Institut Camille Jordan
UMR 5208 du CNRS

Université Claude Bernard Lyon 1

Internship Proposal 2015

WORD PROBLEM, REWRITING AND HOMOLOGY OF MONOIDS

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Duration. Three months.

Subject. A rewriting system is a computational model in which computations are expressed as a succession of elementary transformations, called rewriting step, repeatedly replacing subterms of a syntactical expression by other terms until a normal form is obtained. Rewriting steps are specified by a set of rules that defines the rewriting system. Rewriting appears in several contexts of computer science, but also in algebra in order to give decision procedures, or more recently, computational methods to compute homological invariants. In this internship, we will study string rewriting systems, where the rewriting steps act on strings in free monoids, by using homological and homotopical methods.

In the seventies, difficult decidability problems were studied using rewriting methods, such as the decidability of the word problem in monoids. The word problem for a monoid $\mathbf{M}$ consists in finding a generating set X and a procedure that can determine whether or not any two elements of the free monoid generated by X represent the same element in the monoid $\mathbf{M}$. One way to solve this problem is to exhibit a finite presentation of the monoid $\mathbf{M}$, made of a generating set X and a set R of rules with "good" computational properties: termination (all the computations end eventually) and confluence (when different computations on the same input lead to the same result). One proves that, if a monoid admits a finite convergent presentation, then it has a decidable word problem. The converse implication was still an open problem in the middle of the eighties : does every finitely presented monoid with a decidable word problem admit a finite convergent presentation?

At the end of the eighties, using a homological argument Squier answered the question negatively by showing that there are finitely presented monoids with a decidable word problem which do not have a finite convergent presentation, [Squ87]. He linked the existence of a finite convergent presentation for a finitely presented monoid to the homological type left-FP_3 property. He showed that a monoid needs to satisfy this property to have a finite convergent presentation. As a consequence, if a monoid $\mathbf{M}$ can be presented by a finite convergent string rewriting system, then its third integral homology group $H_3(\mathbf{M}, \mathbb{Z})$ is of finite type.

By giving an example of a finitely presented monoid $\mathbf{M}$ that has a decidable word problem and whose third homology group $H_3(\mathbf{M}, \mathbb{Z})$ is not of finite type, he proved that there are finitely presented monoids with a decidable word problem that cannot be presented by a finite convergent string rewriting system.

Later, he introduced the condition of finite derivation type, which is a homotopical finiteness property on the presentation complex associated to a monoid presentation, [SOK94]. He showed that this condition is an invariant of finite presentations and he gave a constructive way to prove this finiteness property based on the computation of the critical branchings: being of finite derivation type is a necessary condition for a finitely presented monoid to admit a finite convergent presentation.

The aim of this internship is to understand this application of homological methods in string rewriting, and in particular, the homological and homotopical finiteness conditions introduced by Squier, [Squ87, SOK94] and the necessary background, such as

- fundamental notions of rewriting in monoids : string rewriting systems, [BO93],
- homology of monoids, [Squ87],
- the categorical description of string rewriting systems based on 2-categories, [GM14].

One of the generalizations of the constructions of Squier could be studied in this internship, such as the procedure defined by Kobayashi that computes a free resolution for a monoid from a convergent presentation of the monoid, [Kob90], or a similar procedure given by Anick for associative algebras, [Ani86].

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References.

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[BO93] Ronald Book and Friedrich Otto. *String-rewriting systems*. Texts and Monographs in Computer Science. Springer-Verlag, 1993.

[GM14] Yves Guiraud and Philippe Malbos. Polygraphs of finite derivation type. arXiv:1402.2587, 2014.

[Kob90] Yuji Kobayashi. Complete rewriting systems and homology of monoid algebras. *J. Pure Appl. Algebra*, 65(3):263–275, 1990.

[SOK94] Craig C. Squier, Friedrich Otto, and Yuji Kobayashi. A finiteness condition for rewriting systems. *Theoret. Comput. Sci.*, 131(2):271–294, 1994.

[Squ87] Craig C. Squier. Word problems and a homological finiteness condition for monoids. *J. Pure Appl. Algebra*, 49(1-2):201–217, 1987.