its leading monomial must be 0, because $G$ is a Gröbner basis of $I$. By Proposition 5.5.4, we deduce that the linear 2-polygraph $\Lambda(G, \prec)$ is confluent.

As a conclusion to this chapter, the following result summarizes all the characterizations of the confluence property of linear rewriting systems. Note that some equivalences are tautological.

6.3.5. Theorem. Let $\Lambda_1^*$ be a free algebra over a set $\Lambda_1$. Let $\prec$ be a monomial order on $\Lambda_1$. Given an ideal $I$ of $\Lambda_1^*$ and a subset $G$ of $I$, we denote by $\Lambda$ the leading polygraph $\Lambda(G, \prec)$ and by $S$ the reduction system $S(G, \prec)$. The following conditions are equivalent.

i) $G$ is a Gröbner basis with respect to $\prec$.

ii) $\Lambda$ is convergent.

iii) $\Lambda$ is confluent.

iv) $\Lambda$ is locally confluent.

v) All the critical branchings of $\Lambda$ are confluent.

vi) Every composition $(f, g)_w$ is reduced to 0 with respect to the division by $G$.

vii) All the ambiguities of $S$ are resolvable.

viii) All the ambiguities of $S$ are resolvable relative to $\prec$.

ix) $S$ is reduction-unique.

x) $\Lambda_1^* = \Lambda_1^{nf} \oplus I$.

xi) Every 1-cell of $I$ admits 0 as a normal form with respect to $\Lambda_2$.

xii) For any $f$ in $I$, there exists a decomposition $\text{lm}(f) = u \cdot \text{lm}(g) \cdot v$ for some $u, v$ in $\Lambda_1^*$ and $g$ in $G$.

xiii) The set of $G$-reduced monomials forms a linear basis of the algebra $A$. 

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6.3.6. Exercise. Prove the equivalences of Theorem 6.3.5.

6.3.7. Example. Consider the linear 2-polygraph \( \Lambda \) given in Example 5.1.7. For the deglex order \( \prec_{\text{deglex}} \) induced by the alphabetic order \( x \prec y \prec z \), the leading monomial of \( f = z^3 + y^3 + x^3 - xyz \) is \( z^3 \), so that
\[
\Lambda(\{f\}, \prec_{\text{deglex}}) = \langle x, y, z \mid z^3 \alpha_f \Rightarrow xyz - x^3 - y^3 \rangle
\]
The left-monomial linear 2-polygraph \( \Lambda(\{f\}, \prec_{\text{deglex}}) \) is compatible with the monomial order \( \prec_{\text{deglex}} \), hence it is terminating. It is not confluent, because neither of its two critical branchings is confluent:

\[
\begin{align*}
\alpha_f z & \Rightarrow xyz - x^3 - y^3 \\
z^4 & \Rightarrow zxyz - z^3 - zy^3 \\
\alpha_f z^2 & \Rightarrow xy^3 - x^3 z^2 - y^3 z^2 \\
z^5 & \Rightarrow z^2xyz - z^2 x^3 - z^2 y^3 \\
z^2 \alpha_f & \Rightarrow z^2xyz - z^2 x^3 - z^2 y^3
\end{align*}
\]
In particular, \( \{f\} \) does not form a Gröbner basis of the ideal \( I(\Lambda) \) We add to the polygraph \( \Lambda(\{f\}, \prec_{\text{deglex}}) \) the following 2-cell
\[
\beta : zy^3 \Rightarrow zxyz - z^3 + y^3 z + x^3 z - xyz^2.
\]
This new rule makes the two previous critical branchings confluent and create a new critical branching
\[
\begin{align*}
z^2 \beta & \Rightarrow z^3xyz - z^3 x^3 + z^2 y^3 z + z^2 x^3 z - z^2 xyz^2 \\
z^3 y^3 & \Rightarrow xyzy^3 - x^3 y^3 - y^6
\end{align*}
\]
which is also confluent. Finally, the convergent linear 2-polygraph \( \langle x, y, z \mid \alpha_f, \beta \rangle \) is Tietze equivalent to the initial linear 2-polygraph \( \Lambda(\{f\}, \prec_{\text{deglex}}) \). In particular, the set of 1-cells \( \{f, s_1(\beta) - t_1(\beta)\} \) forms a Gröbner basis of the ideal \( I(\Lambda) \) with respect to the order \( \prec_{\text{deglex}} \).

6.3.8. Example. The algebra presented by the following linear 2-polygraph
\[
\langle x, y, z \mid x^2 = 0, \ xy = zx \rangle
\]
does not have a finite Gröbner bases on 3-generators \( x, y \) and \( z \). Indeed, the first relation is oriented as \( x^2 \Rightarrow 0 \) and the orientation \( xy \Rightarrow zx \) induce the addition of the 2-cells \( xz^n x \Rightarrow 0 \), for all integer \( n \geq 1 \). Another way is to orient the relation as \( zx \Rightarrow xy \). But in this case, we need to add the 2-cells \( xy^n x \Rightarrow 0 \), for all integer \( n \geq 1 \).
6.3. Noncommutative Gröbner bases

6.3.9. Exercise. Show that we can compute a Gröbner bases for the algebra given in Example 6.3.8 with four generators. [Hint. Add a generator \( t \) and the relations \( xy \Rightarrow t \) and \( zx \Rightarrow t \).]
In two seminal papers, Anick introduced a method to compute a free resolution for an algebra starting with a Gröbner basis of its ideal of relations. First he gave the construction for monomial algebras in [Ani85] then for associative augmented algebras in [Ani86]. For an algebra presented by a Gröbner basis, the $n$th chains of its Anick resolution are generated by the $n$-fold overlaps of the leading terms of the Gröbner basis, and the differentials are constructed by Noetherian induction. The chains defined by Anick are recall in Subsection 7.2. The construction of the resolution is given in Subsection 7.3.

Resolutions for path algebras using the same method were obtained by Anick and Green in [AG87]. For a deeper discussion on the theory of Gröbner bases for path algebras and how to apply this theory to the construction of free resolutions for path algebras, we refer the reader to [Gre99]. Let us mention that the Anick resolution has been achieved by other methods. In particular, the Anick resolution for a homogeneous algebra can be constructed by a deformation of the resolution computed on the associated monomial algebra, see [DK09 Sec. 2.4.] for details, see also the Backelin construction in [Bac78]. The Anick resolution can be also obtained using algebraic Morse theory with a Morse matching on the bar resolution, see [Skö06 Sec. 3.2.] for details. Morse theory allows to construct, starting from a chain
7.1. Homology of an algebra

complex, a new chain complex such that the homology of the two complexes coincides. This method
was also applied to the computation of minimal resolutions starting from the Anick resolution, [JW09].

Note also that others constructions of free resolutions using convergent rewriting systems were ob-
tained by several authors, [Bro92, Kob90, Gro90, Kob05, GM12b]. Finally, let us mention that noncom-
mutative Gröbner bases where developed by Dotsenko and Khoroshkin for shuffle operads in [DK10],
giving operadic versions of Newman’s lemma and Buchberger’s algorithm. The Anick resolution for
shuffle operads was constructed by Dotsenko and Khoroshkin in [DK09, DK12]. Using this construc-
tion, they prove that a shuffle operad with a quadratic Gröbner basis is Koszul, [DK12].

7.1. Homology of an algebra

In this section, we briefly recall the definition of homology of associative algebras with coefficients in
left modules. For a deeper discussion on basic notions of homological algebra we refer the reader to
[HS97, Rot09].

7.1.1. Functor Tor. Let us recall the definition of the derived functor Torₙᵣ of the tensor product of
modules over a fixed ring R. Let M be a left R-module and N be a right R-module. Given a projective
resolution ℙ of the right R-module N:

\[ ℙ : \cdots \to P_n \xrightarrow{d_{n-1}} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_0} P_0 \xrightarrow{ε} N \to 0 \]

we associate the deleted complex:

\[ ℙ_N : \cdots \to P_n \xrightarrow{d_{n-1}} P_{n-1} \to \cdots \to P_1 \xrightarrow{d_0} P_0 \to 0 \]

obtained by suppressing the module N. Note that, we have not lost any information in the complex
ℙₙ. As \( N = \text{coker}(d_0) \) by exactness of complex ℙ. Then, applying the functor \(- \otimes_R M\), we form a complex
of \( \mathbb{Z}\)-modules, denoted by \( ℙ_N \otimes_R M \):

\[ ℙ_N \otimes_R M : \cdots \to P_n \otimes_R M \xrightarrow{d_{n-1}} P_{n-1} \otimes_R M \to \cdots \to P_1 \otimes_R M \xrightarrow{d_0} P_0 \otimes_R M \to 0 \]

where \( d_{n-1} \) denotes the map \( d_{n-1} \otimes \text{Id}_M \).

For a natural number \( n \geq 0 \), we defined the \( \mathbb{Z}\)-module Torₙᵣ(\( M, N \)) as the \( n \)th homology group of
this complex:

\[ \text{Tor}_n^R(N, M) = H_n(ℙ_N \otimes_R M) = \text{Ker} d_{n-1}/\text{Im} d_n. \]

This definition is functorial in each variables, giving a bifunctor Torₙᵣ from R-modules with values in the
category of \( \mathbb{Z}\)-modules.

7.1.2. Following the definition, the functor Tor₀ᵣ(\( N, - \)) is naturally equivalent to \( N \otimes_R - \) and the functor
Tor₀ᵣ(\(-, M \)) is naturally equivalent to \(- \otimes_R M\). Indeed, we have Tor₀ᵣ(\( N, M \)) = coker(\( d_0 \)). Furthermore,
the functor \( N \otimes_R - \) is right exact, hence

\[ \text{coker}(d_0) = P_0 \otimes_R M/\text{Im} (d_0) = P_0 \otimes_R M/\ker(ε \otimes \text{Id}_M) = N \otimes_R M. \]

This proves that

\[ \text{Tor}_0^R(N, M) = N \otimes_R M. \]
7.1.3. Contracting homotopy. Recall that a method to prove that a complex of \( R \)-modules

\[
\cdots \rightarrow M_{n+1} \xrightarrow{d_n} M_n \xrightarrow{d_{n-1}} M_{n-1} \rightarrow \cdots \rightarrow M_1 \xrightarrow{d_0} M_0 \xrightarrow{\varepsilon} N \rightarrow 0
\]
is acyclic is to construct a contracting homotopy, that is a sequence of morphisms of abelian groups

\[
\cdots \leftarrow M_{n+1} \xleftarrow{i_{n+1}} M_n \xleftarrow{i_n} M_{n-1} \leftarrow \cdots \leftarrow M_1 \xleftarrow{i_1} M_0 \xleftarrow{i_0} N
\]
such that

\[
\varepsilon_{i_0} = \text{Id}_N, \quad d_0 t_1 + t_0 \varepsilon = \text{Id}_{M_0}, \quad d_n t_{n+1} + t_n d_{n-1} = \text{Id}_{M_n},
\]
for every \( n \geq 1 \).

7.1.4. Homology of an algebra. Let \( A \) be an associative algebra over a field \( \mathbb{K} \). For \( n \geq 0 \), the \( n \)-th homology space of the algebra \( A \) with coefficient in a left \( A \)-module \( M \) is defined by

\[
H_n(A, M) = \text{Tor}_n^A(\mathbb{K}, M).
\]

In practice, to compute the \( n \)-th homology spaces \( H_n(A, \mathbb{K}) \), for all \( n \geq 0 \), we construct a free resolution of \( \mathbb{K} \), seen as a trivial right-\( A \)-module:

\[
\mathcal{F}_\mathbb{K} : \cdots \rightarrow F_n \xrightarrow{d_{n-1}} F_{n-1} \rightarrow \cdots \rightarrow F_1 \xrightarrow{d_0} F_0 \xrightarrow{\varepsilon} \mathbb{K} \rightarrow 0
\]
and we compute the homology of the complex \( \mathcal{F}_\mathbb{K} \otimes_A \mathbb{K} \).

7.2. ANICK’S CHAINS

In this subsection, \( \Lambda \) denotes a reduced left-monomial linear 2-polygraph. The set of sources of rules in \( \Lambda_2 \) we will be denoted by \( s_1(\Lambda) = \{ s_1(\alpha) \in \Lambda_1^+ \mid \alpha \in \Lambda_2 \} \). For a monomial \( u \) in \( \Lambda_1^+ \), we denote by \( \deg_{s_1(\Lambda)}(u) \) the number of possible reductions on \( u \) with respect to \( \Lambda_2 \).

7.2.1. Anick’s chains, [Ani86]. For an integer \( n \geq -1 \), the Anick \( n \)-chains of the linear 2-polygraph \( \Lambda \) and their tails are defined by induction as follows.

- The unique \((-1)\)-chain is the empty monomial, denoted by \( 1 \), it is its own tail.
- The 0-chains are the 1-cells in \( \Lambda_1 \), and the tail of a 0-chain \( x \) in \( \Lambda_1 \) is \( x \) itself.
- For \( n \geq 1 \), suppose that \((n-1)\)-chains and their tails constructed. An \( n \)-chain is a monomial \( u \) in \( \Lambda_1^+ \) of the form
  \[
u = vt
\]
such that
  i) \( v \) is \((n-1)\)-chain,
7.2. Anick’s chains

ii) \( t \) is a reduced monomial with respect to \( \Lambda_2 \), called the *tail* of \( u \),

iii) if \( r \) is the tail of \( v \), then \( \text{deg}_{s_1(\Lambda)}(rt) = 1 \),

iv) the unique reduction on \( rt \) is rightmost, that is, given by a 2-cell \( \alpha \) in \( \Lambda_2 \) reducing the ending of the monomial \( rt \):

\[
\begin{array}{c}
\text{u} \\
v \\
\text{t} \\
\hline
\text{r} \\
\hline
\alpha
\end{array}
\]

We will denote by \( \Omega_n(\Lambda) \), or by \( \Omega_n \) if there is no possible confusion, the set of \( n \)-chains of the linear 2-polygraph \( \Lambda \).

### 7.2.2. Anick’s chains and overlapping.

The linear 2-polygraph \( \Lambda \) being reduced, we have the following description of Anick’s chains. We have \( \Omega_1(\Lambda) = s_1(\Lambda) \). Indeed, a 1-chain is a non reduced monomial \( u \) written \( u = xt_1 \), where \( x \) is a 1-cell in \( \Lambda_1 \) and \( t_1 \) is a reduced monomial:

\[
\begin{array}{c}
\text{x} \\
\text{u} \\
\hline
\text{t}_1
\end{array}
\]

and such that there is only one 2-cell in \( \Lambda_2 \) that can be applied on the monomial \( u \). A 2-chain \( u \) is the source of a critical branching. Indeed, \( u = xt_1t_2 \), where \( xt_1 \) is the source of a 2-cell \( \alpha \) in \( \Lambda_2 \) and there is a rightmost reduction \( \tau \) reducing \( t_1t_2 \), and thus overlapping \( \alpha \):

\[
\begin{array}{c}
\text{x} \\
\text{t}_1 \\
\hline
\text{t}_2
\end{array}
\]

Moreover, \( u \) is not the source of a critical triple branching, as we have \( \text{deg}_{s_1(\Lambda)} u = 2 \). In this way, there is a one-to-one correspondence between \( \Omega_2(\Lambda) \) and the set of critical branchings of the 2-polygraph \( \Lambda \).

For \( n \geq 3 \), a \( n \)-chain \( u \) corresponds to a \( n \)-fold overlapping composed by \((n - 1)\) chained critical branchings. It is possible that \( \text{deg}_{s_1(\Lambda)} u > n \), see Example 7.2.5.
7.2.3. Proposition ([Ani86]). Suppose \( n \geq 1 \). If \( u = x_{i_1} \ldots x_{i_s} \) is an \( n \)-chain, then there is a unique \( s \leq t \) such that \( x_{i_1} \ldots x_{i_s} \) is an \((n-1)\)-chain. Moreover, \( x_{i_{s+1}} \ldots x_{i_t} \) is reduced.

Indeed, suppose that there are two \((n-1)\)-chains \( x_{i_1} \ldots x_{i_s} \) and \( x_{i_1} \ldots x_{i_{s'}} \) which factorise \( u \). By uniqueness of the reduction on the tail, condition iii) in (7.2.1), necessarily we have \( s = s' \).

7.2.4. Notation. An \( n \)-chain \( u \), whose \((n-1)\)-chain is \( v \) and tail is \( t \), will be denoted by \( u = v|t \).

Expanding this notation, any \( n \)-chain can be written \( x|t_1|t_2| \ldots |t_n \), where \( x \in s_1(\Lambda) \) and \( x|t_1| \ldots |t_i \) is an \( i \)-chain for any \( 0 < i < n \).

7.2.5. Example, [Ani86]. Let \( \Lambda \) be a reduced left-monomial linear 2-polygraph with \( s_1(\Lambda) = \{x, y, z\} \) and \( s_1(\Lambda) = \{x^3\} \). The 1-cell \( x \) is the unique 0-chain. The monomial \( x^3 = x|x^2 \) is the unique 1-chain, \( xx \) is not a 1-chain because \( \deg s_1(\Lambda) x^2 = 0 \). The monomial \( x^4 = x^3|x \) is the unique 2-chain. Note that \( x^5 = x^3|x^2 \) is not a 2-chain. Indeed, \( \deg s_1(\Lambda) x^4 = 2 \), and on the monomial \( x^5 \) there are three possible reductions, with the first one that intersects the last one, giving a critical triple branching:

\[
\text{xxxxxx}
\]

The monomial \( x^6 = x^4|x^2 \) is the unique 3-chain. Note that \( x^5 = x^4|x \) is not a 3-chain because \( \deg s_1(\Lambda) xx = 0 \). Note that there are four possible reductions on the 3-chain \( x^6 \):

\[
\text{xxxxxx}
\]

Thus we have
\[
\Omega_0 = \Lambda_1, \quad \Omega_1 = s_1(\Lambda), \quad \Omega_2 = \{x^4\}, \quad \Omega_3 = \{x^6\}.
\]

More generally, we show that for any integer \( n \geq 0 \), we have
\[
\Omega_{2n-1} = \{x^{3n}\}, \quad \Omega_{2n} = \{x^{3n+1}\}.
\]

7.2.6. Example, [Ani86]. Suppose that \( \Lambda_1 = \{x, y\} \) and \( s_1(\Lambda) = \{x^2yxy, xyxy^2\} \). Then we have
\[
\Omega_0 = \{x, y\}, \quad \Omega_1 = \{x|xyxy, x|yxy^2\}, \quad \Omega_2 = \{x|xyxy|y, x|xyxy|xy^2\}, \quad \Omega_n = \emptyset, \quad \text{for } n \geq 3.
\]

7.2.7. Exercise, [Ani85]. Let \( \Lambda \) be a linear 2-polygraph such that \( \Lambda_1 = \{x, y, z\} \). Determine Anick’s chains in the following situations
1) \( s_1(\Lambda) = \{xyzxz, zxyy\} \).
2) \( s_1(\Lambda) = \{xyzxz, xyxy\} \). In this case, show that the number of \( n \)-chains equals the \((n + 2)\)nd Fibonacci number when \( n \geq 1 \).

7.3. ANICK’S RESOLUTION

In this subsection, \( \Lambda \) denotes a convergent reduced left-monomial linear 2-polygraph, whose 2-cells are compatible with a monomial order \( \prec \) defined on \( \Lambda_1^1 \). Let denote by \( A \) the algebra presented by \( \Lambda \). We define a section \( t : A \longrightarrow A \) of the canonical projection \( \pi : A_{s_0}^1 \longrightarrow A \), sending every 1-cell \( f \) of \( A \) to the normal form \( \hat{f} \) of any representative 1-cell of \( f \) in \( \Lambda_1^1 \), as in (5.5.5).
7.3. Anick’s resolution

7.3.1. Anick’s resolution. Let $A[\Omega_n(\Lambda)] = K[\Omega_n(\Lambda)] \otimes_K A$ be the free right $A$-module over the set of $n$-chains $\Omega_n(\Lambda)$. We will identify $A[\Omega_0(\Lambda)]$ to $A[\Lambda_1]$ and $A[\Omega_1(\Lambda)]$ to $A$. Anick constructs in [Ani86] a free resolution of right $A$-modules defined by the complex

$$ A(\Lambda) : \cdots \rightarrow A[\Omega_n(\Lambda)] \xrightarrow{d_n} A[\Omega_{n-1}(\Lambda)] \rightarrow \cdots \rightarrow A[\Omega_1(\Lambda)] \xrightarrow{d_1} A[\Lambda_1] \xrightarrow{d_0} A \xrightarrow{\varepsilon} K \rightarrow 0, $$

whose differentials $d_n$ are constructed inductively simultaneously with the contracting homotopy

$$ t_n : \text{Ker } d_{n-1} \rightarrow A[\Omega_n(\Lambda)]. $$

The applications $d_n$ are morphisms of right $A$-modules and the applications $t_n$ are linear maps.

7.3.2. For the first steps of the resolution

$$ A[\Lambda_1] \xrightarrow{d_0} A \xrightarrow{\varepsilon} K \rightarrow 0, $$

we define $\iota_1 : K \hookrightarrow A$ as the embedding of $K$ in $A$, and we define the augmentation map $\varepsilon : A \rightarrow K$ by setting $\varepsilon(x) = 0$, for all $x \in \Lambda_1$. Hence, we have $A = K \oplus \text{Ker } \varepsilon$, and $\varepsilon \iota_1 = \text{Id}_K$. Then, we set

$$ d_0(x \otimes 1) = x, $$

for all $x$ in $\Lambda_1$. By convergence hypothesis, any monomial in $A$ admits a unique normal form in $\Lambda_1^*$ with respect to $\Lambda_2$. For a monomial $u$ in $A$ such that the normal form is written $\hat{u} = x_1x_2\ldots x_k$ in $\Lambda_1^*$, we define

$$ t_0(1 \otimes u) = x_1 \otimes x_2 \ldots x_k. $$

Then, we extend $t_0$ to any $f$ in $A$ by linearity. The map $t_0$ is well defined by uniqueness of the normal form due to the convergence of the linear 2-polygraph $\Lambda$. The exactness of the sequence (7.3.3) in $A$ is a consequence of the two equalities:

$$ \varepsilon d_0(x \otimes 1) = 0 \quad \text{and} \quad d_0 t_0 = \text{id}_{\text{Ker } \varepsilon}. $$

7.3.5. For $n \geq 1$, we define the pair $(d_n, t_n)$ by induction on $n$:

$$ A[\Omega_n(\Lambda)] \xrightarrow{d_n} A[\Omega_{n-1}(\Lambda)] \xrightarrow{t_n} A[\Omega_{n-2}(\Lambda)] \xrightarrow{d_{n-1}} \ldots $$

We suppose that the maps $d_k$ and $t_k : \text{Ker } d_{k-1} \rightarrow A[\Omega_k(\Lambda)]$ are constructed such that

$$ d_{k-1} d_k = 0 \quad \text{and} \quad d_k t_k = \text{Id}_{\text{Ker } d_{k-1}}, $$

for all $k \leq n - 1$. We define inductively $d_n$ on an $n$-chain $v \otimes t$ with tail $t$ by

$$ d_n(v \otimes t) = v \otimes t - t_{n-1} d_{n-1}(v \otimes t). $$
7.3.7. In the definition of $d_n(v \otimes t^1)$, the term $v \otimes t$ will be the leading term with respect to the well-founded order defined on $A[\Omega_n(\Lambda)]$ as follows. We extend the monomial order $\prec$ on $\Lambda^1$ into a well-founded order on $A[\Omega_n(\Lambda)]$ by setting

$$f_1 \otimes u_1 \prec f_2 \otimes u_2 \quad \text{if} \quad f_1 \hat{u}_1 \prec f_2 \hat{u}_2,$$

for all $f_1, f_2$ in $K[\Omega_n(\Lambda)]$ and $u_1, u_2$ in $A$.

7.3.8. Let us define recursively the map

$$\iota_n : \ker d_{n-1} \longrightarrow A[\Omega_n(\Lambda)]$$

as follows. Given $h$ in $\ker d_{n-1} \subset A[\Omega_{n-1}(\Lambda)]$, we denote by $u_{n-1} \otimes t$ the leading term of $h$, that is

$$h = \lambda u_{n-1} \otimes t + \text{(lower terms)},$$

where $\lambda$ in $K$ is non-zero. The $(n-1)$-chain $u_{n-1}$ can be uniquely decomposed in

$$u_{n-1} = u_{n-2} | t',$$

where $u_{n-2}$ is an $(n-2)$-chain and $t'$ is the tail of $u_{n-1}$. By induction, we have

$$d_{n-1}(u_{n-1} \otimes 1) = u_{n-2} \otimes t' + \text{(lower terms)}.$$

As $d_{n-1}$ is a morphism of right $A$-modules, we have

$$d_{n-1}(h) = \lambda d_{n-1}(u_{n-1} \otimes t) + d_{n-1}(\text{lower terms})$$

$$= \lambda u_{n-2} \otimes t' t + \text{(lower terms)}.$$ 

Suppose now that the monomial $t't$ is reduced, then $u_{n-2} \otimes t't$ remain the leading term of $d_{n-1}(h)$, hence $h$ cannot be in $\ker d_{n-1}$ thus contradicting the hypothesis. It follows that $t't$ can be reduced, and we set

$$t't = v'wv,$$

where $w$ is the 1-source of the leftmost reduction with respect to $\Lambda_2$ that can be applied on $t't$:

\[
\begin{array}{ccc}
\uparrow & \\ u_{n-1} \\
\hline
u_{n-2} & t' & t \\
\hline
\hline
\hline
v' & w' & v \\
\hline
w_2 & w_1 \\
\end{array}
\]  

(7.3.9)

Consider the factorization $w = w_2w_1$ and $t = w_1v$ as in the picture (7.3.9). It follows that $u_{n-2}v'w = u_{n-2}t'w_1$ forms an $n$-chain, and $u_{n-2}v'w \otimes v \in A[\Omega_n(\Lambda)]$. We set

$$\iota_n(h) = \iota_n(\lambda u_{n-1} \otimes t + \text{lower terms})$$

$$= \lambda u_{n-2}v'w \otimes v + \iota_n(h - \lambda d_n(u_{n-2}v'w \otimes v)).$$

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This is well defined, because $h - \lambda d_n(u_{n-2}v'w \otimes v) \prec h$ by construction. Indeed

$$d_n(u_{n-2}v'w \otimes v) = d_n(u_{n-2}v'w_1 \otimes v) = u_{n-2}v'w_2 \otimes w_1v + \text{(lower terms)} = u_{n-1} \otimes t + \text{(lower terms)}.$$ 

Moreover, $d_{n-1}(h - \lambda d_n(u_{n-2}v'w \otimes v)) = 0$.

From this construction, we deduce the following result:

7.3.10. **Theorem ([Ani86, Theorem 1.4])**. Let $A$ be an algebra presented by a convergent reduced left-monomial linear 2-polygraph $\Lambda$, compatible with a monomial order $\prec$. The complex of right $A$-modules $A(\Lambda)$ defined by

$$\cdots \rightarrow A[\Omega_n(\Lambda)] \xrightarrow{d_n} A[\Omega_{n-1}(\Lambda)] \rightarrow \cdots \rightarrow A[\Omega_1(\Lambda)] \xrightarrow{d_1} A[\Lambda] \xrightarrow{d_0} A \xrightarrow{\varepsilon} K \rightarrow 0$$

where, for any $n \geq 0$, the morphism $d_n$ is defined on a $n$-chain $v \mid t$ by

$$d_n(v \mid t) = v \otimes t + h,$$

where $\lt(h) \prec v \mid t$, if $h \neq 0$, is a resolution of the trivial right $A$-module $K$.

7.3.11. **Example.** Let consider the algebra $A$ presented by the linear 2-polygraph

$$\Lambda = \langle x, y \mid x^2 \overset{\alpha_0}{\Rightarrow} yx \rangle,$$

compatible with the deglex order $\prec_{\text{deglex}}$ induced by the alphabetic order $y \prec x$. It appears one critical branching

$$\xymatrix{ x \ar@{=>}[rrrr]^{\alpha_0} &&&& xy \ar[llll]_{x^3} }$$

$$\xymatrix{ x \ar[r]_{\alpha_0} & yx \ar[l]^{y^2x} & x^2 \ar[r]_{\alpha_0} & yx \ar[l]^{y\alpha_0} }$$

We complete the linear 2-polygraph $\Lambda$ with the 2-cells

$$\alpha_n : xy^n x \Rightarrow y^{n+1} x,$$

for all $n > 0$. We note that, for any integers $n, m \geq 0$, we have a critical branching

$$\xymatrix{ xy^n \ar[r]_{\alpha_n} & y^{n+1}x \ar[l]_{\alpha_{n,m}} & y^{n+m+1}x \ar[l]_{\alpha_{n,m+1}} \ar[r]_{\alpha_{n+1,m} \alpha_m} & y^{n+m+2}x }$$

Then the linear 2-polygraph $\Lambda'$, whose set of 1-cell is $\Lambda_1$ and $\Lambda'_2 = \{ \alpha_n \mid n \geq 0 \}$ is convergent, compatible with the monomial order $\prec$ and Tietze equivalent to $\Lambda$. Equivalently, the set $\{xy^n x - y^{n+1} x \mid n \geq 0 \}$
forms a Gröbner basis for the ideal \( I(\Lambda) \). Anick’s 1-chains are of the form \( x|y^n|x \) with \( n \geq 0 \) and Anick’s 2-chains are of the form \( x|y^n|x|y^m|x \) with \( n, m \geq 0 \). More generally, for any \( k \geq 2 \), we have

\[
\Omega_k = \{ x|y^n|x|y^n|x| \cdots |y^n|x \text{ for } n_1, \ldots, n_k \geq 0 \},
\]

Let us compute the boundary maps \( d_0, d_1, d_2 \) and \( d_3 \). We have \( d_0(x \otimes 1) = x \), \( d_0(y \otimes 1) = y \) and

\[
d_1(x|y^n|x \otimes 1) = x \otimes y^n|x - t_0 d_0(x \otimes y^n|x),
\]

\[
= x \otimes y^n|x - t_0(1 \otimes xy^n|x),
\]

\[
= x \otimes y^n|x - t_0(1 \otimes y^{n+1}|x),
\]

\[
= x \otimes y^n|x - y \otimes y^n|x.
\]

The last equality is consequence of the definition of the map \( \iota_0 \) in (7.3.4).

\[
d_2(x|y^n|x|y^m|x \otimes 1) = x|y^n|x \otimes y^m|x - t_1 d_1(x|y^n|x \otimes y^m|x),
\]

\[
= x|y^n|x \otimes y^m|x - t_1(x \otimes y^n|xy^m|x - y \otimes y^n|xy^m|x),
\]

\[
= x|y^n|x \otimes y^m|x - t_1(x \otimes y^{n+m+1}|x - y \otimes y^{n+m+1}|x),
\]

\[
= x|y^n|x \otimes y^m|x - x|y^{n+m+1}|x \otimes 1 + t_1(x \otimes y^{n+m+1}|x - y \otimes y^{n+m+1}|x - d_1(x|y^{n+m+1}|x \otimes 1)),
\]

\[
= x|y^n|x \otimes y^m|x - x|y^{n+m+1}|x \otimes 1 + t_1(x \otimes y^{n+m+1}|x - y \otimes y^{n+m+1}|x - x \otimes y^{n+m+1}|x + y \otimes y^{n+m+1}|x)),
\]

\[
= x|y^n|x \otimes y^m|x - x|y^{n+m+1}|x \otimes 1.
\]

\[
d_3(x|y^n|x|y^m|x|y^k|x \otimes 1) = x|y^n|x|y^m|x \otimes y^k|x - t_2 d_2(x|y^n|x|y^m|x \otimes y^k|x),
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - t_2(x|y^n|x|y^m|x|xy^k|x - x|y^{n+m+1}|x \otimes y^k|x),
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - t_2(x|y^n|x|y^m|x|y^m+k+1|x - x|y^{n+m+1}|x \otimes y^k|x),
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - x|y^n|x|y^{m+k+1}|x \otimes 1
\]

\[
- t_2(x|y^n|x|y^{m+k+1}|x - x|y^{n+m+1}|x \otimes y^k|x - x|y^n|x|y^{m+k+1}|x - x|y^{n+m+k+2}|x \otimes 1)
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - x|y^n|x|y^{m+k+1}|x \otimes 1 - t_2(x|y^n|x|y^{m+k+1}|x \otimes y^k|x - x|y^{n+m+1}|x \otimes y^k|x - x|y^{n+m+k+1}|x \otimes 1),
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - x|y^n|x|y^{m+k+1}|x \otimes 1 + x|y^{n+m+1}|x|y^k|x \otimes 1
\]

\[
+ t_2(x|y^{n+m+1}|x \otimes y^k|x - x|y^{n+m+k+1}|x \otimes 1 - d_2(x|y^{n+m+1}|x|y^k|x \otimes 1)),
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - x|y^n|x|y^{m+k+1}|x \otimes 1 + x|y^{n+m+1}|x|y^k|x \otimes 1
\]

\[
+ t_2(x|y^{n+m+1}|x \otimes y^k|x - x|y^{n+m+k+1}|x \otimes 1 - x|y^{n+m+1}|x|y^k|x - x|y^{n+m+k+1}|x \otimes 1)
\]

\[
= x|y^n|x|y^m|x \otimes y^k|x - x|y^n|x|y^{m+k+1}|x \otimes 1 + x|y^{n+m+1}|x|y^k|x \otimes 1.
\]
7.3. Anick’s resolution

7.3.12. Example. Let consider the algebra \( A \) given in \( 7.3.11 \) with the following presentation
\[
\langle x, y \mid yx \Rightarrow x^2 \rangle,
\]
compatible with the deglex order induced by the alphabetic order \( x \prec y \). This polygraph does not have critical branching, thus the sets of Anick’s \( n \)-chains are empty for \( n \geq 2 \). It follows that the associated Anick resolution is
\[
\cdots \rightarrow 0 \rightarrow A[y|x] \xrightarrow{d_1} A[x,y] \xrightarrow{d_0} A \xrightarrow{\cdot} \mathbb{K} \rightarrow 0
\]
with \( d_0(x \otimes 1) = x \), \( d_0(y \otimes 1) = y \) and
\[
d_1(y|x \otimes 1) = y \otimes x - u_0(1 \otimes y x),
\]
\[
= x \otimes y - u_0(1 \otimes x^2),
\]
\[
= x \otimes y - x \otimes x.
\]

7.3.13. Example. Consider Example \( 5.1.7 \) with the algebra \( A \) presented by
\[
\langle x, y, z \mid xyz = x^3 + y^3 + z^3 \rangle.
\]
With the Gröbner basis computed in \( 6.3.7 \),
\[
z^3 \Rightarrow xyz - x^3 - y^3 \quad zy^3 \Rightarrow xyz - xz^3 + y^3 z + x^3 z - xyz^2
\]
Anick’s chains are of the form \( z^n \) and \( \mathbb{A}^n \), for \( n \geq 0 \), so that the Anick resolution, defined in \( [\text{Ani}86] \), is infinite.

7.3.14. Exercise, \([\text{Ani}86, \text{Section} 3]\). Compute the Anick resolution for the algebra presented by the linear 2-polygraph \( (x, y \mid xyxy \Rightarrow xy ) \).

7.3.15. Anick’s resolution for a monomial algebra. We construct the Anick resolution in the case of a monomial algebra \( A \). Recall from \( 5.1.19 \) that such an algebra can be presented by a monomial linear 2-polygraph \( \Lambda \), that is, left-monomial and \( t_1(\alpha) = 0 \) for all \( \alpha \in \Lambda_2 \). Obviously, such a presentation is always convergent. Suppose that the polygraph \( \Lambda \) is reduced. The sets of chains for \( \Lambda \) are \( \Omega_0(\Lambda) = \Lambda_1 \), \( \Omega_1(\Lambda) = s_1(\Lambda) \) and for any \( n \geq 2 \), \( \Omega_n(\Lambda) \) is the set of \( n \)-overlapping \( x|t_1|\ldots|t_{n-1}|t_n \) of branchings of \( \Lambda \) with \( x \in \Lambda_1 \), and \( t_1, \ldots, t_n \in \Lambda_1 \), such that \( x t_1, t_i t_{i+1} \in s_1(\Lambda) \) for any \( 1 \leq i \leq n - 1 \). We have
\[
\widehat{x t_1} = 0 \quad \text{and} \quad \widehat{t_{i-1} t_i} = 0, \text{ for all } 1 \leq i \leq n.
\]
(7.3.16)
Consider the boundary map
\[
d_n : A[\Omega_n(\Lambda)] \rightarrow A[\Omega_{n-1}(\Lambda)]
\]
defined by
\[
d_n(x|t_1|\ldots|t_{n-1}|t_n \otimes 1) = x|t_1|\ldots|t_{n-1} \otimes t_n - t_{n-1} d_n(x|t_1|\ldots|t_{n-1} \otimes t_n).
\]
By definition of \( d_{n-1} \), we have
\[
d_{n-1}(x|t_1|\ldots|t_{n-1} \otimes t_n) = x|t_1|\ldots|t_{n-2} \otimes t_{n-1} t_n - t_{n-2} d_{n-2}(x|t_1|\ldots|t_{n-2} \otimes t_{n-1} t_n)
\]
Using relation in \( 7.3.16 \), we have \( d_n(x|t_1|\ldots|t_{n-1} \otimes t_n) = 0 \), hence
\[
d_n(x|t_1|\ldots|t_{n-1} |t_n \otimes 1) = x|t_1|\ldots|t_{n-1} |t_n.
\]
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7.4. Computing homology with Anick’s resolution

7.4.1. Computing homology. Given an algebra $A$ and a left $A$-module $M$. When the algebra is presented by a convergent reduced left-monomial linear 2-polygraph $\Lambda$, compatible with a monomial order, the Anick resolution $A(\Lambda)$ gives a method to compute the homology groups of $A$ with coefficient in $M$. In this section, we give several examples of computations of homology groups with coefficients in $K$. From the resolution $A(\Lambda)$, we compute the complex $A(\Lambda) \otimes_A K$ given by

$$\cdots \rightarrow K[\Omega_n(\Lambda)] \xrightarrow{d_n} K[\Omega_{n-1}(\Lambda)] \rightarrow \cdots \rightarrow K[\Omega_1(\Lambda)] \xrightarrow{d_1} K[\Lambda_1] \xrightarrow{d_0} K \rightarrow 0$$

where $K[\Omega_n(\Lambda)]$ denotes the free vector space on $\Omega_n(\Lambda)$ and $d_n$ denotes the map $d_n \otimes \text{Id}_K$. These maps satisfy $d_n d_{n+1} = 0$, for all $n \geq 0$, and we have

$$H_0(A, K) = K, \quad \text{and} \quad H_n(A, K) = \ker d_{n-1} / \text{im } d_n.$$ 

As a first application, we have the following finiteness properties.

7.4.2. Proposition. Let $A$ be an algebra presented by a finite convergent left-monomial linear 2-polygraph. The following statements hold.

i) $A$ is of homological type right-$FP_\infty$, that is, there exists an infinite length free finitely generated resolution of the trivial right $A$-module $K$.

ii) For any $n \geq 0$, the vector space $H_n(A, K)$ is finitely generated.

iii) [Ani86, Lemma 3.1] The algebra $A$ has a Poincaré series

$$P_A(t) = \sum_{n=0}^{\infty} \dim_K(H_n(A, K)) t^n,$$

with exponential or slower growth, that is, there are constants $c_1, c_2 > 0$, such that

$$0 \leq \dim_K(H_n(A, K)) \leq c_2(c_1)^n.$$ 

Note that the finiteness conditions i) and ii) were obtained by Kobayashi for monoids. A monoid $M$ is of homological type right-$FP_\infty$ over $K$ if the monoid algebra $K M$ is of homological type right-$FP_\infty$. In [Kob90], by constructing a resolution similar to the Anick resolution, Kobayashi shows that a monoid $M$ having a presentation by a finite convergent rewriting system is of homological type $FP_\infty$. Similar constructions of resolutions of monoids presented by convergent rewriting systems were also obtained by Brown [Bro92] and by Groves [Gro90]. The different constructions are based on distinct ways to describe the $n$-fold critical branchings of a convergent rewriting system.

7.4.3. Exercise. Prove the conditions i) and ii) in Proposition 7.4.2.
7.4. Computing homology with Anick’s resolution

7.4.4. Low-dimensional homology. In the first dimensions, we have the following complex

\[ K[\Omega_2(\Lambda)] \xrightarrow{\bar{d}_2} K[\Omega_1(\Lambda)] \xrightarrow{\bar{d}_1} K[\Lambda_1] \xrightarrow{\bar{d}_0} K \rightarrow 0 \]

The map \( \bar{d}_0 \) is zero, hence

\[ H_1(A, K) = K[\Lambda_1]/\text{Im} \bar{d}_1. \]

A 1-cell \( x \) of \( \Lambda_1 \) in Im \( \bar{d}_1 \) comes from a relation with source or target \( x \). It follows that \( x \) is a redundant generator in the presentation. Indeed, a term \( x \otimes 1 \), with \( x \) in \( \Lambda_1 \) appears in Im \( \bar{d}_1 \) if and only if \( x \) is the source or the target of a 2-cell in \( \Lambda_2 \). Let \( \alpha : x \Rightarrow y_1 \ldots y_k \) be a 2-cell in \( \Lambda_2 \), where by hypothesis \( y_1 \ldots y_k \) is reduced. Thus we have

\[ d_1(1| x \otimes 1) = x \otimes 1 - t_0 d_0(x \otimes 1) \]
\[ = x \otimes 1 - t_0(1 \otimes y_1 \ldots y_k) \]
\[ = x \otimes 1 - y_1 \otimes y_2 \ldots y_k. \]

Hence \( \bar{d}_1(x) = x \). Suppose now that \( x_1 \ldots x_k \overset{\alpha}{\Rightarrow} y \) is a 2-cell in \( \Lambda_2 \). We have

\[ d_1(x_1 \ldots x_k \otimes 1) = x_1 \otimes x_2 \ldots x_k - t_0 d_0(x_1 \otimes x_2 \ldots x_k) \]
\[ = x_1 \otimes x_2 \ldots x_k - t_0(1 \otimes y) \]
\[ = x_1 \otimes x_2 \ldots x_k - y \otimes 1. \]

Hence \( \bar{d}_1(x_1 \ldots x_k) = -y \). Thus, we have \( \bar{d}_1 = 0 \) if and only if the number of generators is minimal. In this way, \( \dim K H_1(A, K) \) is equal to the minimal number of generators for a presentation of the algebra \( A \). For analogous reasons, we show that \( \dim K H_2(A, K) \) is the minimal required number of the defining relations.

7.4.5. Example. Consider the algebra \( A \) presented by the linear 2-polygraph \( \langle x, y \mid yx \Rightarrow x^2 \rangle \). From the Anick resolution computed in 7.3.12, we deduce the complex

\[ \cdots \rightarrow 0 \rightarrow K[y|x] \xrightarrow{\bar{d}_1} K[x,y] \xrightarrow{\bar{d}_0} K \rightarrow 0 \]

whose boundary maps \( \bar{d}_0 \) and \( \bar{d}_1 \) are zero. We deduce

\[ H_n(A, K) = \begin{cases} K & \text{if } n = 0, 2, \\ K^2 & \text{if } n = 1, \\ 0 & \text{if } n \geq 3. \end{cases} \]

7.4.6. Exercise [Ani86, Theorem 3.2]. Let \( A \) be an algebra admitting a presentation by a left-monomial reduced linear 2-polygraph compatible with a monomial order and having no critical branching. Show that \( H_n(A, K) = 0 \), for any \( n \geq 3 \). A presentation without critical branching is called combinatorially free in [Ani86].
7.5. Minimality of Anick’s resolution

7.5.1. Minimal complex. A complex of free right $\Lambda$-modules

$$\cdots \to F_{n+1} \xrightarrow{d_n} F_n \xrightarrow{d_{n-1}} F_{n-1} \to \cdots$$

is minimal if all induced maps \( d_n = d_n \otimes \text{Id}_K : F_{n+1} \otimes_\Lambda K \to F_n \otimes_\Lambda K \) are zero. A resolution is minimal if the associated complex is minimal. Note that a minimal free resolution is one in which each free module has the minimal number of generators as illustrated in the following example.

7.5.2. Example. Let consider the algebra $A$ presented by the linear 2-polygraph $\langle x, y \mid x \Rightarrow y \rangle$, which is compatible with the deglex order induced by $y \prec x$. The Anick resolution is

$$0 \to A[x|1] \xrightarrow{d_1} A[x,y] \xrightarrow{d_0} A \xrightarrow{\varepsilon} K \to 0$$

with $d_0(x \otimes 1) = x, \quad d_0(y \otimes 1) = y, \quad d_1(x|1 \otimes 1) = x \otimes 1 - 1 \otimes y$.

This resolution is not minimal because $d_1 \neq 0$. A minimal resolution for the algebra $A$ can be constructed from the polygraph $\langle x \mid \emptyset \rangle$ with no 2-cell.

7.5.3. Example. Let consider the algebra $A$ presented by the linear 2-polygraph

$$\langle x, y, z, r, s \mid xy \alpha \Rightarrow s, yz \beta \Rightarrow r \rangle$$

compatible with the deglex order induced by the alphabetic order $s \prec r \prec z \prec y \prec x$. There is a critical branching:

$$\xymatrix{ & \alpha z \ar[ld]_{\alpha x} \ar[rd]^{\beta y} & \\
& sz & x \ar[ll]_{\gamma} \ar[ll]_{\delta x} \ar[rr]^{\gamma} & \\
z \ar[ll]_{x z} & x & y }$$

which is confluent by adding the rule $x r \xrightarrow{\gamma} sz$. The linear 2-polygraph $\Lambda' = \langle \Lambda_1 \mid \alpha, \beta, \gamma \rangle$ is compatible with the deglex order considered above, convergent and Tietze equivalent to $\Lambda$. The induced the Anick resolution $A(\Lambda')$ is

$$\cdots \to 0 \to A[xy|z] \xrightarrow{d_2} A[x|y, x|r, y|z] \xrightarrow{d_1} A[x, y, z, r, s] \xrightarrow{d_0} A \xrightarrow{\varepsilon} K \to 0$$

with

$$d_1(x|y \otimes 1) = x \otimes y - s \otimes 1, \quad d_1(x|r \otimes 1) = x \otimes r - s \otimes z, \quad d_1(y|z \otimes 1) = y \otimes z - r \otimes 1,$$

and

$$d_2(x|y|z \otimes 1) = xy \otimes z - xr \otimes 1.$$

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This resolution is not minimal, because the maps $d_1$ and $d_2$ are non zero. Note that

$$H_n(A, \mathbb{K}) = \begin{cases} \mathbb{K} & \text{if } n = 0, \\ \mathbb{K}^3 & \text{if } n = 1, \\ 0 & \text{if } n \geq 2. \end{cases}$$

and a minimal resolution for the algebra $A$ can be constructed from the linear 2-polygraph $\langle x, y, z \mid \emptyset \rangle$ which produces the following resolution

$$\cdots \longrightarrow 0 \longrightarrow A[x, y, z] \xrightarrow{d_3} A \xrightarrow{\epsilon} \mathbb{K} \longrightarrow 0$$

7.5.4. Exercise. Consider the linear 2-polygraph

$$\Lambda = \langle x, y, z, r, s \mid xy \xrightarrow{\alpha} ss, yz \xrightarrow{\beta} sr \rangle.$$

1) Complete the polygraph $\Lambda$ into a convergent polygraph $\Lambda'$.
2) Show that the Anick resolution of $\Lambda'$ is not minimal.
3) Compute the homology of the algebra $A$ presented by $\Lambda$.
4) Compute a minimal Anick’s resolution of the algebra $A$.

7.5.5. Example. Let consider the algebra $A = \langle a, b, c, d, e \mid ab = ee, bc = ed \rangle$.

The alphabetic order $e \prec d \prec c \prec b \prec a$ induces the following orientation:

$$ab \xrightarrow{\alpha} ee, \quad bc \xrightarrow{\beta} ed.$$

There is only one critical branching:

$$abc \xleftarrow{\alpha \gamma} eee \xrightarrow{\alpha \beta} aed$$

completed by adding the rule $aed \xrightarrow{\gamma} eec$. The rewriting system $\{\alpha, \beta, \gamma\}$ is convergent. Anick’s chains are

$$\Omega_{-1} = \{1\}, \quad \Omega_0 = \{a, b, c, d, e\}, \quad \Omega_1 = \{ab, a|ed, b|c\}, \quad \Omega_2 = \{ab|c\}, \quad \Omega_n = \emptyset, \text{ for } n \geq 3.$$

The Anick resolution with this oriented presentation is

$$0 \rightarrow \mathbb{K}(ab|c) \otimes A \xrightarrow{d_3} \mathbb{K}[a|b, a|ed, b|c] \otimes A \xrightarrow{d_1} \mathbb{K}[a, b, c, d, e] \otimes A \xrightarrow{d_0} A \xrightarrow{\epsilon} \mathbb{K} \longrightarrow 0 \quad (7.5.6)$$
with \( d_0(x \otimes 1) = x \), for any \( x \in \Omega_0 \),
\[
d_1(a|b \otimes 1) = a \otimes b - e \otimes e, \quad d_1(a|ed \otimes 1) = a \otimes ed - e \otimes ec, \quad d_1(b|c \otimes 1) = b \otimes c - e \otimes d,
\]
and
\[
d_2(ab|c \otimes 1) = ab \otimes c - aed \otimes 1.
\]

It follows that
\[
H_n(A, \mathbb{K}) = \begin{cases} 
\mathbb{K} & \text{if } n = 0, \\
\mathbb{K}^5 & \text{if } n = 1, \\
\mathbb{K}^2 & \text{if } n = 2, \\
0 & \text{if } n \geq 3.
\end{cases}
\]

Hence the Anick resolution with these presentation is not minimal. A minimal Anick’s resolution for the same algebra \( A \) can be constructed with the following orientation, induced by the alphabetic order with \( b \prec e \prec a \):
\[
ab \xrightarrow{\alpha} ee, \quad ed \xrightarrow{\beta'} bc
\]
which produces the following chains:
\[
\Omega_{-1} = \{1\}, \quad \Omega_0 = \{a, b, c, d, e\}, \quad \Omega_1 = \{a|b, e|d\}, \quad \Omega_2 = \emptyset, \text{ for } n \geq 1.
\]
The Anick resolution with this orientation is
\[
0 \to \mathbb{K}[a|b, e|d] \otimes A \xrightarrow{d_1} \mathbb{K}[a, b, c, d, e] \otimes A \xrightarrow{d_0} A \to \mathbb{K} \to 0 \tag{7.5.7}
\]
with \( d_0(x \otimes 1) = x \), for any \( x \in \Omega_0 \) and
\[
d_1(a|b \otimes 1) = a \otimes b - e \otimes e, \quad d_1(e|d \otimes 1) = e \otimes d - b \otimes c.
\]

This resolution is minimal.

7.5.8. Exercise. Let consider the algebra presented by
\[
\langle x, y, z, r, s \mid xy = ss, \ yz = rr \rangle.
\]
Show that there is no orientation of rules of this presentation giving a convergent linear 2-polygraph, and thus there is no minimal Anick’s resolution for this algebra.

7.5.9. Proposition. Let \( \Lambda \) be a monomial linear 2-polygraph. Let \( A \) be the monomial algebra presented by \( \Lambda \). The following statements hold.

i) The Anick resolution \( A(\Lambda) \) defined in (7.3.1) is a minimal resolution.

ii) There is an isomorphism \( \text{Tor}_n^A(\mathbb{K}, \mathbb{K}) \simeq \mathbb{K}\Omega_{n-1} \), for all \( n \geq 0 \).
7.5. Minimality of Anick’s resolution

Let us mention another consequence for quadratic algebras. Given a monomial linear 2-polygraph \( \Lambda \) which is quadratic, that is its 2-cells are of the form \( x_i x_j \Rightarrow 0 \), with \( x_i, x_j \) in \( \Lambda_1 \). Then the Anick resolution \( A(\Lambda) \) is concentrated in the diagonal in the following sense. The set of 0-chains is \( \Lambda_1 \) and they are of degree 1. The set of 1-chains is \( s_1(\Lambda) \) and they are of degree 2. More generally, an \( n \)-chains \( x|t_1 \ldots |t_{n-1}|t_n \) is of degree \( n+1 \). As a consequence, we have

7.5.10. Theorem. A quadratic monomial algebra is Koszul.

7.5.11. Proposition. Let \( A \) be an algebra and let \( \Lambda \) be a left-monomial reduced convergent linear 2-polygraph compatible with a monomial order that presents \( A \). If the Anick resolution \( A(\Lambda) \) is minimal, then, for any \( n \geq 0 \), we have an isomorphism of spaces

\[ H_n(A, \mathbb{K}) \simeq \mathbb{K}[\Omega_{n-1}(\Lambda)]. \]

7.5.12. Exercise. Prove Proposition 7.5.11.

7.5.13. When Anick’s resolution is minimal. We have seen in Proposition 7.5.9 that the Anick resolution \( A(\Lambda) \) is minimal when the presentation is monomial. Following exercise gives another situation for which the Anick resolution is minimal.

7.5.14. Exercise. Let \( \Lambda \) be a left-monomial reduced linear 2-polygraph compatible with a monomial order. Suppose that \( \Lambda \) is convergent and quadratic, that is, any 2-cell in \( \Lambda_2 \) is of the form \( x_{i_1} x_{i_2} \Rightarrow y_{i_1} y_{i_2} \) with \( x_{i_1}, x_{i_2}, y_{i_1}, y_{i_2} \) in \( \Lambda_1 \). Show that the Anick resolution \( A(\Lambda) \) is minimal.

7.5.15. Exercise. A linear 2-polygraph is cubical if its 2-cells are of the form \( x_{i_1} x_{i_2} x_{i_3} \Rightarrow y_{i_1} y_{i_2} y_{i_3} \). Is the result of Exercise 7.5.14 can be extended to cubical convergent linear 2-polygraphs?

7.5.16. Exercises. Compute homology spaces of the algebras presented by the following linear 2-polygraphs

1) \( \langle x, y \mid xy \Rightarrow yx \rangle \).
2) \( \langle x, y \mid x^2 \Rightarrow 0 \rangle \).
3) \( \langle x, y \mid x^2 \Rightarrow y^2 \rangle \).
4) \( \langle x, y \mid x^2 \Rightarrow xy \rangle \).
5) \( \langle x, y \mid x^2 \Rightarrow xy - y^2 \rangle \).
6) \( \langle x, y \mid xy \Rightarrow yxy \rangle \).
Bibliography


