In this Maple file, we compute the evolution equations for the Painlevé 4 equation using the compatibility equation of the Lax system.
We also obtain the expression of the Lax matrices in the geometric gauge without apparent singularities.

```
> restart:
  with(LinearAlgebra):
  P011:=sX10+sX20;
  P022:=sX10*sX20;
  Pinfty11:=-s12-s22;
  Pinfty01:=-s11-s21;
  Pinfty22:=s12*s22;
  Pinfty12:=s11*s22+s12*s21;
  Pinfty02:=s12*s20+s10*s22+s11*s21;
  CoherenceEquation:=s10+s20+sX10+sX20;

  R1:=unapply( P011/(xi-X1)+Pinfty01+Pinfty11*xi,xi);
  R2:=unapply( P022/(xi-X1)^2+P012/(xi-X1)+Pinfty02+Pinfty12*xi+
  Pinfty22*xi^2,xi);
  tdR2:=unapply(R2(xi)-P012/(xi-X1),xi);

  c2bis:=(beta12*s22-beta22*s12)/(2*(s12-s22));
  c1bis:=(1/2)*(s12*s21-s11*s22)/(s12-s22)^2*(beta12-beta22)+
  (beta11*s22-beta21*s12)/(s12-s22);
  mubis:=(1/2)*(X1*(s12-s22)-s11+s21)*(Q-X1)/(-s22+s12)^2*(beta12-
  beta22)+(beta11-beta21)*(Q-X1)/(s12-s22)+(Q-X1)*betaX1;
  nuMinus1bis:=(beta12-beta22)/(2*(-s22+s12));
  nu0bis:=(1/2)*(s21-s11)/(s12-s22)^2*(beta12-beta22)+(beta11-
  beta21)/(s12-s22);


  dR1dxi:=unapply(diff(R1(xi),xi),xi):
  dR2dxi:=unapply(diff(R2(xi),xi),xi):
  L:=Matrix(2,2,0):
  L[1,1]:=0:
  L[1,2]:=1:
  L[2,1]:=-R2(xi)+P012/(xi-X1) +C/(xi-X1) -h*s12 -h*P/(xi-Q):
  L[2,2]:= R1(xi)-h/(xi-X1)+h/(xi-Q);

  C01:=C:

  A:=Matrix(2,2,0):
  A[1,1]:= c2*xi^2+ c1*xi +c0+rho/(xi-Q):
```

```
A[1,2]:=nuMinus1*xi+nu0+mu/(xi-Q):
A[2,1]:= AA21(xi):
A[2,2]:= AA22(xi):
dAdxi:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dAdxi[i,j]:=diff(A[i,
j],xi):  od: od:

L;
A;
Q2:=unapply(-P*(Q-X1),xi):
J:=Matrix(2,2,0):
J[1,1]:=1:
J[1,2]:=0:
J[2,1]:=Q2(xi)/(xi-Q):
J[2,2]:=(xi-X1)^1/(xi-Q):
J;
dJdxi:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dJdxi[i,j]:=diff(J[i,
j],xi): od: od:
J:

LJ:=Matrix(2,2,0):
LJ[1,1]:=0:
LJ[1,2]:=0:
LJ[2,2]:=diff(J[2,2],Q)*LQ+diff(J[2,2],P)*LP+h*diff(J[2,2],X1)*
betaX1:
LJ[2,1]:=diff(J[2,1],Q)*LQ+diff(J[2,1],P)*LP+h*diff(J[2,1],X1)*
betaX1:
LJ:

checkL:=simplify(Multiply(Multiply(J,L),J^(-1))+h*Multiply(dJdxi,
J^(-1))):
checkA:=simplify(Multiply(Multiply(J,A),J^(-1))+Multiply(LJ,J^
(-1))):
```

$$P011 := sX10 + sX20 \tag{1}$$

$$P022 := sX10\, sX20$$

$$Pinfty11 := -s12 - s22$$

$$Pinfty01 := -s11 - s21$$

$$Pinfty22 := s12\, s22$$

$$Pinfty12 := s11\, s22 + s12\, s21$$

$$Pinfty02 := s10\, s22 + s11\, s21 + s12\, s20$$

$$CoherenceEquation := s10 + s20 + sX10 + sX20$$

$$R1 := \xi \rightarrow \frac{sX10 + sX20}{\xi - X1} - s11 - s21 + (-s12 - s22)\,\xi$$

$$R2 := \xi \to \frac{sX10\, sX20}{(\xi - X1)^2} + \frac{P012}{\xi - X1} + s10\, s22 + s11\, s21 + s12\, s20 + (s11\, s22 + s12\, s21)\, \xi + s12\, s22\, \xi^2$$

$$tdR2 := \xi \to \frac{sX10\, sX20}{(\xi - X1)^2} + s10\, s22 + s11\, s21 + s12\, s20 + (s11\, s22 + s12\, s21)\, \xi + s12\, s22\, \xi^2$$

$$c2bis := \frac{\beta 12\, s22 - \beta 22\, s12}{2\, s12 - 2\, s22}$$

$$c1bis := \frac{1}{2}\, \frac{(-s11\, s22 + s12\, s21)\, (\beta 12 - \beta 22)}{(s12 - s22)^2} + \frac{\beta 11\, s22 - \beta 21\, s12}{s12 - s22}$$

$$mubis := \frac{1}{2}\, \frac{(X1\, (s12 - s22) - s11 + s21)\, (Q - X1)\, (\beta 12 - \beta 22)}{(s12 - s22)^2} + \frac{(\beta 11 - \beta 21)\, (Q - X1)}{s12 - s22} + (Q - X1)\, betaX1$$

$$nuMinus1bis := \frac{\beta 12 - \beta 22}{2\, s12 - 2\, s22}$$

$$nu0bis := \frac{1}{2}\, \frac{(s21 - s11)\, (\beta 12 - \beta 22)}{(s12 - s22)^2} + \frac{\beta 11 - \beta 21}{s12 - s22}$$

$$L_{2,2} := \frac{sX10 + sX20}{\xi - X1} - s11 - s21 + (-s12 - s22)\, \xi - \frac{h}{\xi - X1} + \frac{h}{\xi - Q}$$

$$\left[[0, 1], \left[-\frac{sX10\, sX20}{(\xi - X1)^2} - s10\, s22 - s11\, s21 - s12\, s20 - (s11\, s22 + s12\, s21)\, \xi - s12\, s22\, \xi^2 + \frac{C}{\xi - X1} - h\, s12 - \frac{h\, P}{\xi - Q}, \frac{sX10 + sX20}{\xi - X1} - s11 - s21 + (-s12 - s22)\, \xi - \frac{h}{\xi - X1} + \frac{h}{\xi - Q} \right] \right]$$

$$\begin{bmatrix} c2\, \xi^2 + c1\, \xi + c0 + \frac{\rho}{\xi - Q} & nuMinus1\, \xi + \nu 0 + \frac{\mu}{\xi - Q} \\ AA21(\xi) & AA22(\xi) \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 \\ -\frac{P\, (Q - X1)}{\xi - Q} & \frac{\xi - X1}{\xi - Q} \end{bmatrix}$$

Solving the compatibility equations to obtain the Hamiltonian evolutions.

The compatibility equation is \mathcal{L}L=h\partial_\xi A+[A,L]
Since the first line of L is trivial, we may easily obtain A[2,1] et A[2,2] to obtain the full expression for A

```
> LL:=h*dAdxi+(Multiply(A,L)-Multiply(L,A)):

  Entry11:=LL[1,1]:
  Entry12:=LL[1,2]:

  AA21:=unapply(solve(Entry11=0,AA21(xi)),xi):
  AA21bis:=h*dAdxi[1,1]+A[1,2]*L[2,1]:

  simplify(AA21(xi)-AA21bis);
  AA22:=unapply(solve(Entry12=0,AA22(xi)),xi):
  AA22bis:=h*dAdxi[1,2]+A[1,1]+A[1,2]*L[2,2]:

  simplify(AA22(xi)-AA22bis);
  simplify(Entry11);
  simplify(Entry12);
```

0 **(1.1)**
0
0
0

We now compute the action of \mathcal{L} on L[2,2] et L[2,1] to obtain the evolution equations

Evolution of entry L_{2,2}

```
> Entry22:=simplify(LL[2,2]);
  Entry22TermxiMinusQCube:=factor(residue(Entry22*(xi-Q)^2,xi=Q))
  ;
  Entry22TermxiMinusQSquare:=factor(residue(Entry22*(xi-Q),xi=Q))
  ;
  Entry22TermxiMinusQ:=factor(residue(Entry22,xi=Q));

  Entry22TermxiInfty4:=factor(-residue(Entry22/xi^5,xi=infinity))
  ;
  Entry22TermxiInfty3:=factor(-residue(Entry22/xi^4,xi=infinity))
  ;
  Entry22TermxiInfty2:=factor(-residue(Entry22/xi^3,xi=infinity))
  ;
  Entry22TermxiInfty1:=factor(-residue(Entry22/xi^2,xi=infinity))
  ;
  Entry22TermxiInfty0:=factor(-residue(Entry22/xi,xi=infinity));
  Entry22TermxiInftyMinus1:=factor(-residue(Entry22/xi^2,xi=
  infinity));
  Entry22TermxiInftyMinus2:=factor(-residue(Entry22/xi^3,xi=
  infinity));
  Entry22TermxiTMinus1:=factor(residue(Entry22,xi=X1));
  Entry22TermxiTMinus2:=factor(residue(Entry22*(xi-X1),xi=X1));
```

```
simplify( Entry22-(Entry22TermxiMinusQSquare/(xi-Q)^2+
Entry22TermxiMinusQ/(xi-Q)
+Entry22TermxiInfty0+Entry22TermxiInfty1*xi+
Entry22TermxiInfty2*xi^2+Entry22TermxiInfty3*xi^3+
Entry22TermxiInfty4*xi^4+Entry22TermxiTMinus1/(xi-X1)
+Entry22TermxiTMinus2/(xi-X1)^2) );
L[2,2];
```

$$Entry22 := -\frac{1}{(-\xi+Q)^2(-\xi+X1)^2}\Big(\Big(((2\,s12+2\,s22)\,nuMinus1-4\,c2)\,\xi^5+\Big(((-4\,s12-4\,s22)\,nuMinus1+8\,c2)\,X1+((-4\,s12-4\,s22)\,nuMinus1+8\,c2)\,Q+(s21+s11)\,nuMinus1+(s12+s22)\,v0-2\,c1\Big)\,\xi^4+\Big(((2\,s12+2\,s22)\,nuMinus1-4\,c2)\,X1^2+\Big(((8\,s12+8\,s22)\,nuMinus1-16\,c2)\,Q+(-2\,s11-2\,s21)\,nuMinus1+(-2\,s12-2\,s22)\,v0+4\,c1\Big)\,X1+2\,Q\,\Big(((s12+s22)\,nuMinus1-2\,c2)\,Q+(-s11-s21)\,nuMinus1+(-s12-s22)\,v0+2\,c1\Big)\Big)\,\xi^3+\Big(\Big(((-4\,s12-4\,s22)\,nuMinus1+8\,c2)\,Q+(s21+s11)\,nuMinus1+(s12+s22)\,v0-2\,c1\Big)\,X1^2+\Big(((-4\,s12-4\,s22)\,nuMinus1+8\,c2)\,Q^2+\Big((4\,s11+4\,s21)\,nuMinus1+(4\,s12+4\,s22)\,v0-8\,c1\Big)\,Q+nuMinus1\,(sX10+sX20-h)\Big)\,X1+\Big((s21+s11)\,nuMinus1+(s12+s22)\,v0-2\,c1\Big)\,Q^2+\Big(nuMinus1\,h-\mu\,(s12+s22)\Big)\,Q+(sX10+sX20)\,v0+(-s11-s21)\,\mu+2\,\rho\Big)\,\xi^2+\Big(2\,Q\,\Big(((s12+s22)\,nuMinus1-2\,c2)\,Q+(-s11-s21)\,nuMinus1+(-s12-s22)\,v0+2\,c1\Big)\,X1^2+\Big(\Big((-2\,s11-2\,s21)\,nuMinus1+(-2\,s12-2\,s22)\,v0+4\,c1\Big)\,Q^2+\Big((-2\,sX10-2\,sX20)\,nuMinus1+2\,\mu\,(s12+s22)\Big)\,Q-2\,h\,v0+(2\,s11+2\,s21)\,\mu-4\,\rho\Big)\,X1-2\,(sX10+sX20-h)\,(Q\,v0-\mu)\Big)\,\xi+\Big(\Big((s21+s11)\,nuMinus1+(s12+s22)\,v0-2\,c1\Big)\,Q^2+\Big(nuMinus1\,h-\mu\,(s12+s22)\Big)\,Q+h\,v0+(-s11-s21)\,\mu+2\,\rho\Big)\,X1^2+\Big(Q^2\,nuMinus1-\mu\Big)\,(sX10+sX20-h)\,X1+Q\,(sX10+sX20-h)\,(Q\,v0-\mu)\Big)\,h\Big) \tag{1.2}$$

$$Entry22TermxiMinusQCube := 0$$

$$Entry22TermxiMinusQSquare := -\frac{1}{Q-X1}\Big(\Big(Q^2\,h\,nuMinus1-Q^2\,\mu\,s12-Q^2\,\mu\,s22-Q\,X1\,h\,nuMinus1+Q\,X1\,\mu\,s12+Q\,X1\,\mu\,s22+Q\,h\,v0-Q\,\mu\,s11-Q\,\mu\,s21-X1\,h\,v0+X1\,\mu\,s11+X1\,\mu\,s21+2\,Q\,\rho-2\,X1\,\rho-h\,\mu+\mu\,sX10+\mu\,sX20\Big)\,h\Big)$$

$$Entry22TermxiMinusQ := 0$$

$$Entry22TermxiInfty4 := 0$$

$$Entry22TermxiInfty3 := 0$$

$$Entry22TermxiInfty2 := 0$$

$$Entry22TermxiInfty1 := 2\,(-s12\,nuMinus1-s22\,nuMinus1+2\,c2)\,h$$

$$Entry22TermxiInfty0 := h\,\Big(-v0\,s12-v0\,s22-s11\,nuMinus1-s21\,nuMinus1+2\,c1\Big)$$

$$Entry22TermxiInftyMinus1 := 2\,(-s12\,nuMinus1-s22\,nuMinus1+2\,c2)\,h$$

$$Entry22TermxiInftyMinus2 := 0$$

$$Entry22TermxiTMinus1 := 0$$

$$Entry22TermxiTMinus2 := \frac{1}{Q - X1}\left(h\,(-sX10 - sX20 + h)\,\left(Q\,X1\,nuMinus1 - X1^2\,nuMinus1 + Q\,\nu 0 - X1\,\nu 0 - \mu\right)\right)$$

$$0$$

$$\frac{sX10 + sX20}{\xi - X1} - s11 - s21 + (-s12 - s22)\,\xi - \frac{h}{\xi - X1} + \frac{h}{\xi - Q}$$

Since the deformation operator is \hbar (beta12\partial_{t_{\infty^{(1)},2} +beta22\partial_{t_{\infty^{(2)},2}+beta11\partial_{t_{\infty^{(1)},1} +beta21\partial_{t_{\infty^{(2)},1})) we can obtain L[Q]

```
> L22Orderxi2:=-residue(L[2,2]/xi^3,xi=infinity);
  L22Orderxi1:=-residue(L[2,2]/xi^2,xi=infinity);
  L22Orderxi0:=-residue(L[2,2]/xi^1,xi=infinity);
  L22OrderxiMinus1:=-residue(L[2,2]/xi^2,xi=infinity);
  L22OrderS1:=residue(L[2,2],xi=X1):
  factor(simplify(h*(beta12*diff(L22Orderxi2,s12)+beta22*diff
  (L22Orderxi2,s22)
  +beta11*diff(L22Orderxi2,s11)+beta21*diff(L22Orderxi2,s21)   +
  betaX1*diff(L22Orderxi2,X1) )
  - Entry22TermxiInfty2));

  Equation0:=Entry22TermxiTMinus2-h*(sX10+sX20-h)*betaX1;

  Equation1:=factor(simplify(h*(beta12*diff(L22Orderxi1,s12)+
  beta22*diff(L22Orderxi1,s22)
  +beta11*diff(L22Orderxi1,s11)+beta21*diff(L22Orderxi1,s21)+
  betaX1*diff(L22Orderxi1,X1))
  - Entry22TermxiInfty1));

  Equation2:=simplify(h*(beta12*diff(L22Orderxi0,s12)+beta22*diff
  (L22Orderxi0,s22)
  +beta11*diff(L22Orderxi0,s11)+beta21*diff(L22Orderxi0,s21)+
  betaX1*diff(L22Orderxi0,X1))
  - Entry22TermxiInfty0);

  Equation3:=factor(simplify(h*(beta12*diff(L22OrderxiMinus1,s12)
  +beta22*diff(L22OrderxiMinus1,s22)
  +beta11*diff(L22OrderxiMinus1,s11)+beta21*diff
  (L22OrderxiMinus1,s21)+betaX1*diff(L22OrderxiMinus1,X1))
  - Entry22TermxiInftyMinus1));

  Equation3bis:=factor(simplify(h*(beta12*diff(L22OrderS1,s12)+
  beta22*diff(L22OrderS1,s22)
  +beta11*diff(L22OrderS1,s11)+beta21*diff(L22OrderS1,s21)+
```

```
betaX1*diff(L22OrderS1,X1))
- Entry22TermxiTMinus1));
```

$$L22Orderxi2 := 0$$
$$L22Orderxi1 := -s12 - s22$$
$$L22Orderxi0 := -s11 - s21$$
$$L22OrderxiMinus1 := -s12 - s22$$
$$0$$
$$Equation0 := \frac{1}{Q - X1}\left(h\,(-sX10 - sX20 + h)\,\left(Q\,X1\,nuMinus1 - X1^2\,nuMinus1 + Q\,\nu0 - X1\,\nu0 - \mu\right)\right) - h\,(sX10 + sX20 - h)\,betaX1$$
$$Equation1 := -h\left(-2\,s12\,nuMinus1 - 2\,s22\,nuMinus1 + \beta12 + \beta22 + 4\,c2\right)$$
$$Equation2 := h\left((s21 + s11)\,nuMinus1 + (s12 + s22)\,\nu0 - 2\,c1 - \beta11 - \beta21\right)$$
$$Equation3 := -h\left(-2\,s12\,nuMinus1 - 2\,s22\,nuMinus1 + \beta12 + \beta22 + 4\,c2\right)$$
$$Equation3bis := 0 \tag{1.3}$$

```
> LQ:=factor(Entry22TermxiMinusQSquare/h):
nu0:=solve(Equation0,nu0);
```

$$\nu0 := -\frac{Q\,X1\,nuMinus1 - X1^2\,nuMinus1 + Q\,betaX1 - X1\,betaX1 - \mu}{Q - X1} \tag{1.4}$$

We now look at \mathcal{L}[L[2,1]]

```
> Entry21:=simplify(LL[2,1]):
Entry21TermxiMinusQCube:=factor(residue(Entry21*(xi-Q)^2,xi=Q))
;
Entry21TermxiMinusQSquare:=factor(residue(Entry21*(xi-Q),xi=Q))
;
Entry21TermxiMinusQ:=factor(residue(Entry21,xi=Q));
Entry21TermxiInfty2:=factor(-residue(Entry21/xi^3,xi=infinity))
;
Entry21TermxiInfty1:=factor(-residue(Entry21/xi^2,xi=infinity))
;
Entry21TermxiInfty0:=factor(-residue(Entry21/xi,xi=infinity));
Entry21TermxiS1:=factor(residue(Entry21,xi=X1));
Entry21TermxiS2:=factor(residue(Entry21*(xi-X1),xi=X1));

simplify( Entry21-(Entry21TermxiMinusQCube/(xi-Q)^3+
Entry21TermxiMinusQSquare/(xi-Q)^2+Entry21TermxiMinusQ/(xi-Q)
+Entry21TermxiInfty0+Entry21TermxiInfty1*
xi+Entry21TermxiInfty2*xi^2
+Entry21TermxiS1/(xi-X1)
) );
L[2,1];
```

$$Entry21TermxiMinusQCube := 3\,(P\,\mu + \rho)\,h^2 \tag{1.5}$$

$$Entry21TermxiMinusQSquare := -\frac{1}{(Q-X1)^2}\Big(\big(-2\,Q^4\,\mu\,s12\,s22 + 4\,Q^3\,X1\,\mu\,s12\,s22 - 2\,Q^2\,X1^2\,\mu\,s12\,s22 - P\,Q^3\,h\,nuMinus1 + 3\,P\,Q^2\,X1\,h\,nuMinus1 - 3\,P\,Q\,X1^2\,h\,nuMinus1 + P\,X1^3\,h\,nuMinus1 - 2\,Q^3\,\mu\,s11\,s22 - 2\,Q^3\,\mu\,s12\,s21 + 4\,Q^2\,X1\,\mu\,s11\,s22 + 4\,Q^2\,X1\,\mu\,s12\,s21 - 2\,Q\,X1^2\,\mu\,s11\,s22 - 2\,Q\,X1^2\,\mu\,s12\,s21 + P\,Q^2\,betaX1\,h - 2\,P\,Q\,X1\,betaX1\,h + P\,X1^2\,betaX1\,h + Q^3\,\rho\,s12 + Q^3\,\rho\,s22 - 2\,Q^2\,X1\,\rho\,s12 - 2\,Q^2\,X1\,\rho\,s22 - 2\,Q^2\,h\,\mu\,s12 - 2\,Q^2\,\mu\,s10\,s22 - 2\,Q^2\,\mu\,s11\,s21 - 2\,Q^2\,\mu\,s12\,s20 + Q\,X1^2\,\rho\,s12 + Q\,X1^2\,\rho\,s22 + 4\,Q\,X1\,h\,\mu\,s12 + 4\,Q\,X1\,\mu\,s10\,s22 + 4\,Q\,X1\,\mu\,s11\,s21 + 4\,Q\,X1\,\mu\,s12\,s20 - 2\,X1^2\,h\,\mu\,s12 - 2\,X1^2\,\mu\,s10\,s22 - 2\,X1^2\,\mu\,s11\,s21 - 2\,X1^2\,\mu\,s12\,s20 - P\,Q\,h\,\mu + P\,X1\,h\,\mu + Q^2\,\rho\,s11 + Q^2\,\rho\,s21 - 2\,Q\,X1\,\rho\,s11 - 2\,Q\,X1\,\rho\,s21 + X1^2\,\rho\,s11 + X1^2\,\rho\,s21 + 2\,C\,Q\,\mu - 2\,C\,X1\,\mu + Q\,h\,\rho - Q\,\rho\,sX10 - Q\,\rho\,sX20 - X1\,h\,\rho + X1\,\rho\,sX10 + X1\,\rho\,sX20 - 2\,\mu\,sX10\,sX20\big)\,h\Big)$$

$$Entry21TermxiMinusQ := \frac{1}{(Q-X1)^3}\Big(h\,\big(2\,Q^4\,\mu\,s12\,s22 - 6\,Q^3\,X1\,\mu\,s12\,s22 + 6\,Q^2\,X1^2\,\mu\,s12\,s22 - 2\,Q\,X1^3\,\mu\,s12\,s22 - P\,Q^3\,h\,nuMinus1 + 3\,P\,Q^2\,X1\,h\,nuMinus1 - 3\,P\,Q\,X1^2\,h\,nuMinus1 + P\,X1^3\,h\,nuMinus1 - 2\,Q^4\,c2\,h + 6\,Q^3\,X1\,c2\,h + Q^3\,\mu\,s11\,s22 + Q^3\,\mu\,s12\,s21 - 6\,Q^2\,X1^2\,c2\,h - 3\,Q^2\,X1\,\mu\,s11\,s22 - 3\,Q^2\,X1\,\mu\,s12\,s21 + 2\,Q\,X1^3\,c2\,h + 3\,Q\,X1^2\,\mu\,s11\,s22 + 3\,Q\,X1^2\,\mu\,s12\,s21 - X1^3\,\mu\,s11\,s22 - X1^3\,\mu\,s12\,s21 - Q^3\,c1\,h - Q^3\,\rho\,s12 - Q^3\,\rho\,s22 + 3\,Q^2\,X1\,c1\,h + 3\,Q^2\,X1\,\rho\,s12 + 3\,Q^2\,X1\,\rho\,s22 - 3\,Q\,X1^2\,c1\,h - 3\,Q\,X1^2\,\rho\,s12 - 3\,Q\,X1^2\,\rho\,s22 + X1^3\,c1\,h + X1^3\,\rho\,s12 + X1^3\,\rho\,s22 + C\,Q\,\mu - C\,X1\,\mu + Q\,h\,\rho - Q\,\rho\,sX10 - Q\,\rho\,sX20 - X1\,h\,\rho + X1\,\rho\,sX10 + X1\,\rho\,sX20 - 2\,\mu\,sX10\,sX20\big)\Big)$$

$$Entry21TermxiInfty2 := 2\,(-2\,s12\,s22\,nuMinus1 + c2\,s12 + c2\,s22)\,h$$

$$Entry21TermxiInfty1 := \frac{1}{Q-X1}\Big(h\,\big(2\,Q\,X1\,s12\,s22\,nuMinus1 - 2\,X1^2\,s12\,s22\,nuMinus1 + 2\,Q\,betaX1\,s12\,s22 - 3\,Q\,s11\,s22\,nuMinus1 - 3\,Q\,s12\,s21\,nuMinus1 - 2\,X1\,betaX1\,s12\,s22 + 3\,X1\,s11\,s22\,nuMinus1 + 3\,X1\,s12\,s21\,nuMinus1 + Q\,c1\,s12 + Q\,c1\,s22 + 2\,Q\,c2\,s11 + 2\,Q\,c2\,s21 - X1\,c1\,s12 - X1\,c1\,s22 - 2\,X1\,c2\,s11 - 2\,X1\,c2\,s21 - 2\,\mu\,s12\,s22\big)\Big)$$

$$Entry21TermxiInfty0 := \frac{1}{Q-X1}\Big(h\,\big(Q\,X1\,s11\,s22\,nuMinus1 + Q\,X1\,s12\,s21\,nuMinus1 - X1^2\,s11\,s22\,nuMinus1 - X1^2\,s12\,s21\,nuMinus1 + Q\,betaX1\,s11\,s22 + Q\,betaX1\,s12\,s21 - 2\,Q\,h\,s12\,nuMinus1 - 2\,Q\,s10\,s22\,nuMinus1 - 2\,Q\,s11\,s21\,nuMinus1 - 2\,Q\,s12\,s20\,nuMinus1 - X1\,betaX1\,s11\,s22 - X1\,betaX1\,s12\,s21 + 2\,X1\,h\,s12\,nuMinus1 + 2\,X1\,s10\,s22\,nuMinus1 + 2\,X1\,s11\,s21\,nuMinus1 + 2\,X1\,s12\,s20\,nuMinus1 + Q\,c1\,s11 + Q\,c1\,s21 + 2\,Q\,c2\,h - 2\,Q\,c2\,sX10 - 2\,Q\,c2\,sX20 - X1\,c1\,s11 - X1\,c1\,s21 - 2\,X1\,c2\,h + 2\,X1\,c2\,sX10 + 2\,X1\,c2\,sX20 - \mu\,s11\,s22 - \mu\,s12\,s21\big)\Big)$$

$$Entry21TermxiS1 := \frac{1}{(Q-X1)^3}\Big(\big(2\,Q^3\,X1\,c2\,h - 2\,Q^3\,X1\,c2\,sX10 - 2\,Q^3\,X1\,c2\,sX20 - 6\,Q^2\,X1^2\,c2\,h + 6\,Q^2\,X1^2\,c2\,sX10 + 6\,Q^2\,X1^2\,c2\,sX20 + 6\,Q\,X1^3\,c2\,h$$

$$-6\,Q\,X1^3\,c2\,sX10-6\,Q\,X1^3\,c2\,sX20-2\,X1^4\,c2\,h+2\,X1^4\,c2\,sX10+2\,X1^4\,c2\,sX20$$
$$+C\,Q^3\,numMinus1-3\,C\,Q^2\,X1\,numMinus1+3\,C\,Q\,X1^2\,numMinus1-C\,X1^3\,numMinus1$$
$$+Q^3\,c1\,h-Q^3\,c1\,sX10-Q^3\,c1\,sX20-3\,Q^2\,X1\,c1\,h+3\,Q^2\,X1\,c1\,sX10$$
$$+3\,Q^2\,X1\,c1\,sX20+3\,Q\,X1^2\,c1\,h-3\,Q\,X1^2\,c1\,sX10-3\,Q\,X1^2\,c1\,sX20-X1^3\,c1\,h$$
$$+X1^3\,c1\,sX10+X1^3\,c1\,sX20-C\,Q\,\mu+C\,X1\,\mu-Q\,h\,\rho+Q\,\rho\,sX10+Q\,\rho\,sX20$$
$$+X1\,h\,\rho-X1\,\rho\,sX10-X1\,\rho\,sX20+2\,\mu\,sX10\,sX20\big)\,h\big)$$

$$Entry21TermxiS2 := C\,h\,betaX1$$

$$\frac{betaX1\,h\,\left(\left(-\xi+X1\right)C+2\,sX10\,sX20\right)}{\left(-\xi+X1\right)^3}$$

$$-\frac{sX10\,sX20}{\left(\xi-X1\right)^2}-s10\,s22-s11\,s21-s12\,s20-\left(s11\,s22+s12\,s21\right)\xi-s12\,s22\,\xi^2$$
$$+\frac{C}{\xi-X1}-h\,s12-\frac{h\,P}{\xi-Q}$$

```
> rho:=factor(solve(Entry21TermxiMinusQCube,rho));
  simplify(rho-(-P*mu));
  simplify(Entry21TermxiMinusQCube);
```

$$\rho := -P\,\mu \qquad \textbf{(1.6)}$$
$$0$$
$$0$$

```
> L21Orderxi3:=-residue(L[2,1]/xi^4,xi=infinity);
  L21Orderxi2:=-residue(L[2,1]/xi^3,xi=infinity);
  L21Orderxi1:=-residue(L[2,1]/xi^2,xi=infinity);
  L21Orderxi0:=-residue(L[2,1]/xi^1,xi=infinity);
  L21OrderxiMinus1:=-residue(L[2,1]/xi^2,xi=infinity);
  L21OrderxiMinus2:=-residue(L[2,1]/xi^3,xi=infinity);
  L21TOrder2:=factor(residue(L[2,1]*(xi-X1),xi=X1));
  L21TOrder1:=residue(L[2,1],xi=X1);
  Equation4:=simplify(h*(beta12*diff(L21Orderxi2,s12)+beta22*diff
  (L21Orderxi2,s22)+beta11*diff(L21Orderxi2,s11)+beta21*diff
  (L21Orderxi2,s21)+betaX1*diff(L21Orderxi2,X1))-
  Entry21TermxiInfty2);
  Equation5:=simplify(h*(beta12*diff(L21Orderxi1,s12)+beta22*diff
  (L21Orderxi1,s22)+beta11*diff(L21Orderxi1,s11)+beta21*diff
  (L21Orderxi1,s21)+betaX1*diff(L21Orderxi1,X1))-
  Entry21TermxiInfty1);
  Equation6:=simplify(h*(beta12*diff(L21TOrder2,s12)+beta22*diff
  (L21TOrder2,s22)+beta11*diff(L21TOrder2,s11)+beta21*diff
  (L21TOrder2,s21)+betaX1*diff(L21TOrder2,X1))- Entry21TermxiS2);
```

$$L21Orderxi3 := 0 \qquad \textbf{(1.7)}$$
$$L21Orderxi2 := -s12\,s22$$
$$L21Orderxi1 := -s11\,s22-s12\,s21$$
$$L21Orderxi0 := -h\,s12-s10\,s22-s11\,s21-s12\,s20$$
$$L21OrderxiMinus1 := -s11\,s22-s12\,s21$$

$$L21OrderxiMinus2 := -s12\, s22$$
$$L21TOrder2 := -sX10\, sX20$$
$$L21TOrder1 := \frac{C\, Q - C\, X1}{Q - X1}$$
$$Equation4 := -2\left(\left(-2\, s22\, nuMinus1 + c2 + \frac{1}{2}\, \beta 22\right) s12 + s22\left(c2 + \frac{1}{2}\, \beta 12\right)\right) h$$
$$Equation5 := -\frac{1}{Q - X1}\big(\big(-2\, X1^2\, s12\, s22\, nuMinus1 + \big(2\, Q\, s12\, s22\, nuMinus1 + \big(-2\, betaX1\, s22 + 3\, s21\, nuMinus1 - \beta 21 - c1\big)\, s12 + \big(3\, s11\, nuMinus1 - \beta 11 - c1\big)\, s22 + \big(-\beta 22 - 2\, c2\big)\, s11 - s21\, \big(\beta 12 + 2\, c2\big)\big)\, X1 + \big(\big(2\, betaX1\, s22 - 3\, s21\, nuMinus1 + \beta 21 + c1\big)\, s12 + \big(-3\, s11\, nuMinus1 + \beta 11 + c1\big)\, s22 + \big(\beta 22 + 2\, c2\big)\, s11 + s21\, \big(\beta 12 + 2\, c2\big)\big)\, Q - 2\, \mu\, s12\, s22\big)\, h\big)$$
$$Equation6 := -C\, h\, betaX1$$

```
> c2:=factor(solve(Equation1,c2));
  mu:=factor(solve(Equation2,mu));
  nuMinus1:=factor(solve(Equation4,nuMinus1));
  c1:=factor(solve(Equation5,c1));
  c2:=simplify(c2);
  mu:=factor(mu);
```

$$c2 := \frac{1}{2}\, s12\, nuMinus1 + \frac{1}{2}\, s22\, nuMinus1 - \frac{1}{4}\, \beta 12 - \frac{1}{4}\, \beta 22 \tag{1.8}$$
$$\mu := \frac{1}{s12 + s22}\big(\big(X1\, s12\, nuMinus1 + X1\, s22\, nuMinus1 + betaX1\, s12 + betaX1\, s22 - s11\, nuMinus1 - s21\, nuMinus1 + \beta 11 + \beta 21 + 2\, c1\big)\, (Q - X1)\big)$$
$$nuMinus1 := \frac{1}{2}\, \frac{\beta 12 - \beta 22}{s12 - s22}$$
$$c1 := \frac{1}{2}\, \frac{1}{(s12 - s22)^2}\big(2\, \beta 11\, s12\, s22 - 2\, \beta 11\, s22^2 - \beta 12\, s11\, s22 + \beta 12\, s12\, s21 - 2\, \beta 21\, s12^2 + 2\, \beta 21\, s12\, s22 + \beta 22\, s11\, s22 - \beta 22\, s12\, s21\big)$$
$$c2 := \frac{\beta 12\, s22 - \beta 22\, s12}{2\, s12 - 2\, s22}$$
$$\mu := \frac{1}{2}\, \frac{1}{(s12 - s22)^2}\big((Q - X1)\, \big(X1\, \beta 12\, s12 - X1\, \beta 12\, s22 - X1\, \beta 22\, s12 + X1\, \beta 22\, s22 + 2\, betaX1\, s12^2 - 4\, betaX1\, s12\, s22 + 2\, betaX1\, s22^2 + 2\, \beta 11\, s12 - 2\, \beta 11\, s22 - \beta 12\, s11 + \beta 12\, s21 - 2\, \beta 21\, s12 + 2\, \beta 21\, s22 + \beta 22\, s11 - \beta 22\, s21\big)\big)$$

```
> simplify(Equation1);
  simplify(Equation2);
  simplify(Equation3);
  simplify(Equation4);
  simplify(Equation5);
  simplify(c1-c1bis);
  simplify(c2-c2bis);
  simplify(mu-mubis);
  simplify(nu0-nu0bis);
```

```
simplify(nuMinus1-nuMinus1bis);
```

$$0$$ **(1.9)**

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

$$0$$

```
> simplify(Entry21TermxiMinusQSquare-(-h*P*LQ)):
> LPfunction:=unapply(-Entry21TermxiMinusQ/h,C):
> Equation7:=unapply( simplify(Entry21TermxiMinusQSquare-(-h*P*LQ)),C):
Cter:=(Q^4*s12*s22-2*Q^3*X1*s12*s22+Q^2*X1^2*s12*s22+P*Q^3*s12+P*Q^3*s22-2*P*Q^2*X1*s12-2*P*Q^2*X1*s22+P*Q*X1^2*s12+P*Q*X1^2*s22+Q^3*s11*s22+Q^3*s12*s21-2*Q^2*X1*s11*s22-2*Q^2*X1*s12*s21+Q*X1^2*s11*s22+Q*X1^2*s12*s21+h*Q^2*s12-2*h*Q*X1*s12+h*X1^2*s12+P^2*Q^2-2*P^2*Q*X1+P^2*X1^2+P*Q^2*s11+P*Q^2*s21-2*P*Q*X1*s11-2*P*Q*X1*s21+P*X1^2*s11+P*X1^2*s21+Q^2*s10*s22+Q^2*s11*s21+Q^2*s12*s20-2*Q*X1*s10*s22-2*Q*X1*s11*s21-2*Q*X1*s12*s20+X1^2*s10*s22+X1^2*s11*s21+X1^2*s12*s20+h*P*Q-h*P*X1-P*Q*sX10-P*Q*sX20+P*sX10*X1+P*sX20*X1+sX10*sX20)/(Q-X1):
simplify(Equation7(Cter));
solve(Equation7(C),C):

Cbis:=(Q-X1)*P^2+h*P-(Q-X1)*R1(Q)*P+(Q-X1)*R2(Q)-P012+h*s12*(Q-X1);
simplify(Equation7(Cbis));
simplify(series(Cter-Cbis,P));
C:=Cbis:
```

$$0$$ **(1.10)**

$$Cbis := (Q-X1)\,P^2 + h\,P - (Q-X1)\left(\frac{sX10+sX20}{Q-X1} - s11 - s21 + (-s12 - s22)\,Q\right)P + (Q-X1)\left(\frac{sX10\,sX20}{(Q-X1)^2} + \frac{P012}{Q-X1} + s10\,s22 + s11\,s21 + s12\,s20 + (s11\,s22+s12\,s21)\,Q + s12\,s22\,Q^2\right) - P012 + h\,s12\,(Q-X1)$$

$$0$$

$$0$$

```
> LP:=factor(simplify(LPfunction(Cbis))):
LPbis:=mu*(P*diff(R1(Q),Q)+P*h*1/(Q-X1)^2-diff(tdR2(Q),Q)- C01/(Q-X1)^2)+h*nuMinus1*P +h*c1+2*h*c2*Q:
factor(series(LP-LPbis,P012=0));
```

$$0 \quad \textbf{(1.11)}$$

```
> LQbis:=2*mu*(P-R1(Q)/2+1/2*h*1/(Q-X1))-h*nu0-h*nuMinus1*Q:
  simplify(LQ-LQbis);
```

$$0 \quad \textbf{(1.12)}$$

```
> nuMinus1:=nuMinus1;
  nu0:=nu0;
  c1:=c1;
  c2:=c2;
```

$$nuMinus1 := \frac{1}{2}\frac{\beta 12-\beta 22}{s12-s22} \quad \textbf{(1.13)}$$

$$\nu 0 := -\frac{1}{Q-X1}\left(\frac{1}{2}\frac{Q\,X1\,(\beta 12-\beta 22)}{s12-s22}-\frac{1}{2}\frac{X1^2\,(\beta 12-\beta 22)}{s12-s22}+Q\,betaX1 - X1\,betaX1 - \frac{1}{2}\frac{1}{(s12-s22)^2}\big((Q-X1)\,(X1\,\beta 12\,s12 - X1\,\beta 12\,s22 - X1\,\beta 22\,s12 + X1\,\beta 22\,s22 + 2\,betaX1\,s12^2 - 4\,betaX1\,s12\,s22 + 2\,betaX1\,s22^2 + 2\,\beta 11\,s12 - 2\,\beta 11\,s22 - \beta 12\,s11 + \beta 12\,s21 - 2\,\beta 21\,s12 + 2\,\beta 21\,s22 + \beta 22\,s11 - \beta 22\,s21)\big)\right)$$

$$c1 := \frac{1}{2}\frac{1}{(s12-s22)^2}\left(2\,\beta 11\,s12\,s22 - 2\,\beta 11\,s22^2 - \beta 12\,s11\,s22 + \beta 12\,s12\,s21 - 2\,\beta 21\,s12^2 + 2\,\beta 21\,s12\,s22 + \beta 22\,s11\,s22 - \beta 22\,s12\,s21\right)$$

$$c2 := \frac{\beta 12\,s22-\beta 22\,s12}{2\,s12-2\,s22}$$

We thus get that
L[Q]=2*mu*Q*(P-R1(Q)/2)-(1/2)*h*Q/(s12-s22)*(beta12-beta22) -h*mu
L[P] = -mu*P^2+mu*diff(P*Q*R1(Q)-Q*R2(Q),Q)-h*s12*mu +1/2*h*(beta12-beta22)/(s12-s22)*P+ h*c1+2*h*c2*Q
with

$$\boldsymbol{nuMinus1 := \frac{\alpha 12-\alpha 22}{2\,(s12-s22)}}$$

$$\boldsymbol{\nu 0 := -\frac{1}{(Q-X1)\,h}\left(\frac{h\,Q\,X1\,(\alpha 12-\alpha 22)}{2\,(s12-s22)}-\frac{h\,X1^2\,(\alpha 12-\alpha 22)}{2\,(s12-s22)}+betaX1\,Q - betaX1\,X1 - \frac{1}{2\,(s12-s22)^2}\big((Q-X1)\,(\alpha 12\,h\,X1\,s12 - \alpha 12\,h\,X1\,s22 - \alpha 22\,h\,X1\,s12 + \alpha 22\,h\,X1\,s22 + 2\,betaX1\,s12^2 - 4\,betaX1\,s12\,s22 + 2\,s22^2\,betaX1 + 2\,\alpha 11\,h\,s12 - 2\,\alpha 11\,h\,s22 - \alpha 12\,h\,s11 + \alpha 12\,h\,s21 - 2\,\alpha 21\,h\,s12 + 2\,\alpha 21\,h\,s22 + \alpha 22\,h\,s11 - \alpha 22\,h\,s21)\big)\right)}$$

$$\boldsymbol{c1 := \frac{1}{2\,(s12-s22)^2}\left(2\,s12\,\alpha 11\,s22 - 2\,s22^2\,\alpha 11 - s11\,\alpha 12\,s22 + s21\,\alpha 12\,s12 - 2\,s12^2\,\alpha 21 + 2\,s12\,\alpha 21\,s22 + s11\,\alpha 22\,s22 - s21\,\alpha 22\,s12\right)}$$

$$\boldsymbol{c2 := \frac{\alpha 12\,s22-\alpha 22\,s12}{2\,s12-2\,s22}}$$

```
> Hamiltonianbis:= mu*(P^2-R1(Q)*P+h*P/(Q-X1) +tdR2(Q)+h*s12 )-h*
```

```
nu0*P-h*nuMinus1*Q*P-h*c1*Q-h*c2*Q^2
:
factor(simplify(LP-(-diff(Hamiltonianbis,Q))));
simplify(LQ-(diff(Hamiltonianbis,P)));
```

$$0 \tag{1.14}$$
$$0$$

In order to match the notation with the article, we shall take Hamiltonianbis as the Hamiltonian including the purely time dependent terms.

```
> simplify(series(mu,betaX1));
```

$$-\frac{1}{2}\frac{1}{(s12-s22)^2}\Big(((Q-X1)\,(((\beta 22-\beta 12)\,X1-2\,\beta 11+2\,\beta 21)\,s12+(X1\,(\beta 12-\beta 22)+2\,\beta 11-2\,\beta 21)\,s22-(-s21+s11)\,(\beta 22-\beta 12)))+(Q-X1)\,betaX1 \tag{1.15}$$

```
> simplify(L[2,1]-(-h*P/(xi-Q)-tdR2(xi)-h*s12+((Q-X1)*(P^2-R1(Q)*
P+h/(Q-X1)*P+tdR2(Q)+h*s12)   )/(xi-X1) ));
```

$$0 \tag{1.16}$$

Expression of the Lax matrix in the geometric gauge and normalisation at infinity

```
> simplify(checkL[1,1]);
simplify(checkL[1,2]);
checkL22bis:=R1(xi)-P*(Q-X1)/(xi-X1);
simplify(checkL[2,2]-checkL22bis);
checkL21:=factor(checkL[2,1]);
simplify(series(checkL[2,1],xi=X1)):
checkL21bis:=(P*(Q-X1)-sX10)*(P*(Q-X1)-sX20)/((Q-X1)*(xi-X1))-
s12*s22*xi^2+(((-Q+X1)*s22-s21)*s12-s11*s22)*xi+((-Q^2+Q*X1)*
s22+(-s21-P)*Q+(s21+P)*X1-h-s20)*s12+((-P-s11)*Q+(s11+P)*X1-
s10)*s22-s11*s21:
simplify(checkL21-checkL21bis);
```

$$-\frac{P\,(Q-X1)}{-\xi+X1} \tag{2.1}$$

$$\frac{-\xi+Q}{-\xi+X1}$$

$$checkL22bis := \frac{sX10+sX20}{\xi-X1}-s11-s21+(-s12-s22)\,\xi-\frac{P\,(Q-X1)}{\xi-X1}$$

$$0$$

$$checkL21 := -\frac{1}{(-\xi+X1)\,(Q-X1)}\big(Q^3\,X1\,s12\,s22-Q^3\,s12\,s22\,\xi-2\,Q^2\,X1^2\,s12\,s22+3\,Q^2\,X1\,s12\,s22\,\xi-Q^2\,s12\,s22\,\xi^2+Q\,X1^3\,s12\,s22-3\,Q\,X1^2\,s12\,s22\,\xi+3\,Q\,X1\,s12\,s22\,\xi^2-Q\,s12\,s22\,\xi^3+X1^3\,s12\,s22\,\xi-2\,X1^2\,s12\,s22\,\xi^2+X1\,s12\,s22\,\xi^3+P\,Q^2\,X1\,s12+P\,Q^2\,X1\,s22-P\,Q^2\,s12\,\xi-P\,Q^2\,s22\,\xi-2\,P\,Q\,X1^2\,s12-2\,P\,Q\,X1^2\,s22+2\,P\,Q\,X1\,s12\,\xi+2\,P\,Q\,X1\,s22\,\xi+P\,X1^3\,s12+P\,X1^3\,s22$$

$$- P\, XI^2\, s12\, \xi - P\, XI^2\, s22\, \xi + Q^2\, XI\, s11\, s22 + Q^2\, XI\, s12\, s21 - Q^2\, s11\, s22\, \xi$$
$$- Q^2\, s12\, s21\, \xi - 2\, Q\, XI^2\, s11\, s22 - 2\, Q\, XI^2\, s12\, s21 + 3\, Q\, XI\, s11\, s22\, \xi$$
$$+ 3\, Q\, XI\, s12\, s21\, \xi - Q\, s11\, s22\, \xi^2 - Q\, s12\, s21\, \xi^2 + XI^3\, s11\, s22 + XI^3\, s12\, s21$$
$$- 2\, XI^2\, s11\, s22\, \xi - 2\, XI^2\, s12\, s21\, \xi + XI\, s11\, s22\, \xi^2 + XI\, s12\, s21\, \xi^2 + P^2\, Q^2$$
$$- 2\, P^2\, Q\, XI + P^2\, XI^2 + Q\, XI\, h\, s12 + Q\, XI\, s10\, s22 + Q\, XI\, s11\, s21 + Q\, XI\, s12\, s20$$
$$- Q\, h\, s12\, \xi - Q\, s10\, s22\, \xi - Q\, s11\, s21\, \xi - Q\, s12\, s20\, \xi - XI^2\, h\, s12 - XI^2\, s10\, s22$$
$$- XI^2\, s11\, s21 - XI^2\, s12\, s20 + XI\, h\, s12\, \xi + XI\, s10\, s22\, \xi + XI\, s11\, s21\, \xi + XI\, s12\, s20\, \xi$$
$$- P\, Q\, sX10 - P\, Q\, sX20 + P\, XI\, sX10 + P\, XI\, sX20 + sX10\, sX20 \Big)$$
$$0$$

```
> Verification:=simplify(LL-h*dAdxi-(Multiply(A,L)-Multiply(L,A))
);
checkL:=simplify(checkL):
```

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad (2.2)$$

```
> LcheckL:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do LcheckL[i,j]:=diff
(checkL[i,j],Q)*LQbis+diff(checkL[i,j],P)*LPbis+h*diff(checkL
[i,j],s12)*beta12+ h*diff(checkL[i,j],s22)*beta22+h*diff(checkL
[i,j],s11)*beta11+h*diff(checkL[i,j],s21)*beta21+h*diff(checkL
[i,j],X1)*betaX1: od: od:

checkA:=simplify(checkA):
dcheckAdxi:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dcheckAdxi[i,j]:=diff
(checkA[i,j],xi): od: od:

Verification:=simplify(LcheckL-h*dcheckAdxi-(Multiply(checkA,
checkL)-Multiply(checkL,checkA)));
```

$$\begin{bmatrix} 0 & 0 \\ \dfrac{(\beta 12\, s22 - \beta 22\, s12)\, (s10 + s20 + sX10 + sX20)\, h}{s12 - s22} & 0 \end{bmatrix} \quad (2.3)$$

```
> G1:=Matrix(2,2,0):
G1[1,1]:=1:
G1[2,2]:=1:
G1[1,2]:=0:
G1[2,1]:=s12*xi+eta0:
eta0:=(Q-X1)*s12+s11;
```

```
dG1dxi:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dG1dxi[i,j]:=diff(G1
[i,j],xi): od: od:

LG1:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do LG1[i,j]:=diff(G1[i,
j],Q)*LQ+diff(G1[i,j],P)*LP+h*diff(G1[i,j],s12)*beta12+ h*diff
(G1[i,j],s22)*beta22+h*diff(G1[i,j],s11)*beta11+h* diff(G1[i,
j],s21)*beta21+h*diff(G1[i,j],X1)*betaX1: od: od:

tdL:=simplify(Multiply(Multiply(G1,checkL),G1^(-1))+h*Multiply
(dG1dxi,G1^(-1))):

LtdL:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do LtdL[i,j]:=diff
(tdL[i,j],Q)*LQ+diff(tdL[i,j],P)*LP+h*diff(tdL[i,j],s12)*
beta12+ h*diff(tdL[i,j],s22)*beta22+h*diff(tdL[i,j],s11)*
beta11+h* diff(tdL[i,j],s21)*beta21+ h*diff(tdL[i,j],X1)*
betaX1: od: od:

tdA:=simplify(Multiply(Multiply(G1,checkA),G1^(-1))+Multiply
(LG1,G1^(-1))):
dtdAdxi:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dtdAdxi[i,j]:=diff
(tdA[i,j],xi): od: od:

Verification1:=simplify(LtdL-h*dtdAdxi-(Multiply(tdA,tdL)-
Multiply(tdL,tdA)));
```

$$\eta 0 := (Q - X1)\, s12 + s11 \tag{2.4}$$

$$\begin{bmatrix} 0 & 0 \\ \dfrac{(\beta 12\, s22 - \beta 22\, s12)\,(s10 + s20 + sX10 + sX20)\, h}{s12 - s22} & 0 \end{bmatrix}$$

```
> simplify(tdL);
series(tdL[1,1],xi=infinity,1);
series(tdL[1,2],xi=infinity,1);
series(tdL[2,1],xi=infinity,1);

tdL11bis:=((Q-X1)*(Q*s12+P+s11))/(xi-X1)-s12*xi-s11;
tdL12bis:=1+(X1-Q)/(xi-X1);
tdL21bis:=(Q^2*s12+(-X1*s12+P+s11)*Q+(-P-s11)*X1-sX10)*(Q^2*
s12+(-X1*s12+P+s11)*Q+(-P-s11)*X1-sX20)/(Q-X1)/(xi-X1)+(s12-
```

```
s22)*(Q-X1)*P+(s12-s22)*s12*Q^2-(s12-s22)*(X1*s12-s11)*Q-(X1*
s11+sX10+sX20+s20)*s12+s22*(X1*s11-s10);
tdL22bis:=(-Q^2*s12+(X1*s12-P-s11)*Q+(s11+P)*X1+sX10+sX20)/(xi-
X1)-xi*s22-s21;
simplify(tdL[1,1]-tdL11bis);
simplify(tdL[1,2]-tdL12bis);
simplify(tdL[2,1]-tdL21bis);
simplify(tdL[2,2]-tdL22bis);

L21bis:=-h*P/(xi-Q)-tdR2(xi)-h*s12+(Q-X1)/(xi-X1)*( P^2-R1(Q)*
P+h/(Q-X1)*P+tdR2(Q)+h*s12):
simplify( L[2,1]-L21bis);
L22bis:=h/(xi-Q)+R1(xi):
simplify(series(tdL[2,2]-tdL22bis,Q=0));
```

$$\left[\left[\frac{-(-\xi+Q)\,(\xi+Q-X1)\,s12+X1\,P+s11\,\xi-(P+s11)\,Q}{-\xi+X1},\ \frac{-\xi+Q}{-\xi+X1}\right],\right. \tag{2.5}$$

$$\left[\frac{1}{(-\xi+X1)\,(Q-X1)}\Big((s12-s22)\,(Q\,s12+P+s11)\,X1^3+\big((-3\,s12^2+2\,s12\,s22)\,Q^2+(-\xi\,s12^2+(s22\,\xi-4\,P-4\,s11)\,s12+2\,s22\,(P+s11))\,Q+((-P-s11)\,\xi+sX10+sX20+s20)\,s12+s22\,(P+s11)\,\xi-P^2-2\,P\,s11+s10\,s22-s11^2\big)\,X1^2+\big((3\,s12^2-s12\,s22)\,Q^3+(2\,\xi\,s12^2+(-2\,s22\,\xi+5\,P+5\,s11)\,s12-s22\,(P+s11))\,Q^2+(((2\,P+2\,s11)\,\xi-2\,sX10-2\,sX20-s20)\,s12-2\,s22\,(P+s11)\,\xi+2\,P^2+4\,P\,s11-s10\,s22+2\,s11^2)\,Q-\xi\,(sX10+sX20+s20)\,s12-\xi\,s10\,s22-(sX10+sX20)\,(P+s11)\big)\,X1-Q^4\,s12^2-2\,s12\left(\frac{1}{2}\,s12\,\xi-\frac{1}{2}\,s22\,\xi+P+s11\right)Q^3+(((-P-s11)\,\xi+sX10+sX20)\,s12-(P+s11)\,(-s22\,\xi+P+s11))\,Q^2+(\xi\,(sX10+sX20+s20)\,s12+\xi\,s10\,s22+(sX10+sX20)\,(P+s11))\,Q-sX10\,sX20\Big),\ \frac{1}{-\xi+X1}\Big((-Q\,s12-s22\,\xi-P-s11-s21)\,X1+\xi^2\,s22+s21\,\xi+Q^2\,s12+(P+s11)\,Q-sX20-sX10\Big)\right]\Bigg]$$

$$-s12\,\xi - s11 + \mathrm{O}\left(\frac{1}{\xi}\right)$$

$$\mathrm{O}(1)$$

$$\frac{1}{Q - X1}\Big(-\big((-s12^2 + s12\,s22)\,Q + (-P - s11)\,s12 + s22\,(P + s11)\big)\,X1^2 - \big((2\,s12^2 - 2\,s12\,s22)\,Q^2 + ((2\,P + 2\,s11)\,s12 - 2\,s22\,(P + s11))\,Q - (sX10 + sX20 + s20)\,s12 - s10\,s22\big)\,X1 + 2\,s12\left(\frac{1}{2}\,s12 - \frac{1}{2}\,s22\right)Q^3 - ((-P - s11)\,s12 + s22\,(P + s11))\,Q^2 - ((sX10 + sX20 + s20)\,s12 + s10\,s22)\,Q\Big) + \mathrm{O}\left(\frac{1}{\xi}\right)$$

$$tdL11bis := \frac{(Q - X1)\,(Q\,s12 + P + s11)}{\xi - X1} - s12\,\xi - s11$$

$$tdL12bis := 1 + \frac{-Q + X1}{\xi - X1}$$

$$tdL21bis := \frac{1}{(Q - X1)\,(\xi - X1)}\Big(\big(Q^2\,s12 + (-X1\,s12 + P + s11)\,Q + (-P - s11)\,X1 - sX10\big)\,\big(Q^2\,s12 + (-X1\,s12 + P + s11)\,Q + (-P - s11)\,X1 - sX20\big)\Big) + (s12 - s22)\,(Q - X1)\,P + (s12 - s22)\,s12\,Q^2 - (s12 - s22)\,(X1\,s12 - s11)\,Q - (X1\,s11 + s20 + sX10 + sX20)\,s12 + s22\,(X1\,s11 - s10)$$

$$tdL22bis := \frac{-Q^2\,s12 + (X1\,s12 - P - s11)\,Q + (P + s11)\,X1 + sX10 + sX20}{\xi - X1} - s22\,\xi - s21$$

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> GeneralSpectralCurvefunction:=unapply(simplify(Determinant(y*
IdentityMatrix(2)-tdL)),xi,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10,
sX20):

GeneralSpectralCurve:=GeneralSpectralCurvefunction(xi,Q,P,s12,
s22,s11,s21,X1,s10,s20,sX10,sX20);

GeneralSpectralCurvebis:=y^2 -R1(xi)*y+tdR2(xi) +(Q-X1)*(-P^2+
R1(Q)*P-tdR2(Q))/(xi-X1) ;
simplify(GeneralSpectralCurve-GeneralSpectralCurvebis);
```

$$GeneralSpectralCurve := \frac{1}{(-\xi + X1)^2\,(Q - X1)}\Big(\big(-s12\,s22\,\xi^2 + ((-s21 - y)\,s12 - s22\,(y + s11))\,\xi + s12\,s22\,Q^2 + ((s21 + P)\,s12 + s22\,(P + s11))\,Q + (P - y)\,(P + s21 + s11 + y)\big)\,X1^3 + \big(2\,s22\,\xi^3\,s12 + (Q\,s12\,s22 + (2\,s21 + 2\,y)\,s12 + 2\,s22\,(y + s11))\,\xi^2 + \big(-s12\,s22\,Q^2 - (s12 + s22)\,(P - y)\,Q + s12\,s20 + s10\,s22 - P^2 + (\tag{2.6}$$

$$-s11 - s21) P + 2 y^2 + (2 s11 + 2 s21) y + s11 s21\big) \xi - 2 s12 s22 Q^3 + ((-2 P - 2 s21) s12 - 2 s22 (P + s11)) Q^2 + \big(-s12 s20 - s10 s22 - 2 P^2 + (-2 s11 - 2 s21) P + y^2 + (s21 + s11) y - s11 s21\big) Q + (sX10 + sX20) (P - y)\big) X1^2 + \big(-s12 s22 \xi^4 + (-2 Q s12 s22 + (-s21 - y) s12 - s22 (y + s11)) \xi^3 + (((-2 s21 - 2 y) s12 - 2 s22 (y + s11)) Q - s12 s20 - s10 s22 - (y + s11) (s21 + y)) \xi^2 + \big(2 s12 s22 Q^3 + ((2 s21 + 2 P) s12 + 2 s22 (P + s11)) Q^2 + 2 (P - y) (P + s21 + s11 + y) Q - (sX10 + sX20) (P - y)\big) \xi + \big(s12 s22 Q^3 + ((s21 + P) s12 + s22 (P + s11)) Q^2 + (s12 s20 + s10 s22 + (P + s11) (s21 + P)) Q - (sX10 + sX20) (P - y)\big) Q\big) X1 + Q s12 s22 \xi^4 + Q ((s21 + y) s12 + s22 (y + s11)) \xi^3 + (s12 s20 + s10 s22 + (y + s11) (s21 + y)) Q \xi^2 + \big(-Q^4 s12 s22 + ((-s21 - P) s12 - s22 (P + s11)) Q^3 + (-s12 s20 - s10 s22 - (P + s11) (s21 + P)) Q^2 + (sX10 + sX20) (P - y) Q - sX10 sX20\big) \xi + Q sX10 sX20\big)$$

$$GeneralSpectralCurvebis := y^2 - \left(\frac{sX10 + sX20}{\xi - X1} - s11 - s21 + (-s12 - s22) \xi\right) y + \frac{sX10 sX20}{(\xi - X1)^2} + s10 s22 + s11 s21 + s12 s20 + (s11 s22 + s12 s21) \xi + s12 s22 \xi^2 + \frac{1}{\xi - X1} \left((Q - X1) \left(-P^2 + \left(\frac{sX10 + sX20}{Q - X1} - s11 - s21 + (-s12 - s22) Q\right) P - \frac{sX10 sX20}{(Q - X1)^2} - s10 s22 - s11 s21 - s12 s20 - (s11 s22 + s12 s21) Q - s12 s22 Q^2\right)\right)$$

$$0$$

Computation of the Hamiltonian flows in various direction

```
> LQfunction:=unapply(LQ,beta12,beta22,beta11,beta21,betaX1):
  LPfunction:=unapply(LP,beta12,beta22,beta11,beta21,betaX1):
  Hamiltonianfunction:=unapply(Hamiltonianbis,s12,s22,s11,s21,X1,
  beta12,beta22,beta11,beta21,betaX1):
  nu0:=simplify(nu0);
  c1bis:=-(1/2)*(s11*s22-s12*s21)*(beta12-beta22)/(s12-s22)^2+
  (beta11*s22-beta21*s12)/(s12-s22);
  factor(series(simplify(c1-c1bis),beta12));
  mubis:=(Q-X1)*(betaX1 +(beta11-beta21)/(s12-s22)+ (X1*(s12-s22)
  -s11+s21)*(beta12-beta22)/(2*(s12-s22)^2)    ):
  simplify(series(mu-mubis,beta11=0));
```

$$\nu0 := \frac{1}{2} \frac{(2 \beta11 - 2 \beta21) s12 + (-2 \beta11 + 2 \beta21) s22 - (-s21 + s11) (\beta12 - \beta22)}{(s12 - s22)^2} \tag{2.7}$$

$$c1bis := -\frac{1}{2} \frac{(s11 s22 - s12 s21) (\beta12 - \beta22)}{(s12 - s22)^2} + \frac{\beta11 s22 - \beta21 s12}{s12 - s22}$$

$$0$$

$$0$$

Trivial directions

```
> simplify(LQfunction(1,1,0,0,0));
```

```
simplify(LPfunction(1,1,0,0,0));
simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,1,1,0,0,0));
simplify(LQfunction(0,0,1,1,0));
simplify(LPfunction(0,0,1,1,0));
simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,1,1,0));
simplify(LQfunction(2*s12,2*s22,s11,s21,-X1));
simplify(LPfunction(2*s12,2*s22,s11,s21,-X1));
simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,2*s12,2*s22,
s11,s21,-X1));
simplify(LQfunction(0,0,s12,s22,-1));
simplify(LPfunction(0,0,s12,s22,-1));
simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,s12,s22,-1)
);
```

$$0 \tag{2.8}$$
$$-Q\,h$$
$$\frac{1}{2}\,Q^2\,h$$
$$0$$
$$-h$$
$$Q\,h$$
$$-Q\,h$$
$$h\,P$$
$$-h\,P\,Q$$
$$-h$$
$$0$$
$$-h\,P$$

Direction X1

```
> simplify(LQfunction(0,0,0,0,1));
simplify(LPfunction(0,0,0,0,1));
HamX1:=unapply( (Q-X1)*(P^2-P*R1(Q)+h*P/(Q-X1)+tdR2(Q)+h*s12),
Q,P);
simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,0,0,1)
-HamX1(Q,P));
simplify(LQfunction(0,0,0,0,1)-diff(HamX1(Q,P),P));
simplify(LPfunction(0,0,0,0,1)+diff(HamX1(Q,P),Q));
```

$$(s12+s22)\,Q^2+((-s12-s22)\,X1+2\,P+s11+s21)\,Q+(-2\,P-s11-s21)\,X1 - sX20+h-sX10 \tag{2.9}$$

$$\frac{1}{(Q-X1)^2}\Big(-3\,Q^4\,s12\,s22+(8\,X1\,s12\,s22+(-2\,P-2\,s21)\,s12-2\,s22\,(P+s11))\,Q^3 + \big(-7\,X1^2\,s12\,s22+((5\,P+5\,s21)\,s12+5\,s22\,(P+s11))\,X1+(-s20-h)\,s12 - P^2+(-s11-s21)\,P-s10\,s22-s11\,s21\big)\,Q^2+2\,\big(X1^2\,s12\,s22+((-2\,P - 2\,s21)\,s12-2\,s22\,(P+s11))\,X1+(s20+h)\,s12+P^2+(s21+s11)\,P+s10\,s22 + s11\,s21\big)\,X1\,Q+((s21+P)\,s12+s22\,(P+s11))\,X1^3+\big((-s20-h)\,s12-P^2+(-s11-s21)\,P-s10\,s22-s11\,s21\big)\,X1^2+sX10\,sX20\Big)$$

$$HamX1 := (Q, P) \to (Q - X1) \left(P^2 - \left(\frac{sX10 + sX20}{Q - X1} - s11 - s21 + (-s12 - s22)\, Q\right) P + \frac{h\, P}{Q - X1} + \frac{sX10\, sX20}{(Q - X1)^2} + s10\, s22 + s11\, s21 + s12\, s20 + (s11\, s22 + s12\, s21)\, Q + s12\, s22\, Q^2 + h\, s12\right)$$

$$0$$
$$0$$
$$0$$

Direction s11

```
> simplify(LQfunction(0,0,1,0,0));
  simplify(LPfunction(0,0,1,0,0));
  Hams11bis:=unapply( (-Q^4*s12*s22+(2*X1*s12*s22-s11*s22-s12*
  s21)*Q^3+(-s12*s22*X1^2+(2*s11*s22+2*s12*s21)*X1+(-s20-h)*s12+
  (h-s10)*s22-s11*s21)*Q^2-((s11*s22+s12*s21)*X1+(-s20-h)*s12+(h-
  s10)*s22-s11*s21)*X1*Q-sX10*sX20)/((Q-X1)*(s22-s12))+((s12+s22)
  *Q^2+((-s12-s22)*X1+s11+s21)*Q+(-s11-s21)*X1-sX10-sX20)*P/(s12-
  s22)+(Q-X1)*P^2/(s12-s22)
  -((s20+h)*s12+s10*s22+s11*s21)*X1/(s12-s22)
  ,Q,P,s11,s12,s21,s22,X1);
  simplify(LQfunction(0,0,1,0,0)-diff(Hams11bis(Q,P,s11,s12,s21,
  s22,X1),P));
  simplify(LPfunction(0,0,1,0,0)+diff(Hams11bis(Q,P,s11,s12,s21,
  s22,X1),Q));
  simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,1,0,0)-
  Hams11bis(Q,P,s11,s12,s21,s22,X1));
```

$$\frac{1}{s12 - s22} \left((s12 + s22)\, Q^2 + ((-s12 - s22)\, X1 + 2\, P + s11 + s21)\, Q + (-2\, P - s11 - s21)\, X1 - sX10 - sX20\right) \tag{2.10}$$

$$\frac{1}{(Q - X1)^2\, (-s12 + s22)} \Big(3\, Q^4\, s12\, s22 + (-8\, X1\, s12\, s22 + (2\, s21 + 2\, P)\, s12 + 2\, s22\, (P + s11))\, Q^3 + \big(7\, X1^2\, s12\, s22 + ((-5\, P - 5\, s21)\, s12 - 5\, s22\, (P + s11))\, X1 + (s20 + h)\, s12 + (-h + s10)\, s22 + (P + s11)\, (s21 + P)\big)\, Q^2 - 2\, X1\, \big(X1^2\, s12\, s22 + ((-2\, P - 2\, s21)\, s12 - 2\, s22\, (P + s11))\, X1 + (s20 + h)\, s12 + (-h + s10)\, s22 + (P + s11)\, (s21 + P)\big)\, Q + ((-s21 - P)\, s12 - s22\, (P + s11))\, X1^3 + ((s20 + h)\, s12 + (-h + s10)\, s22 + (P + s11)\, (s21 + P))\, X1^2 - sX10\, sX20\Big)$$

$$Hams11bis := (Q, P, s11, s12, s21, s22, X1) \to \frac{1}{(Q - X1)\, (-s12 + s22)} \Big(-Q^4\, s12\, s22 + (2\, X1\, s12\, s22 - s22\, s11 - s12\, s21)\, Q^3 + \big(-X1^2\, s12\, s22 + (2\, s22\, s11 + 2\, s12\, s21)\, X1 + (-s20 - h)\, s12 + (h - s10)\, s22 - s11\, s21\big)\, Q^2 - ((s22\, s11 + s12\, s21)\, X1 + (-s20 - h)\, s12 + (h - s10)\, s22 - s11\, s21)\, X1\, Q - sX10\, sX20\Big) + \frac{1}{s12 - s22} \Big(((s12 + s22)\, Q^2 + ((-s12 - s22)\, X1 + s11 + s21)\, Q + (-s11$$

$$-s21)\ X1-sX10-sX20)\ P\big)+\frac{(Q-X1)\ P^2}{s12-s22}$$
$$-\frac{((s20+h)\ s12+s10\ s22+s11\ s21)\ X1}{s12-s22}$$
$$0$$
$$0$$
$$0$$

Non trivial direction

```
> solve({Sinfty1=s11+s21,Sinfty2=s12+s22, S2=sqrt(s12-s22)/sqrt
  (2), S1=(s11-s21)/sqrt(2)/sqrt(s12-s22),
  tdX1function=S2*X1+S1},{s11,s12,s21,s22,X1});
```

$$\left\{X1=-\frac{S1-tdX1function}{S2},\ s11=\frac{1}{2}\ Sinfty1+S1\ S2,\ s12=\frac{1}{2}\ Sinfty2+S2^2,\ s21=-S1\ S2+\frac{1}{2}\ Sinfty1,\ s22=-S2^2+\frac{1}{2}\ Sinfty2\right\} \quad \textbf{(2.11)}$$

```
> hpartialtdX1functionQ:=simplify(LQfunction(0,0,0,0,sqrt(2)/sqrt
  (s12-s22)    ));
  hpartialtdX1functionP:=simplify(LPfunction(0,0,0,0,sqrt(2)/sqrt
  (s12-s22)    ));
  HamtdX1function:=unapply(2*P*((s12+s22)*Q^2+((-s12-s22)*X1+P+
  s11+s21)*Q+(-P-s11-s21)*X1+h-sX10-sX20)/sqrt(2*s12-2*s22)+(2*Q*
  (Q*(Q-X1)*s22+Q*s21-X1*s21+h+s20)*(Q-X1)*s12+2*Q*(Q-X1)*(Q*s11-
  X1*s11+s10)*s22+2*Q^2*s11*s21-2*Q*X1*s11*s21+2*sX10*sX20)/(sqrt
  (2*s12-2*s22)*(Q-X1))
  -(2*((s20+h)*s12+s10*s22+s11*s21))*X1/sqrt(2*s12-2*s22)
  ,Q,P);
  simplify(hpartialtdX1functionQ-diff(HamtdX1function(Q,P),P));
  simplify(hpartialtdX1functionP+diff(HamtdX1function(Q,P),Q));
  simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,0,0,sqrt(2)
  /sqrt(s12-s22))-HamtdX1function(Q,P,s11));
```

$$hpartialtdX1functionQ:=\frac{1}{\sqrt{s12-s22}}\left(2\left(\left(\left(\frac{1}{2}\ s12+\frac{1}{2}\ s22\right)Q^2+\left(\left(-\frac{1}{2}\ s12-\frac{1}{2}\ s22\right)X1+P+\frac{1}{2}\ s11+\frac{1}{2}\ s21\right)Q+\left(-P-\frac{1}{2}\ s11-\frac{1}{2}\ s21\right)X1-\frac{1}{2}\ sX20+\frac{1}{2}\ h-\frac{1}{2}\ sX10\right)\sqrt{2}\right) \quad \textbf{(2.12)}$$

$$hpartialtdX1functionP:=-\frac{1}{\sqrt{s12-s22}\ (Q-X1)^2}\left(\left((Q-X1)^2 P^2+2\ (Q-X1)^2\left(\left(Q-\frac{1}{2}\ X1\right)s12+\left(Q-\frac{1}{2}\ X1\right)s22+\frac{1}{2}\ s11+\frac{1}{2}\ s21\right)P+3\ (Q-X1)^2\left(Q\left(Q-\frac{2}{3}\ X1\right)s22+\left(\frac{2}{3}\ Q-\frac{1}{3}\ X1\right)s21+\frac{1}{3}\ h+\frac{1}{3}\ s20\right)s12+2\ (Q-X1)^2\left(\left(Q-\frac{1}{2}\ X1\right)s11+\frac{1}{2}\ s10\right)s22+s21\ (Q-X1)^2\ s11-sX10\ sX20\right)\sqrt{2}\right)$$

$$HamtdX1function := (Q, P) \rightarrow \frac{1}{\sqrt{2\,s12 - 2\,s22}} \Big(2\,P\,\big((s12 + s22)\,Q^2 + ((-s12 - s22)\,X1 + P + s11 + s21)\,Q + (-P - s11 - s21)\,X1 + h - sX10 - sX20\big)\Big) + \frac{1}{\sqrt{2\,s12 - 2\,s22}\,(Q - X1)} \big(2\,Q\,(Q\,(Q - X1)\,s22 + Q\,s21 - X1\,s21 + h + s20)\,(Q - X1)\,s12 + 2\,Q\,(Q - X1)\,(Q\,s11 - X1\,s11 + s10)\,s22 + 2\,Q^2\,s11\,s21 - 2\,Q\,X1\,s11\,s21 + 2\,sX10\,sX20\big) - \frac{2\,((s20 + h)\,s12 + s10\,s22 + s11\,s21)\,X1}{\sqrt{2\,s12 - 2\,s22}}$$

$$0$$

$$0$$

$$0$$

```
> KOldCoordinates := unapply( (1/2)*(sX10+s20)*(sX10+s10)*ln((s12
-s22)/2)+(1/2)*X1*((X1*s12+2*s11)*s10+s20*(X1*s22+2*s21)),s11,
s21,s12,s22,X1);
```

$$KOldCoordinates := (s11, s21, s12, s22, X1) \rightarrow \frac{1}{2}\,(sX10 + s20)\,(sX10 + s10)\,\ln\left(\frac{1}{2}\,s12 - \frac{1}{2}\,s22\right) + \frac{1}{2}\,X1\,((s12\,X1 + 2\,s11)\,s10 + s20\,(s22\,X1 + 2\,s21)) \qquad \textbf{(2.13)}$$

```
> Hams11:=simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,1,
0,0))+
U11(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s11);
Hams21:=simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,0,
1,0))+
U21(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s21);
Hams12:=simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,1,0,0,
0,0))+
U12(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s12);
Hams22:=simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,1,0,
0,0))+
U22(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s22);
HamX1:=simplify(Hamiltonianfunction(s12,s22,s11,s21,X1,0,0,0,0,
1))+
UX1(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),X1)
;

Hams11formula := (-Q^4*s12*s22+(2*X1*s12*s22+(-s12-s22)*P-s11*
s22-s12*s21)*Q^3+(-X1^2*s12*s22+((2*s12+2*s22)*P+2*s11*s22+2*
s12*s21)*X1-P^2+(-s11-s21)*P+(-s20-h)*s12+(h-s10)*s22-s11*s21)*
```

```
Q^2+(((-s12-s22)*P-s12*s21-s11*s22)*X1^2+(2*P^2+(2*s11+2*s21)*
P+(2*s20+2*h)*s12+(-h+2*s10)*s22+2*s11*s21)*X1+P*(sX10+sX20))*
Q+((-s20-h)*s12-P^2+(-s11-s21)*P-s10*s22-s11*s21)*X1^2-P*(sX10+
sX20)*X1-sX10*sX20)/((Q-X1)*(-s12+s22))
+
U11(s11,s21,s12,s22,X1) +diff(KOldCoordinates(s11,s21,s12,s22,
X1),s11):

Hams21formula := (Q^4*s12*s22+(-2*X1*s12*s22+(s12+s22)*P+s11*
s22+s12*s21)*Q^3+(X1^2*s12*s22+((-2*s12-2*s22)*P-2*s11*s22-2*
s12*s21)*X1+P^2+(s21+s11)*P+s10*s22+s11*s21+s12*s20)*Q^2+((
(s12+s22)*P+s11*s22+s12*s21)*X1^2+(-2*P^2+(-2*s11-2*s21)*P+(-
h-2*s20)*s12-2*s10*s22-2*s11*s21)*X1-P*(sX10+sX20))*Q+((s20+h)*
s12+P^2+(s21+s11)*P+s10*s22+s11*s21)*X1^2+P*(sX10+sX20)*X1+
sX10*sX20)/((Q-X1)*(-s12+s22))
+
U21(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s21):

Hams12formula := (-(s12*s22*Q^2+((s21+P)*s12+s22*(P+s11))*Q+
(s20+h)*s12+s10*s22+(P+s11)*(s21+P))*(-s12+s22)*X1^3+(2*s22*
s12*(-s12+s22)*Q^3+((-2*P-2*s21)*s12^2-3*s22*(-s21+s11)*s12+2*
s22^2*(P+s11))*Q^2+((-2*s20-2*h)*s12^2+((2*h-2*s10+2*s20)*s22-
(2*(P+3*s11*(1/2)-(1/2)*s21))*(s21+P))*s12+(2*(s10*s22+(P+s11)*
(P-(1/2)*s11+3*s21*(1/2))))*s22)*Q+((sX10+sX20-h)*P-(-s21+s11)*
(s20+h))*s12+((-sX10-sX20+h)*P-s10*(-s21+s11))*s22-(-s21+s11)*
(s21+P)*(P+s11))*X1^2+(-s22*s12*(-s12+s22)*Q^4+((s21+P)*
s12^2+3*s22*(-s21+s11)*s12-s22^2*(P+s11))*Q^3+((s20+h)*s12^2+(
(s10-s20)*s22+(s21+P)*(P+3*s11-2*s21))*s12-s22*((h+s10)*s22+(P+
s11)*(P-2*s11+3*s21)))*Q^2+(((-sX10-sX20+2*h)*P+(2*s20+2*h)*s11
-s21*(h+2*s20))*s12+((sX10+sX20-2*h)*P+(-h+2*s10)*s11-2*s21*
s10)*s22+(2*(-s21+s11))*(s21+P)*(P+s11))*Q+sX10*sX20*s12-sX10*
sX20*s22-P*(-s21+s11)*(sX10+sX20))*X1-s22*(-s21+s11)*s12*Q^4+(
(-s22*h-(-s21+s11)*(s21+P))*s12-(-s22*h+(-s21+s11)*(P+s11))*
s22)*Q^3+((-h*P+(-s20-h)*s11+s20*s21)*s12+(h*P+(h-s10)*s11+s21*
s10)*s22-(-s21+s11)*(s21+P)*(P+s11))*Q^2+P*(-s21+s11)*(sX10+
sX20)*Q-sX10*sX20*(-s21+s11))/(2*(-s12+s22)^2*(Q-X1))
+
U12(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s12):

Hams22formula := ((s12*s22*Q^2+((s21+P)*s12+s22*(P+s11))*Q+
```

```
(s20+h)*s12+s10*s22+(P+s11)*(s21+P))*(-s12+s22)*X1^3+(-2*s22*
s12*(-s12+s22)*Q^3+((2*s21+2*P)*s12^2+3*s22*(-s21+s11)*s12-2*
s22^2*(P+s11))*Q^2+((2*s20+2*h)*s12^2+((-2*h+2*s10-2*s20)*s22+
(2*(P+3*s11*(1/2)-(1/2)*s21))*(s21+P))*s12-(2*(s10*s22+(P+s11)*
(P-(1/2)*s11+3*s21*(1/2))))*s22)*Q+((-sX10-sX20+h)*P+(-s21+s11)
*(s20+h))*s12+((sX10+sX20-h)*P+s10*(-s21+s11))*s22+(-s21+s11)*
(s21+P)*(P+s11))*X1^2+(s22*s12*(-s12+s22)*Q^4+((-s21-P)*s12^2
-3*s22*(-s21+s11)*s12+s22^2*(P+s11))*Q^3+((-2*h-s20)*s12^2+((2*
h-s10+s20)*s22-(s21+P)*(P+3*s11-2*s21))*s12+s22*(s10*s22+(P+
s11)*(P-2*s11+3*s21)))*Q^2+(((sX10+sX20-2*h)*P+(-2*s20-2*h)*
s11+s21*(h+2*s20))*s12+((-sX10-sX20+2*h)*P+(h-2*s10)*s11+2*s21*
s10)*s22-(2*(-s21+s11))*(s21+P)*(P+s11))*Q-sX10*sX20*s12+sX10*
sX20*s22+P*(-s21+s11)*(sX10+sX20))*X1+s22*(-s21+s11)*s12*Q^4+
(s12^2*h+(-s22*h+(-s21+s11)*(s21+P))*s12+s22*(-s21+s11)*(P+s11)
)*Q^3+((h*P+(s20+h)*s11-s20*s21)*s12+(-h*P+(-h+s10)*s11-s21*
s10)*s22+(-s21+s11)*(s21+P)*(P+s11))*Q^2-P*(-s21+s11)*(sX10+
sX20)*Q+sX10*sX20*(-s21+s11))/(2*(-s12+s22)^2*(Q-X1))
+
U22(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),s22):

HamX1formula := (Q^4*s12*s22+(-2*X1*s12*s22+(s12+s22)*P+s11*
s22+s12*s21)*Q^3+(X1^2*s12*s22+((-2*s12-2*s22)*P-2*s11*s22-2*
s12*s21)*X1+(s20+h)*s12+P^2+(s21+s11)*P+s10*s22+s11*s21)*Q^2+((
(s12+s22)*P+s11*s22+s12*s21)*X1^2+(-2*P^2+(-2*s11-2*s21)*P+(-2*
s20-2*h)*s12-2*s10*s22-2*s11*s21)*X1-(sX10+sX20-h)*P)*Q+((s20+
h)*s12+P^2+(s21+s11)*P+s10*s22+s11*s21)*X1^2+(sX10+sX20-h)*P*
X1+sX10*sX20)/(Q-X1)
+
UX1(s11,s21,s12,s22,X1)+diff(KOldCoordinates(s11,s21,s12,s22,
X1),X1):

simplify(HamX1-HamX1formula);
simplify(Hams11-Hams11formula);
simplify(Hams21-Hams21formula);
simplify(Hams12-Hams12formula);
simplify(Hams22-Hams22formula);
```

$$Hams11 := \frac{1}{(Q-X1)\,(-s12+s22)}\Big(-Q^4\,s12\,s22+(2\,X1\,s12\,s22+(-s12-s22)\,P - s11\,s22-s12\,s21)\,Q^3+\big(-X1^2\,s12\,s22+((2\,s12+2\,s22)\,P+2\,s11\,s22 + 2\,s12\,s21)\,X1-P^2+(-s11-s21)\,P+(-s20-h)\,s12+(h-s10)\,s22 - s11\,s21\big)\,Q^2+\big(((-s12-s22)\,P-s12\,s21-s11\,s22)\,X1^2+\big(2\,P^2+(2\,s11 + 2\,s21)\,P+(2\,s20+2\,h)\,s12+(-h+2\,s10)\,s22+2\,s11\,s21\big)\,X1+P\,(sX10 \tag{2.14}$$

$$+ sX20)) Q + ((-s20 - h) s12 - P^2 + (-s11 - s21) P - s10 s22 - s11 s21) X1^2 - P (sX10 + sX20) X1 - sX10 sX20) + U11(s11, s21, s12, s22, X1) + X1 s10$$

$$Hams21 := \frac{1}{(Q - X1)(-s12 + s22)} (Q^4 s12 s22 + (-2 X1 s12 s22 + (s12 + s22) P + s11 s22 + s12 s21) Q^3 + (X1^2 s12 s22 + ((-2 s12 - 2 s22) P - 2 s11 s22 - 2 s12 s21) X1 + P^2 + (s21 + s11) P + s10 s22 + s11 s21 + s12 s20) Q^2 + (((s12 + s22) P + s11 s22 + s12 s21) X1^2 + (-2 P^2 + (-2 s11 - 2 s21) P + (-h - 2 s20) s12 - 2 s10 s22 - 2 s11 s21) X1 - P (sX10 + sX20)) Q + ((s20 + h) s12 + P^2 + (s21 + s11) P + s10 s22 + s11 s21) X1^2 + P (sX10 + sX20) X1 + sX10 sX20) + U21(s11, s21, s12, s22, X1) + X1 s20$$

$$Hams12 := \frac{1}{2} \frac{1}{(-s12 + s22)^2 (Q - X1)} \Big(-(-s12 + s22) (s12 s22 Q^2 + ((s21 + P) s12 + s22 (P + s11)) Q + (s20 + h) s12 + s10 s22 + (P + s11)(s21 + P)) X1^3 + \Big(2 s22 s12 (-s12 + s22) Q^3 + ((-2 P - 2 s21) s12^2 - 3 s22 (-s21 + s11) s12 + 2 s22^2 (P + s11)) Q^2 + \Big((-2 s20 - 2 h) s12^2 + \Big((2 h - 2 s10 + 2 s20) s22 - 2 \Big(P + \frac{3}{2} s11 - \frac{1}{2} s21\Big)(s21 + P)\Big) s12 + 2 s22 \Big(s10 s22 + (P + s11) \Big(P - \frac{1}{2} s11 + \frac{3}{2} s21\Big)\Big)\Big) Q + ((sX10 + sX20 - h) P - (-s21 + s11)(s20 + h)) s12 + ((-sX10 - sX20 + h) P - s10 (-s21 + s11)) s22 - (-s21 + s11)(s21 + P)(P + s11)\Big) X1^2 + (-s22 s12 (-s12 + s22) Q^4 + ((s21 + P) s12^2 + 3 s22 (-s21 + s11) s12 - s22^2 (P + s11)) Q^3 + ((s20 + h) s12^2 + ((s10 - s20) s22 + (s21 + P)(P + 3 s11 - 2 s21)) s12 - s22 ((h + s10) s22 + (P + s11)(P - 2 s11 + 3 s21))) Q^2 + (((-sX10 - sX20 + 2 h) P + (2 s20 + 2 h) s11 - s21 (h + 2 s20)) s12 + ((sX10 + sX20 - 2 h) P + (-h + 2 s10) s11 - 2 s21 s10) s22 + 2 (-s21 + s11)(s21 + P)(P + s11)) Q + sX10 sX20 s12 - sX10 sX20 s22 - P (-s21 + s11)(sX10 + sX20)) X1 - s22 (-s21 + s11) s12 Q^4 + ((-s22 h - (-s21 + s11)(s21 + P)) s12 - s22 (-s22 h + (-s21 + s11)(P + s11))) Q^3 + ((-h P + (-s20 - h) s11 + s20 s21) s12 + (h P + (h - s10) s11 + s21 s10) s22 - (-s21 + s11)(s21 + P)(P + s11)) Q^2 + P (-s21 + s11)(sX10 + sX20) Q - sX10 sX20 (-s21 + s11)\Big) + U12(s11, s21, s12, s22, X1) + \frac{1}{4} \frac{(sX10 + s20)(sX10 + s10)}{\frac{1}{2} s12 - \frac{1}{2} s22} + \frac{1}{2} X1^2 s10$$

$$Hams22 := \frac{1}{2} \frac{1}{(-s12 + s22)^2 (Q - X1)} \Big((-s12 + s22) (s12 s22 Q^2 + ((s21 + P) s12 + s22 (P + s11)) Q + (s20 + h) s12 + s10 s22 + (P + s11)(s21 + P)) X1^3 + \Big(-2 s22 s12 (-s12 + s22) Q^3 + ((2 s21 + 2 P) s12^2 + 3 s22 (-s21 + s11) s12 - 2 s22^2 (P + s11)) Q^2 + \Big((2 s20 + 2 h) s12^2 + \Big((-2 h + 2 s10 - 2 s20) s22 + 2 \Big(P$$

$$+\frac{3}{2}\,s11-\frac{1}{2}\,s21\Big)\,(s21+P)\Big)\,s12-2\,s22\Big(s10\,s22+(P+s11)\,\Big(P-\frac{1}{2}\,s11$$
$$+\frac{3}{2}\,s21\Big)\Big)\Big)\,Q+((-sX10-sX20+h)\,P+(-s21+s11)\,(s20+h))\,s12+((sX10$$
$$+sX20-h)\,P+s10\,(-s21+s11))\,s22+(-s21+s11)\,(s21+P)\,(P+s11)\Big)\,X1^2$$
$$+\big(s22\,s12\,(-s12+s22)\,Q^4+\big((-s21-P)\,s12^2-3\,s22\,(-s21+s11)\,s12+s22^2\,(P$$
$$+s11)\big)\,Q^3+\big((-2\,h-s20)\,s12^2+((2\,h-s10+s20)\,s22-(s21+P)\,(P+3\,s11$$
$$-2\,s21))\,s12+s22\,(s10\,s22+(P+s11)\,(P-2\,s11+3\,s21))\big)\,Q^2+(((sX10$$
$$+sX20-2\,h)\,P+(-2\,s20-2\,h)\,s11+s21\,(h+2\,s20))\,s12+((-sX10-sX20$$
$$+2\,h)\,P+(h-2\,s10)\,s11+2\,s21\,s10)\,s22-2\,(-s21+s11)\,(s21+P)\,(P+s11))$$
$$Q-sX10\,sX20\,s12+sX10\,sX20\,s22+P\,(-s21+s11)\,(sX10+sX20))\,X1+s22\,($$
$$-s21+s11)\,s12\,Q^4+\big(s12^2\,h+(-s22\,h+(-s21+s11)\,(s21+P))\,s12+s22\,(-s21$$
$$+s11)\,(P+s11)\big)\,Q^3+((h\,P+(s20+h)\,s11-s20\,s21)\,s12+(-h\,P+(-h$$
$$+s10)\,s11-s21\,s10)\,s22+(-s21+s11)\,(s21+P)\,(P+s11))\,Q^2-P\,(-s21$$
$$+s11)\,(sX10+sX20)\,Q+sX10\,sX20\,(-s21+s11)\big)+U22(s11,s21,s12,s22,X1)$$
$$-\frac{1}{4}\,\frac{(sX10+s20)\,(sX10+s10)}{\frac{1}{2}\,s12-\frac{1}{2}\,s22}+\frac{1}{2}\,X1^2\,s20$$

$$HamX1:=\frac{1}{Q-X1}\big(Q^4\,s12\,s22+(-2\,X1\,s12\,s22+(s12+s22)\,P+s11\,s22$$
$$+s12\,s21)\,Q^3+\big(X1^2\,s12\,s22+((-2\,s12-2\,s22)\,P-2\,s11\,s22-2\,s12\,s21)\,X1$$
$$+(s20+h)\,s12+P^2+(s21+s11)\,P+s10\,s22+s11\,s21\big)\,Q^2+\big(((s12+s22)\,P$$
$$+s11\,s22+s12\,s21)\,X1^2+\big(-2\,P^2+(-2\,s11-2\,s21)\,P+(-2\,s20-2\,h)\,s12$$
$$-2\,s10\,s22-2\,s11\,s21\big)\,X1-(sX10+sX20-h)\,P\big)\,Q+\big((s20+h)\,s12+P^2$$
$$+(s21+s11)\,P+s10\,s22+s11\,s21\big)\,X1^2+(sX10+sX20-h)\,P\,X1+sX10\,sX20\big)$$
$$+UX1(s11,s21,s12,s22,X1)+\frac{1}{2}\,(X1\,s12+2\,s11)\,s10+\frac{1}{2}\,s20\,(X1\,s22+2\,s21)$$
$$+\frac{1}{2}\,X1\,(s10\,s12+s20\,s22)$$

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> U11 := unapply( ((X1*s22+s21)*s10+(X1*s12+s11)*s20)/(s12-s22)
  +s22*X1/(s12-s22)*h,s11,s21,s12,s22,X1);
  U21 := unapply(-((X1*s22+s21)*s10+(X1*s12+s11)*s20)/(s12-s22) -
  s12*X1/(s12-s22)*h,s11,s21,s12,s22,X1);
  U12 := unapply( (X1*(s12-s22)-s11+s21)*((s10*s22+s12*s20)*X1+
  s21*s10+s11*s20)/(2*(s12-s22)^2)
  +(-s22^2*X1+(X1*s12-s11)*s22+s12*s21)*X1/(2*(s12-s22)^2)*h
  ,s11,s21,s12,s22,X1);
  U22 := unapply(-(X1*(s12-s22)-s11+s21)*((s10*s22+s12*s20)*X1+
  s21*s10+s11*s20)/(2*(s12-s22)^2)
```

```
-h*(X1*s12^2+(-X1*s22+s21)*s12-s11*s22)*X1/(2*(s12-s22)^2)
,s11,s21,s12,s22,X1);
UX1 := unapply( (X1*s22+s21)*s10+(X1*s12+s11)*s20,s11,s21,s12,
s22,X1);
```

$$U11 := (s11, s21, s12, s22, X1) \rightarrow \frac{(s22\,X1 + s21)\,s10 + (s12\,X1 + s11)\,s20}{s12 - s22} + \frac{s22\,X1\,h}{s12 - s22} \quad \textbf{(2.15)}$$

$$U21 := (s11, s21, s12, s22, X1) \rightarrow -\frac{(s22\,X1 + s21)\,s10 + (s12\,X1 + s11)\,s20}{s12 - s22} - \frac{s12\,X1\,h}{s12 - s22}$$

$$U12 := (s11, s21, s12, s22, X1) \rightarrow \frac{1}{2}\,\frac{(X1\,(s12 - s22) - s11 + s21)\,((s10\,s22 + s12\,s20)\,X1 + s10\,s21 + s11\,s20)}{(s12 - s22)^2} + \frac{1}{2}\,\frac{\left(-s22^2\,X1 + (s12\,X1 - s11)\,s22 + s12\,s21\right)\,X1\,h}{(s12 - s22)^2}$$

$$U22 := (s11, s21, s12, s22, X1) \rightarrow -\frac{1}{2}\,\frac{(X1\,(s12 - s22) - s11 + s21)\,((s10\,s22 + s12\,s20)\,X1 + s10\,s21 + s11\,s20)}{(s12 - s22)^2} - \frac{1}{2}\,\frac{h\left(X1\,s12^2 + (-s22\,X1 + s21)\,s12 - s11\,s22\right)\,X1}{(s12 - s22)^2}$$

$$UX1 := (s11, s21, s12, s22, X1) \rightarrow (s22\,X1 + s21)\,s10 + (s12\,X1 + s11)\,s20$$

JMU differential

First part: Contributions at xi=X_1

```
> ExpMonodromiesX1:=Matrix(2,2,0):
ExpMonodromiesX1[1,1]:=exp(sX10*ln(xi-X1)/h):
ExpMonodromiesX1[2,2]:=exp(sX20*ln(xi-X1)/h):
ExpMonodromiesX1;

MonodromiesX1:=Matrix(2,2,0):
MonodromiesX1[1,1]:=sX10/(xi-X1):
MonodromiesX1[2,2]:=sX20/(xi-X1):
MonodromiesX1:

LambdaX1:=Matrix(2,2,0):
LambdaX1[1,1]:=sX10*ln(xi-X1):
LambdaX1[2,2]:=sX20*ln(xi-X1):
LambdaX1:

N0:=Matrix(2,2,0):
N1:=Matrix(2,2,0):
N2:=Matrix(2,2,0):
N3:=Matrix(2,2,0):
N4:=Matrix(2,2,0):
```

```
for i from 1 to 2 do for j from 1 to 2 do
N0[i,j]:=n0[i,j]:
N1[i,j]:=n1[i,j]:
N2[i,j]:=n2[i,j]:
N3[i,j]:=n3[i,j]:
N4[i,j]:=n4[i,j]:
od: od:

HatPsiRegX1:=IdentityMatrix(2)+N1*(xi-X1)+N2*(xi-X1)^2+N3*(xi-
X1)^3+N4*(xi-X1)^4:
tdPsiX1:=Multiply(N0,Multiply(HatPsiRegX1,ExpMonodromiesX1)):

dHatPsiRegdxiX1:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dHatPsiRegdxiX1[i,j]
:=diff(HatPsiRegX1[i,j],xi): od: od:

dtdPsidxiX1:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dtdPsidxiX1[i,j]:=
diff(tdPsiX1[i,j],xi): od: od:

dLambdaX1dt:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dLambdaX1dt[i,j]:=
simplify(diff(LambdaX1[i,j],X1)*dX1): od: od:
dLambdaX1dt;

omegaJMUPartX1:=simplify(-h*residue(Trace(Multiply(Multiply
(HatPsiRegX1^(-1),dHatPsiRegdxiX1), dLambdaX1dt)),xi=X1));

ToCancelX1:=simplify(h*dtdPsidxiX1-Multiply(tdL,tdPsiX1)):
```

$$\begin{bmatrix} e^{\frac{sX10 \ln(\xi - X1)}{h}} & 0 \\ 0 & e^{\frac{sX20 \ln(\xi - X1)}{h}} \end{bmatrix} \tag{2.16}$$

$$\begin{bmatrix} \frac{sX10\, dX1}{-\xi + X1} & 0 \\ 0 & \frac{sX20\, dX1}{-\xi + X1} \end{bmatrix}$$

$$omegaJMUPartX1 := h\, dX1 \left(sX10\, n1_{1,1} + sX20\, n1_{2,2}\right)$$

```
> SingularParttdLxiX1:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do SingularParttdLxiX1
[i,j]:=simplify(residue(tdL[i,j],xi=X1)): od: od:
```

```
MatrixEigenvectorsSingularPartX1:=Matrix(2,2,0):
MatrixEigenvectorsSingularPartX1[1,1]:=Q-X1:
MatrixEigenvectorsSingularPartX1[1,2]:=Q-X1:
MatrixEigenvectorsSingularPartX1[2,1]:=Q^2*s12-Q*X1*s12+P*Q-P*
X1+Q*s11-X1*s11-sX10:
MatrixEigenvectorsSingularPartX1[2,2]:=Q^2*s12-Q*X1*s12+P*Q-P*
X1+Q*s11-X1*s11-sX20:
simplify(Multiply(Multiply(MatrixEigenvectorsSingularPartX1^
(-1),SingularParttdLxiX1),MatrixEigenvectorsSingularPartX1));
GX10:=MatrixEigenvectorsSingularPartX1;

n0[1,1]:=Q-X1;
n0[1,2]:=Q-X1;
```

$$\begin{bmatrix} sX10 & 0 \\ 0 & sX20 \end{bmatrix} \tag{2.17}$$

$$[[Q - X1, Q - X1], [Q^2 s12 - Q\,X1\,s12 + P\,Q - P\,X1 + Q\,s11 - X1\,s11 - sX10, Q^2 s12 - Q\,X1\,s12 + P\,Q - P\,X1 + Q\,s11 - X1\,s11 - sX20]]$$

$$n0_{1,\,1} := Q - X1$$

$$n0_{1,\,2} := Q - X1$$

```
> ToCancelX1Entry11:=simplify(series(simplify(ToCancelX1[1,1]*
  (xi-X1)^(-(sX10)/h)),xi=X1)):
  ToCancelX1Entry12:=simplify(series(simplify(ToCancelX1[1,2]*
  (xi-X1)^(-(sX20)/h)),xi=X1)):
  ToCancelX1Entry21:=simplify(series(simplify(ToCancelX1[2,1]*
  (xi-X1)^(-(sX10)/h)),xi=X1)):
  ToCancelX1Entry22:=simplify(series(simplify(ToCancelX1[2,2]*
  (xi-X1)^(-(sX20)/h)),xi=X1)):
```

```
> n0[2,1]:=simplify(solve(residue( ToCancelX1Entry11,xi=X1),n0[2,
  1]));
  n0[2,2]:=simplify(solve(residue( ToCancelX1Entry12,xi=X1),n0[2,
  2]));
  simplify(residue(ToCancelX1Entry21,xi=X1));
  simplify(residue(ToCancelX1Entry22,xi=X1));
  n021:=P*(Q-X1)+s12*Q^2+(s11-X1*s12)*Q-X1*s11-sX10;
  n022:=P*(Q-X1)+s12*Q^2+(s11-X1*s12)*Q-X1*s11-sX20;
  simplify(n0[2,1]-n021);
  simplify(n0[2,2]-n022);
```

$$n0_{2,\,1} := Q^2 s12 + (-X1\,s12 + P + s11)\,Q + (-P - s11)\,X1 - sX10 \tag{2.18}$$

$$n0_{2,\,2} := Q^2 s12 + (-X1\,s12 + P + s11)\,Q + (-P - s11)\,X1 - sX20$$

$$0$$
$$0$$
$$n021 := P\,(Q - X1) + Q^2\,s12 + (-X1\,s12 + s11)\,Q - X1\,s11 - sX10$$
$$n022 := P\,(Q - X1) + Q^2\,s12 + (-X1\,s12 + s11)\,Q - X1\,s11 - sX20$$
$$0$$
$$0$$

```
> n1[1,1]:=(Q^4*s12*s22+(-2*X1*s12*s22+(s12+s22)*P+s11*s22+s12*
s21)*Q^3+(X1^2*s12*s22+((-2*s12-2*s22)*P-2*s11*s22-2*s12*s21)*
X1+P^2+(s21+s11)*P+s10*s22+s11*s21+s12*s20)*Q^2+(((s12+s22)*P+
s11*s22+s12*s21)*X1^2+(-2*P^2+(-2*s11-2*s21)*P+(-sX10-2*s20)*
s12+(-sX10-2*s10)*s22-2*s11*s21)*X1+(-sX10-sX20)*P-sX10*(s21+
s11))*Q+(P^2+(s21+s11)*P+(sX10+s20)*s12+s22*(sX10+s10)+s11*s21)
*X1^2+(P*(sX10+sX20)+sX10*(s21+s11))*X1+sX10*sX20)/((Q-X1)*h*
(sX10-sX20)):

n1[1,2]:=(-Q^4*s12*s22+(2*X1*s12*s22+(-s21-P)*s12-s22*(P+s11))*
Q^3+(-X1^2*s12*s22+((2*s21+2*P)*s12+2*s22*(P+s11))*X1+(-sX10+
sX20-s20)*s12-P^2+(-s11-s21)*P-s10*s22-s11*s21)*Q^2+(((-s21-P)*
s12-s22*(P+s11))*X1^2+((2*sX10-sX20+2*s20)*s12+2*P^2+(2*s11+2*
s21)*P+(sX20+2*s10)*s22+2*s11*s21)*X1+2*sX20*(P+(1/2)*s11+(1/2)
*s21))*Q+((-sX10-s20)*s12-P^2+(-s11-s21)*P+(-sX20-s10)*s22-s11*
s21)*X1^2-2*sX20*(P+(1/2)*s11+(1/2)*s21)*X1-sX20^2)/((sX10-
sX20)*(-h+sX10-sX20)*(Q-X1)):

n1[2,1]:=(-Q^4*s12*s22+(2*X1*s12*s22+(-s21-P)*s12-s22*(P+s11))*
Q^3+(-X1^2*s12*s22+((2*s21+2*P)*s12+2*s22*(P+s11))*X1+(sX10-
sX20-s20)*s12-P^2+(-s11-s21)*P-s10*s22-s11*s21)*Q^2+(((-s21-P)*
s12-s22*(P+s11))*X1^2+((-sX10+2*sX20+2*s20)*s12+2*P^2+(2*s11+2*
s21)*P+(sX10+2*s10)*s22+2*s11*s21)*X1+(2*(P+(1/2)*s11+(1/2)*
s21))*sX10)*Q+((-sX20-s20)*s12-P^2+(-s11-s21)*P+(-sX10-s10)*s22
-s11*s21)*X1^2-(2*(P+(1/2)*s11+(1/2)*s21))*sX10*X1-sX10^2)/((h+
sX10-sX20)*(sX10-sX20)*(Q-X1)):

n1[2,2]:=-(Q^4*s12*s22-2*Q^3*X1*s12*s22+Q^2*X1^2*s12*s22+P*Q^3*
s12+P*Q^3*s22-2*P*Q^2*X1*s12-2*P*Q^2*X1*s22+P*Q*X1^2*s12+P*Q*
X1^2*s22+Q^3*s11*s22+Q^3*s12*s21-2*Q^2*X1*s11*s22-2*Q^2*X1*s12*
s21+Q*X1^2*s11*s22+Q*X1^2*s12*s21+P^2*Q^2-2*P^2*Q*X1+P^2*X1^2+
P*Q^2*s11+P*Q^2*s21-2*P*Q*X1*s11-2*P*Q*X1*s21+P*X1^2*s11+P*
X1^2*s21+Q^2*s10*s22+Q^2*s11*s21+Q^2*s12*s20-2*Q*X1*s10*s22-2*
Q*X1*s11*s21-2*Q*X1*s12*s20-Q*X1*s12*sX20-Q*X1*s22*sX20+X1^2*
s10*s22+X1^2*s11*s21+X1^2*s12*s20+X1^2*s12*sX20+X1^2*s22*sX20-
P*Q*sX10-P*Q*sX20+P*X1*sX10+P*X1*sX20-Q*s11*sX20-Q*s21*sX20+X1*
s11*sX20+X1*s21*sX20+sX10*sX20)/((Q-X1)*h*(sX10-sX20)):
```

```
simplify(residue( ToCancelX1Entry11/(xi-X1),xi=X1));
simplify(residue( ToCancelX1Entry12/(xi-X1),xi=X1));
simplify(residue( ToCancelX1Entry21/(xi-X1),xi=X1));
simplify(residue( ToCancelX1Entry22/(xi-X1),xi=X1));
```

$$0 \tag{2.19}$$
$$0$$
$$0$$
$$0$$

```
> omegaJMUPartX1:=simplify(omegaJMUPartX1):
```

Second part: Contributions at infinity

```
> ExpXiInfinity:=Matrix(2,2,0):
ExpXiInfinity[1,1]:=exp(-(s11*xi+s12*xi^2/2)/h):
ExpXiInfinity[2,2]:=exp(-(s21*xi+s22*xi^2/2)/h):
ExpXiInfinity;

XiInfinity:=Matrix(2,2,0):
XiInfinity[1,1]:=-(s10*ln(xi)+s11*xi+s12*xi^2/2):
XiInfinity[2,2]:=-(s20*ln(xi)+s21*xi+s22*xi^2/2):
XiInfinity:

dXiInfinitydxi:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dXiInfinitydxi[i,j]:=
diff(XiInfinity[i,j],xi): od: od:
dXiInfinitydxi:

ExpMonodromiesInf:=Matrix(2,2,0):
ExpMonodromiesInf[1,1]:=exp(-s10*ln(xi)/h):
ExpMonodromiesInf[2,2]:=exp(-s20*ln(xi)/h):
ExpMonodromiesInf;

MonodromiesInf:=Matrix(2,2,0):
MonodromiesInf[1,1]:=-s10/xi:
MonodromiesInf[2,2]:=-s20/xi:
MonodromiesInf:

J1:=Matrix(2,2,0):
J2:=Matrix(2,2,0):
J3:=Matrix(2,2,0):
J4:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do J1[i,j]:=j1[i,j]:
J2[i,j]:=j2[i,j]:
J3[i,j]:=j3[i,j]:
J4[i,j]:=j4[i,j]:
```

```
od: od:


HatPsiRegInf:=IdentityMatrix(2)+J1/xi+J2/xi^2+J3/xi^3+J4/xi^4:
tdPsiInf:=Multiply(Multiply(HatPsiRegInf,ExpXiInfinity),
ExpMonodromiesInf):

dHatPsiRegdxiInf:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dHatPsiRegdxiInf[i,j]
:=diff(HatPsiRegInf[i,j],xi): od: od:

dtdPsidxiInf:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dtdPsidxiInf[i,j]:=
diff(tdPsiInf[i,j],xi): od: od:

dXiInfinitydt:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dXiInfinitydt[i,j]:=
simplify(diff(XiInfinity[i,j],s11)*ds11+diff(XiInfinity[i,j],
s21)*ds21+diff(XiInfinity[i,j],s12)*ds12+diff(XiInfinity[i,j],
s22)*ds22) od: od:
dXiInfinitydt:

omegaJMUPartInfinity:=-h*residue(Trace(Multiply(Multiply
(HatPsiRegInf^(-1),dHatPsiRegdxiInf), dXiInfinitydt)),xi=
infinity);

ToCancelInf:=simplify(h*dtdPsidxiInf-Multiply(tdL,tdPsiInf)):
```

$$\begin{bmatrix} e^{-\frac{s11\,\xi+\frac{1}{2}\xi^2 s12}{h}} & 0 \\ 0 & e^{-\frac{s21\,\xi+\frac{1}{2}\xi^2 s22}{h}} \end{bmatrix} \tag{2.20}$$

$$\begin{bmatrix} e^{-\frac{s10\ln(\xi)}{h}} & 0 \\ 0 & e^{-\frac{s20\ln(\xi)}{h}} \end{bmatrix}$$

$$omegaJMUPartInfinity := -h\left(-jl_{1,1}\,ds11+\frac{1}{2}\left(jl_{1,1}^2+jl_{1,2}\,jl_{2,1}-2\,j2_{1,1}\right)ds12 -jl_{2,2}\,ds21+\frac{1}{2}\left(jl_{1,2}\,jl_{2,1}+jl_{2,2}^2-2\,j2_{2,2}\right)ds22\right)$$

```
> series(simplify(ToCancelInf[1,1]*exp((xi^2*s12+2*s10*ln(xi)+2*
```

```
s11*xi)/(2*h))),xi=infinity):
j1[2,1]:=simplify(solve(residue( simplify(ToCancelInf[1,1]*exp(
(xi^2*s12+2*s10*ln(xi)+2*s11*xi)/(2*h)))
,xi=infinity),j1[2,1]));
j2[2,1]:=simplify(solve(residue( simplify(ToCancelInf[1,1]*exp(
(xi^2*s12+2*s10*ln(xi)+2*s11*xi)/(2*h)))*xi
,xi=infinity),j2[2,1]));
j3[2,1]:=simplify(solve(residue( simplify(ToCancelInf[1,1]*exp(
(xi^2*s12+2*s10*ln(xi)+2*s11*xi)/(2*h)))*xi^2
,xi=infinity),j3[2,1])):
```

$$j1_{2,1} := -Q^2\, s12 + (X1\, s12 - P - s11)\, Q + (P + s11)\, X1 - s10 \qquad \textbf{(2.21)}$$

$$j2_{2,1} := -s12\, Q^3 + \left(X1\, s12 - s12\, j1_{1,1} - P - s11\right) Q^2 + \left((X1\, s12 - P - s11)\, j1_{1,1} + (P + s11)\, X1 - s10\right) Q + ((P + s11)\, X1 - h - s10)\, j1_{1,1} + X1\, s10$$

```
> series(simplify(ToCancelInf[1,2]*exp((xi^2*s22+2*s20*ln(xi)+2*
s21*xi)/(2*h))),xi=infinity):
j1[1,2]:=simplify(solve(residue( simplify(ToCancelInf[1,2]*exp(
(xi^2*s22+2*s20*ln(xi)+2*s21*xi)/(2*h)))/xi
,xi=infinity),j1[1,2]));
j2[1,2]:=simplify(solve(residue( simplify(ToCancelInf[1,2]*exp(
(xi^2*s22+2*s20*ln(xi)+2*s21*xi)/(2*h)))
,xi=infinity),j2[1,2]));
j3[1,2]:=simplify(solve(residue( simplify(ToCancelInf[1,2]*exp(
(xi^2*s22+2*s20*ln(xi)+2*s21*xi)/(2*h)))*xi
,xi=infinity),j3[1,2])):
```

$$j1_{1,2} := \frac{1}{s12 - s22} \qquad \textbf{(2.22)}$$

$$j2_{1,2} := \frac{\left(j1_{2,2} - Q + X1\right) s12 + \left(Q - X1 - j1_{2,2}\right) s22 - s11 + s21}{(s12 - s22)^2}$$

```
> sX20:=-s10-s20-sX10:
series(simplify(ToCancelInf[2,1]*exp((xi^2*s12+2*s10*ln(xi)+2*
s11*xi)/(2*h))) -s12*(s10+s20+sX10+sX20),xi=infinity):
j1[1,1]:=simplify(solve(residue( simplify(ToCancelInf[2,1]*exp(
(xi^2*s12+2*s10*ln(xi)+2*s11*xi)/(2*h)))
,xi=infinity),j1[1,1]));
j2[1,1]:=simplify(solve(residue( simplify(ToCancelInf[2,1]*exp(
(xi^2*s12+2*s10*ln(xi)+2*s11*xi)/(2*h)))*xi
,xi=infinity),j2[1,1])):
```

$$j1_{1,1} := \frac{1}{(Q - X1)\,(-s12 + s22)\, h} \Big(-Q^4\, s12\, s22 + (2\, X1\, s12\, s22 + (-s12 - s22)\, P - s11\, s22 - s12\, s21)\, Q^3 + \big(-X1^2\, s12\, s22 + ((2\, s12 + 2\, s22)\, P + 2\, s11\, s22 + 2\, s12\, s21)\, X1 - P^2 + (-s11 - s21)\, P - s10\, s22 - s11\, s21 - s12\, s20 \big) Q^2 + \big(((-s12 - s22)\, P - s12\, s21 - s11\, s22)\, X1^2 + \big(2\, P^2 + (2\, s11 + 2\, s21)\, P + (-s10 + s20)\, s12 \qquad \textbf{(2.23)}$$

$$+ 2\, s10\, s22 + 2\, s11\, s21 \big)\, X1 + P\,(-s10 - s20) - s21\, s10 - s11\, s20 \big)\, Q + \big(s10\, s12 - P^2 + (-s11 - s21)\, P - s11\, s21 - s10\, s22 \big)\, X1^2 + \big((s10 + s20)\, P + s21\, s10 + s11\, s20 \big)\, X1 + sX10\,(sX10 + s10 + s20) \big)$$

```
> series(simplify(ToCancelInf[2,2]*exp((xi^2*s22+2*s20*ln(xi)+2*s21*xi)/(2*h))),xi=infinity):
j1[2,2]:=simplify(solve(residue( simplify(ToCancelInf[2,2]*exp((xi^2*s22+2*s20*ln(xi)+2*s21*xi)/(2*h)))*xi
,xi=infinity),j1[2,2]));
j2[2,2]:=simplify(solve(residue( simplify(ToCancelInf[2,2]*exp((xi^2*s22+2*s20*ln(xi)+2*s21*xi)/(2*h)))*xi^2
,xi=infinity),j2[2,2])):
```

$$j1_{2,2} := \frac{1}{(-s12 + s22)\,(Q - X1)\, h} \Big(Q^4\, s12\, s22 + \big(-2\, X1\, s12\, s22 + (s12 + s22)\, P + s11\, s22 + s12\, s21 \big)\, Q^3 + \big(X1^2\, s12\, s22 + \big((-2\, s12 - 2\, s22)\, P - 2\, s11\, s22 - 2\, s12\, s21 \big)\, X1 + P^2 + (s21 + s11)\, P + s10\, s22 + s11\, s21 + s12\, s20 \big)\, Q^2 + \Big(\big((s12 + s22)\, P + s11\, s22 + s12\, s21 \big)\, X1^2 + \big(-2\, P^2 + (-2\, s11 - 2\, s21)\, P + (-s10 + s20)\, s22 - 2\, s11\, s21 - 2\, s12\, s20 \big)\, X1 + (s10 + s20)\, P + s21\, s10 + s11\, s20 \Big)\, Q + \big(P^2 + (s21 + s11)\, P - s20\, s22 + s11\, s21 + s12\, s20 \big)\, X1^2 + \big(P\,(-s10 - s20) - s21\, s10 - s11\, s20 \big)\, X1 - sX10\,(sX10 + s10 + s20) \Big) \tag{2.24}$$

```
> omegaJMUPartInfinity:=simplify(omegaJMUPartInfinity):
```

```
> omegaJMUP4:=omegaJMUPartX1+omegaJMUPartInfinity:
```

```
> omegaJMUP4ds11:=simplify(residue(omegaJMUP4/ds11^2,ds11=0)):
omegaJMUP4ds21:=simplify(residue(omegaJMUP4/ds21^2,ds21=0)):
omegaJMUP4ds12:=simplify(residue(omegaJMUP4/ds12^2,ds12=0)):
omegaJMUP4ds22:=simplify(residue(omegaJMUP4/ds22^2,ds22=0)):
omegaJMUP4dX1:=simplify(residue(omegaJMUP4/dX1^2,dX1=0)):
```

```
> simplify(omegaJMUP4ds11-residue(Hams11/h,h=0));
simplify(omegaJMUP4ds21-residue(Hams21/h,h=0));
simplify(omegaJMUP4ds12-residue(Hams12/h,h=0));
simplify(omegaJMUP4ds22-residue(Hams22/h,h=0));
simplify(omegaJMUP4dX1 -residue(HamX1/h,h=0));
```

$$0 \tag{2.25}$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> omegaJMUP4:=simplify(omegaJMUP4):
omegaJMUP4formula:= ((2*(-(1/2)*(ds12-ds22)*(-s12+s22)*X1+dX1*
s12^2+(-2*dX1*s22+ds11-ds21)*s12+dX1*s22^2+(-ds11+ds21)*s22-
(1/2)*(-s21+s11)*(ds12-ds22)))*s22*s12*Q^4+(2*(-(1/2)*(ds12-
ds22)*(-s12+s22)*X1+dX1*s12^2+(-2*dX1*s22+ds11-ds21)*s12+dX1*
s22^2+(-ds11+ds21)*s22-(1/2)*(-s21+s11)*(ds12-ds22)))*(-2*X1*
s12*s22+(s21+P)*s12+s22*(P+s11))*Q^3+(2*(-(1/2)*(ds12-ds22)*(-
```

```
s12+s22)*X1+dX1*s12^2+(-2*dX1*s22+ds11-ds21)*s12+dX1*s22^2+(-
ds11+ds21)*s22-(1/2)*(-s21+s11)*(ds12-ds22)))*(X1^2*s12*s22+(
(-2*P-2*s21)*s12-2*s22*(P+s11))*X1+s12*s20+s10*s22+(P+s11)*
(s21+P))*Q^2+(-(-s12+s22)*((s21+P)*s12+s22*(P+s11))*(ds12-ds22)
*X1^3+(2*dX1*(s21+P)*s12^3+(-2*dX1*(P-s11+2*s21)*s22+(2*ds11-2*
ds21)*P+(2*ds11-2*ds21)*s21+(s10-s20)*ds12+2*ds22*s20)*s12^2+
(-2*dX1*(P+2*s11-s21)*s22^2+((2*ds11-2*ds21)*s11+(-2*ds11+2*
ds21)*s21+(-3*s10+s20)*ds12+ds22*(s10-3*s20))*s22-(2*(P+3*s11*
(1/2)-(1/2)*s21))*(ds12-ds22)*(s21+P))*s12+2*s22*(dX1*(P+s11)*
s22^2+((-ds11+ds21)*P+(-ds11+ds21)*s11+ds12*s10-(1/2)*ds22*(s10
-s20))*s22+(P-(1/2)*s11+3*s21*(1/2))*(ds12-ds22)*(P+s11)))*
X1^2+(2*dX1*(s10-s20)*s12^3+(-6*dX1*(s10-s20)*s22-4*P^2*dX1-4*
dX1*(s21+s11)*P-4*dX1*s11*s21+2*ds11*s10-2*s20*(ds11-2*ds21))*
s12^2+(6*dX1*(s10-s20)*s22^2+(8*P^2*dX1+8*dX1*(s21+s11)*P+8*
dX1*s11*s21+(-6*ds11+2*ds21)*s10+2*s20*(ds11-3*ds21))*s22+(-4*
ds11+4*ds21)*P^2+((-4*ds11+4*ds21)*s11+(-4*ds11+4*ds21)*s21+
(s10+s20)*(ds12-ds22))*P+((-4*ds11+4*ds21)*s21+2*s20*(ds12-
ds22))*s11+s21*(s10-s20)*(ds12-ds22))*s12-2*dX1*(s10-s20)*
s22^3+(-4*P^2*dX1-4*dX1*(s21+s11)*P-4*dX1*s11*s21+(4*ds11-2*
ds21)*s10+2*ds21*s20)*s22^2+((4*ds11-4*ds21)*P^2+((4*ds11-4*
ds21)*s11+(4*ds11-4*ds21)*s21-(s10+s20)*(ds12-ds22))*P+((4*ds11
-4*ds21)*s21+(s10-s20)*(ds12-ds22))*s11-2*s21*s10*(ds12-ds22))*
s22+(2*(-s21+s11))*(s21+P)*(P+s11)*(ds12-ds22))*X1+2*dX1*(s10+
s20)*(P+s11+s21)*s12^2+(-4*dX1*(s10+s20)*(P+s11+s21)*s22+(2*
(s10+s20))*(ds11-ds21)*P+2*s20*(ds11-ds21)*s11+2*s10*(ds11-
ds21)*s21+(sX10+s20)*(sX10+s10)*(ds12-ds22))*s12+2*dX1*(s10+
s20)*(P+s11+s21)*s22^2+(-(2*(s10+s20))*(ds11-ds21)*P-2*s20*
(ds11-ds21)*s11-2*s10*(ds11-ds21)*s21-(sX10+s20)*(sX10+s10)*
(ds12-ds22))*s22-(ds12-ds22)*((s10+s20)*P+s21*s10+s11*s20)*(-
s21+s11))*Q-(-s12+s22)*((-ds12*s10-ds22*s20)*s12+(ds12*s10+
ds22*s20)*s22+(s21+P)*(P+s11)*(ds12-ds22))*X1^3+(-2*dX1*s10*
s12^3+((4*(s10-(1/2)*s20))*dX1*s22+2*P^2*dX1+2*dX1*(s21+s11)*
P+2*dX1*s11*s21-2*ds11*s10-2*ds21*s20)*s12^2+(-2*dX1*(s10-2*
s20)*s22^2+(-4*P^2*dX1-4*dX1*(s21+s11)*P-4*dX1*s11*s21+4*ds11*
s10+4*ds21*s20)*s22+(2*ds11-2*ds21)*P^2+((2*ds11-2*ds21)*s11+
(2*ds11-2*ds21)*s21-(s10+s20)*(ds12-ds22))*P+((2*ds11-2*ds21)*
s21-s20*(ds12-ds22))*s11-s21*s10*(ds12-ds22))*s12-2*s22^3*dX1*
s20+(2*P^2*dX1+2*dX1*(s21+s11)*P+2*dX1*s11*s21-2*ds11*s10-2*
ds21*s20)*s22^2+((-2*ds11+2*ds21)*P^2+((-2*ds11+2*ds21)*s11+
(-2*ds11+2*ds21)*s21+(s10+s20)*(ds12-ds22))*P+((-2*ds11+2*ds21)
*s21+s20*(ds12-ds22))*s11+s21*s10*(ds12-ds22))*s22-(-s21+s11)*
(s21+P)*(P+s11)*(ds12-ds22))*X1^2+(-2*dX1*(s10+s20)*(P+s11+s21)
```

```
*s12^2+(4*dX1*(s10+s20)*(P+s11+s21)*s22-(2*(s10+s20))*(ds11-
ds21)*P-2*s20*(ds11-ds21)*s11-2*s10*(ds11-ds21)*s21-(2*((sX10+
(1/2)*s20)*s10+sX10*(sX10+s20)))*(ds12-ds22))*s12-2*dX1*(s10+
s20)*(P+s11+s21)*s22^2+((2*(s10+s20))*(ds11-ds21)*P+2*s20*(ds11
-ds21)*s11+2*s10*(ds11-ds21)*s21+(2*((sX10+(1/2)*s20)*s10+sX10*
(sX10+s20)))*(ds12-ds22))*s22+(ds12-ds22)*((s10+s20)*P+s21*s10+
s11*s20)*(-s21+s11))*X1-2*sX10*(sX10+s10+s20)*(dX1*s12^2+(-2*
dX1*s22+ds11-ds21)*s12+dX1*s22^2+(-ds11+ds21)*s22-(1/2)*(-s21+
s11)*(ds12-ds22)))/((2*(Q-X1))*(-s12+s22)^2):
simplify(omegaJMUP4-omegaJMUP4formula);
K0:=unapply(KOldCoordinates(s11,s21,s12,s22,X1),s11,s21,s12,
s22,X1);
dK0:=unapply( diff(K0(s11,s21,s12,s22,X1),s11)*ds11+ diff(K0
(s11,s21,s12,s22,X1),s21)*ds21+diff(K0(s11,s21,s12,s22,X1),s12)
*ds12+diff(K0(s11,s21,s12,s22,X1),s22)*ds22+diff(K0(s11,s21,
s12,s22,X1),X1)*dX1,s11,s21,s12,s22,X1);
simplify(omegaJMUP4-(residue(Hams11/h,h=0)*ds11+residue
(Hams21/h,h=0)*ds21 +residue(Hams12/h,h=0)*ds12+residue
(Hams22/h,h=0)*ds22+residue(HamX1/h,h=0)*dX1));
```

$$0 \tag{2.26}$$

$$K0 := (s11, s21, s12, s22, X1) \rightarrow \frac{1}{2}\,(sX10+s20)\,(sX10+s10)\ln\left(\frac{1}{2}\,s12-\frac{1}{2}\,s22\right) + \frac{1}{2}\,X1\,((s12\,X1+2\,s11)\,s10+s20\,(s22\,X1+2\,s21))$$

$$dK0 := (s11, s21, s12, s22, X1) \rightarrow X1\,s10\,ds11 + X1\,s20\,ds21 + \left(\frac{1}{4}\,\frac{(sX10+s20)\,(sX10+s10)}{\frac{1}{2}\,s12-\frac{1}{2}\,s22}+\frac{1}{2}\,X1^2\,s10\right) ds12 + \left(-\frac{1}{4}\,\frac{(sX10+s20)\,(sX10+s10)}{\frac{1}{2}\,s12-\frac{1}{2}\,s22}+\frac{1}{2}\,X1^2\,s20\right) ds22 + \left(\frac{1}{2}\,(s12\,X1+2\,s11)\,s10 + \frac{1}{2}\,s20\,(s22\,X1+2\,s21)+\frac{1}{2}\,X1\,(s10\,s12+s20\,s22)\right) dX1$$

$$0$$

Definition of the shifted Darboux coordinates

```
> checkQfunction:=S2*Q+S1;
checkPfunction=1/S2*(P-1/2*R1(Q));
SolQ:=-(S1-checkQ)/S2;
SolP:=checkP*S2+(1/2)*R1(-(S1-checkQ)/S2);
simplify(checkQ-(S2*SolQ+S1));
simplify(checkP-(1/S2*(SolP-1/2*R1(SolQ))));
```

$$checkQfunction := Q\,S2+S1 \tag{2.27}$$

$$checkPfunction = \frac{P - \frac{1}{2}\,\frac{-s10-s20}{Q-X1} + \frac{1}{2}\,s11 + \frac{1}{2}\,s21 - \frac{1}{2}\,(-s12-s22)\,Q}{S2}$$

$$SolQ := -\frac{S1 - checkQ}{S2}$$

$$SolP := checkP\,S2 + \frac{1}{2}\,\frac{-s10-s20}{-\frac{S1-checkQ}{S2} - X1} - \frac{1}{2}\,s11 - \frac{1}{2}\,s21 - \frac{1}{2}\,\frac{(-s12-s22)\,(S1-checkQ)}{S2}$$

$$0$$

$$0$$

```
> Sinfty2function:=s12+s22;
  Sinfty1function:=s11+s21;
  S2function:=sqrt(s12-s22)/sqrt(2);
  S1function:=(s11-s21)/sqrt(2)/sqrt(s12-s22);
  tdX1function:=X1*S2function+S1function;
  solve( {s12+s22=Sinfty2,s11+s21=Sinfty1, S2=sqrt(s12-s22)/sqrt
  (2),
  S1=(s11-s21)/sqrt(2)/sqrt(s12-s22),
  tdX1=X1*S2function+S1function},{s12,s22,s11,s21,X1});
  X1function := unapply( -(S1-tdX1)/S2 ,Sinfty1,Sinfty2,S1,S2,
  tdX1);
  s11function:= unapply( S2*S1+(1/2)*Sinfty1, Sinfty1,Sinfty2,S1,
  S2,tdX1);
  s12function:= unapply( S2^2+(1/2)*Sinfty2, Sinfty1,Sinfty2,S1,
  S2,tdX1);
  s21function:= unapply( -S2*S1+(1/2)*Sinfty1, Sinfty1,Sinfty2,
  S1,S2,tdX1);
  s22function:= unapply( -S2^2+(1/2)*Sinfty2, Sinfty1,Sinfty2,S1,
  S2,tdX1 );
  simplify(X1function(Sinfty1function,Sinfty2function,S1function,
  S2function,tdX1function));
  simplify(s11function(Sinfty1function,Sinfty2function,
  S1function,S2function,tdX1function));
  simplify(s12function(Sinfty1function,Sinfty2function,
  S1function,S2function,tdX1function));
  simplify(s21function(Sinfty1function,Sinfty2function,
  S1function,S2function,tdX1function));
  simplify(s22function(Sinfty1function,Sinfty2function,
  S1function,S2function,tdX1function));

  partialtdX1function:=simplify(diff(s11function(Sinfty1,Sinfty2,
  S1,S2,tdX1),tdX1))*partials11+
```

```
simplify(diff(s21function(Sinfty1,Sinfty2,S1,S2,tdX1),tdX1))
*partials21
+simplify(diff(s12function(Sinfty1,Sinfty2,S1,S2,tdX1),tdX1))
*partials12
+simplify(diff(s22function(Sinfty1,Sinfty2,S1,S2,tdX1),tdX1))
*partials22
+simplify(diff(X1function(Sinfty1,Sinfty2,S1,S2,tdX1),tdX1))*
partialX1;
```

$$Sinfty2function := s12 + s22 \tag{2.28}$$

$$Sinfty1function := s21 + s11$$

$$S2function := \frac{1}{2}\sqrt{s12 - s22}\sqrt{2}$$

$$S1function := \frac{1}{2}\frac{(-s21 + s11)\sqrt{2}}{\sqrt{s12 - s22}}$$

$$tdX1function := \frac{1}{2}X1\sqrt{s12 - s22}\sqrt{2} + \frac{1}{2}\frac{(-s21 + s11)\sqrt{2}}{\sqrt{s12 - s22}}$$

$$\left\{X1 = -\frac{S1 - tdX1}{S2}, s11 = \frac{1}{2}Sinfty1 + S1\,S2, s12 = \frac{1}{2}Sinfty2 + S2^2, s21 = -S1\,S2 + \frac{1}{2}Sinfty1, s22 = -S2^2 + \frac{1}{2}Sinfty2\right\}$$

$$X1function := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow -\frac{S1 - tdX1}{S2}$$

$$s11function := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow \frac{1}{2}Sinfty1 + S1\,S2$$

$$s12function := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow \frac{1}{2}Sinfty2 + S2^2$$

$$s21function := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow -S1\,S2 + \frac{1}{2}Sinfty1$$

$$s22function := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow -S2^2 + \frac{1}{2}Sinfty2$$

$$X1$$
$$s11$$
$$s12$$
$$s21$$
$$s22$$

$$partialtdX1function := \frac{partialX1}{S2}$$

```
> ds11function:=diff(s11function(Sinfty1,Sinfty2,S1,S2,tdX1),
Sinfty1)*dSinfty1+ diff(s11function(Sinfty1,Sinfty2,S1,S2,
tdX1),Sinfty2)*dSinfty2+diff(s11function(Sinfty1,Sinfty2,S1,S2,
tdX1),S1)*dS1+ diff(s11function(Sinfty1,Sinfty2,S1,S2,tdX1),S2)
*dS2 +diff(s11function(Sinfty1,Sinfty2,S1,S2,tdX1),tdX1)*dtdX1;
ds21function:=diff(s21function( Sinfty1,Sinfty2,S1,S2,tdX1),
Sinfty1)*dSinfty1+ diff(s21function( Sinfty1,Sinfty2,S1,S2,
tdX1),Sinfty2)*dSinfty2+diff(s21function( Sinfty1,Sinfty2,S1,
```

```
S2,tdX1),S1)*dS1+ diff(s21function( Sinfty1,Sinfty2,S1,S2,
tdX1),S2)*dS2 +diff(s21function( Sinfty1,Sinfty2,S1,S2,tdX1),
tdX1)*dtdX1;
ds12function:=diff(s12function( Sinfty1,Sinfty2,S1,S2,tdX1),
Sinfty1)*dSinfty1+ diff(s12function( Sinfty1,Sinfty2,S1,S2,
tdX1),Sinfty2)*dSinfty2+diff(s12function( Sinfty1,Sinfty2,S1,
S2,tdX1),S1)*dS1+ diff(s12function( Sinfty1,Sinfty2,S1,S2,
tdX1),S2)*dS2 +diff(s12function( Sinfty1,Sinfty2,S1,S2,tdX1),
tdX1)*dtdX1;
ds22function:=diff(s22function( Sinfty1,Sinfty2,S1,S2,tdX1),
Sinfty1)*dSinfty1+ diff(s22function( Sinfty1,Sinfty2,S1,S2,
tdX1),Sinfty2)*dSinfty2+diff(s22function( Sinfty1,Sinfty2,S1,
S2,tdX1),S1)*dS1+ diff(s22function( Sinfty1,Sinfty2,S1,S2,
tdX1),S2)*dS2 +diff(s22function( Sinfty1,Sinfty2,S1,S2,tdX1),
tdX1)*dtdX1;
dX1function:=diff(X1function( Sinfty1,Sinfty2,S1,S2,tdX1),
Sinfty1)*dSinfty1+ diff(X1function( Sinfty1,Sinfty2,S1,S2,
tdX1),Sinfty2)*dSinfty2+diff(X1function( Sinfty1,Sinfty2,S1,S2,
tdX1),S1)*dS1+ diff(X1function( Sinfty1,Sinfty2,S1,S2,tdX1),S2)
*dS2 +diff(X1function( Sinfty1,Sinfty2,S1,S2,tdX1),tdX1)*dtdX1;

solve({ds11function=ds11,ds21function=ds21,ds12function=
ds12,ds22function=ds22,dX1function=dX1,ds11function=ds11},{dS1,
dS2,dSinfty1,dSinfty2,dtdX1});

dS1function := ((2*ds11-2*ds21)*S2-S1*(ds12-ds22))/(4*S2^2);
dS2function := (ds12-ds22)/(4*S2);
dtdX1function :=(4*S2^3*dX1+(2*ds11-2*ds21)*S2-(2*(S1-(1/2)*
tdX1))*(ds12-ds22))/(4*S2^2);
dSinfty1function := ds11+ds21;
dSinfty2function := ds12+ds22;
```

$$ds11function := \frac{1}{2}\,dSinfty1 + S2\,dS1 + S1\,dS2 \tag{2.29}$$

$$ds21function := \frac{1}{2}\,dSinfty1 - S2\,dS1 - S1\,dS2$$

$$ds12function := \frac{1}{2}\,dSinfty2 + 2\,S2\,dS2$$

$$ds22function := \frac{1}{2}\,dSinfty2 - 2\,S2\,dS2$$

$$dX1function := -\frac{dS1}{S2} + \frac{(S1 - tdX1)\,dS2}{S2^2} + \frac{dtdX1}{S2}$$

$$\left\{ dS1 = -\frac{1}{4}\,\frac{S1\,ds12 - S1\,ds22 - 2\,S2\,ds11 + 2\,S2\,ds21}{S2^2},\ dS2 = \frac{1}{4}\,\frac{ds12 - ds22}{S2},\ dtdX1 = \right.$$

$$- \frac{1}{4} \frac{1}{S2^2} \left(-4\, S2^3\, dX1 + 2\, S1\, ds12 - 2\, S1\, ds22 - 2\, S2\, ds11 + 2\, S2\, ds21 - ds12\, tdX1 + ds22\, tdX1 \right),\ dSinfty1 = ds11 + ds21,\ dSinfty2 = ds12 + ds22 \Big\}$$

$$dS1function := \frac{1}{4} \frac{(2\, ds11 - 2\, ds21)\, S2 - S1\, (ds12 - ds22)}{S2^2}$$

$$dS2function := \frac{1}{4} \frac{ds12 - ds22}{S2}$$

$$dtdX1function := \frac{1}{4} \frac{1}{S2^2} \left(4\, S2^3\, dX1 + (2\, ds11 - 2\, ds21)\, S2 - 2 \left(S1 - \frac{1}{2}\, tdX1 \right) (ds12 - ds22) \right)$$

$$dSinfty1function := ds11 + ds21$$

$$dSinfty2function := ds12 + ds22$$

Rewriting the JMU differential in terms of (\check{Q},\check{P},S_1,S_2,S_{\infty,1},S_{\infty,2},\td{X}_1)

```
> omegaJMUP4function1:=unapply(omegaJMUP4,Q,P):
  omegaJMUP4function2:=unapply(simplify
  (omegaJMUP4function1(SolQ,SolP)),s11,s21,s12,s22,X1,ds11,ds21,
  ds12,ds22,dX1):
  omegaJMUP4function3:=simplify( omegaJMUP4function2(s11function
  (Sinfty1,Sinfty2,S1,S2,tdX1),s21function(Sinfty1,Sinfty2,S1,S2,
  tdX1),s12function(Sinfty1,Sinfty2,S1,S2,tdX1),s22function
  (Sinfty1,Sinfty2,S1,S2,tdX1),X1function(Sinfty1,Sinfty2,S1,S2,
  tdX1),
  ds11function,ds21function,ds12function,ds22function,dX1function)
   ):
```

```
> omegaJMUP4NewCoordinates:=simplify(omegaJMUP4function3):
  CoeffdS1:=simplify(residue(omegaJMUP4NewCoordinates/dS1^2,dS1=
  0));
  CoeffdS2:=simplify(residue(omegaJMUP4NewCoordinates/dS2^2,dS2=
  0));
  CoeffdSinfty1:=simplify(residue
  (omegaJMUP4NewCoordinates/dSinfty1^2,dSinfty1=0));
  CoeffdSinfty2:=simplify(residue
  (omegaJMUP4NewCoordinates/dSinfty2^2,dSinfty2=0));
  CoeffdtdX1:=simplify(residue(omegaJMUP4NewCoordinates/dtdX1^2,
  dtdX1=0));
```

$$CoeffdS1 := \frac{1}{2} \frac{1}{S2^2} \left(-2\, S1\, (s10 - s20)\, S2^2 - Sinfty1\, (s10 + s20)\, S2 + Sinfty2\, (s10 + s20)\, (S1 - tdX1) \right) \tag{2.30}$$

$$CoeffdS2 := \frac{1}{2} \frac{1}{S2^3} \left(2\, (sX10 + s20)\, (sX10 + s10)\, S2^2 + Sinfty1\, (s10 + s20)\, (S1 - tdX1)\, S2 - Sinfty2\, (S1 - tdX1)^2\, (s10 + s20) \right)$$

$$CoeffdSinfty1 := -\frac{1}{2}\frac{(s10+s20)(S1-tdX1)}{S2}$$

$$CoeffdSinfty2 := \frac{1}{4}\frac{(S1-tdX1)^2(s10+s20)}{S2^2}$$

$$CoeffdtdX1 := \frac{1}{4}\frac{1}{S2^2(tdX1-checkQ)}\Big(\Big(\big(-4\,checkP^2+4\,checkQ^2+4\,s10-4\,s20\big)\,tdX1^2-8\,checkQ\,\big(-checkP^2+checkQ^2+s10-s20\big)\,tdX1+4\,checkQ^4+\big(-4\,checkP^2+4\,s10-4\,s20\big)\,checkQ^2+(s10+s20+2\,sX10)^2\Big)\,S2^2+2\,Sinfty1\,(tdX1-checkQ)\,(s10+s20)\,S2-2\,Sinfty2\,(tdX1-checkQ)\,(s10+s20)\,(S1-tdX1)\Big)$$

```
> CoeffdtdX1bis:=(checkQ-tdX1)*checkP^2-checkQ^3+tdX1*checkQ^2-
  (s10-s20)*checkQ
  +(-(sX10-sX20)^2)/4/(checkQ-tdX1):
  PartConstantCoefftdX1bis:=simplify(CoeffdtdX1-CoeffdtdX1bis);
```

$$PartConstantCoefftdX1bis := \frac{1}{2}\frac{1}{S2^2}\big(2\,tdX1\,(s10-s20)\,S2^2+Sinfty1\,(s10+s20)\,S2-Sinfty2\,(s10+s20)\,(S1-tdX1)\big) \qquad \textbf{(2.31)}$$

```
> pdsolve({CoeffdS1=diff(MM(Sinfty1,Sinfty2,S1,S2,tdX1),S1),
  CoeffdS2=diff(MM(Sinfty1,Sinfty2,S1,S2,tdX1),S2),
  CoeffdSinfty1=diff(MM(Sinfty1,Sinfty2,S1,S2,tdX1),Sinfty1),
  CoeffdSinfty2=diff(MM(Sinfty1,Sinfty2,S1,S2,tdX1),Sinfty2),
  PartConstantCoefftdX1bis=diff(MM(Sinfty1,Sinfty2,S1,S2,tdX1),
  tdX1)
  },{MM(Sinfty1,Sinfty2,S1,S2,tdX1)});

  M:=unapply( (4*S2^2*(sX10+s20)*(sX10+s10)*ln(S2)+((-2*S1^2+2*
  tdX1^2)*s10+(2*S1^2-2*tdX1^2)*s20)*S2^2-2*Sinfty1*(s10+s20)*(S1
  -tdX1)*S2+Sinfty2*(S1-tdX1)^2*(s10+s20))/4/S2^2,Sinfty1,
  Sinfty2,S1,S2,tdX1);
  simplify( CoeffdS1-diff(M(Sinfty1,Sinfty2,S1,S2,tdX1),S1));
  simplify( CoeffdS2-diff(M(Sinfty1,Sinfty2,S1,S2,tdX1),S2));
  simplify( CoeffdSinfty1-diff(M(Sinfty1,Sinfty2,S1,S2,tdX1),
  Sinfty1));
  simplify( CoeffdSinfty2-diff(M(Sinfty1,Sinfty2,S1,S2,tdX1),
  Sinfty2));
  simplify( PartConstantCoefftdX1bis-diff(M(Sinfty1,Sinfty2,S1,
  S2,tdX1),tdX1));
```

$$\Big\{MM(Sinfty1, Sinfty2, S1, S2, tdX1) = \frac{1}{4}\frac{1}{S2^2}\Big(4\,S2^2\,(sX10+s20)\,(sX10+s10)\ln(S2)+\big(\big(-2\,S1^2+2\,tdX1^2\big)\,s10+\big(2\,S1^2-2\,tdX1^2\big)\,s20+4\,\_C1\big)\,S2^2-2\,Sinfty1\,(s10+s20)\,(S1-tdX1)\,S2+Sinfty2\,(S1-tdX1)^2\,(s10+s20)\Big)\Big\} \qquad \textbf{(2.32)}$$

$$M := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow \frac{1}{4}\frac{1}{S2^2}\big(4\,S2^2\,(sX10+s20)\,(sX10+s10)\ln(S2)$$

$+\left(\left(-2\,S1^2+2\,tdX1^2\right)s10+\left(2\,S1^2-2\,tdX1^2\right)s20\right)S2^2-2\,Sinfty1\,(s10+s20)\,(S1-tdX1)\,S2+Sinfty2\,(S1-tdX1)^2\,(s10+s20)\Big)$

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

```
> KNewCoordinates:=unapply( simplify(KOldCoordinates(s11function
  (Sinfty1,Sinfty2,S1,S2,tdX1),s21function(Sinfty1,Sinfty2,S1,S2,
  tdX1),s12function(Sinfty1,Sinfty2,S1,S2,tdX1),s22function
  (Sinfty1,Sinfty2,S1,S2,tdX1),X1function(Sinfty1,Sinfty2,S1,S2,
  tdX1)),symbolic) ,Sinfty1,Sinfty2,S1,S2,tdX1);
  simplify(KNewCoordinates(Sinfty1,Sinfty2,S1,S2,tdX1)-M(Sinfty1,
  Sinfty2,S1,S2,tdX1));
  KNewCoordinates2:=unapply((sX10+s20)*(sX10+s10)*ln(S2)-(1/2)*
  (S1-tdX1)*(s10+s20)*Sinfty1/S2
  +(1/4)*(S1-tdX1)^2*(s10+s20)*Sinfty2/S2^2-(1/2)*(S1-tdX1)*(s10-
  s20)*(S1+tdX1)
  ,Sinfty1,Sinfty2,S1,S2,tdX1);
  simplify(KNewCoordinates(Sinfty1,Sinfty2,S1,S2,tdX1)
  -KNewCoordinates2(Sinfty1,Sinfty2,S1,S2,tdX1),symbolic);
```

$$KNewCoordinates := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow \frac{1}{4}\,\frac{1}{S2^2}\Bigg(4\,S2^2\,(sX10+s20)\,(sX10+s10)\,\ln(S2)-2\,(S1-tdX1)\left((s10-s20)\,(S1+tdX1)\,S2^2+Sinfty1\,(s10+s20)\,S2-\frac{1}{2}\,Sinfty2\,(s10+s20)\,(S1-tdX1)\right)\Bigg) \quad \textbf{(2.33)}$$

$$0$$

$$KNewCoordinates2 := (Sinfty1, Sinfty2, S1, S2, tdX1) \rightarrow (sX10+s20)\,(sX10+s10)\,\ln(S2)-\frac{1}{2}\,\frac{(s10+s20)\,(S1-tdX1)\,Sinfty1}{S2}+\frac{1}{4}\,\frac{(S1-tdX1)^2\,(s10+s20)\,Sinfty2}{S2^2}-\frac{1}{2}\,(S1-tdX1)\,(s10-s20)\,(S1+tdX1)$$

$$0$$

Obtaining the only non-trivial Hamiltonians for (checkQ,checkP)

```
> assume(S2>0):
  hpartialtdX1functioncheckQfunction:= unapply(diff(S2*Q+S1,Q)*
  hpartialtdX1functionQ+diff(S2*Q+S1,P)*hpartialtdX1functionP
  +h/S2*(diff(S2function*Q+S1function,X1)),Q,P):
  hpartialtdX1functioncheckQfunction2:=unapply( simplify
  (hpartialtdX1functioncheckQfunction(SolQ,SolP)),s12,s22,s11,
  s21,X1):
  hpartialtdX1functioncheckQ:=simplify
  (hpartialtdX1functioncheckQfunction2(s12function(Sinfty1,
```

```
Sinfty2,S1,S2,tdX1),s22function(Sinfty1,Sinfty2,S1,S2,
tdX1),s11function(Sinfty1,Sinfty2,S1,S2,tdX1),s21function
(Sinfty1,Sinfty2,S1,S2,tdX1),X1function(Sinfty1,Sinfty2,S1,S2,
tdX1))):

hpartialtdX1functioncheckPfunction:= unapply(diff(1/S2*(P-1/2*
R1(Q)),Q)*hpartialtdX1functionQ+diff(1/S2*(P-1/2*R1(Q)),P)*
hpartialtdX1functionP
+h/S2*(diff(1/S2function*(P-1/2*R1(Q)),X1)),Q,P):
hpartialtdX1functioncheckPfunction2:=unapply( simplify
(hpartialtdX1functioncheckPfunction(SolQ,SolP)),s12,s22,s11,
s21,X1):
hpartialtdX1functioncheckP:=simplify
(hpartialtdX1functioncheckPfunction2(s12function(Sinfty1,
Sinfty2,S1,S2,tdX1),s22function(Sinfty1,Sinfty2,S1,S2,tdX1),
s11function(Sinfty1,Sinfty2,S1,S2,tdX1),s21function(Sinfty1,
Sinfty2,S1,S2,tdX1),X1function(Sinfty1,Sinfty2,S1,S2,tdX1))
+1/(4*S2^2*(tdX1-checkQ)^2)*2*Sinfty2*(tdX1-checkQ)^2*(s10+s20+
sX10+sX20)
):
HamtdX1checkQcheckP:=unapply(((checkQ-tdX1)*checkP+h)*checkP
+(4*checkQ^4-8*tdX1*checkQ^3+(4*tdX1^2-4*h+4*s10-4*s20)*
checkQ^2+4*tdX1*(h+s20-s10)*checkQ+sX10^2+(-2*sX20)*sX10-2*
sX20*(-(1/2)*sX20))/4/(tdX1-checkQ)
,checkQ,checkP):
simplify(hpartialtdX1functioncheckQ-diff(HamtdX1checkQcheckP
(checkQ,checkP),checkP));
simplify(series(simplify(hpartialtdX1functioncheckP+diff
(HamtdX1checkQcheckP(checkQ,checkP),checkQ)),h=0));

HamtdX1checkQcheckPbis:=unapply((checkQ-tdX1)*checkP^2+h*
checkP-checkQ^3+tdX1*checkQ^2-(s10-s20-h)*checkQ
+(-(sX10-sX20)^2)/4/(checkQ-tdX1)   ,checkQ,checkP);
simplify(HamtdX1checkQcheckP(checkQ,checkP)
-HamtdX1checkQcheckPbis(checkQ,checkP));
```

$$0 \qquad \textbf{(2.34)}$$

$$0$$

$$HamtdX1checkQcheckPbis := (checkQ, checkP) \rightarrow (-tdX1 + checkQ)\, checkP^2 + h\, checkP - checkQ^3 + tdX1\, checkQ^2 - (-h + s10 - s20)\, checkQ - \frac{1}{4}\,\frac{(s10 + s20 + 2\, sX10)^2}{-tdX1 + checkQ}$$

$$0$$

```
> K0NewCoordinates:=unapply(KNewCoordinates(Sinfty1, Sinfty2, S1,
S2, tdX1),Sinfty1, Sinfty2, S1, S2, tdX1): dK0NewCoordinates:=
```

```
unapply( diff(K0NewCoordinates(Sinfty1, Sinfty2, S1, S2, tdX1),
Sinfty1)*dSinfty1+ diff(K0NewCoordinates(Sinfty1, Sinfty2, S1,
S2, tdX1),Sinfty2)*dSinfty2+diff(K0NewCoordinates(Sinfty1,
Sinfty2, S1, S2, tdX1),S1)*dS1
+diff(K0NewCoordinates(Sinfty1, Sinfty2, S1, S2, tdX1),S2)*
dS2+diff(K0NewCoordinates(Sinfty1, Sinfty2, S1, S2, tdX1),tdX1)
*dtdX1
,Sinfty1, Sinfty2, S1, S2, tdX1):
simplify(omegaJMUP4NewCoordinates-(residue
(HamtdX1checkQcheckPbis(checkQ,checkP)/h,h=0)*dtdX1
+dK0NewCoordinates(Sinfty1, Sinfty2, S1, S2, tdX1)));
```

$$0 \tag{2.35}$$

```
> tdL:=tdL:
tdA:=simplify(tdA):
```

General expressions for reduction

```
> c0:=0:
tdL11function:=unapply(tdL[1,1],s12,s22,s11,s21,X1):
tdL12function:=unapply(tdL[1,2],s12,s22,s11,s21,X1):
tdL21function:=unapply(tdL[2,1],s12,s22,s11,s21,X1):
tdL22function:=unapply(tdL[2,2],s12,s22,s11,s21,X1):

tdA11function:=unapply(tdA[1,1],s12,s22,s11,s21,X1,beta12,
beta22,beta11,beta21,betaX1):
tdA12function:=unapply(tdA[1,2],s12,s22,s11,s21,X1,beta12,
beta22,beta11,beta21,betaX1):
tdA21function:=unapply(tdA[2,1],s12,s22,s11,s21,X1,beta12,
beta22,beta11,beta21,betaX1):
tdA22function:=unapply(tdA[2,2],s12,s22,s11,s21,X1,beta12,
beta22,beta11,beta21,betaX1):
SolQfunction:=unapply(simplify(SolQ), s12,s22,s11,s21,X1);
SolPfunction:=unapply(simplify(SolP), s12,s22,s11,s21,X1);
tdX1functionfunction:=unapply(tdX1function,s12,s22,s11,s21,X1);

S1function:=unapply(S1,s12,s22,s11,s21,X1);
S2function:=unapply(S2,s12,s22,s11,s21,X1);
R1function:=unapply(R1(xi),xi,s12,s22,s11,s21,X1);

LQfunction:=unapply(LQ,s12,s22,s11,s21,X1,beta12,beta22,beta11,
beta21,betaX1):
LPfunction:=unapply(LP,s12,s22,s11,s21,X1,beta12,beta22,beta11,
```

```
beta21,betaX1):
Hamiltonianfunction:=unapply(Hamiltonianbis,s12,s22,s11,s21,X1,
beta12,beta22,beta11,beta21,betaX1):
```

$$SolQfunction := (s12, s22, s11, s21, X1) \rightarrow \frac{-S1 + checkQ}{S2\sim} \tag{3.1}$$

$$SolPfunction := (s12, s22, s11, s21, X1) \rightarrow \frac{1}{2} \frac{1}{(S2\sim X1 + S1 - checkQ)\, S2\sim} \big(2\, S2\sim^3 X1\, checkP + (-2\, checkP\, checkQ + 2\, S1\, checkP + (-s11 - s21)\, X1 + s10 + s20)\, S2\sim^2 + ((s22 + s12)\, X1 - s11 - s21)\, (S1 - checkQ)\, S2\sim + (S1 - checkQ)^2\, (s22 + s12)\big)$$

$$tdX1functionfunction := (s12, s22, s11, s21, X1) \rightarrow \frac{1}{2} X1 \sqrt{s12 - s22}\, \sqrt{2} + \frac{1}{2} \frac{(s11 - s21)\sqrt{2}}{\sqrt{s12 - s22}}$$

$$S1function := (s12, s22, s11, s21, X1) \rightarrow S1$$

$$S2function := (s12, s22, s11, s21, X1) \rightarrow S2\sim$$

$$R1function := (\xi, s12, s22, s11, s21, X1) \rightarrow \frac{-s10 - s20}{\xi - X1} - s11 - s21 + (-s12 - s22)\, \xi$$

Jimbo-Miwa case: We take s12=1 and s22=-1 and X_1=0 and s21=-s11. The time is sigma=s11. The direction is beta21=-1 and beta11=1.

```
> simplify(SolQfunction(1,-1,sigma,-sigma,0));
  simplify(series(SolPfunction(1,-1,sigma,-sigma,0),checkP));
  tdX1functionfunction(1,-1,sigma,-sigma,0);
  S1function(1,-1,sigma,-sigma,0);
  S2function(1,-1,sigma,-sigma,0);
  R1function(xi,1,-1,sigma,-sigma,0);
  solve({SolQfunction(1,-1,sigma,-sigma,0)=QQ,SolPfunction(1,
  -1,sigma,-sigma,0)=PP},{checkQ,checkP});
  tdLReduced:=Matrix(2,2,0):
  tdL11function0:=unapply( tdL11function(1,-1,sigma,-sigma,0),Q,
  P):
  tdL12function0:=unapply( tdL12function(1,-1,sigma,-sigma,0),Q,
  P):
  tdL21function0:=unapply( tdL21function(1,-1,sigma,-sigma,0),Q,
  P):
  tdL22function0:=unapply( tdL22function(1,-1,sigma,-sigma,0),Q,
  P):
  tdLReduced[1,1]:=simplify(tdL11function(1,-1,sigma,-sigma,0) ):
  tdLReduced[1,2]:=simplify(tdL12function(1,-1,sigma,-sigma,0) ):
  tdLReduced[2,1]:=simplify(tdL21function(1,-1,sigma,-sigma,0) ):
```

```
tdLReduced[2,2]:=simplify(tdL22function(1,-1,sigma,-sigma,0) ):
tdLReduced:
tdLReduced2:=Matrix(2,2,0):
tdLReduced2[1,1]:=simplify(tdL11function0(SolQfunction(1,
-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) ):
tdLReduced2[1,2]:=simplify(tdL12function0(SolQfunction(1,
-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) ):
tdLReduced2[2,1]:=simplify(tdL21function0(SolQfunction(1,
-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) ):
tdLReduced2[2,2]:=simplify(tdL22function0(SolQfunction(1,
-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) ):
tdLReduced2;
```

$$\frac{-S1 + checkQ}{S2\sim}$$

$$\frac{(s10 + s20)\, S2\sim}{2\, S1 - 2\, checkQ} + S2\sim\, checkP$$

$$\sigma$$

$$S1$$

$$S2\sim$$

$$\frac{-s10 - s20}{\xi} \tag{4.1}$$

$$\left\{checkP = \frac{1}{2}\,\frac{2\,PP\,QQ + s10 + s20}{QQ\,S2\sim},\ checkQ = QQ\,S2\sim + S1\right\}$$

$$\Bigg[\Bigg[\frac{1}{2}\,\frac{1}{\xi\,S2\sim^2}\Big(\big(-2\,S1\,checkP + 2\,checkP\,checkQ - 2\,\sigma\,\xi - 2\,\xi^2 - s10 - s20\big)\,S2\sim^2 - 2\,\sigma\,(S1 - checkQ)\,S2\sim + 2\,(S1 - checkQ)^2\Big),\ \frac{S2\sim\,\xi + S1 - checkQ}{\xi\,S2\sim}\Bigg],$$

$$\Bigg[\frac{1}{4}\,\frac{1}{S2\sim^3\,\xi\,(S1 - checkQ)}\Big(-4\,\Big(S1\,checkP - checkP\,checkQ + sX10 + \frac{1}{2}\,s10 + \frac{1}{2}\,s20\Big)\Big(S1\,checkP - checkP\,checkQ - sX10 - \frac{1}{2}\,s10 - \frac{1}{2}\,s20\Big)\,S2\sim^4 - 8\,(S1 - checkQ)\,\Big(-checkP\,(\xi + \sigma)\,checkQ + checkP\,(\xi + \sigma)\,S1 - \frac{1}{2}\,\xi\,(s10 - s20)\Big)\,S2\sim^3 + 8\,(S1 - checkQ)^2\,\Big(-checkP\,checkQ + S1\,checkP - \Big(\xi + \frac{1}{2}\,\sigma\Big)\,\sigma\Big)\,S2\sim^2 + 8\,(S1 - checkQ)^3\,(\xi + \sigma)\,S2\sim - 4\,(S1 - checkQ)^4\Big),$$

$$\frac{1}{2}\,\frac{1}{\xi\,S2\sim^2}\Big(\big(2\,S1\,checkP - 2\,checkP\,checkQ + 2\,\sigma\,\xi + 2\,\xi^2 - s10 - s20\big)\,S2\sim^2 + 2\,\sigma\,(S1 - checkQ)\,S2\sim - 2\,(S1 - checkQ)^2\Big)\Bigg]\Bigg]$$

```
> series(tdLReduced2[1,1],xi=infinity);
  series(tdLReduced2[1,2],xi=infinity);
  series(tdLReduced2[2,2],xi=infinity);
  series(tdLReduced2[2,1],xi=0):
  tdLReduced221bis:=((checkP+checkQ)^2*(checkQ-sigma)-(sX10-sX20)
  ^2/4/(checkQ-sigma))/xi
  +2*(checkQ-sigma)*(checkP+checkQ)+s10-s20 ;
  simplify(series(tdLReduced2[2,1]-tdLReduced221bis,xi=0));
```

$$-\xi-\sigma+\frac{1}{2}\,\frac{1}{S2\sim^2\,\xi}\left(\left(-2\,S1\,checkP+2\,checkP\,checkQ-s10-s20\right)S2\sim^2-2\,\sigma\,\left(S1-checkQ\right)S2\sim+2\,\left(S1-checkQ\right)^2\right)\tag{4.2}$$

$$1+\frac{S1-checkQ}{S2\sim\,\xi}$$

$$\xi+\sigma+\frac{1}{2}\,\frac{1}{S2\sim^2\,\xi}\left(\left(2\,S1\,checkP-2\,checkP\,checkQ-s10-s20\right)S2\sim^2+2\,\sigma\,\left(S1-checkQ\right)S2\sim-2\,\left(S1-checkQ\right)^2\right)$$

$$tdLReduced221bis:=\frac{\left(checkP+checkQ\right)^2\left(checkQ-\sigma\right)-\frac{1}{4}\,\frac{\left(s10+s20+2\,sX10\right)^2}{checkQ-\sigma}}{\xi}+2\,\left(checkQ-\sigma\right)\left(checkP+checkQ\right)+s10-s20$$

$$\frac{1}{\xi}\left(\frac{1}{4}\,\frac{1}{\left(S1-checkQ\right)S2\sim^3\left(-checkQ+\sigma\right)}\left(-4\,\left(-checkQ+\sigma\right)\left(S1\,checkP-checkP\,checkQ-sX10-\frac{1}{2}\,s10-\frac{1}{2}\,s20\right)\left(S1\,checkP-checkP\,checkQ+sX10+\frac{1}{2}\,s10+\frac{1}{2}\,s20\right)S2\sim^4-8\,\left(-\frac{1}{2}\,checkQ^4+\left(\sigma-checkP\right)checkQ^3+\left(-\frac{1}{2}\,\sigma^2+3\,checkP\,\sigma-\frac{1}{2}\,checkP^2\right)checkQ^2-\sigma\,checkP\,\left(S1+2\,\sigma-checkP\right)checkQ+checkP\,\left(S1-\frac{1}{2}\,checkP\right)\sigma^2+\frac{1}{8}\,\left(s10+s20+2\,sX10\right)^2\right)\left(S1-checkQ\right)S2\sim^3+8\,\left(-checkQ+\sigma\right)\left(S1-checkQ\right)^2\left(-checkP\,checkQ+S1\,checkP-\frac{1}{2}\,\sigma^2\right)S2\sim^2+8\,\sigma\,\left(S1-checkQ\right)^3\left(-checkQ+\sigma\right)S2\sim-4\,\left(S1-checkQ\right)^4\left(-checkQ+\sigma\right)\right)\right)+\frac{1}{S2\sim^2}\left(\left(-2\,checkQ^2+2\,\sigma\,checkQ-2\,checkP\,\left(S1-\sigma\right)\right)S2\sim^2-2\,\sigma\,\left(S1-checkQ\right)S2\sim+2\,\left(S1-checkQ\right)^2\right)$$

```
> tdAReduced:=Matrix(2,2,0):
  tdA11function0:=unapply( tdA11function(1,-1,sigma,-sigma,0,0,0,
  1,-1,0),Q,P):
  tdA12function0:=unapply( tdA12function(1,-1,sigma,-sigma,0,0,0,
  1,-1,0),Q,P):
  tdA21function0:=unapply( tdA21function(1,-1,sigma,-sigma,0,0,0,
```

```
1,-1,0),Q,P):
tdA22function0:=unapply( tdA22function(1,-1,sigma,-sigma,0,0,0,
1,-1,0),Q,P):
tdAReduced[1,1]:=simplify(tdA11function(1,-1,sigma,-sigma,0,0,
0,1,-1,0) ):
tdAReduced[1,2]:=simplify(tdA12function(1,-1,sigma,-sigma,0,0,
0,1,-1,0) ):
tdAReduced[2,1]:=simplify(tdA21function(1,-1,sigma,-sigma,0,0,
0,1,-1,0) ):
tdAReduced[2,2]:=simplify(tdA22function(1,-1,sigma,-sigma,0,0,
0,1,-1,0) ):
tdAReduced:
tdAReduced2:=Matrix(2,2,0):
tdAReduced2[1,1]:=factor(simplify(tdA11function0(SolQfunction
(1,-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) )):
tdAReduced2[1,2]:=factor(simplify(tdA12function0(SolQfunction
(1,-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) )):
tdAReduced2[2,1]:=factor(simplify(tdA21function0(SolQfunction
(1,-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) )):
tdAReduced2[2,2]:=factor(simplify(tdA22function0(SolQfunction
(1,-1,sigma,-sigma,0),SolPfunction(1,-1,sigma,-sigma,0)) )):
tdAReduced2;
factor(2*checkP*checkQ-2*checkP*sigma+2*checkQ^2-2*checkQ*
sigma);
```

$$\left[\left[\frac{-S2\sim\sigma - S2\sim\xi + S1 - checkQ}{S2\sim}, 1\right], \left[\frac{1}{S2\sim^2}\left(-2\,S1\,S2\sim^2\,checkP + 2\,S2\sim^2\,checkP\,checkQ - 2\,S1\,S2\sim\sigma + S2\sim^2\,s10 - S2\sim^2\,s20 + 2\,S2\sim\,checkQ\,\sigma + 2\,S1^2 - 4\,S1\,checkQ + 2\,checkQ^2\right), -\frac{-S2\sim\sigma - S2\sim\xi + S1 - checkQ}{S2\sim}\right]\right] \tag{4.3}$$

$$2\,(checkQ - \sigma)\,(checkP + checkQ)$$

```
> tdR2function:=unapply(tdR2(xi),s12,s22,s11,s21,X1):
tdR2function(1,-1,sigma,-sigma,0);
R1function:=unapply(R1(xi),s12,s22,s11,s21,X1):
R1function(1,-1,sigma,-sigma,0);
L22function:=unapply(simplify(L[2,2]),s12,s22,s11,s21,X1):
simplify(L22function(1,-1,sigma,-sigma,0)-h/(xi-Q));

A11function:=unapply( simplify(A[1,1]), s12,s22,s11,s21,X1,
```

```
beta12,beta22,beta11,beta21,betaX1):
A12function:=unapply( simplify(A[1,2]), s12,s22,s11,s21,X1,
beta12,beta22,beta11,beta21,betaX1):
A21function:=unapply( simplify(A[2,1]), s12,s22,s11,s21,X1,
beta12,beta22,beta11,beta21,betaX1):
A22function:=unapply( simplify(A[2,2]), s12,s22,s11,s21,X1,
beta12,beta22,beta11,beta21,betaX1):
A11function(1,-1,sigma,-sigma,0,0,0,1,-1,0);
A12function(1,-1,sigma,-sigma,0,0,0,1,-1,0);
simplify(A21function(1,-1,sigma,-sigma,0,0,0,1,-1,0)):
simplify(A22function(1,-1,sigma,-sigma,0,0,0,1,-1,0)):
factor(residue(simplify(A21function(1,-1,sigma,-sigma,0,0,0,1,
-1,0)),xi=0));
A21bis:=P*(P*Q-sX10-sX20)/(xi-Q)+sX10*sX20/Q/xi+xi^2+xi*Q+2*xi*
sigma+Q^2+2*Q*sigma+sigma^2-h+s10-s20;
factor(A21function(1,-1,sigma,-sigma,0,0,0,1,-1,0)-A21bis);
```

$$\frac{sX10\,(-s10-s20-sX10)}{\xi^2}-s10-\sigma^2+s20-2\,\sigma\,\xi-\xi^2 \quad \textbf{(4.4)}$$

$$\frac{-s10-s20}{\xi}$$

$$\frac{-s10-s20-h}{\xi}$$

$$\frac{P\,Q}{-\xi+Q}$$

$$-\frac{\xi}{-\xi+Q}$$

$$-\frac{sX10\,(sX10+s10+s20)}{Q}$$

$$A21bis := \frac{P\,(P\,Q+s10+s20)}{\xi-Q}+\frac{sX10\,(-s10-s20-sX10)}{Q\,\xi}+\xi^2+\xi\,Q+2\,\sigma\,\xi+Q^2+2\,Q\,\sigma+\sigma^2-h+s10-s20$$

$$0$$

```
> simplify(LQfunction(1,-1,X1,-X1,0,0,0,1,-1,0));
simplify(LPfunction(1,-1,X1,-X1,0,0,0,1,-1,0));
simplify(Hamiltonianfunction(1,-1,X1,-X1,0,0,0,1,-1,0));
```

$$2\,P\,Q+s10+s20 \quad \textbf{(4.5)}$$

$$\frac{3\,Q^4+4\,Q^3\,X1+\left(-P^2+X1^2-h+s10-s20\right)Q^2-sX10\,(sX10+s10+s20)}{Q^2}$$

$$\frac{1}{Q}\left(-Q^4-2\,Q^3\,X1+\left(P^2-X1^2+h-s10+s20\right)Q^2+(s10+s20)\,P\,Q-sX10\,(sX10+s10+s20)\right)$$

Case of t_{1,2}=1, t_{2,2}=-1 , t_{1,1}=0, t_{2,1}=0 and sigma= X_1 position of the pole.

We take s12=1 and s22=-1 and s11=0 and s21=0

```
> SolQfunction(1,-1,0,0,sigma);
  simplify(series(SolPfunction(1,-1,0,0,sigma),checkP));
  tdX1functionfunction(1,-1,0,0,sigma);
  S1function(1,-1,0,0,sigma);
  S2function(1,-1,0,0,sigma);
  R1function(xi,1,-1,0,0,sigma);
  solve({SolQfunction(1,-1,0,0,sigma)=QQ,SolPfunction(1,-1,0,
  0,sigma)=PP},{checkQ,checkP});
  tdLReduced:=Matrix(2,2,0):
  tdL11function0:=unapply( tdL11function(1,-1,0,0,sigma),Q,P):
  tdL12function0:=unapply( tdL12function(1,-1,0,0,sigma),Q,P):
  tdL21function0:=unapply( tdL21function(1,-1,0,0,sigma),Q,P):
  tdL22function0:=unapply( tdL22function(1,-1,0,0,sigma),Q,P):
  tdLReduced[1,1]:=simplify(tdL11function(1,-1,0,0,sigma) ):
  tdLReduced[1,2]:=simplify(tdL12function(1,-1,0,0,sigma) ):
  tdLReduced[2,1]:=simplify(tdL21function(1,-1,0,0,sigma) ):
  tdLReduced[2,2]:=simplify(tdL22function(1,-1,0,0,sigma) ):
  tdLReduced;
  tdLReduced2:=Matrix(2,2,0):
  tdLReduced2[1,1]:=simplify(tdL11function0(SolQfunction(1,-1,0,
  0,sigma),SolPfunction(1,-1,0,0,sigma)) ):
  tdLReduced2[1,2]:=simplify(tdL12function0(SolQfunction(1,-1,0,
  0,sigma),SolPfunction(1,-1,0,0,sigma)) ):
  tdLReduced2[2,1]:=simplify(tdL21function0(SolQfunction(1,-1,0,
  0,sigma),SolPfunction(1,-1,0,0,sigma)) ):
  tdLReduced2[2,2]:=simplify(tdL22function0(SolQfunction(1,-1,0,
  0,sigma),SolPfunction(1,-1,0,0,sigma)) ):
  tdLReduced2;
```

$$\frac{checkQ - S1}{S2\sim} \qquad (5.1)$$

$$\frac{(s10 + s20)\, S2\sim}{2\, S2\sim\, \sigma + 2\, S1 - 2\, checkQ} + S2\sim\, checkP$$

$$\sigma$$

$$S1$$

$$S2\sim$$

$$\frac{-s10 - s20}{\xi} + 1 + (-\xi - 1)\, \xi$$

$$\left\{checkP = \frac{1}{2}\,\frac{2\,PP\,QQ - 2\,PP\,\sigma + s10 + s20}{S2\sim\,(QQ - \sigma)},\ checkQ = QQ\,S2\sim + S1\right\}$$

$$\left[\left[\frac{(-Q + \xi - P)\,\sigma - \xi^2 + Q\,(Q + P)}{\xi - \sigma},\ \frac{-\xi + Q}{-\xi + \sigma}\right],\right.$$

$$\left[\frac{1}{(\xi - \sigma)\,(Q - \sigma)}\Big(Q^4 + (2\,P - 4\,\sigma + 2\,\xi)\,Q^3 + \big(5\,\sigma^2 + (-6\,P - 4\,\xi)\,\sigma + P^2\right.$$

$$+ 2\,\xi\,P + s10 + s20\big)\,Q^2 + \big(-2\,\sigma^3 + (6\,P + 2\,\xi)\,\sigma^2 + \big(-2\,P^2 - 4\,P\,\xi - 3\,s10$$

$$- s20\big)\,\sigma + (s10 + s20)\,P + 2\,s10\,\xi\big)\,Q - 2\,\sigma^3\,P + \big(P^2 + 2\,P\,\xi + 2\,s10\big)\,\sigma^2 + ($$

$$-2\,s10\,\xi + P\,(-s10 - s20)\,)\,\sigma - sX10\,(sX10 + s10 + s20)\Big),$$

$$\left.\left.\frac{(Q - \xi + P)\,\sigma - P\,Q - Q^2 + \xi^2 - s10 - s20}{\xi - \sigma}\right]\right]$$

$$\left[\left[\frac{1}{2}\,\frac{1}{(\xi - \sigma)\,S2\sim^2}\Big(-2\,S2\sim^3\,\sigma\,checkP + \big(-2\,S1\,checkP + 2\,checkP\,checkQ + 2\,\sigma\,\xi\right.\right.$$

$$- 2\,\xi^2 - s10 - s20\big)\,S2\sim^2 + 2\,\sigma\,(S1 - checkQ)\,S2\sim + 2\,(S1 - checkQ)^2\Big),$$

$$\left.\frac{S2\sim\,\xi + S1 - checkQ}{(\xi - \sigma)\,S2\sim}\right],$$

$$\left[\frac{1}{4}\,\frac{1}{S2\sim^3\,(\xi - \sigma)\,(S2\sim\,\sigma + S1 - checkQ)}\Big(-4\,S2\sim^6\,checkP^2\,\sigma^2 - 8\,checkP\,\sigma\,\big(\right.$$

$$-\sigma^2 + \sigma\,\xi + checkP\,(S1 - checkQ)\big)\,S2\sim^5 + \Big((24\,S1\,checkP - 24\,checkP\,checkQ$$

$$- 4\,s10 + 4\,s20)\,\sigma^2 - 16\,\left(S1\,checkP - checkP\,checkQ - \frac{1}{4}\,s10 + \frac{1}{4}\,s20\right)\,\xi\,\sigma$$

$$- 4\,\left(S1\,checkP - checkP\,checkQ + sX10 + \frac{1}{2}\,s10 + \frac{1}{2}\,s20\right)\,\Big(S1\,checkP$$

$$- checkP\,checkQ - sX10 - \frac{1}{2}\,s10 - \frac{1}{2}\,s20\Big)\Big)\,S2\sim^4 - 8\,(S1 - checkQ)\,\Big(\sigma^3 - \sigma^2\,\xi$$

$$+ \left(-3\,S1\,checkP + 3\,checkP\,checkQ + \frac{1}{2}\,s10 - \frac{1}{2}\,s20\right)\,\sigma + \xi\,\Big(S1\,checkP$$

$$- checkP\,checkQ - \frac{1}{2}\,s10 + \frac{1}{2}\,s20\Big)\Big)\,S2\sim^3 + 8\,(S1 - checkQ)^2\,\Big(-\frac{5}{2}\,\sigma^2 + 2\,\sigma\,\xi$$

$$+ checkP\,(S1 - checkQ)\Big)\,S2\sim^2 + 8\,(S1 - checkQ)^3\,(\xi - 2\,\sigma)\,S2\sim - 4\,(S1$$

$- checkQ)^4\Big), \frac{1}{2} \frac{1}{(\xi - \sigma)\, S2\text{\textasciitilde}^2} \Big(2\, S2\text{\textasciitilde}^3\, \sigma\, checkP + \big(2\, S1\, checkP$

$- 2\, checkP\, checkQ - 2\, \sigma\, \xi + 2\, \xi^2 - s10 - s20\big)\, S2\text{\textasciitilde}^2 - 2\, \sigma\, (S1 - checkQ)\, S2\text{\textasciitilde}$

$- 2\, (S1 - checkQ)^2\Big)\Big]\Big]$

```
> tdAReduced:=Matrix(2,2,0):
  tdA11function0:=unapply( tdA11function(1,-1,0,0,sigma,0,0,0,0,
  1),Q,P):
  tdA12function0:=unapply( tdA12function(1,-1,0,0,sigma,0,0,0,0,
  1),Q,P):
  tdA21function0:=unapply( tdA21function(1,-1,0,0,sigma,0,0,0,0,
  1),Q,P):
  tdA22function0:=unapply( tdA22function(1,-1,0,0,sigma,0,0,0,0,
  1),Q,P):
  tdAReduced[1,1]:=simplify(tdA11function(1,-1,0,0,sigma,0,0,0,0,
  1) ):
  tdAReduced[1,2]:=simplify(tdA12function(1,-1,0,0,sigma,0,0,0,0,
  1) ):
  tdAReduced[2,1]:=simplify(tdA21function(1,-1,0,0,sigma,0,0,0,0,
  1) ):
  tdAReduced[2,2]:=simplify(tdA22function(1,-1,0,0,sigma,0,0,0,0,
  1) ):
  tdAReduced;
  tdAReduced2:=Matrix(2,2,0):
  tdAReduced2[1,1]:=factor(simplify(tdA11function0(SolQfunction
  (1,-1,0,0,sigma),SolPfunction(1,-1,0,0,sigma)) )):
  tdAReduced2[1,2]:=factor(simplify(tdA12function0(SolQfunction
  (1,-1,0,0,sigma),SolPfunction(1,-1,0,0,sigma)) )):
  tdAReduced2[2,1]:=factor(simplify(tdA21function0(SolQfunction
  (1,-1,0,0,sigma),SolPfunction(1,-1,0,0,sigma)) )):
  tdAReduced2[2,2]:=factor(simplify(tdA22function0(SolQfunction
  (1,-1,0,0,sigma),SolPfunction(1,-1,0,0,sigma)) )):
  tdAReduced2;
```

$$\Bigg[\Bigg[-\frac{(Q - \sigma)\, (Q + P + \xi - \sigma)}{-\sigma + \xi}, \frac{Q - \sigma}{-\sigma + \xi}\Bigg],$$
$$\Bigg[-\frac{\big(Q^2 + (-\sigma + P)\, Q - P\, \sigma - sX20\big)\, \big(Q^2 + (-\sigma + P)\, Q - P\, \sigma - sX10\big)}{(-\sigma + \xi)\, (Q - \sigma)},$$
$$\frac{\sigma^2 + (-P - 2\, Q - \xi)\, \sigma + Q^2 + (\xi + P)\, Q - sX10 - sX20}{-\sigma + \xi}\Bigg]\Bigg] \qquad (5.2)$$

$\Big[\Big[\frac{1}{2} \frac{1}{-\xi + \sigma} \big(2\, checkP\, checkQ - 2\, checkP\, \sigma + 2\, checkQ^2 - 4\, checkQ\, \sigma + 2\, checkQ\, \xi$

$$+2\sigma^2-2\sigma\xi+sX10+sX20\Big),\ -\frac{checkQ-\sigma}{-\xi+\sigma}\Bigg],$$

$$\Bigg[\frac{1}{4}\,\frac{1}{(checkQ-\sigma)\,(-\xi+\sigma)}\Big(\big(2\,checkP\,checkQ-2\,checkP\,\sigma+2\,checkQ^2$$

$$-2\,checkQ\,\sigma+sX10-sX20\big)\,\big(2\,checkP\,checkQ-2\,checkP\,\sigma+2\,checkQ^2$$

$$-2\,checkQ\,\sigma-sX10+sX20\big)\Big),\ -\frac{1}{2}\,\frac{1}{-\xi+\sigma}\Big(2\,checkP\,checkQ-2\,checkP\,\sigma$$

$$+2\,checkQ^2-4\,checkQ\,\sigma+2\,checkQ\,\xi+2\sigma^2-2\sigma\xi-sX10-sX20\Big)\Bigg]\Bigg]$$

```
> simplify(LQfunction(1,-1,0,0,sigma,0,0,0,0,1));
  simplify(LPfunction(1,-1,0,0,sigma,0,0,0,0,1));
  simplify(Hamiltonianfunction(1,-1,0,0,sigma,0,0,0,0,1));
```

$$(2\,Q-2\,\sigma)\,P+h-sX10-sX20 \tag{5.3}$$

$$\frac{1}{(Q-\sigma)^2}\Big(3\,Q^4-8\,Q^3\,\sigma+\big(-P^2+7\,\sigma^2-h+s10-s20\big)\,Q^2+2\,\sigma\,\big(P^2-\sigma^2+h-s10+s20\big)\,Q+\big(-P^2-h+s10-s20\big)\,\sigma^2+sX10\,sX20\Big)$$

$$\frac{1}{Q-\sigma}\Big(-Q^4+2\,Q^3\,\sigma+\big(P^2-\sigma^2+h-s10+s20\big)\,Q^2+\Big(\big(\big(-2\,P^2-2\,h+2\,s10-2\,s20\big)\,\sigma+(-sX10-sX20+h)\,P\big)\,Q+\big(P^2+h-s10+s20\big)\,\sigma^2-P\,(-sX10-sX20+h)\,\sigma+sX10\,sX20\Big)$$