

Method 1: Comparing the JMU differentials using their reduced forms.

Loading the gl_2 side results

```
> restart;
checkQfunction:=S2*Q+S1;
checkPfunction=1/S2*(P-1/2*R1(Q));

SolQ:=- (S1-checkQ)/S2;
SolP:=checkP*S2+(1/2)*R1(- (S1-checkQ)/S2);
simplify(checkQ-(S2*SolQ+S1));
simplify(checkP-(1/S2*(SolP-1/2*R1(SolQ))));
Sinfty2function:=s12+s22;
Sinfty1function:=s11+s21;
S2function:=sqrt(s12-s22)/sqrt(2);
S1function:=(s11-s21)/sqrt(2)/sqrt(s12-s22);
tdX1function:=X1*S2function+S1function;
solve({s12+s22=Sinfty2,s11+s21=Sinfty1, S2=sqrt(s12-s22)/sqrt
(2),
S1=(s11-s21)/sqrt(2)/sqrt(s12-s22),
tdX1=X1*S2function+S1function},{s12,s22,s11,s21,X1});
X1function := unapply(- (S1-tdX1)/S2 ,Sinfty1,Sinfty2,S1,S2,
tdX1);
s11function:= unapply(S2*S1+(1/2)*Sinfty1, Sinfty1,Sinfty2,S1,
S2,tdX1);
s12function:= unapply(S2^2+(1/2)*Sinfty2, Sinfty1,Sinfty2,S1,
S2,tdX1);
s21function:= unapply(-S2*S1+(1/2)*Sinfty1, Sinfty1,Sinfty2,
S1,S2,tdX1);
s22function:= unapply(-S2^2+(1/2)*Sinfty2, Sinfty1,Sinfty2,S1,
S2,tdX1);
simplify(X1function(Sinfty1function,Sinfty2function,S1function,
S2function,tdX1function));
simplify(s11function(Sinfty1function,Sinfty2function,
S1function,S2function,tdX1function));
simplify(s12function(Sinfty1function,Sinfty2function,
S1function,S2function,tdX1function));
simplify(s21function(Sinfty1function,Sinfty2function,
S1function,S2function,tdX1function));
simplify(s22function(Sinfty1function,Sinfty2function,
S1function,S2function,tdX1function));
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partialtdX1function:=simplify(diff(s11function(Sinfy1,Sinfy2,
S1,S2,tdX1),tdX1))*partials11+
simplify(diff(s21function(Sinfy1,Sinfy2,S1,S2,tdX1),tdX1))*
partials21
+simplify(diff(s12function(Sinfy1,Sinfy2,S1,S2,tdX1),tdX1))*
partials12
+simplify(diff(s22function(Sinfy1,Sinfy2,S1,S2,tdX1),tdX1))*
partials22
+simplify(diff(X1function(Sinfy1,Sinfy2,S1,S2,tdX1),tdX1))*
partialX1;

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K0OldCoordinates := unapply( (1/2)*(sX10+s20)*(sX10+s10)*ln(
(s12-s22)/2)+(1/2)*X1*((X1*s12+2*s11)*s10+s20*(X1*s22+2*s21)),
s11,s21,s12,s22,X1);

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K0NewCoordinates2:=unapply((sX10+s20)*(sX10+s10)*ln(S2)-(1/2)*
(S1-tdX1)*(s10+s20)*Sinfy1/S2
+(1/4)*(S1-tdX1)^2*(s10+s20)*Sinfy2/S2^2-(1/2)*(S1-tdX1)*(s10-
s20)*(S1+tdX1)
,Sinfy1,Sinfy2,S1,S2,tdX1);

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dtdX1function:=diff(tdX1function,s11)*ds11+diff(tdX1function,
s21)*ds21+diff(tdX1function,s12)*ds12+diff(tdX1function,s22)*
ds22+diff(tdX1function,X1)*dX1;

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$$checkQfunction := Q S2 + S1 \quad (1.1)$$

$$checkPfunction = \frac{P - \frac{1}{2} RI(Q)}{S2}$$

$$SolQ := -\frac{S1 - checkQ}{S2}$$

$$SolP := checkP S2 + \frac{1}{2} RI\left(-\frac{S1 - checkQ}{S2}\right)$$

$$Sinfy2function := s12 + s22$$

$$Sinfy1function := s11 + s21$$

$$S2function := \frac{1}{2} \sqrt{s12 - s22} \sqrt{2}$$

$$S1function := \frac{1}{2} \frac{(s11 - s21) \sqrt{2}}{\sqrt{s12 - s22}}$$

$$tdX1function := \frac{1}{2} X1 \sqrt{s12 - s22} \sqrt{2} + \frac{1}{2} \frac{(s11 - s21) \sqrt{2}}{\sqrt{s12 - s22}}$$

$$\left\{ X1 = -\frac{S1 - tdX1}{S2}, s11 = S1 S2 + \frac{1}{2} Sinfy1, s12 = S2^2 + \frac{1}{2} Sinfy2, s21 = -S1 S2 \right.$$

$$+ \frac{1}{2} \text{Sinfty1}, s22 = -S2^2 + \frac{1}{2} \text{Sinfty2} \}$$

$$X1function := (\text{Sinfty1}, \text{Sinfty2}, S1, S2, tdX1) \rightarrow -\frac{S1 - tdX1}{S2}$$

$$s11function := (\text{Sinfty1}, \text{Sinfty2}, S1, S2, tdX1) \rightarrow S1 S2 + \frac{1}{2} \text{Sinfty1}$$

$$s12function := (\text{Sinfty1}, \text{Sinfty2}, S1, S2, tdX1) \rightarrow S2^2 + \frac{1}{2} \text{Sinfty2}$$

$$s21function := (\text{Sinfty1}, \text{Sinfty2}, S1, S2, tdX1) \rightarrow -S1 S2 + \frac{1}{2} \text{Sinfty1}$$

$$s22function := (\text{Sinfty1}, \text{Sinfty2}, S1, S2, tdX1) \rightarrow -S2^2 + \frac{1}{2} \text{Sinfty2}$$

X1
s11
s12
s21
s22

$$\text{partialtdX1function} := \frac{\text{partialX1}}{S2}$$

$$K0OldCoordinates := (s11, s21, s12, s22, X1) \rightarrow \frac{1}{2} (sX10 + s20) (sX10 + s10) \ln\left(\frac{1}{2} s12 - \frac{1}{2} s22\right) + \frac{1}{2} X1 ((X1 s12 + 2 s11) s10 + s20 (X1 s22 + 2 s21))$$

$$K0NewCoordinates2 := (\text{Sinfty1}, \text{Sinfty2}, S1, S2, tdX1) \rightarrow (sX10 + s20) (sX10 + s10) \ln(S2) - \frac{1}{2} \frac{(S1 - tdX1) (s10 + s20) \text{Sinfty1}}{S2} + \frac{1}{4} \frac{(S1 - tdX1)^2 (s10 + s20) \text{Sinfty2}}{S2^2} - \frac{1}{2} (S1 - tdX1) (s10 - s20) (S1 + tdX1)$$

$$dtdX1function := \frac{1}{2} \frac{\sqrt{2} ds11}{\sqrt{s12 - s22}} - \frac{1}{2} \frac{\sqrt{2} ds21}{\sqrt{s12 - s22}} + \left(\frac{1}{4} \frac{X1 \sqrt{2}}{\sqrt{s12 - s22}} - \frac{1}{4} \frac{(s11 - s21) \sqrt{2}}{(s12 - s22)^{3/2}} \right) ds12 + \left(-\frac{1}{4} \frac{X1 \sqrt{2}}{\sqrt{s12 - s22}} + \frac{1}{4} \frac{(s11 - s21) \sqrt{2}}{(s12 - s22)^{3/2}} \right) ds22 + \frac{1}{2} \sqrt{s12 - s22} \sqrt{2} dX1$$

Loading the gl_3 side results

> t10 := -t20 - t30;

P1 := unapply((t12+t22+t32)*lambda+t11+t21+t31, lambda);

P2 := unapply((t12*t22+t12*t32+t22*t32)*lambda^2+(t21+t31)*t12+(t11+t31)*t22+t32*(t11+t21))*lambda-t10*t12-t20*t22-t30*t32+t21*t11+t31*t11+t21*t31, lambda);

P3 := unapply(t12*t22*t32*lambda^3+(t12*t22*t31+t12*t32*t21+t22*t32*t11)*lambda^2+(t12*t22*t30+t12*t32*t20+t22*t32*t10+t12*t21*t31+t22*t11*t31+t32*t11*t21)*lambda, lambda);

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taufunction:=unapply( ((t21-t31)*t12+(t31-t11)*t22+(t11-t21)*
t32)/(sqrt((t22-t12)*(t12-t32)*(t32-t22))), t11,t21,t31,t12,
t22,t32);

checkqfunction:=unapply( sqrt((t12-t32)/((t22-t12)*(t32-t22)))*
(-p+t22*q+t21), q,p);
checkpfunction:=unapply( sqrt((t32-t22)/((t12-t32)*(t22-t12)))*
(p-t12*q-t11), q,p);
TT1:=unapply(t12+t22+t32, t11, t21, t31, t12, t22, t32);
TT2:=unapply(t11+t21+t31, t11, t21, t31, t12, t22, t32);
TT3:=unapply(t22, t11, t21, t31, t12, t22, t32);
TT4:=unapply(t11, t11, t21, t31, t12, t22, t32);
TT5:=unapply(t32, t11, t21, t31, t12, t22, t32);

solve({TT1(t11, t21, t31, t12, t22, t32)=T1, TT2(t11, t21, t31, t12, t22,
t32)=T2, TT3(t11, t21, t31, t12, t22, t32)=T3, TT4(t11, t21, t31, t12,
t22, t32)=T4, TT5(t11, t21, t31, t12, t22, t32)=T5, taufunction(t11,
t21, t31, t12, t22, t32)=tau }, {t11, t21, t31, t12, t22, t32});

t11function := T4;
t12function := -T3-T5+T1;
t21function := (tau*sqrt((-2*T3-T5+T1)*(-T3-2*T5+T1)*(T3-T5))+
T1*T2-T1*T4-2*T3*T2-T5*T2+3*T4*T3)/(2*T1-3*T3-3*T5);
t22function := T3;
t31function := -(tau*sqrt((-2*T3-T5+T1)*(-T3-2*T5+T1)*(T3-T5))-
T1*T2+T1*T4+T3*T2+2*T5*T2-3*T4*T5)/(2*T1-3*T3-3*T5);
t32function := T5;
simplify(TT1(t11function, t21function, t31function, t12function,
t22function, t32function));
simplify(TT2(t11function, t21function, t31function, t12function,
t22function, t32function));
simplify(TT3(t11function, t21function, t31function, t12function,
t22function, t32function));
simplify(TT4(t11function, t21function, t31function, t12function,
t22function, t32function));
simplify(TT5(t11function, t21function, t31function, t12function,
t22function, t32function));
simplify(taufunction(t11function, t21function, t31function,
t12function, t22function, t32function));

SOLL:=solve({checkqfunction(q,p)=checkq, checkpfunction(q,p)=
checkp}, {q,p}):

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Solp:=simplify(rhs(SOLL[1])):
Solq:=simplify(rhs(SOLL[2])):
simplify(checkqfunction(Solq,Solp)-checkq);
simplify(checkpfunction(Solq,Solp)-checkp);

HamtauchekCoordinatesTheo:=unapply(-(checkq*checkp^2+checkq^2*
checkp-tau*checkq*checkp-t20*checkp+(t10+h)*checkq),checkq,
checkp);
GSolOldCoordinatesbis:=1/2*(t30*t10*ln(-t12+t32)+t10*t20*ln
(t22-t12)+ln(-t32+t22)*t20*t30)
-t10*t11^2/2/(t12-t32)-t20*t21^2/2/(t22-t32)
+t30*t31^2/2/(t12-t32)
-(1/2)*(t12-t22)*t31^2*t20/((t22-t32)*(t12-t32))
+t31*t11*t10/(t12-t32)+t20*t31*t21/(t22-t32);

GSolbis:=1/2*(-(t20+t30)*(t20*ln(2*T3+T5-T1)+t30*ln(T3+2*T5-T1)
)+t20*t30*ln(T3-T5))
+(T2-3*T4)^2*((T1-2*T3-T5)*t20+t30*(T1-T3-2*T5))/(2*(2*T1-3*T3
-3*T5)^2)
+sqrt(T1-T3-2*T5)*sqrt(T1-2*T3-T5)*sqrt(T3-T5)*(T2-3*T4)*(t20-
t30)*tau/(2*T1-3*T3-3*T5)^2
-(((4*T1-5*T3-7*T5)*t20-t30*(T3-T5))*((T1-2*T3-T5)*tau^2/2/(2*
T1-3*T3-3*T5)^2);

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$$t10 := -t20 - t30 \quad (1.2)$$

$$P1 := \lambda \rightarrow (t12 + t22 + t32) \lambda + t11 + t21 + t31$$

$$P2 := \lambda \rightarrow (t12 t22 + t12 t32 + t22 t32) \lambda^2 + ((t21 + t31) t12 + (t11 + t31) t22 + t32 (t11 + t21)) \lambda - (-t20 - t30) t12 - t20 t22 - t30 t32 + t21 t11 + t31 t11 + t21 t31$$

$$P3 := \lambda \rightarrow t12 t22 t32 \lambda^3 + (t11 t22 t32 + t12 t21 t32 + t12 t22 t31) \lambda^2 + ((-t20 - t30) t22 t32 + t32 t11 t21 + t22 t11 t31 + t12 t32 t20 + t12 t21 t31 + t12 t22 t30) \lambda$$

$$\text{taufunction} := (t11, t21, t31, t12, t22, t32)$$

$$\rightarrow \frac{(t21 - t31) t12 + (t31 - t11) t22 + (t11 - t21) t32}{\sqrt{(t22 - t12) (t12 - t32) (t32 - t22)}}$$

$$\text{checkqfunction} := (q, p) \rightarrow \sqrt{\frac{t12 - t32}{(t22 - t12) (t32 - t22)}} (q t22 - p + t21)$$

$$\text{checkpfunction} := (q, p) \rightarrow \sqrt{\frac{t32 - t22}{(t22 - t12) (t12 - t32)}} (-q t12 + p - t11)$$

$$TT1 := (t11, t21, t31, t12, t22, t32) \rightarrow t12 + t22 + t32$$

$$TT2 := (t11, t21, t31, t12, t22, t32) \rightarrow t11 + t21 + t31$$

$$TT3 := (t11, t21, t31, t12, t22, t32) \rightarrow t22$$

$$TT4 := (t11, t21, t31, t12, t22, t32) \rightarrow t11$$

$$TT5 := (t11, t21, t31, t12, t22, t32) \rightarrow t32$$

$$\left\{ \begin{aligned} t11 &= T4, t12 = -T3 - T5 + T1, t21 \\ &= \frac{1}{2 T1 - 3 T3 - 3 T5} \left(\tau \sqrt{(-2 T3 - T5 + T1) (-T3 - 2 T5 + T1) (-T5 + T3)} \right. \\ &\quad \left. + T1 T2 - T1 T4 - 2 T3 T2 - T5 T2 + 3 T3 T4 \right), t22 = T3, t31 = \\ &= \frac{1}{2 T1 - 3 T3 - 3 T5} \left(\tau \sqrt{(-2 T3 - T5 + T1) (-T3 - 2 T5 + T1) (-T5 + T3)} \right. \\ &\quad \left. - T1 T2 + T1 T4 + T3 T2 + 2 T5 T2 - 3 T5 T4 \right), t32 = T5 \end{aligned} \right\}$$

$$t11function := T4$$

$$t12function := -T3 - T5 + T1$$

$$t21function := \frac{1}{2 T1 - 3 T3 - 3 T5} \left(\tau \sqrt{(-2 T3 - T5 + T1) (-T3 - 2 T5 + T1) (-T5 + T3)} + T1 T2 - T1 T4 - 2 T3 T2 - T5 T2 + 3 T3 T4 \right)$$

$$t22function := T3$$

$$t31function := \frac{1}{2 T1 - 3 T3 - 3 T5} \left(\tau \sqrt{(-2 T3 - T5 + T1) (-T3 - 2 T5 + T1) (-T5 + T3)} - T1 T2 + T1 T4 + T3 T2 + 2 T5 T2 - 3 T5 T4 \right)$$

$$t32function := T5$$

$$T1$$

$$T2$$

$$T3$$

$$T4$$

$$T5$$

$$\tau$$

$$0$$

$$0$$

$$HamtauchekCoordinatesTheo := (checkq, checkp) \rightarrow -checkq checkp^2 - checkq^2 checkp + \tau checkq checkp + t20 checkp - (-t20 - t30 + h) checkq$$

$$GSolOldCoordinatesbis := \frac{1}{2} t30 (-t20 - t30) \ln(-t12 + t32) + \frac{1}{2} (-t20 - t30) t20 \ln(t22 - t12) + \frac{1}{2} \ln(-t32 + t22) t20 t30 - \frac{1}{2} \frac{(-t20 - t30) t11^2}{t12 - t32} - \frac{1}{2} \frac{t20 t21^2}{-t32 + t22} + \frac{1}{2} \frac{t30 t31^2}{t12 - t32} - \frac{1}{2} \frac{(-t22 + t12) t31^2 t20}{(-t32 + t22) (t12 - t32)} + \frac{t31 t11 (-t20 - t30)}{t12 - t32} + \frac{t20 t31 t21}{-t32 + t22}$$

$$GSolbis := -\frac{1}{2} (t20 + t30) (t20 \ln(2 T3 + T5 - T1) + t30 \ln(T3 + 2 T5 - T1)) + \frac{1}{2} t20 t30 \ln(-T5 + T3) + \frac{1}{2} \frac{(T2 - 3 T4)^2 ((-2 T3 - T5 + T1) t20 + t30 (-T3 - 2 T5 + T1))}{(2 T1 - 3 T3 - 3 T5)^2}$$

$$+ \frac{\sqrt{-T3 - 2 T5 + T1} \sqrt{-2 T3 - T5 + T1} \sqrt{-T5 + T3} (T2 - 3 T4) (t20 - t30) \tau}{(2 T1 - 3 T3 - 3 T5)^2}$$

$$- \frac{1}{2} \frac{((4 T1 - 5 T3 - 7 T5) t20 - t30 (-T5 + T3)) (-2 T3 - T5 + T1) \tau^2}{(2 T1 - 3 T3 - 3 T5)^2}$$

Loading the spectral duality correspondence

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> t11functionduality:=-s21/s22;
t21functionduality:=X1;
t31functionduality:=-s11/s12;
t12functionduality:=t22-1/s22;
t22functionduality:=t22;
t32functionduality:=t22-1/s12;
t20functionduality:=-s10-s20;
t30functionduality:=s10;
dt11duality:=diff(t11functionduality,s11)*ds11+diff
(t11functionduality,s21)*ds21+diff(t11functionduality,s12)*
ds12+diff(t11functionduality,s22)*ds22+diff(t11functionduality,
X1)*dX1+diff(t11functionduality,t22)*dt22;
dt21duality:=diff(t21functionduality,s11)*ds11+diff
(t21functionduality,s21)*ds21+diff(t21functionduality,s12)*
ds12+diff(t21functionduality,s22)*ds22+diff(t21functionduality,
X1)*dX1+diff(t21functionduality,t22)*dt22;
dt31duality:=diff(t31functionduality,s11)*ds11+diff
(t31functionduality,s21)*ds21+diff(t31functionduality,s12)*
ds12+diff(t31functionduality,s22)*ds22+diff(t31functionduality,
X1)*dX1+diff(t31functionduality,t22)*dt22;
dt12duality:=diff(t12functionduality,s11)*ds11+diff
(t12functionduality,s21)*ds21+diff(t12functionduality,s12)*
ds12+diff(t12functionduality,s22)*ds22+diff(t12functionduality,
X1)*dX1+diff(t12functionduality,t22)*dt22;
dt22duality:=diff(t22functionduality,s11)*ds11+diff
(t22functionduality,s21)*ds21+diff(t22functionduality,s12)*
ds12+diff(t22functionduality,s22)*ds22+diff(t22functionduality,
X1)*dX1+diff(t22functionduality,t22)*dt22;
dt32duality:=diff(t32functionduality,s11)*ds11+diff
(t32functionduality,s21)*ds21+diff(t32functionduality,s12)*
ds12+diff(t32functionduality,s22)*ds22+diff(t32functionduality,
X1)*dX1+diff(t32functionduality,t22)*dt22;

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$$t11functionduality := - \frac{s21}{s22}$$

$$t21functionduality := X1$$

(1.3)

$$\begin{aligned}
t31functionduality &:= -\frac{s11}{s12} \\
t12functionduality &:= t22 - \frac{1}{s22} \\
t22functionduality &:= t22 \\
t32functionduality &:= t22 - \frac{1}{s12} \\
t20functionduality &:= -s10 - s20 \\
t30functionduality &:= s10 \\
dt11duality &:= -\frac{ds21}{s22} + \frac{s21 ds22}{s22^2} \\
dt21duality &:= dX1 \\
dt31duality &:= -\frac{ds11}{s12} + \frac{s11 ds12}{s12^2} \\
dt12duality &:= \frac{ds22}{s22^2} + dt22 \\
dt22duality &:= dt22 \\
dt32duality &:= \frac{ds12}{s12^2} + dt22
\end{aligned}$$

Decomposing the dG_0 term in the dual coordinates

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> dG0:=unapply( diff(GSolOldCoordinatesbis,t11)*dt11+diff
(GSolOldCoordinatesbis,t21)*dt21+diff(GSolOldCoordinatesbis,
t31)*dt31+
diff(GSolOldCoordinatesbis,t12)*dt12+diff
(GSolOldCoordinatesbis,t22)*dt22+diff(GSolOldCoordinatesbis,
t32)*dt32,
t11,t21,t31,t12,t22,t32,dt11,dt21,dt31,dt12,dt22,dt32,t20,t30);

dG0Intermediate:=unapply(simplify(dG0(t11functionduality,
t21functionduality,t31functionduality,t12functionduality,
t22functionduality,t32functionduality,dt11duality,dt21duality,
dt31duality,dt12duality,dt22duality,dt32duality,
t20functionduality,t30functionduality))
,s11,s21,s12,s22,X1):
dG0Termds11:=simplify(residue(dG0Intermediate(s11,s21,s12,s22,
X1)/ds11^2,ds11=0));
dG0Termds21:=simplify(residue(dG0Intermediate(s11,s21,s12,s22,
X1)/ds21^2,ds21=0));
dG0Termds12:=simplify(residue(dG0Intermediate(s11,s21,s12,s22,
X1)/ds12^2,ds12=0));
dG0Termds22:=simplify(residue(dG0Intermediate(s11,s21,s12,s22,
X1)/ds22^2,ds22=0));
dG0TermdX1:=simplify(residue(dG0Intermediate(s11,s21,s12,s22,
X1)/dX1^2,dX1=0));

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dG0Termdt22 := simplify (residue (dG0Intermediate (s11, s21, s12, s22, x1) / dt22^2, dt22=0)) ;

$$dG0 := (t11, t21, t31, t12, t22, t32, dt11, dt21, dt31, dt12, dt22, dt32, t20, t30) \rightarrow \quad (1.4)$$

$$\begin{aligned} & - \left(\frac{(-t20 - t30) t11}{t12 - t32} + \frac{t31 (-t20 - t30)}{t12 - t32} \right) dt11 + \left(- \frac{t20 t21}{-t32 + t22} \right. \\ & + \left. \frac{t20 t31}{-t32 + t22} \right) dt21 + \left(\frac{t30 t31}{t12 - t32} - \frac{(-t22 + t12) t31 t20}{(-t32 + t22) (t12 - t32)} \right. \\ & + \left. \frac{(-t20 - t30) t11}{t12 - t32} + \frac{t20 t21}{-t32 + t22} \right) dt31 + \left(- \frac{1}{2} \frac{t30 (-t20 - t30)}{-t12 + t32} \right. \\ & - \frac{1}{2} \frac{(-t20 - t30) t20}{t22 - t12} + \frac{1}{2} \frac{(-t20 - t30) t11^2}{(t12 - t32)^2} - \frac{1}{2} \frac{t30 t31^2}{(t12 - t32)^2} \\ & - \frac{1}{2} \frac{t31^2 t20}{(-t32 + t22) (t12 - t32)} + \frac{1}{2} \frac{(-t22 + t12) t31^2 t20}{(-t32 + t22) (t12 - t32)^2} \\ & \left. - \frac{t31 t11 (-t20 - t30)}{(t12 - t32)^2} \right) dt12 + \left(\frac{1}{2} \frac{(-t20 - t30) t20}{t22 - t12} + \frac{1}{2} \frac{t20 t30}{-t32 + t22} \right. \\ & + \frac{1}{2} \frac{t20 t21^2}{(-t32 + t22)^2} + \frac{1}{2} \frac{t31^2 t20}{(-t32 + t22) (t12 - t32)} \\ & + \left. \frac{1}{2} \frac{(-t22 + t12) t31^2 t20}{(-t32 + t22)^2 (t12 - t32)} - \frac{t20 t31 t21}{(-t32 + t22)^2} \right) dt22 + \left(\frac{1}{2} \frac{t30 (-t20 - t30)}{-t12 + t32} \right. \\ & - \frac{1}{2} \frac{t20 t30}{-t32 + t22} - \frac{1}{2} \frac{(-t20 - t30) t11^2}{(t12 - t32)^2} - \frac{1}{2} \frac{t20 t21^2}{(-t32 + t22)^2} + \frac{1}{2} \frac{t30 t31^2}{(t12 - t32)^2} \\ & - \frac{1}{2} \frac{(-t22 + t12) t31^2 t20}{(-t32 + t22)^2 (t12 - t32)} - \frac{1}{2} \frac{(-t22 + t12) t31^2 t20}{(-t32 + t22) (t12 - t32)^2} \\ & \left. + \frac{t31 t11 (-t20 - t30)}{(t12 - t32)^2} + \frac{t20 t31 t21}{(-t32 + t22)^2} \right) dt32 \end{aligned}$$

$$dG0Termds11 := \frac{1}{s12 (s12 - s22)} (X1 (s10 + s20) s12^2 + ((-X1 s22 + s11) s10 - s20 (X1 s22 - s11 + s21)) s12 - s10 s11 s22)$$

$$dG0Termds21 := - \frac{(s11 s22 - s12 s21) s20}{s22 (s12 - s22)}$$

$$\begin{aligned} dG0Termds12 := & \frac{1}{2} \frac{1}{s12^2 (s12 - s22)^2} \left(X1^2 (s10 + s20) s12^4 - 2 \left(X1^2 s22 \right. \right. \\ & \left. \left. - \frac{1}{2} s10 \right) (s10 + s20) s12^3 + (-2 s22 s10^2 + (X1^2 s22^2 - s11^2 - s20 s22) s10 \right. \\ & + s20 (X1 s22 - s11 + s21) (X1 s22 + s11 - s21)) s12^2 + (s10^2 s22^2 \\ & \left. + 2 s10 s11^2 s22) s12 - s10 s11^2 s22^2 \right) \end{aligned}$$

$$dG0Termds22 := - \frac{1}{2} \frac{1}{s22^2 (s12 - s22)^2} (s20 ((-s10 - s20) s22^3 + ((s10 + 2 s20) s12 - s11 (s11 - 2 s21)) s22^2 + (-s12^2 s20 - 2 s12 s21^2) s22 + s21^2 s12^2))$$

$$dG0TermdX1 := (s10 + s20) (X1 s12 + s11)$$

$$dG0Termdt22 := 0$$

Computing the dK0 differential

```

> dK0:=unapply( simplify(diff(K0OldCoordinates(s11,s21,s12,s22,
X1),s11)*ds11+ diff(K0OldCoordinates(s11,s21,s12,s22,X1),s21)*
ds21+diff(K0OldCoordinates(s11,s21,s12,s22,X1),s12)*ds12+diff
(K0OldCoordinates(s11,s21,s12,s22,X1),s22)*ds22+diff
(K0OldCoordinates(s11,s21,s12,s22,X1),X1)*dX1,s11,s21,s12,s22,
X1)):
dK0Termds11:=simplify(residue(dK0(s11,s21,s12,s22,X1)/ds11^2,
ds11=0));
dK0Termds21:=simplify(residue(dK0(s11,s21,s12,s22,X1)/ds21^2,
ds21=0));
dK0Termds12:=simplify(residue(dK0(s11,s21,s12,s22,X1)/ds12^2,
ds12=0));
dK0Termds22:=simplify(residue(dK0(s11,s21,s12,s22,X1)/ds22^2,
ds22=0));
dK0TermdX1:=simplify(residue(dK0(s11,s21,s12,s22,X1)/dX1^2,dX1=
0));
dK0Termdt22:=simplify(residue(dK0(s11,s21,s12,s22,X1)/dt22^2,
dt22=0));

```

$$\begin{aligned}
 dK0Termds11 &:= s10 X1 & (1.5) \\
 dK0Termds21 &:= s20 X1 \\
 dK0Termds12 &:= \frac{(X1^2 s12 - X1^2 s22 + s20 + sX10) s10 + sX10 (sX10 + s20)}{2 s12 - 2 s22} \\
 dK0Termds22 &:= \frac{(X1^2 s12 - X1^2 s22 - s10 - sX10) s20 - sX10 (sX10 + s10)}{2 s12 - 2 s22} \\
 dK0TermdX1 &:= (X1 s12 + s11) s10 + s20 (X1 s22 + s21) \\
 dK0Termdt22 &:= 0
 \end{aligned}$$

Comparing the difference of the JMU differentials on both sides

```

> dG0MinusdK0:=dG0Intermediate(s11,s21,s12,s22,X1)-dK0(s11,s21,
s12,s22,X1):
omegaJMUMinusomegaJMUP4:=dG0MinusdK0-2*s20*tdX1function*
dtdX1function:
omegaJMUMinusomegaJMUP4Termds11:=simplify(residue
(omegaJMUMinusomegaJMUP4/ds11^2,ds11=0));
omegaJMUMinusomegaJMUP4Termds21:=simplify(residue
(omegaJMUMinusomegaJMUP4/ds21^2,ds21=0));
omegaJMUMinusomegaJMUP4Termds12:=simplify(residue
(omegaJMUMinusomegaJMUP4/ds12^2,ds12=0));
omegaJMUMinusomegaJMUP4Termds22:=simplify(residue
(omegaJMUMinusomegaJMUP4/ds22^2,ds22=0));
omegaJMUMinusomegaJMUP4TermdX1:=simplify(residue
(omegaJMUMinusomegaJMUP4/dX1^2,dX1=0));
omegaJMUMinusomegaJMUP4Termdt22:=simplify(residue
(omegaJMUMinusomegaJMUP4/dt22^2,dt22=0));

```

$$\begin{aligned} \omega_{JMUMinus\omega_{JMUP4Termds11}} &:= \frac{s_{10} s_{11}}{s_{12}} & (1.6) \\ \omega_{JMUMinus\omega_{JMUP4Termds21}} &:= \frac{s_{20} s_{21}}{s_{22}} \end{aligned}$$

$$\omega_{JMUMinus\omega_{JMUP4Termds12}} := \frac{1}{2} \frac{1}{(s_{12} - s_{22}) s_{12}^2} \left((s_{10}^2 - s_{X10} s_{10} - s_{X10} (s_{X10} + s_{20})) s_{12}^2 + (-s_{10}^2 s_{22} - s_{10} s_{11}^2) s_{12} + s_{10} s_{11}^2 s_{22} \right)$$

$$\omega_{JMUMinus\omega_{JMUP4Termds22}} := \frac{1}{2} \frac{1}{(s_{12} - s_{22}) s_{22}^2} \left((-s_{20}^2 + s_{20} s_{X10} + s_{X10} (s_{X10} + s_{10})) s_{22}^2 + (s_{12} s_{20}^2 + s_{20} s_{21}^2) s_{22} - s_{12} s_{20} s_{21}^2 \right)$$

$$\begin{aligned} \omega_{JMUMinus\omega_{JMUP4TermdX1}} &:= 0 \\ \omega_{JMUMinus\omega_{JMUP4Termdt22}} &:= 0 \end{aligned}$$

Imposing $s_{X20}=0$ and obtaining the differential df

> $s_{X20} := -s_{X10} - s_{20} - s_{10} :$
 $s_{X10} := -s_{10} - s_{20} :$

```
pdsolve({diff(ff(s11, s21, s12, s22, X1), s11) =
omegaJMUMinusomegaJMUP4Termds11,
diff(ff(s11, s21, s12, s22, X1), s21) =
omegaJMUMinusomegaJMUP4Termds21,
diff(ff(s11, s21, s12, s22, X1), s12) =
omegaJMUMinusomegaJMUP4Termds12,
diff(ff(s11, s21, s12, s22, X1), s22) =
omegaJMUMinusomegaJMUP4Termds22,
diff(ff(s11, s21, s12, s22, X1), X1) = omegaJMUMinusomegaJMUP4TermdX1
}, {ff(s11, s21, s12, s22, X1)});
```

```
f := unapply(s10*s11^2/(2*s12)+s20*s21^2/(2*s22)+(1/2)*s10^2*ln
(s12)+(1/2)*s20^2*ln(s22), s11, s21, s12, s22, X1);
df := unapply(diff(f(s11, s21, s12, s22, X1), s11)*ds11+ diff(f(s11,
s21, s12, s22, X1), s21)*ds21+diff(f(s11, s21, s12, s22, X1), s12)*ds12+
diff(f(s11, s21, s12, s22, X1), s22)*ds22+diff(f(s11, s21, s12, s22,
X1), X1)*dX1, s11, s21, s12, s22, X1);
```

$$\left\{ ff(s_{11}, s_{21}, s_{12}, s_{22}, X_1) = \frac{1}{2} \frac{s_{10} s_{11}^2}{s_{12}} + \frac{1}{2} \frac{s_{20} s_{21}^2}{s_{22}} + \frac{1}{2} s_{10}^2 \ln(s_{12}) + \frac{1}{2} s_{20}^2 \ln(s_{22}) + _C1 \right\} \quad (1.7)$$

$$f := (s_{11}, s_{21}, s_{12}, s_{22}, X_1) \rightarrow \frac{1}{2} \frac{s_{10} s_{11}^2}{s_{12}} + \frac{1}{2} \frac{s_{20} s_{21}^2}{s_{22}} + \frac{1}{2} s_{10}^2 \ln(s_{12}) + \frac{1}{2} s_{20}^2 \ln(s_{22})$$

$$df := (s_{11}, s_{21}, s_{12}, s_{22}, X_1) \rightarrow \frac{s_{10} s_{11} ds_{11}}{s_{12}} + \frac{s_{20} s_{21} ds_{21}}{s_{22}} + \left(\frac{1}{2} \frac{s_{10}^2}{s_{12}} \right)$$

$$-\frac{1}{2} \frac{s_{10} s_{11}^2}{s_{12}^2} \Big) ds_{12} + \left(\frac{1}{2} \frac{s_{20}^2}{s_{22}} - \frac{1}{2} \frac{s_{20} s_{21}^2}{s_{22}^2} \right) ds_{22}$$

Rewriting f and df in terms of the gl_3 coordinates

```

> solve({t11functionduality=t11,t21functionduality=
t21,t31functionduality=t31,t12functionduality=
t12,t32functionduality=t32},{s11,s21,s12,s22,X1});
X1duality := t21;
s11duality:= -t31/(t22-t32);
s12duality:= 1/(t22-t32);
s21duality:= t11/(t12-t22);
s22duality:= -1/(t12-t22);
t22duality:= t22;
solve({t20functionduality=t20,t30functionduality=t30},{s10,s20}
);
s10duality:= t30;
s20duality:= t10;

ftCoordinates:=f(s11duality,s21duality,s12duality,s22duality,
X1duality,s10duality,s20duality);
ftCoordinatesbis:=1/2*(s10*t31^2/(t22-t32)-s20*t11^2/(t12-t22)-
s10^2*ln(t22-t32)-s20^2*ln(t12-t22));
{X1=t21,s11=-\frac{t31}{-t32+t22},s12=\frac{1}{-t32+t22},s21=\frac{t11}{-t22+t12},s22=
-\frac{1}{-t22+t12}}

```

(1.8)

$$\begin{aligned}
X1duality &:= t21 \\
s11duality &:= -\frac{t31}{-t32+t22} \\
s12duality &:= \frac{1}{-t32+t22} \\
s21duality &:= \frac{t11}{-t22+t12} \\
s22duality &:= -\frac{1}{-t22+t12} \\
t22duality &:= t22 \\
\{s10=t30,s20=-t20-t30\} \\
s10duality &:= t30 \\
s20duality &:= -t20-t30
\end{aligned}$$

$$\begin{aligned}
ftCoordinates &:= \frac{1}{2} \frac{s_{10} t_{31}^2}{-t_{32} + t_{22}} - \frac{1}{2} \frac{s_{20} t_{11}^2}{-t_{22} + t_{12}} + \frac{1}{2} s_{10}^2 \ln\left(\frac{1}{-t_{32} + t_{22}}\right) \\
&\quad + \frac{1}{2} s_{20}^2 \ln\left(-\frac{1}{-t_{22} + t_{12}}\right)
\end{aligned}$$

$$ftCoordinatesbis := \frac{1}{2} \frac{s_{10} t_{31}^2}{-t_{32} + t_{22}} - \frac{1}{2} \frac{s_{20} t_{11}^2}{-t_{22} + t_{12}} - \frac{1}{2} s_{10}^2 \ln(-t_{32} + t_{22})$$

$$-\frac{1}{2} s20^2 \ln(-t22 + t12)$$

Method 2: Direct verifications from the full JMU differentials

> omegaJMUdt11:=1/((t12-t22)*(-t12+t32))*(-p^3+(t11+t21+t31+(t12+t22+t32)*q)*p^2+((-t22-t32)*t12-t22*t32)*q^2+((-t31-t21)*t12+(-t11-t31)*t22-t32*(t11+t21))*q+t12*(-t20-t30)+t22*t20+t30*t32+(-t31-t21)*t11-t21*t31)*p+q^3*t12*t22*t32+(t21*t32+t22*t31)*t12+t22*t32*t11)*q^2+((t20*t32+t21*t31+t22*t30)*t12+((-t20-t30)*t32+t11*t31)*t22+t32*t11*t21)*q+(t20+t30)*(t31-t11+t21)*t12+(t11*t30-t31*(t20+t30))*t22+(t11*t20-t21*(t20+t30))*t32+t11*t21*t31):

omegaJMUdt21:=1/((t12-t22)*(-t22+t32))*(p^3+((-t12-t22-t32)*q-t31-t11-t21)*p^2+((t12+t32)*t22+t12*t32)*q^2+(t12*(t21+t31)+t22*(t11+t31)+t32*(t11+t21))*q-t22*t20-t30*t32+(t20+t30)*t12+(t11+t31)*t21+t11*t31)*p-q^3*t12*t22*t32+((-t11*t32-t12*t31)*t22-t32*t12*t21)*q^2+((t20+t30)*t32-t11*t31-t12*t30)*t22+(-t11*t21-t12*t20)*t32-t12*t21*t31)*q+t20*(t31+t11-t21)*t22+(t21*(t20+t30)-t11*t20)*t32+(-t20*t31-t21*t30)*t12-t11*t21*t31):

omegaJMUdt31:=1/((-t22+t32)*(-t12+t32))*(p^3+((-t12-t22-t32)*q-t31-t11-t21)*p^2+((t12+t22)*t32+t12*t22)*q^2+(t12*(t21+t31)+t22*(t11+t31)+t32*(t11+t21))*q-t30*t32-t22*t20+(t20+t30)*t12+t31*(t11+t21)+t11*t21)*p-q^3*t12*t22*t32+((-t11*t22-t12*t21)*t32-t31*t12*t22)*q^2+((t20+t30)*t22-t11*t21-t12*t20)*t32+(-t11*t31-t12*t30)*t22-t12*t21*t31)*q-t30*(t31-t11-t21)*t32+(-t11*t30+t31*(t20+t30))*t22+(-t20*t31-t21*t30)*t12-t11*t21*t31):

omegaJMUdt12:=1/(2*(t12-t22)^2*(-t12+t32)^2)*(-(t20+t30)^2*t12^3+((-q^3*(t31-2*t11+t21)*t32+2*q*(-p+t31)*q+t30)*t11-(t21+t31)*(-p+t31)*q^2-t30*(t21+t31)*q+(t20+2*t30)*(t20+t30))*t22+(-2*q*(p-t21)*q-t20)*t11+(p-t21)*(t21+t31)*q^2-t20*(t21+t31)*q+2*t20^2+3*t30*t20+t30^2)*t32+(-t20-t30)*t11^2+(-(2*(p-t21))*(-p+t31)*q+(2*(t20+t30))*(t31-p+t21))*t11+(p-t21)*(t21+t31)*(-p+t31)*q-(t20+t30)*((-t31-t21)*p+t31^2+t21*t31+t21^2))*t12^2+(q^3*(-t11+t31)*t32-q*(-p+t31)*q+t30)*t11+t31*(-p+t31)*q^2+t31*t30*q-t30*(t20+t30))*t22^2+(-q^3*(t11-t21)*t32^2+(2*q^2*t11^2-2*q*(t21+t31)*q+3*t20*(1/2)+3*t30*(1/2))*t11+2*q^2*t21*t31+(t20+2*t30)*t21+2*t31*(t20+(1/2)*t30))*q-2*(t20+t30)^2)*

$$\begin{aligned}
& t32 + ((2*t31 - 2*p) * q + 2*t30) * t11^2 + (-(-p + t31) * (t31 + 2*t21 + p) * q + \\
& (t30 + 3*t20) * p - t21 * t30 - (3 * (t20 + 4*t30 * (1/3))) * t31) * t11 + (-p^2 * t21 + \\
& t21 * t31^2) * q + (-t21 * t20 - 2*t31 * (t20 + (1/2) * t30)) * p + (2 * (t20 + t30)) * \\
& t31 * ((1/2) * t21 + t31) * t22 + (q * ((p - t21) * q - t20) * t11 - (p - t21) * t21 * \\
& q^2 + t21 * t20 * q - t20 * (t20 + t30)) * t32^2 + ((-2*p + 2*t21) * q + 2*t20) * \\
& t11^2 + (2 * (p - t21)) * ((1/2) * p + t31 + (1/2) * t21) * q + (t20 + 3*t30) * p + (-4 * \\
& t20 - 3*t30) * t21 - t31 * t20) * t11 + (-p^2 * t31 + t21^2 * t31) * q + ((-t20 - 2 * \\
& t30) * t21 - t31 * t30) * p + t21 * (t20 + t30) * (t31 + 2*t21) * t32 - (p - t21) * (p - \\
& t11) * (t31 - 2*t11 + t21) * (-p + t31) * t12 + ((-q^2 * t11^2 + q * ((t31 + p) * q + \\
& t20 + t30) * t11 - q^2 * p * t31 - (t20 + t30) * t31 * q + t30 * (t20 + t30)) * t32 - (- \\
& t11 + t31) * ((p - t31) * q - t30) * t11 + p * (-p + t31) * q - p * t20 + t31 * (t20 + t30) \\
&) * t22^2 + ((-q^2 * t11^2 + (p + t21) * q + t20 + t30) * q * t11 - q^2 * p * t21 - t21 * \\
& (t20 + t30) * q + t20 * (t20 + t30)) * t32^2 + (-q * (t31 - 2*p + t21) * t11^2 + (-2 * \\
& p^2 + 2*t21 * t31) * q + (-t20 - t30) * p + t31 * t20 + t21 * t30) * t11 + ((t21 + t31) * \\
& p - 2*t21 * t31) * p * q + (t20 * t21 + t30 * t31) * p - t31 * t21 * (t20 + t30) * t32 + (p - \\
& t11) * (-t11 + t31) * (-p + t31) * (p - t21) * t22 - (((-p + t21) * q + t20) * t11 + p * \\
& (p - t21) * q + p * t30 - t21 * (t20 + t30)) * t32 + (p - t21) * (p - t11) * (-p + t31) * \\
& (t11 - t21) * t32) :
\end{aligned}$$

$$\begin{aligned}
\text{omegaJMUdt22} := & 1 / (2 * (t12 - t22)^2 * (-t22 + t32)^2) * (-t20^2 * t22^3 + ((- \\
& q^3 * (t31 + t11 - 2*t21) * t12 - 2*q * ((p - t11) * q + t20 + t30) * t21 + (p - t11) * \\
& (t11 + t31) * q^2 + (t20 + t30) * (t11 + t31) * q + 2*t20^2 + t30 * t20) * t32 + (2*q * \\
& (-p + t31) * q + t30) * t21 - (t11 + t31) * (-p + t31) * q^2 - t30 * (t11 + t31) * q + t20 * \\
& (t20 - t30)) * t12 + t20 * t21^2 + (-2 * (p - t11)) * (-p + t31) * q - 2*t20 * (t31 - p + \\
& t11) * t21 + (p - t11) * (t11 + t31) * (-p + t31) * q + t20 * ((-t11 - t31) * p + t31^2 + \\
& t11 * t31 + t11^2)) * t22^2 + (q^3 * (t11 - t21) * t12 + q * ((p - t11) * q + t20 + t30) \\
& * t21 - (p - t11) * t11 * q^2 - t11 * (t20 + t30) * q - t20 * (t20 + t30)) * t32^2 + (q^3 * \\
& (-t21 + t31) * t12^2 + (2*q^2 * t21^2 - (2 * ((t11 + t31) * q - 3*t20 * (1/2)))) * q * \\
& t21 + 2*q^2 * t11 * t31 + (t11 * (-t20 + t30) - 2*t31 * (t20 + (1/2) * t30)) * q - 2 * \\
& t20^2) * t12 + ((-2*p + 2*t11) * q - 2*t20 - 2*t30) * t21^2 + (2 * ((1/2) * p + t31 + \\
& (1/2) * t11)) * (p - t11) * q + (2*t30 - t20) * p + (t30 + 4*t20) * t11 + t31 * (t20 + \\
& t30) * t21 + (-p^2 * t31 + t11^2 * t31) * q + ((t20 - t30) * t11 - t31 * t30) * p - t11 * \\
& t20 * (t31 + 2*t11) * t32 + (-q * ((-p + t31) * q + t30) * t21 + t31 * (-p + t31) * q^2 + \\
& t31 * t30 * q + t30 * t20) * t12^2 + (((2*t31 - 2*p) * q + 2*t30) * t21^2 + (-(-p + \\
& t31) * (t31 + 2*t11 + p) * q + (-2*t30 - 3*t20) * p - t11 * t30 + 3*t31 * (t20 - (1/3) * \\
& t30)) * t21 + (-p^2 * t11 + t11 * t31^2) * q + ((t20 + t30) * t11 + 2*t31 * (t20 + \\
& (1/2) * t30)) * p - (2 * (t31 + (1/2) * t11)) * t31 * t20) * t12 - (p - t21) * (p - t11) * \\
& (t31 + t11 - 2*t21) * (-p + t31) * t22 + ((-q^2 * t21^2 + (p + t11) * q - t20) * q * \\
& t21 - q^2 * p * t11 + t20 * t11 * q + t20 * (t20 + t30)) * t12 + (t11 - t21) * (((-p + t11) \\
& * q - t20 - t30) * t21 + p * (p - t11) * q + p * t30 + t11 * t20) * t32^2 + ((-q^2 * t21^2 + \\
& q * ((t31 + p) * q - t20) * t21 - q^2 * p * t31 + t31 * t20 * q - t30 * t20) * t12^2 + (-q * \\
& (-2*p + t31 + t11) * t21^2 + (-2*p^2 + 2*t11 * t31) * q + p * t20 + t11 * t30 - t31 *
\end{aligned}$$

$$\begin{aligned}
& s_{22} - (1/2) * (-s_{21} + s_{11}) * (ds_{12} - ds_{22}) * (X_1^2 * s_{12} * s_{22} + ((-2 * P - 2 * s_{21}) * \\
& s_{12} - 2 * s_{22} * (P + s_{11})) * X_1 + s_{12} * s_{20} + s_{10} * s_{22} + (P + s_{11}) * (s_{21} + P)) * Q^2 + (-(- \\
& s_{12} + s_{22}) * ((s_{21} + P) * s_{12} + s_{22} * (P + s_{11})) * (ds_{12} - ds_{22}) * X_1^3 + (2 * dX_1 * \\
& (s_{21} + P) * s_{12}^3 + (-2 * dX_1 * (P - s_{11} + 2 * s_{21}) * s_{22} + (2 * ds_{11} - 2 * ds_{21}) * P + (2 * \\
& ds_{11} - 2 * ds_{21}) * s_{21} + (s_{10} - s_{20}) * ds_{12} + 2 * ds_{22} * s_{20}) * s_{12}^2 + (-2 * dX_1 * (P + 2 * \\
& s_{11} - s_{21}) * s_{22}^2 + ((2 * ds_{11} - 2 * ds_{21}) * s_{11} + (-2 * ds_{11} + 2 * ds_{21}) * s_{21} + (-3 * \\
& s_{10} + s_{20}) * ds_{12} + ds_{22} * (s_{10} - 3 * s_{20})) * s_{22} - (2 * (P + 3 * s_{11} * (1/2) - (1/2) * \\
& s_{21})) * (ds_{12} - ds_{22}) * (s_{21} + P)) * s_{12} + 2 * s_{22} * (dX_1 * (P + s_{11}) * s_{22}^2 + ((- \\
& ds_{11} + ds_{21}) * P + (-ds_{11} + ds_{21}) * s_{11} + ds_{12} * s_{10} - (1/2) * ds_{22} * (s_{10} - s_{20})) * \\
& s_{22} + (P - (1/2) * s_{11} + 3 * s_{21} * (1/2)) * (ds_{12} - ds_{22}) * (P + s_{11})) * X_1^2 + (2 * \\
& dX_1 * (s_{10} - s_{20}) * s_{12}^3 + (-6 * dX_1 * (s_{10} - s_{20}) * s_{22} - 4 * P^2 * dX_1 - 4 * dX_1 * (s_{21} + \\
& s_{11}) * P - 4 * dX_1 * s_{11} * s_{21} + 2 * ds_{11} * s_{10} - 2 * s_{20} * (ds_{11} - 2 * ds_{21})) * s_{12}^2 + (6 * \\
& dX_1 * (s_{10} - s_{20}) * s_{22}^2 + (8 * P^2 * dX_1 + 8 * dX_1 * (s_{21} + s_{11}) * P + 8 * dX_1 * s_{11} * s_{21} + \\
& (-6 * ds_{11} + 2 * ds_{21}) * s_{10} + 2 * s_{20} * (ds_{11} - 3 * ds_{21})) * s_{22} + (-4 * ds_{11} + 4 * ds_{21}) * \\
& P^2 + ((-4 * ds_{11} + 4 * ds_{21}) * s_{11} + (-4 * ds_{11} + 4 * ds_{21}) * s_{21} + (s_{10} + s_{20}) * (ds_{12} - \\
& ds_{22})) * P + ((-4 * ds_{11} + 4 * ds_{21}) * s_{21} + 2 * s_{20} * (ds_{12} - ds_{22})) * s_{11} + s_{21} * (s_{10} - \\
& s_{20}) * (ds_{12} - ds_{22})) * s_{12} - 2 * dX_1 * (s_{10} - s_{20}) * s_{22}^3 + (-4 * P^2 * dX_1 - 4 * dX_1 * \\
& (s_{21} + s_{11}) * P - 4 * dX_1 * s_{11} * s_{21} + (4 * ds_{11} - 2 * ds_{21}) * s_{10} + 2 * ds_{21} * s_{20}) * \\
& s_{22}^2 + ((4 * ds_{11} - 4 * ds_{21}) * P^2 + ((4 * ds_{11} - 4 * ds_{21}) * s_{11} + (4 * ds_{11} - 4 * ds_{21}) \\
& * s_{21} - (s_{10} + s_{20}) * (ds_{12} - ds_{22})) * P + ((4 * ds_{11} - 4 * ds_{21}) * s_{21} + (s_{10} - s_{20}) * \\
& (ds_{12} - ds_{22})) * s_{11} - 2 * s_{21} * s_{10} * (ds_{12} - ds_{22})) * s_{22} + (2 * (-s_{21} + s_{11})) * \\
& (s_{21} + P) * (P + s_{11}) * (ds_{12} - ds_{22})) * X_1 + 2 * dX_1 * (s_{10} + s_{20}) * (P + s_{11} + s_{21}) * \\
& s_{12}^2 + (-4 * dX_1 * (s_{10} + s_{20}) * (P + s_{11} + s_{21}) * s_{22} + (2 * (s_{10} + s_{20})) * (ds_{11} - \\
& ds_{21}) * P + 2 * s_{20} * (ds_{11} - ds_{21}) * s_{11} + 2 * s_{10} * (ds_{11} - ds_{21}) * s_{21} + (s_{X10} + s_{20}) * \\
& (s_{X10} + s_{10}) * (ds_{12} - ds_{22})) * s_{12} + 2 * dX_1 * (s_{10} + s_{20}) * (P + s_{11} + s_{21}) * s_{22}^2 + \\
& (- (2 * (s_{10} + s_{20})) * (ds_{11} - ds_{21}) * P - 2 * s_{20} * (ds_{11} - ds_{21}) * s_{11} - 2 * s_{10} * (ds_{11} \\
& - ds_{21}) * s_{21} - (s_{X10} + s_{20}) * (s_{X10} + s_{10}) * (ds_{12} - ds_{22})) * s_{22} - (ds_{12} - ds_{22}) * (\\
& (s_{10} + s_{20}) * P + s_{21} * s_{10} + s_{11} * s_{20}) * (-s_{21} + s_{11})) * Q - (-s_{12} + s_{22}) * ((-ds_{12} * \\
& s_{10} - ds_{22} * s_{20}) * s_{12} + (ds_{12} * s_{10} + ds_{22} * s_{20}) * s_{22} + (s_{21} + P) * (P + s_{11}) * (ds_{12} \\
& - ds_{22})) * X_1^3 + (-2 * dX_1 * s_{10} * s_{12}^3 + ((4 * (s_{10} - (1/2) * s_{20})) * dX_1 * s_{22} + 2 * \\
& P^2 * dX_1 + 2 * dX_1 * (s_{21} + s_{11}) * P + 2 * dX_1 * s_{11} * s_{21} - 2 * ds_{11} * s_{10} - 2 * ds_{21} * s_{20}) * \\
& s_{12}^2 + (-2 * dX_1 * (s_{10} - 2 * s_{20}) * s_{22}^2 + (-4 * P^2 * dX_1 - 4 * dX_1 * (s_{21} + s_{11}) * \\
& P - 4 * dX_1 * s_{11} * s_{21} + 4 * ds_{11} * s_{10} + 4 * ds_{21} * s_{20}) * s_{22} + (2 * ds_{11} - 2 * ds_{21}) * P^2 + \\
& ((2 * ds_{11} - 2 * ds_{21}) * s_{11} + (2 * ds_{11} - 2 * ds_{21}) * s_{21} - (s_{10} + s_{20}) * (ds_{12} - ds_{22})) \\
& * P + ((2 * ds_{11} - 2 * ds_{21}) * s_{21} - s_{20} * (ds_{12} - ds_{22})) * s_{11} - s_{21} * s_{10} * (ds_{12} - \\
& ds_{22})) * s_{12} - 2 * s_{22}^3 * dX_1 * s_{20} + (2 * P^2 * dX_1 + 2 * dX_1 * (s_{21} + s_{11}) * P + 2 * dX_1 * \\
& s_{11} * s_{21} - 2 * ds_{11} * s_{10} - 2 * ds_{21} * s_{20}) * s_{22}^2 + ((-2 * ds_{11} + 2 * ds_{21}) * P^2 + (\\
& (-2 * ds_{11} + 2 * ds_{21}) * s_{11} + (-2 * ds_{11} + 2 * ds_{21}) * s_{21} + (s_{10} + s_{20}) * (ds_{12} - ds_{22}) \\
&) * P + ((-2 * ds_{11} + 2 * ds_{21}) * s_{21} + s_{20} * (ds_{12} - ds_{22})) * s_{11} + s_{21} * s_{10} * (ds_{12} - \\
& ds_{22})) * s_{22} - (-s_{21} + s_{11}) * (s_{21} + P) * (P + s_{11}) * (ds_{12} - ds_{22})) * X_1^2 + (-2 * \\
& dX_1 * (s_{10} + s_{20}) * (P + s_{11} + s_{21}) * s_{12}^2 + (4 * dX_1 * (s_{10} + s_{20}) * (P + s_{11} + s_{21}) * \\
& s_{22} - (2 * (s_{10} + s_{20})) * (ds_{11} - ds_{21}) * P - 2 * s_{20} * (ds_{11} - ds_{21}) * s_{11} - 2 * s_{10} *
\end{aligned}$$

```
(ds11-ds21)*s21-(2*((sX10+(1/2)*s20)*s10+sX10*(sX10+s20)))*
(ds12-ds22))*s12-2*dX1*(s10+s20)*(P+s11+s21)*s22^2+((2*(s10+
s20))*(ds11-ds21)*P+2*s20*(ds11-ds21)*s11+2*s10*(ds11-ds21)*
s21+(2*((sX10+(1/2)*s20)*s10+sX10*(sX10+s20)))*(ds12-ds22))*
s22+(ds12-ds22)*((s10+s20)*P+s21*s10+s11*s20)*(-s21+s11))*X1-2*
sX10*(sX10+s10+s20)*(dX1*s12^2+(-2*dX1*s22+ds11-ds21)*s12+dX1*
s22^2+(-ds11+ds21)*s22-(1/2)*(-s21+s11)*(ds12-ds22)))/((2*(Q-
X1))*(-s12+s22)^2):
```

```
> omegaJMUfunction:=unapply(omegaJMU,q,p):
omegaJMUfunction2:=unapply(simplify(omegaJMUfunction(P,Q+t22*P)
),t11,t21,t31,t12,t22,t32,t20,t30,dt11,dt21,dt31,dt12,dt22,
dt32):
omegaJMUduality:=simplify(omegaJMUfunction2(t11functionduality,
t21functionduality,t31functionduality,t12functionduality,
t22functionduality,t32functionduality,t20functionduality,
t30functionduality,dt11duality,dt21duality,dt31duality,
dt12duality,dt22duality,dt32duality)):
> DifferenceomegaJMU:=simplify(omegaJMUduality-omegaJMUP4):
DifferenceomegaJMUds11:=residue(DifferenceomegaJMU/ds11^2,ds11=
0);
DifferenceomegaJMUds21:=residue(DifferenceomegaJMU/ds21^2,ds21=
0);
DifferenceomegaJMUds12:=simplify(residue
(DifferenceomegaJMU/ds12^2,ds12=0));
DifferenceomegaJMUds22:=simplify(residue
(DifferenceomegaJMU/ds22^2,ds22=0));
DifferenceomegaJMUdX1:=simplify(residue
(DifferenceomegaJMU/dX1^2,dX1=0));
```

```
simplify(DifferenceomegaJMU-omegaJMUMinusomegaJMUP4);
```

$$\text{DifferenceomegaJMUds11} := \frac{s_{10}s_{11}}{s_{12}} \quad (2.1)$$

$$\text{DifferenceomegaJMUds21} := \frac{s_{20}s_{21}}{s_{22}}$$

$$\text{DifferenceomegaJMUds12} := \frac{1}{2} \frac{s_{10}(s_{10}s_{12} - s_{11}^2)}{s_{12}^2}$$

$$\text{DifferenceomegaJMUds22} := \frac{1}{2} \frac{s_{20}(s_{20}s_{22} - s_{21}^2)}{s_{22}^2}$$

$$\text{DifferenceomegaJMUdX1} := 0$$

```
> pdsolve({DifferenceomegaJMUds11=diff(gg(s11,s21,s12,s22,X1),
s11),
DifferenceomegaJMUds21=diff(gg(s11,s21,s12,s22,X1),s21),
```

```

DifferenceomegaJMUds12:=diff (gg (s11 , s21 , s12 , s22 , X1) , s12) ,
DifferenceomegaJMUds22:=diff (gg (s11 , s21 , s12 , s22 , X1) , s22) ,
DifferenceomegaJMUdX1:=diff (gg (s11 , s21 , s12 , s22 , X1) , X1)
} , {gg (s11 , s21 , s12 , s22 , X1) } ) ;
g:=unapply (s10*s11^2/ (2*s12)+s20*s21^2/ (2*s22)+(1/2)*s10^2*ln
(s12)+(1/2)*s20^2*ln (s22) , s11 , s21 , s12 , s22 , X1 , s10 , s20) ;
dg:=unapply ( diff (g (s11 , s21 , s12 , s22 , X1 , s10 , s20) , s11)*ds11+ diff
(g (s11 , s21 , s12 , s22 , X1 , s10 , s20) , s21)*ds21+diff (g (s11 , s21 , s12 ,
s22 , X1 , s10 , s20) , s12)*ds12+diff (g (s11 , s21 , s12 , s22 , X1 , s10 , s20) ,
s22)*ds22+diff (g (s11 , s21 , s12 , s22 , X1 , s10 , s20) , X1)*dX1 , s11 , s21 ,
s12 , s22 , X1) :
simplify (omegaJMUduality-omegaJMUP4-dg (s11 , s21 , s12 , s22 , X1) ) ;

```

$$\left\{ \begin{aligned} gg(s11, s21, s12, s22, X1) &= \frac{1}{2} \frac{s10 s11^2}{s12} + \frac{1}{2} \frac{s20 s21^2}{s22} + \frac{1}{2} s10^2 \ln(s12) \\ &+ \frac{1}{2} s20^2 \ln(s22) + _CI \end{aligned} \right\} \quad (2.2)$$

$$g := (s11, s21, s12, s22, X1, s10, s20) \rightarrow \frac{1}{2} \frac{s10 s11^2}{s12} + \frac{1}{2} \frac{s20 s21^2}{s22} + \frac{1}{2} s10^2 \ln(s12) + \frac{1}{2} s20^2 \ln(s22)$$

0

```

> gtcoordinates:=g (s11duality , s21duality , s12duality , s22duality ,
X1duality , s10duality , s20duality) ;
gtcoordinates:= 1/2* (t30*t31^2/(t22-t32)-t10*t11^2/(t12-t22)-
t30^2*ln (t22-t32)-t10^2*ln (t12-t22)) ;

```

$$gtcoordinates := \frac{1}{2} \frac{t30 t31^2}{-t32 + t22} - \frac{1}{2} \frac{(-t20 - t30) t11^2}{-t22 + t12} + \frac{1}{2} t30^2 \ln\left(\frac{1}{-t32 + t22}\right) + \frac{1}{2} (-t20 - t30)^2 \ln\left(-\frac{1}{-t22 + t12}\right) \quad (2.3)$$

$$gtcoordinates := \frac{1}{2} \frac{t30 t31^2}{-t32 + t22} - \frac{1}{2} \frac{(-t20 - t30) t11^2}{-t22 + t12} - \frac{1}{2} t30^2 \ln(-t32 + t22) - \frac{1}{2} (-t20 - t30)^2 \ln(-t22 + t12)$$

```

> simplify (g (s11 , s21 , s12 , s22 , X1 , s10 , s20)-f (s11 , s21 , s12 ,
s22 , X1) ) ;

```

0

(2.4)