```
> restart:
  with(LinearAlgebra):
```

1. gl_2 case: Loading the general spectral curve. The times are (s12, s22, s11,s21, X1) and monodromies are (s10,s20, sX10, sX20).

```
> CoherenceEquation:=sX10+sX20+s10+s20;
  sX20:=-(sX10+s10+s20);

  R1:=unapply((-s10-s20)/(xi-X1)-s11-s21+(-s12-s22)*xi,xi):
  R2:=unapply(sX10*(-s10-s20-sX10)/(xi-X1)^2+s10*s22+s11*s21+s12*
  s20+(s11*s22+s12*s21)*xi+s12*s22*xi^2,xi);


  SpectralCurveGl2 := unapply(((-s12*s22*lambda^2+((-s21-y)*s12-
  s22*(s11+y))*lambda+s12*s22*q^2+((s21+p)*s12+s22*(p+s11))*q+(p-y)
  *(p+s21+s11+y))*X1^3+(2*s22*lambda^3*s12+(q*s12*s22+(2*s21+2*y)*
  s12+2*s22*(s11+y))*lambda^2+(-s12*s22*q^2-(s12+s22)*(p-y)*q+s12*
  s20+s10*s22-p^2+(-s11-s21)*p+2*y^2+(2*s11+2*s21)*y+s11*s21)*
  lambda-2*s12*s22*q^3+((-2*s21-2*p)*s12-2*s22*(p+s11))*q^2+(-s12*
  s20-s10*s22-2*p^2+(-2*s11-2*s21)*p+y^2+(s21+s11)*y-s11*s21)*q+
  (sX10+sX20)*(p-y))*X1^2+(-s12*s22*lambda^4+(-2*q*s12*s22+(-s21-y)
  *s12-s22*(s11+y))*lambda^3+(((-2*s21-2*y)*s12-2*s22*(s11+y))*q-
  s12*s20-s10*s22-(s11+y)*(s21+y))*lambda^2+(2*s12*s22*q^3+((2*
  s21+2*p)*s12+2*s22*(p+s11))*q^2+(2*(p-y))*(p+s21+s11+y)*q-(sX10+
  sX20)*(p-y))*lambda+q*(s12*s22*q^3+((s21+p)*s12+s22*(p+s11))*q^2+
  (s12*s20+s10*s22+(p+s11)*(s21+p))*q-(sX10+sX20)*(p-y)))*X1+
  lambda^4*q*s12*s22+q*((s21+y)*s12+s22*(s11+y))*lambda^3+q*(s12*
  s20+s10*s22+(s11+y)*(s21+y))*lambda^2+(-q^4*s12*s22+((-s21-p)*s12
  -s22*(p+s11))*q^3+(-s12*s20-s10*s22-(p+s11)*(s21+p))*q^2+(sX10+
  sX20)*(p-y)*q-sX10*sX20)*lambda+q*sX10*sX20)/((lambda-X1)^2*(q-
  X1)),
  lambda,y,q,p,s12,s22,s11,s21,X1,s10,s20,sX10);
```

$$CoherenceEquation := sX10 + sX20 + s10 + s20 \tag{1}$$

$$sX20 := -sX10 - s10 - s20$$

$$R2 := \xi \to \frac{sX10\,(-sX10 - s10 - s20)}{(\xi - X1)^2} + s10\,s22 + s11\,s21 + s12\,s20 + (s11\,s22 + s12\,s21)\,\xi + s12\,s22\,\xi^2$$

$$SpectralCurveGl2 := (\lambda, y, q, p, s12, s22, s11, s21, X1, s10, s20, sX10) \to \frac{1}{(\lambda - X1)^2\,(q - X1)}\Big(\big(-s12\,s22\,\lambda^2 + ((-s21 - y)\,s12 - s22\,(s11 + y))\,\lambda + s12\,s22\,q^2 + ((s21 + p)\,s12 + s22\,(p + s11))\,q + (p - y)\,(p + s21 + s11 + y)\big)\,X1^3 + \big(2\,s22\,\lambda^3\,s12 + (q\,s12\,s22 + (2\,s21 + 2\,y)\,s12 + 2\,s22\,(s11 + y))\,\lambda^2 + \big(-s12\,s22\,q^2$$

$$-(s12+s22)\,(p-y)\,q+s12\,s20+s10\,s22-p^2+(-s11-s21)\,p+2\,y^2+(2\,s11+2\,s21)\,y+s11\,s21)\,\lambda-2\,s12\,s22\,q^3+((-2\,s21-2\,p)\,s12-2\,s22\,(p+s11))\,q^2+(-s12\,s20-s10\,s22-2\,p^2+(-2\,s11-2\,s21)\,p+y^2+(s21+s11)\,y-s11\,s21)\,q+(-s10-s20)\,(p-y))\,X1^2+(-s12\,s22\,\lambda^4+(-2\,q\,s12\,s22+(-s21-y)\,s12-s22\,(s11+y))\,\lambda^3+(((-2\,s21-2\,y)\,s12-2\,s22\,(s11+y))\,q-s12\,s20-s10\,s22-(s11+y)\,(s21+y))\,\lambda^2+(2\,s12\,s22\,q^3+((2\,s21+2\,p)\,s12+2\,s22\,(p+s11))\,q^2+2\,(p-y)\,(p+s21+s11+y)\,q-(-s10-s20)\,(p-y))\,\lambda+q\,(s12\,s22\,q^3+((s21+p)\,s12+s22\,(p+s11))\,q^2+(s12\,s20+s10\,s22+(p+s11)\,(s21+p))\,q-(-s10-s20)\,(p-y)))\,X1+\lambda^4\,q\,s12\,s22+q\,((s21+y)\,s12+s22\,(s11+y))\,\lambda^3+q\,(s12\,s20+s10\,s22+(s11+y)\,(s21+y))\,\lambda^2+(-q^4\,s12\,s22+((-s21-p)\,s12-s22\,(p+s11))\,q^3+(-s12\,s20-s10\,s22-(p+s11)\,(s21+p))\,q^2+(-s10-s20)\,(p-y)\,q-sX10\,(-sX10-s10-s20))\,\lambda+q\,sX10\,(-sX10-s10-s20))$$

```
> simplify(SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,
  sX10)-(Y^2-R1(xi)*Y+R2(xi)-(Q-X1)/(xi-X1)*(P^2-R1(Q)*P+R2(Q)) ));
  simplify(residue(simplify((xi-X1)^2*SpectralCurveGl2(xi,Y,Q,P,
  s12,s22,s11,s21,X1,s10,s20,sX10)    ),xi=X1));
  Coeffpolytopexi2Y2:=simplify(-residue(residue((xi-X1)^2*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^3,Y=
  0)/xi^3,xi=infinity));
  Coeffpolytopexi1Y2:=simplify(-residue(residue((xi-X1)
  ^2*SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)
  /Y^3,Y=0)/xi^2,xi=infinity));
  Coeffpolytopexi0Y2:=simplify(-residue(residue((xi-X1)
  ^2*SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)
  /Y^3,Y=0)/xi^1,xi=infinity));

  Coeffpolytopexi4Y1:=simplify(-residue(residue((xi-X1)^2*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^5,xi=infinity));
  Coeffpolytopexi3Y1:=simplify(-residue(residue((xi-X1)^2*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^4,xi=infinity));
  Coeffpolytopexi2Y1:=simplify(-residue(residue((xi-X1)
  ^2*SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)
  /Y^2,Y=0)/xi^3,xi=infinity));
  Coeffpolytopexi1Y1:=simplify(-residue(residue((xi-X1)
  ^2*SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)
  /Y^2,Y=0)/xi^2,xi=infinity));
  Coeffpolytopexi0Y1:=simplify(-residue(residue((xi-X1)
  ^2*SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)
```

```
/Y^2,Y=0)/xi^1,xi=infinity));

Coeffpolytopexi0Y0:=simplify(-residue(residue((xi-X1)^2*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^1,xi=infinity));
Coeffpolytopexi1Y0:=simplify(-residue(residue((xi-X1)^2*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^2,xi=infinity));
Coeffpolytopexi2Y0:=simplify(-residue(residue((xi-X1)^2*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^3,xi=infinity));
Coeffpolytopexi3Y0:=simplify(-residue(residue((xi-X1)^2*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^4,xi=infinity));
Coeffpolytopexi4Y0:=simplify(-residue(residue((xi-X1)^2*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^5,xi=infinity));
Coeffpolytopexi5Y0:=simplify(-residue(residue((xi-X1)^2*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^6,xi=infinity));
```

$$0$$
$$0$$

$$Coeffpolytopexi2Y2 := 1$$
$$Coeffpolytopexi1Y2 := -2\,X1$$
$$Coeffpolytopexi0Y2 := X1^2$$
$$Coeffpolytopexi4Y1 := 0$$
$$Coeffpolytopexi3Y1 := s12 + s22$$
$$Coeffpolytopexi2Y1 := (-2\,s12 - 2\,s22)\,X1 + s11 + s21$$
$$Coeffpolytopexi1Y1 := (s12 + s22)\,X1^2 + (-2\,s11 - 2\,s21)\,X1 + s10 + s20$$
$$Coeffpolytopexi0Y1 := X1\,((s21 + s11)\,X1 - s10 - s20)$$

$$Coeffpolytopexi0Y0 := \frac{1}{Q - X1}\Big(\big(s12\,s22\,Q^2 + ((s12 + s22)\,P + s22\,s11 + s12\,s21)\,Q + P\,(P + s11 + s21)\big)\,X1^3 + \big(-2\,s12\,s22\,Q^3 + ((-2\,s12 - 2\,s22)\,P - 2\,s22\,s11 - 2\,s12\,s21)\,Q^2 + \big(-2\,P^2 + (-2\,s11 - 2\,s21)\,P - s10\,s22 - s11\,s21 - s12\,s20\big)\,Q - P\,(s10 + s20)\big)\,X1^2 + Q\,\big(s12\,s22\,Q^3 + ((s12 + s22)\,P + s22\,s11 + s12\,s21)\,Q^2 + \big(P^2 + (s21 + s11)\,P + s10\,s22 + s11\,s21 + s12\,s20\big)\,Q + P\,(s10 + s20)\big)\,X1 - sX10\,Q\,(sX10 + s10 + s20)\Big)$$

$$Coeffpolytopexi1Y0 := \frac{1}{Q - X1}\Big(-Q^4\,s12\,s22 + (2\,s12\,s22\,X1 + (-s12 - s22)\,P - s22\,s11 - s12\,s21)\,Q^3 + \big(-X1^2\,s12\,s22 + ((2\,s12 + 2\,s22)\,P + 2\,s22\,s11 + 2\,s12\,s21)\,X1 - P^2 + (-s11 - s21)\,P - s10\,s22 - s11\,s21 - s12\,s20\big)\,Q^2 + 2\,\Big(\Big(-\frac{1}{2}\,s12 - \frac{1}{2}\,s22\Big)\,X1^2 + (P + s11 + s21)\,X1 - \frac{1}{2}\,s10 - \frac{1}{2}\,s20\Big)\,P\,Q + (-s11\,s22 - s12\,s21)\,X1^3 + \big(-P^2 + ($$

(2)

$$-s11 - s21)\,P + s10\,s22 + s11\,s21 + s12\,s20)\,X1^2 + P\,(s10 + s20)\,X1 + sX10\,(sX10 + s10 + s20))$$

$$Coeffpolytopexi2Y0 := (X1^2\,s22 - 2\,X1\,s21 + s20)\,s12 + (-2\,X1\,s11 + s10)\,s22 + s11\,s21$$

$$Coeffpolytopexi3Y0 := (-2\,X1\,s22 + s21)\,s12 + s22\,s11$$

$$Coeffpolytopexi4Y0 := s12\,s22$$

$$Coeffpolytopexi5Y0 := 0$$

```
> sX10:=0:
  simplify(residue(simplify((xi-X1)^2*SpectralCurveGl2(xi,Y,Q,P,
  s12,s22,s11,s21,X1,s10,s20,sX10)    ),xi=X1));
  Coeffpolytopexi2Y2:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^3,Y=
  0)/xi^3,xi=infinity));
  Coeffpolytopexi1Y2:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^3,Y=
  0)/xi^2,xi=infinity));
  Coeffpolytopexi0Y2:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^3,Y=
  0)/xi^1,xi=infinity));

  Coeffpolytopexi4Y1:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^5,xi=infinity));
  Coeffpolytopexi3Y1:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^4,xi=infinity));
  Coeffpolytopexi2Y1:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^3,xi=infinity));
  Coeffpolytopexi1Y1:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^2,xi=infinity));
  Coeffpolytopexi0Y1:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y^2,Y=
  0)/xi^1,xi=infinity));

  Coeffpolytopexi0Y0:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
  /xi^1,xi=infinity));
  Coeffpolytopexi1Y0:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
  /xi^2,xi=infinity));
  Coeffpolytopexi2Y0:=simplify(-residue(residue((xi-X1)*
  SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
```

```
/xi^3,xi=infinity));
Coeffpolytopexi3Y0:=simplify(-residue(residue((xi-X1)*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^4,xi=infinity));
Coeffpolytopexi4Y0:=simplify(-residue(residue((xi-X1)*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^5,xi=infinity));
Coeffpolytopexi5Y0:=simplify(-residue(residue((xi-X1)*
SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,s20,sX10)/Y,Y=0)
/xi^6,xi=infinity));
```

$$0 \tag{3}$$

$$Coeffpolytopexi2Y2 := 0$$
$$Coeffpolytopexi1Y2 := 1$$
$$Coeffpolytopexi0Y2 := -X1$$
$$Coeffpolytopexi4Y1 := 0$$
$$Coeffpolytopexi3Y1 := 0$$
$$Coeffpolytopexi2Y1 := s12 + s22$$
$$Coeffpolytopexi1Y1 := (-s12 - s22)\,X1 + s11 + s21$$
$$Coeffpolytopexi0Y1 := (-s11 - s21)\,X1 + s10 + s20$$
$$Coeffpolytopexi0Y0 := -s12\,s22\,Q^3 + ((-s12 - s22)\,P + s12\,s22\,X1 - s22\,s11 - s12\,s21)\,Q^2 + \left(-P^2 + ((s12 + s22)\,X1 - s11 - s21)\,P + (s11\,s22 + s12\,s21)\,X1 - s10\,s22 - s11\,s21 - s12\,s20\right)Q + P\,(X1\,P + (s21 + s11)\,X1 - s10 - s20)$$
$$Coeffpolytopexi1Y0 := (-X1\,s22 + s21)\,s11 + (-X1\,s21 + s20)\,s12 + s10\,s22$$
$$Coeffpolytopexi2Y0 := (-X1\,s22 + s21)\,s12 + s22\,s11$$
$$Coeffpolytopexi3Y0 := s12\,s22$$
$$Coeffpolytopexi4Y0 := 0$$
$$Coeffpolytopexi5Y0 := 0$$

```
> R1(xi);
R2(xi);
simplify(series(SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,
s20,sX10),Y=0));
simplify(series(SpectralCurveGl2(xi,Y,Q,P,s12,s22,s11,s21,X1,s10,
s20,sX10),Y=0));
RHSReduced:=-simplify(SpectralCurveGl2(xi,tdY+1/2*R1(xi),Q,P,s12,
s22,s11,s21,X1,s10,s20,sX10) -tdY^2):
RHSReduced2:=simplify(1/4*R1(xi)^2-R2(xi)):
simplify(RHSReduced-RHSReduced2-(Q-X1)*(P^2-R1(Q)*P+R2(Q))/(xi-
X1));
RHSReduced2bis:=1/4*(s12-s22)^2*xi^2+(1/2)*(s12-s22)*(s11-s21)*
xi+(s11-s21)^2/4+(1/2)*(s12-s22)*(s10-s20) +(1/4)*(s10+s20)^2/
(xi-X1)^2+(1/2)*(s10+s20)*((s12+s22)*X1+s11+s21)/(xi-X1):
simplify(series(simplify((xi-X1)^2*(RHSReduced2-RHSReduced2bis)),
xi=X1));
```

$$\frac{-s10 - s20}{\xi - X1} - s11 - s21 + (-s12 - s22)\,\xi \tag{4}$$

$$s10\,s22 + s11\,s21 + s12\,s20 + (s11\,s22 + s12\,s21)\,\xi + s12\,s22\,\xi^2$$

$$\frac{1}{X1-\xi}\left(s12\,s22\,Q^3 + ((-X1\,s22 + P + s21)\,s12 + s22\,(P + s11))\,Q^2 + (((-s21 - P)\,X1 + s20)\,s12 + ((-P - s11)\,X1 + s10)\,s22 + (P + s11)\,(s21 + P))\,Q + \xi\left(\xi\left(X1 - \xi\right)s22 + X1\,s21 - s21\,\xi - s20\right)s12 + \xi\left(X1\,s11 - s11\,\xi - s10\right)s22 - P\,(P + s11 + s21)\,X1 + P\,(s10 + s20) - s11\,s21\,\xi\right) + \frac{(-s12 - s22)\,\xi^2 + ((s12 + s22)\,X1 - s11 - s21)\,\xi + (s21 + s11)\,X1 - s10 - s20}{X1 - \xi}\,Y + Y^2$$

$$\frac{1}{X1-\xi}\left(s12\,s22\,Q^3 + ((-X1\,s22 + P + s21)\,s12 + s22\,(P + s11))\,Q^2 + (((-s21 - P)\,X1 + s20)\,s12 + ((-P - s11)\,X1 + s10)\,s22 + (P + s11)\,(s21 + P))\,Q + \xi\left(\xi\left(X1 - \xi\right)s22 + X1\,s21 - s21\,\xi - s20\right)s12 + \xi\left(X1\,s11 - s11\,\xi - s10\right)s22 - P\,(P + s11 + s21)\,X1 + P\,(s10 + s20) - s11\,s21\,\xi\right) + \frac{(-s12 - s22)\,\xi^2 + ((s12 + s22)\,X1 - s11 - s21)\,\xi + (s21 + s11)\,X1 - s10 - s20}{X1 - \xi}\,Y + Y^2$$

$$0$$
$$0$$

Identification with the one-matrix model

```
> Vprime:=unapply(a*xi+b+c/(xi-X1),xi);
  simplify(simplify((Vprime(x)-Vprime(ti))/(x-ti)));
  RHSMatrixModel:=Vprime(xi)^2/4- (c0/(xi-X1)+a);
  a:=-(s12-s22);
  b:=-(s11-s21);
  c:=-(s10+s20);
  s20:=-1;
  c0:=-(Q-X1)*(P^2-R1(Q)*P+R2(Q))-(s10+s20)*(X1*s22+s21);
  simplify(series(simplify((RHSReduced-RHSMatrixModel)*(xi-X1)^2),
  xi=X1));
  Vprime(xi);
```

$$Vprime := \xi \to a\,\xi + b + \frac{c}{\xi - X1} \qquad \textbf{(5)}$$

$$\frac{-X1^2\,a + (x + ti)\,a\,X1 - a\,ti\,x + c}{(x - X1)\,(-ti + X1)}$$

$$RHSMatrixModel := \frac{1}{4}\left(a\,\xi + b + \frac{c}{\xi - X1}\right)^2 - \frac{c0}{\xi - X1} - a$$

$$a := s22 - s12$$
$$b := -s11 + s21$$
$$c := -s10 - s20$$
$$s20 := -1$$

$$c0 := -(Q - X1)\left(P^2 - \left(\frac{-s10 + 1}{Q - X1} - s11 - s21 + (-s12 - s22)\,Q\right)P + s10\,s22 + s11\,s21\right.$$

$$-s12+Q\,(s11\,s22+s12\,s21)+s12\,s22\,Q^2\Big)-(s10-1)\,(X1\,s22+s21)$$

$$0$$

$$(s22-s12)\,\xi-s11+s21+\frac{-s10+1}{\xi-X1}$$

2. gl_3 case: Loading the spectral curve in terms of (Q,P) and the irregular times (t12,t22,t32,t11,t21,t31) and the monodromies (t10,t20, t30)

```
> P1:=unapply( (t12+t22+t32)*lambda+t11+t21+t31,lambda);
  P2:=unapply( (t12*t22+t12*t32+t22*t32)*lambda^2+((t21+t31)*t12+
  (t11+t31)*t22+t32*(t11+t21))*lambda-t10*t12-t20*t22-t30*t32+t21*
  t11 +t31*t11+t21*t31,lambda);
  P3:=unapply(t12*t22*t32*lambda^3+(t12*t22*t31+t12*t32*t21+t22*
  t32*t11)*lambda^2+(t12*t22*t30+t12*t32*t20+t22*t32*t10  +t12*t21*
  t31+t22*t11*t31+t32*t11*t21)*lambda,lambda);

  CoherenceEquation2:=t10+t20+t30;
  t10:=-(t20+t30);
  SpectralCurveGl3:=unapply(
  -p^3+(t11+t21+t31+(t12+t22+t32)*q)*p^2+(((-t12-t32)*t22-t12*t32)*
  q^2+((-t11-t31)*t22+(-t11-t21)*t32-t12*(t21+t31))*q+t20*t22+t30*
  t32+t12*(-t20-t30)+(-t31-t21)*t11-t21*t31)*p-(lambda-q)*(q^2*t12*
  t22*t32+(((lambda*t12+t11)*t32+t12*t31)*t22+t32*t12*t21)*q+(
  (lambda^2*t12+lambda*t11-t20-t30)*t32+(lambda*t31+t30)*t12+t11*
  t31)*t22+((lambda*t21+t20)*t12+t11*t21)*t32+t12*t21*t31)+(((t22+
  t32)*t12+t22*t32)*lambda^2+(t12*(t21+t31)+t22*(t11+t31)+t32*(t11+
  t21))*lambda+(t20+t30)*t12-t20*t22-t30*t32+t11*(t21+t31)+t21*t31)
  *y+((-t12-t22-t32)*lambda-t11-t21-t31)*y^2+y^3
  ,lambda,y,q,p,t12,t22,t32,t11,t21,t31,t20,t30);
```

$$P1 := \lambda \to (t12+t22+t32)\,\lambda+t11+t21+t31 \tag{6}$$

$$P2 := \lambda \to (t12\,t22+t12\,t32+t22\,t32)\,\lambda^2+((t21+t31)\,t12+(t11+t31)\,t22+t32\,(t11+t21))\,\lambda-t10\,t12-t20\,t22-t30\,t32+t21\,t11+t31\,t11+t21\,t31$$

$$P3 := \lambda \to t12\,t22\,t32\,\lambda^3+(t11\,t22\,t32+t12\,t21\,t32+t12\,t22\,t31)\,\lambda^2+(t10\,t22\,t32+t11\,t21\,t32+t11\,t22\,t31+t12\,t20\,t32+t12\,t21\,t31+t12\,t22\,t30)\,\lambda$$

$$CoherenceEquation2 := t10+t20+t30$$

$$t10 := -t20-t30$$

$$SpectralCurveGl3 := (\lambda, y, q, p, t12, t22, t32, t11, t21, t31, t20, t30) \to -p^3+(t11+t21+t31+(t12+t22+t32)\,q)\,p^2+(((-t12-t32)\,t22-t12\,t32)\,q^2+((-t11-t31)\,t22+(-t11-t21)\,t32-(t21+t31)\,t12)\,q+t20\,t22+t30\,t32+t12\,(-t20-t30)+(-t31-t21)\,t11-t21\,t31)\,p-(\lambda-q)\,(q^2\,t12\,t22\,t32+(((\lambda\,t12+t11)\,t32+t12\,t31)\,t22+t12\,t32\,t21)\,q+((\lambda^2\,t12+\lambda\,t11-t20-t30)\,t32+(\lambda\,t31+t30)\,t12+t31\,t11)\,t22$$

$$+\left(\left(\left(\lambda\, t21+t20\right) t12+t21\, t11\right) t32+t12\, t21\, t31\right)+\left(\left(\left(t22+t32\right) t12+t22\, t32\right)\lambda^{2}\right.$$
$$\left.+\left(\left(t21+t31\right) t12+\left(t11+t31\right) t22+t32\left(t11+t21\right)\right)\lambda+\left(t20+t30\right) t12-t20\, t22\right.$$
$$\left.-t30\, t32+t11\left(t21+t31\right)+t21\, t31\right) y+\left(\left(-t12-t22-t32\right)\lambda-t11-t21-t31\right) y^{2}$$
$$+y^{3}$$

```
> series(residue(SpectralCurveGl3(lambda,y,q,p,t12,t22,t32,t11,t21,
  t31,t20,t30)/y^2,y=0),lambda=0);
  series(residue(SpectralCurveGl3(lambda,y,q,p,t12,t22,t32,t11,t21,
  t31,t20,t30)/y^1,y=0),lambda=0);
```

$$t11\, t21+t11\, t31+t12\, t20+t12\, t30-t20\, t22+t21\, t31-t30\, t32+\left(t11\, t22+t11\, t32\right. \tag{7}$$
$$\left.+t12\, t21+t12\, t31+t21\, t32+t22\, t31\right)\lambda+\left(t12\, t22+t12\, t32+t22\, t32\right)\lambda^{2}$$
$$q^{3}\, t12\, t22\, t32-p\, q^{2}\, t12\, t22-p\, q^{2}\, t12\, t32-p\, q^{2}\, t22\, t32+q^{2}\, t11\, t22\, t32+q^{2}\, t12\, t21\, t32$$
$$+q^{2}\, t12\, t22\, t31+p^{2}\, q\, t12+p^{2}\, q\, t22+p^{2}\, q\, t32-p\, q\, t11\, t22-p\, q\, t11\, t32-p\, q\, t12\, t21$$
$$-p\, q\, t12\, t31-p\, q\, t21\, t32-p\, q\, t22\, t31+q\, t11\, t21\, t32+q\, t11\, t22\, t31+q\, t12\, t20\, t32$$
$$+q\, t12\, t21\, t31+q\, t12\, t22\, t30-q\, t20\, t22\, t32-q\, t22\, t30\, t32-p^{3}+p^{2}\, t11+p^{2}\, t21$$
$$+p^{2}\, t31-p\, t11\, t21-p\, t11\, t31-p\, t12\, t20-p\, t12\, t30+p\, t20\, t22-p\, t21\, t31+p\, t30\, t32$$
$$+\left(-t11\, t21\, t32-t11\, t22\, t31-t12\, t20\, t32-t12\, t21\, t31-t12\, t22\, t30+t20\, t22\, t32\right.$$
$$\left.+t22\, t30\, t32\right)\lambda+\left(-t11\, t22\, t32-t12\, t21\, t32-t12\, t22\, t31\right)\lambda^{2}-t12\, t22\, t32\,\lambda^{3}$$

```
> SpectralcurveGl3dual:=simplify(series( SpectralCurveGl3(y,lambda,
  q,p,t12,t22,t32,t11,t21,t31,t20,t30),y=0));
  Coeffy2:=series(simplify(residue(SpectralcurveGl3dual/y^3,y=0)),
  lambda=0);
  Coeffy1:=series(simplify(residue(SpectralcurveGl3dual/y^2,y=0)),
  lambda=0);
  Coeffy0:=series(simplify(residue(SpectralcurveGl3dual/y^1,y=0)),
  lambda=0);
  simplify(series(simplify(SpectralcurveGl3dual/(-t12*t22*t32)
  -SpectralCurveGl3(lambda,y,p,q,1/t12,1/t22,1/t32,-t11/t12,-
  t21/t22,-t31/t32,-t20,-t30)),lambda=0));
```

$$SpectralcurveGl3dual:=-p^{3}+\left(t11+t21+t31+\left(t12+t22+t32\right) q\right) p^{2}+\left(\left(\left(-t32\right.\right.\right. \tag{8}$$
$$\left.-t22\right) t12-t22\, t32\Big) q^{2}+\left(\left(-t31-t21\right) t12+\left(-t11-t31\right) t22-t32\left(t11+t21\right)\right) q$$
$$+t12\left(-t20-t30\right)+t20\, t22+t30\, t32+\left(-t31-t21\right) t11-t21\, t31\Big) p+q^{3}\, t12\, t22\, t32$$
$$+\left(\left(t21\, t32+t22\, t31\right) t12+t22\, t32\, t11\right) q^{2}+\left(\left(t20\, t32+t21\, t31+t22\, t30\right) t12+\left(\left(\right.\right.\right.$$
$$\left.\left.-t20-t30\right) t32+t31\, t11\right) t22+t32\, t11\, t21\Big) q-\lambda\left(t12\left(-t20-t30\right)+t20\, t22+t30\, t32\right.$$
$$\left.-\lambda^{2}+\left(t11+t21+t31\right)\lambda+\left(-t31-t21\right) t11-t21\, t31\right)+\left(\left(-t12-t22-t32\right)\lambda^{2}\right.$$
$$\left.+\left(\left(t21+t31\right) t12+\left(t11+t31\right) t22+t32\left(t11+t21\right)\right)\lambda+\left(\left(t20+t30\right) t32-t31\, t11\right.\right.$$
$$\left.\left.-t12\, t30\right) t22+\left(-t11\, t21-t12\, t20\right) t32-t12\, t21\, t31\Big) y+\left(\left(\left(-t31+\lambda\right) t22\right.\right.$$
$$\left.\left.-t32\left(t21-\lambda\right)\right) t12-t22\, t32\left(t11-\lambda\right)\right) y^{2}-t12\, t22\, t32\, y^{3}$$
$$Coeffy2:=\left(-t21\, t32-t22\, t31\right) t12-t22\, t32\, t11+\left(\left(t22+t32\right) t12+t22\, t32\right)\lambda$$
$$Coeffy1:=\left(\left(t20+t30\right) t32-t31\, t11-t12\, t30\right) t22+\left(-t11\, t21-t12\, t20\right) t32-t12\, t21\, t31$$
$$+\left(\left(t21+t31\right) t12+\left(t11+t31\right) t22+t32\left(t11+t21\right)\right)\lambda+\left(-t12-t22-t32\right)\lambda^{2}$$
$$Coeffy0:=-p^{3}+\left(t11+t21+t31+\left(t12+t22+t32\right) q\right) p^{2}+\left(\left(\left(-t32-t22\right) t12\right.\right.$$

$$-t22\,t32)\,q^2+((-t31-t21)\,t12+(-t11-t31)\,t22-t32\,(t11+t21))\,q+t12\,(-t20-t30)+t20\,t22+t30\,t32+(-t31-t21)\,t11-t21\,t31)\,p+q^3\,t12\,t22\,t32+((t21\,t32+t22\,t31)\,t12+t22\,t32\,t11)\,q^2+((t20\,t32+t21\,t31+t22\,t30)\,t12+((-t20-t30)\,t32+t31\,t11)\,t22+t32\,t11\,t21)\,q+(-t12\,(-t20-t30)-t20\,t22-t30\,t32-(-t31-t21)\,t11+t21\,t31)\,\lambda+(-t11-t21-t31)\,\lambda^2+\lambda^3$$

$$0$$

```
> SpectralCurveGl3Shifted:=unapply( simplify(series
  (SpectralCurveGl3(lambda,y-(beta1*lambda),q,p,t12,t22,t32,t11,
  t21,t31,t20,t30),lambda=0)),lambda,y,q,p,t12,t22,t32,t11,t21,t31,
  t20,t30);
```

$$SpectralCurveGl3Shifted := (\lambda, y, q, p, t12, t22, t32, t11, t21, t31, t20, t30) \rightarrow -p^3+(t11+t21+t31+(t12+t22+t32)\,q)\,p^2+(((-t32-t22)\,t12-t22\,t32)\,q^2+((-t31-t21)\,t12+(-t11-t31)\,t22-t32\,(t11+t21))\,q+t12\,(-t20-t30)+t20\,t22+t30\,t32+(-t31-t21)\,t11-t21\,t31)\,p+q^3\,t12\,t22\,t32+((t21\,t32+t31\,t22)\,t12+t22\,t32\,t11)\,q^2+((t20\,t32+t21\,t31+t22\,t30)\,t12+((-t20-t30)\,t32+t31\,t11)\,t22+t32\,t11\,t21)\,q-(t12\,(-t20-t30)+t20\,t22+t30\,t32+(-t31+y-t21)\,t11+(y-t21)\,(-y+t31))\,y+((-3\,\beta 1-t12-t22-t32)\,y^2+((2\,t11+2\,t21+2\,t31)\,\beta 1+(t21+t31)\,t12+(t11+t31)\,t22+t32\,(t11+t21))\,y+(t12\,(-t20-t30)+t20\,t22+t30\,t32+(-t31-t21)\,t11-t21\,t31)\,\beta 1+(-t20\,t32-t21\,t31-t22\,t30)\,t12+((t20+t30)\,t32-t31\,t11)\,t22-t32\,t11\,t21)\,\lambda+\left((3\,y-t11-t21-t31)\,\beta 1^2+\left((2\,y-t21-t31)\,t12+(2\,y-t11-t31)\,t22+2\,t32\left(y-\frac{1}{2}\,t11-\frac{1}{2}\,t21\right)\right)\beta 1+((y-t31)\,t22+t32\,(y-t21))\,t12+t22\,t32\,(y-t11)\right)\lambda^2-(t32+\beta 1)\,(t22+\beta 1)\,(t12+\beta 1)\,\lambda^3 \tag{9}$$

```
> beta1:=-t22;
  SpectralCurveGl3Shiftedbis:=simplify(series
  (SpectralCurveGl3Shifted(lambda,y,q,p,t12,t22,t32,t11,t21,t31,
  t20,t30),y=0));
  Termy2lambda2:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^3,y=0)/lambda^3,lambda=infinity));
  Termy2lambda1:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^3,y=0)/lambda^2,lambda=infinity));
  Termy2lambda0:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^3,y=0)/lambda^1,lambda=infinity));
  Termy1lambda3:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^2,y=0)/lambda^4,lambda=infinity));
  Termy1lambda2:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^2,y=0)/lambda^3,lambda=infinity));
  Termy1lambda1:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^2,y=0)/lambda^2,lambda=infinity));
  Termy1lambda0:=simplify(-residue(residue
  (SpectralCurveGl3Shiftedbis/y^2,y=0)/lambda^1,lambda=infinity));
```

```
Termy0lambda3:=simplify(-residue(residue
(SpectralCurveGl3Shiftedbis/y^1,y=0)/lambda^4,lambda=infinity));
Termy0lambda2:=simplify(-residue(residue
(SpectralCurveGl3Shiftedbis/y^1,y=0)/lambda^3,lambda=infinity));
Termy0lambda1:=simplify(series(simplify(-residue(residue
(SpectralCurveGl3Shiftedbis/y^1,y=0)/lambda^2,lambda=infinity)),
t20));
```

$$\beta 1 := -t22 \tag{10}$$

$$SpectralCurveGl3Shiftedbis := -p^3 + (t11+t21+t31+(t12+t22+t32)\,q)\,p^2 + (((-t12-t32)\,t22 - t12\,t32)\,q^2 + ((-t31-t21)\,t12 + (-t11-t31)\,t22 - t32\,(t11+t21))\,q + t20\,t22 + t12\,(-t20-t30) + t30\,t32 + (-t11-t31)\,t21 - t31\,t11)\,p + q^3\,t12\,t22\,t32 + ((t11\,t32 + t12\,t31)\,t22 + t12\,t32\,t21)\,q^2 + (((-t20-t30)\,t32 + t12\,t30 + t31\,t11)\,t22 + (t20\,t32 + t21\,t31)\,t12 + t32\,t11\,t21)\,q - ((\lambda\,t21 + t20)\,t22^2 + ((-\lambda\,t21 - t20)\,t12 + (-\lambda\,t21 - t20)\,t32 - (t11+t31)\,t21)\,t22 + ((\lambda\,t21+t20)\,t32 + t21\,t31)\,t12 + t32\,t11\,t21)\,\lambda + (-(t22-t32)\,(t12-t22)\,\lambda^2 + ((-t11-2\,t21 - t31)\,t22 + t32\,(t11+t21) + (t21+t31)\,t12)\,\lambda - t20\,t22 + (t20+t30)\,t12 + (t11 + t31)\,t21 + t31\,t11 - t30\,t32)\,y + (-t11-t21-t31+(2\,t22-t12-t32)\,\lambda)\,y^2 + y^3$$

$$Termy2lambda2 := 0$$

$$Termy2lambda1 := 2\,t22 - t12 - t32$$

$$Termy2lambda0 := -t11 - t21 - t31$$

$$Termy1lambda3 := 0$$

$$Termy1lambda2 := -(t22-t32)\,(t12-t22)$$

$$Termy1lambda1 := (-2\,t22 + t12 + t32)\,t21 + (-t11-t31)\,t22 + t32\,t11 + t12\,t31$$

$$Termy1lambda0 := (t20+t30)\,t12 - t20\,t22 - t30\,t32 + t11\,(t21+t31) + t21\,t31$$

$$Termy0lambda3 := 0$$

$$Termy0lambda2 := t21\,(t22-t32)\,(t12-t22)$$

$$Termy0lambda1 := ((t22-t32)\,t11 - t31\,(t12-t22))\,t21 + (t22-t32)\,(t12-t22)\,t20$$

Identification with the two-matrix model

```
> V1prime:=unapply(a1*lambda+a2,lambda);
V2prime:=unapply(b0*y+b1+b2/(y-X1),y);
E0part1:=unapply(  (V1prime(lambda)-y)*(V2prime(y)-lambda),
lambda,y);
E0:=unapply(E0part1(lambda,y)+1 -a1*(b0-b2/(y-X1)*C0),lambda,y);
series(simplify(E0(lambda,y)-E0part1(lambda,y)),y=X1);
```

$$V1prime := \lambda \to a1\,\lambda + a2 \tag{11}$$

$$V2prime := y \to b0\,y + b1 + \frac{b2}{y - X1}$$

$$E0part1 := (\lambda, y) \to (a1\,\lambda + a2 - y)\left(b0\,y + b1 + \frac{b2}{y-X1} - \lambda\right)$$

$$E0 := (\lambda, y) \to (a1\,\lambda + a2 - y)\left(b0\,y + b1 + \frac{b2}{y-X1} - \lambda\right) + 1 - a1\left(b0 - \frac{b2\,C0}{y-X1}\right)$$

$$\frac{C0\,a1\,b2}{y-X1} - a1\,b0 + 1$$

```
> simplify(series(residue(-1/b0*(y-X1)*E0(lambda,y)/y^4,y=0),
lambda=0));
```

```
simplify(series(residue(-1/b0*(y-X1)*E0(lambda,y)/y^3,y=0),
lambda=0));
simplify(series(residue(-1/b0*(y-X1)*E0(lambda,y)/y^2,y=0),
lambda=0));
simplify(series(residue(-1/b0*(y-X1)*E0(lambda,y)/y^1,y=0),
lambda=0));
series((-a1*b0+a2*b1+1)*X1-b2*(C0*a1+a2),C0);
```

(12)

$$\frac{1}{\frac{(-X1-a2)\,b0+b1}{b0}+\frac{-a1\,b0-1}{b0}\,\lambda}$$

$$\frac{(X1\,a2+a1)\,b0-b1\,X1-a2\,b1+b2-1}{b0}+\frac{(X1\,b0-b1)\,a1+X1+a2}{b0}\,\lambda+\frac{a1}{b0}\,\lambda^2$$

$$\frac{(-a1\,b0+a2\,b1+1)\,X1-b2\,(C0\,a1+a2)}{b0}+\frac{(a1\,b1-a2)\,X1-a1\,b2}{b0}\,\lambda-\frac{X1\,a1}{b0}\,\lambda^2$$

$$(-a1\,b0+a2\,b1+1)\,X1-a2\,b2-a1\,b2\,C0$$

```
> SpectralCurveGl3(lambda,y,q,p,t12,t22,t32,t11,t21,t31,t20,t30):
simplify(series(residue((SpectralCurveGl3(lambda,y+t22*lambda,q,
p,t12,t22,t32,t11,t21,t31,t20,t30)+1/b0*(y-X1)* E0(lambda,
y))/y^4,y=0),lambda=0));
simplify(series(residue((SpectralCurveGl3(lambda,y+t22*lambda,q,
p,t12,t22,t32,t11,t21,t31,t20,t30)+1/b0*(y-X1)*E0(lambda,y))/y^3,
y=0),lambda=0));
simplify(series(residue((SpectralCurveGl3(lambda,y+t22*lambda,q,
p,t12,t22,t32,t11,t21,t31,t20,t30)+1/b0*(y-X1)*E0(lambda,y))/y^2,
y=0),lambda=0));
simplify(series(residue((SpectralCurveGl3(lambda,y+t22*lambda,q,
p,t12,t22,t32,t11,t21,t31,t20,t30)+1/b0*(y-X1)*E0(lambda,y))/y^1,
y=0),lambda=0));
```

(13)

$$\frac{0}{\frac{(X1+a2-t11-t21-t31)\,b0-b1}{b0}+\frac{1+(a1-t12+2\,t22-t32)\,b0}{b0}\,\lambda}$$

$$\frac{1}{b0}\,((t11\,(t21+t31)+(t20+t30)\,t12-X1\,a2+t21\,t31-t20\,t22-t30\,t32-a1)\,b0$$
$$+b1\,X1+a2\,b1-b2+1)+\frac{1}{b0}\,(((-2\,t22+t12+t32)\,t21+(-t11-t31)\,t22$$
$$-X1\,a1+t12\,t31+t32\,t11)\,b0+a1\,b1-X1-a2)\,\lambda$$
$$+\frac{-(t12-t22)\,(t22-t32)\,b0-a1}{b0}\,\lambda^2$$

$$\frac{1}{b0}\,((-p^3+(t11+t21+t31+(t12+t22+t32)\,q)\,p^2+(((-t32-t22)\,t12-t22\,t32)\,q^2$$
$$+((-t31-t21)\,t12+(-t11-t31)\,t22-t32\,(t11+t21))\,q+t12\,(-t20-t30)$$
$$+t20\,t22+t30\,t32+(-t31-t21)\,t11-t21\,t31)\,p+q^3\,t12\,t22\,t32+((t21\,t32$$
$$+t22\,t31)\,t12+t22\,t32\,t11)\,q^2+((t20\,t32+t21\,t31+t22\,t30)\,t12+((-t20-t30)\,t32$$
$$+t31\,t11)\,t22+t32\,t11\,t21)\,q+X1\,a1)\,b0+(-a2\,b1-1)\,X1+b2\,(C0\,a1+a2))$$

$$+\frac{1}{b0}\left(\left(-t20\,t22^2+\left(\left(t12+t32\right)t20+\left(t11+t31\right)t21\right)t22-t12\,t32\,t20-t21\left(t11\,t32+t12\,t31\right)\right)b0+\left(-a1\,b1+a2\right)X1+a1\,b2\right)\lambda+\frac{t21\left(t22-t32\right)\left(t12-t22\right)b0+X1\,a1}{b0}\lambda^2$$

```
> simplify(series(residue(simplify(series
  (SpectralCurveGl3Shiftedbis+1/b0*(y-X1)*E0(lambda,y),y=0))/y^4,y=
  0),lambda=0));
  simplify(series(residue(simplify(series
  (SpectralCurveGl3Shiftedbis+1/b0*(y-X1)*E0(lambda,y),y=0))/y^3,y=
  0),lambda=0));
  simplify(series(residue(simplify(series
  (SpectralCurveGl3Shiftedbis+1/b0*(y-X1)*E0(lambda,y),y=0))/y^2,y=
  0),lambda=0));
  simplify(series(residue(simplify(series
  (SpectralCurveGl3Shiftedbis+1/b0*(y-X1)*E0(lambda,y),y=0))/y^1,y=
  0),lambda=0));
```

$$\frac{\left(X1+a2-t11-t21-t31\right)b0-b1}{b0}\,0+\frac{1+\left(a1-t12+2\,t22-t32\right)b0}{b0}\lambda+\frac{1}{b0}\left(\left(t11\left(t21+t31\right)+\left(t20+t30\right)t12-X1\,a2+t21\,t31-t20\,t22-t30\,t32-a1\right)b0+b1\,X1+a2\,b1-b2+1\right)+\frac{1}{b0}\left(\left(\left(-2\,t22+t12+t32\right)t21+\left(-t11-t31\right)t22-X1\,a1+t12\,t31+t32\,t11\right)b0+a1\,b1-X1-a2\right)\lambda+\frac{-\left(t12-t22\right)\left(t22-t32\right)b0-a1}{b0}\lambda^2+\frac{1}{b0}\left(\left(-p^3+\left(t11+t21+t31+\left(t12+t22+t32\right)q\right)p^2+\left(\left(\left(-t32-t22\right)t12-t22\,t32\right)q^2+\left(\left(-t31-t21\right)t12+\left(-t11-t31\right)t22-t32\left(t11+t21\right)\right)q+t12\left(-t20-t30\right)+t20\,t22+t30\,t32+\left(-t31-t21\right)t11-t21\,t31\right)p+q^3\,t12\,t22\,t32+\left(\left(t21\,t32+t22\,t31\right)t12+t22\,t32\,t11\right)q^2+\left(\left(t20\,t32+t21\,t31+t22\,t30\right)t12+\left(\left(-t20-t30\right)t32+t31\,t11\right)t22+t32\,t11\,t21\right)q+X1\,a1\right)b0+\left(-a2\,b1-1\right)X1+b2\left(C0\,a1+a2\right)\right)+\frac{1}{b0}\left(\left(-t20\,t22^2+\left(\left(t12+t32\right)t20+\left(t11+t31\right)t21\right)t22-t12\,t32\,t20-t21\left(t11\,t32+t12\,t31\right)\right)b0+\left(-a1\,b1+a2\right)X1+a1\,b2\right)\lambda+\frac{t21\left(t22-t32\right)\left(t12-t22\right)b0+X1\,a1}{b0}\lambda^2 \tag{14}$$

```
> EQ1:=simplify(residue(residue(
  (SpectralCurveGl3Shiftedbis+1/b0*(y-X1)*E0(lambda,y))/y^3,y=0)
  /lambda^2,lambda=0));
  EQ2:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
  (y-X1)*E0(lambda,y))/y^3,y=0)/lambda^1,lambda=0));
  EQ3:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
```

```
(y-X1)*E0(lambda,y))/y^2,y=0)/lambda^3,lambda=0));
EQ4:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
(y-X1)*E0(lambda,y))/y^2,y=0)/lambda^2,lambda=0));
EQ5:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
(y-X1)*E0(lambda,y))/y^2,y=0)/lambda^1,lambda=0));
EQ6:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
(y-X1)*E0(lambda,y))/y^1,y=0)/lambda^3,lambda=0));
EQ7:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
(y-X1)*E0(lambda,y))/y^1,y=0)/lambda^2,lambda=0));
EQ8:=simplify(residue(residue( (SpectralCurveGl3Shiftedbis+1/b0*
(y-X1)*E0(lambda,y))/y^1,y=0)/lambda^1,lambda=0)):
```

$$EQ1 := \frac{1 + (a1 - t12 + 2\,t22 - t32)\,b0}{b0} \tag{15}$$

$$EQ2 := \frac{(X1 + a2 - t11 - t21 - t31)\,b0 - b1}{b0}$$

$$EQ3 := \frac{-(t12 - t22)\,(t22 - t32)\,b0 - a1}{b0}$$

$$EQ4 := \frac{1}{b0}\left(\left(\left(-2\,t22 + t12 + t32\right)t21 + \left(-t11 - t31\right)t22 - X1\,a1 + t12\,t31 + t32\,t11\right)b0 + a1\,b1 - X1 - a2\right)$$

$$EQ5 := \frac{1}{b0}\left(\left(t11\,(t21 + t31) + (t20 + t30)\,t12 - X1\,a2 + t21\,t31 - t20\,t22 - t30\,t32 - a1\right)b0 + b1\,X1 + a2\,b1 - b2 + 1\right)$$

$$EQ6 := \frac{t21\,(t22 - t32)\,(t12 - t22)\,b0 + X1\,a1}{b0}$$

$$EQ7 := \frac{1}{b0}\left(\left(-t20\,t22^2 + \left((t12 + t32)\,t20 + (t11 + t31)\,t21\right)t22 - t12\,t32\,t20 - t21\,(t11\,t32 + t12\,t31)\right)b0 + \left(-a1\,b1 + a2\right)X1 + a1\,b2\right)$$

```
> a1:=-(t12-t22)*(t22-t32)*b0;
  simplify(EQ3);
  b0:=1/(t32-t22);
  simplify(EQ1);
  X1:=t21;
  simplify(EQ6);
  a2:=b1*(t32-t22)+t11+t31;
  simplify(EQ2);
  simplify(EQ4);
  b1:=-t31/(t32-t22);
  simplify(EQ4);
  b2:=t20;
  simplify(EQ7);
  t30:=1-t20;
  simplify(EQ5);
```

```
a1:=simplify(a1);
a2:=simplify(a2);
b0:=simplify(b0);
b1:=simplify(b1);
b2:=simplify(b2);
X1:=simplify(X1);
```

$$a1 := -(t12 - t22)\,(t22 - t32)\,b0 \tag{16}$$
$$0$$
$$b0 := \frac{1}{t32 - t22}$$
$$0$$
$$X1 := t21$$
$$0$$
$$a2 := b1\,(t32 - t22) + t11 + t31$$
$$0$$
$$-(-t32 + t12)\,(b1\,(t22 - t32) - t31)$$
$$b1 := -\frac{t31}{t32 - t22}$$
$$0$$
$$b2 := t20$$
$$0$$
$$t30 := 1 - t20$$
$$0$$
$$a1 := t12 - t22$$
$$a2 := t11$$
$$b0 := -\frac{1}{t22 - t32}$$
$$b1 := \frac{t31}{t22 - t32}$$
$$b2 := t20$$
$$X1 := t21$$

```
> simplify(V1prime(lambda));
  int(V1prime(lambda),lambda);
  V2prime(y);
  int(V2prime(y),y);
```

$$(t12 - t22)\,\lambda + t11 \tag{17}$$
$$\frac{1}{2}\,(t12 - t22)\,\lambda^2 + \lambda\,t11$$
$$-\frac{y}{t22 - t32} + \frac{t31}{t22 - t32} + \frac{t20}{y - t21}$$
$$t20\ln(y - t21) + \frac{t31\,y}{t22 - t32} - \frac{1}{2}\,\frac{y^2}{t22 - t32}$$

```
> C0:=1/(t20*(t12-t22)*(t32-t22))* (   (p^3-P1(q)*p^2+P2(q)*p-P3(q))
  -t11*t21*t31+t11*t20*(t22-t32)-t21*(t12-t32));
  simplify(EQ8);
```

$$C0 := \frac{1}{t20\,(t12-t22)\,(t32-t22)}\left(p^3-(t11+t21+t31+(t12+t22+t32)\,q)\,p^2\right. \tag{18}$$
$$+\left((t12\,t22+t12\,t32+t22\,t32)\,q^2+((t21+t31)\,t12+(t11+t31)\,t22+t32\,(t11\right.$$
$$\left.+t21))\,q+t12-t20\,t22-(1-t20)\,t32+t21\,t11+t31\,t11+t21\,t31\right)p$$
$$-q^3\,t12\,t22\,t32-(t11\,t22\,t32+t12\,t21\,t32+t12\,t22\,t31)\,q^2-(-t22\,t32+t32\,t11\,t21$$
$$+t22\,t11\,t31+t12\,t32\,t20+t12\,t21\,t31+t12\,t22\,(1-t20))\,q-t11\,t21\,t31$$
$$\left.+t11\,t20\,(t22-t32)-t21\,(-t32+t12)\right)$$
$$0$$

3. Dual spectral curve and dual spectral curve shifted

```
> DualSpectralCurveShifted:=simplify(series
  (SpectralCurveGl3Shifted(y,lambda,Q,P,t12,t22,t32,t11,t21,t31,
  t20,t30),y=0)):
  simplify(series(SpectralCurveGl2(lambda,y,q,p,s12,s22,s11,s21,X1,
  s10,s20,sX10),y=0)):
  ToCancelShifted:=simplify(series(DualSpectralCurveShifted-(t22-
  t32)*(t12-t22)*(t21-lambda)*SpectralCurveGl2(lambda,y,q,p,s12,
  s22,s11,s21,X1,s10,s20,sX10),y=0)):
  ToCancelShiftedfunction:=unapply(ToCancelShifted,Q,P):
  TermProportionalToySquareShifted:=series(simplify(series((lambda-
  X1)*residue(ToCancelShifted/y^3,y=0),lambda=0)),lambda=0);
  TermProportionalToyShifted:=series(simplify(series((lambda-X1)*
  residue(ToCancelShifted/y^2,y=0),lambda=0)),lambda=0);
  TermConstantShifted:=series(simplify(series((lambda-X1)^2*residue
  (ToCancelShifted/y,y=0),lambda=0,10)),lambda=0,10):

  simplify(residue(TermProportionalToyShifted/lambda^5,lambda=0));
  EQ1:=simplify(residue(TermProportionalToyShifted/lambda^4,lambda=
  0));
  EQ2:=simplify(residue(TermProportionalToyShifted/lambda^3,lambda=
  0)):
  EQ3:=simplify(residue(TermProportionalToyShifted/lambda^2,lambda=
  0)):
  EQ4:=simplify(residue(TermProportionalToyShifted/lambda,lambda=0)
  ):
  simplify(residue(TermConstantShifted/lambda^7,lambda=0));
  EQ5:=simplify(residue(TermConstantShifted/lambda^6,lambda=0));
  EQ6:=simplify(residue(TermConstantShifted/lambda^5,lambda=0)):
  EQ7:=simplify(residue(TermConstantShifted/lambda^4,lambda=0)):
```

```
EQ8:=simplify(residue(TermConstantShifted/lambda^3,lambda=0)):
EQ9:=simplify(residue(TermConstantShifted/lambda^2,lambda=0)):
EQ10:=simplify(residue(TermConstantShifted/lambda,lambda=0)):
```

$$TermProportionalToySquareShifted := 0 \tag{19}$$

$$\begin{aligned} TermProportionalToyShifted := {} & t21\,(((-s11-s21)\,t21+s10+t20-1)\,t22^2+(((s21+s11)\,t12+(s21+s11)\,t32-t11-t31)\,t21-(t12+t32)\,(s10+t20-1))\,t22+(((-s11-s21)\,t32+t31)\,t12+t32\,t11)\,t21+t32\,t12\,(s10+t20-1))+((((-s12-s22)\,t21^2+(2\,s11+2\,s21)\,t21-s10-t20+1)\,t22^2+(((s12+s22)\,t12+2+(s12+s22)\,t32)\,t21^2+((-2\,s11-2\,s21)\,t12+(-2\,s11-2\,s21)\,t32+2\,t31+2\,t11)\,t21+(t12+t32)\,(s10+t20-1))\,t22+((-1+(-s12-s22)\,t32)\,t12-t32)\,t21^2+(((2\,s11+2\,s21)\,t32-2\,t31)\,t12-2\,t32\,t11)\,t21-t32\,t12\,(s10+t20-1))\,\lambda+(((2\,s12+2\,s22)\,t21-s11-s21)\,t22^2+(((-2\,s12-2\,s22)\,t12-4+(-2\,s12-2\,s22)\,t32)\,t21+(s21+s11)\,t12+(s21+s11)\,t32-t11-t31)\,t22+((2+(2\,s12+2\,s22)\,t32)\,t12+2\,t32)\,t21+((-s11-s21)\,t32+t31)\,t12+t32\,t11)\,\lambda^2+((-s12-s22)\,t22^2+((s12+s22)\,t12+2+(s12+s22)\,t32)\,t22+(-1+(-s12-s22)\,t32)\,t12-t32)\,\lambda^3 \end{aligned}$$

$$0$$

$$EQ1 := (-s12-s22)\,t22^2+((s12+s22)\,t12+2+(s12+s22)\,t32)\,t22+(-1+(-s12-s22)\,t32)\,t12-t32$$

$$0$$

$$EQ5 := 1+(t12-t22)\,(t22-t32)\,s22\,s12$$

```
> solve({EQ1,EQ5},{t12,t32});
```

$$\left\{t12=\frac{s12\,t22-1}{s12},\ t32=\frac{s22\,t22-1}{s22}\right\},\ \left\{t12=\frac{s22\,t22-1}{s22},\ t32=\frac{s12\,t22-1}{s12}\right\} \tag{20}$$

First Solution

```
> t12 := (t22*s12-1)/s12;
  t32 := (t22*s22-1)/s22;
  simplify(EQ1);
  simplify(EQ5);
  EQ2:=simplify(EQ2);
  EQ6:=simplify(EQ6);
```

$$t12 := \frac{t22\,sinfty12-1}{sinfty12} \tag{1.1}$$

$$t32 := \frac{t22\,sinfty22-1}{sinfty22}$$

$$0$$

$$0$$

$$EQ2 := \frac{-t11\,sinfty12-t31\,sinfty22-sinfty11-sinfty21}{sinfty22\,sinfty12}$$

$$EQ6 := \frac{((-t11-t31)\,sinfty22-sinfty21)\,sinfty12-sinfty11\,sinfty22}{sinfty12\,sinfty22}$$

```
> solve({EQ2,EQ6},{t11,t31});
```

```
t11 := -s11/s12;
t31 := -s21/s22;
simplify(EQ2);
simplify(EQ6);
EQ3:=simplify(EQ3);
EQ7:=simplify(EQ7);
```

$$\left\{t11 = -\frac{sinfty11}{sinfty12}, t31 = -\frac{sinfty21}{sinfty22}\right\} \tag{1.2}$$

$$t11 := -\frac{sinfty11}{sinfty12}$$

$$t31 := -\frac{sinfty21}{sinfty22}$$

$$0$$

$$0$$

$$EQ3 := \frac{-t20 - sinfty10 - sinfty20}{sinfty22\ sinfty12}$$

$$EQ7 := \frac{(-t20 - t30 - sinfty10)\ sinfty22 + sinfty12\ (t30 - sinfty20)}{sinfty12\ sinfty22}$$

```
> solve({EQ3,EQ7},{t20,t30});
t20 := -s10-s20;
t30 := s20;
simplify(EQ3);
simplify(EQ7);
EQ4:=simplify(unapply(EQ4,Q,P)):
EQ8:=simplify(unapply(EQ8,Q,P)):
EQ9:=simplify(unapply(EQ9,Q,P)):
EQ10:=simplify(unapply(EQ10,Q,P)):
```

$$\{t20 = -sinfty10 - sinfty20, t30 = sinfty20\} \tag{1.3}$$

$$t20 := -sinfty10 - sinfty20$$

$$t30 := sinfty20$$

$$0$$

$$0$$

```
> t21:=X1;
simplify(EQ4(p,q+t22*p));
series(simplify(EQ8(p,q+t22*p)),p=0);
```

$$t21 := X1 \tag{1.4}$$

$$0$$

$$0$$

We find sX10*sX20=0

```
> sX10:=0;
simplify(ToCancelShiftedfunction(p,q+t22*p));
```

$$sX10 := 0 \tag{1.5}$$

$$0$$

```
> t12:=t12;
t32:=t32;
```

```
t22:=t22;
t11:=t11;
t21:=t21;
t31:=t31;
t20:=t20;
t30:=t30;
sX10:=sX10;
```

$$t12 := \frac{t22\ sinfty12 - 1}{sinfty12} \tag{1.6}$$
$$t32 := \frac{t22\ sinfty22 - 1}{sinfty22}$$
$$t22 := t22$$
$$t11 := -\frac{sinfty11}{sinfty12}$$
$$t21 := X1$$
$$t31 := -\frac{sinfty21}{sinfty22}$$
$$t20 := -sinfty10 - sinfty20$$
$$t30 := sinfty20$$
$$sX10 := 0$$

Second Solution

```
> t12 := (t22*s22-1)/s22;
  t32 := (t22*s12-1)/s12;
```

$$t12 := \frac{t22\ sinfty22 - 1}{sinfty22} \tag{2.1}$$
$$t32 := \frac{t22\ sinfty12 - 1}{sinfty12}$$

```
> simplify(EQ1);
  simplify(EQ5);
  EQ2:=simplify(EQ2);
  EQ6:=simplify(EQ6);
```

$$0$$
$$0 \tag{2.2}$$
$$EQ2 := \frac{-t11\ sinfty22 - t31\ sinfty12 - sinfty11 - sinfty21}{sinfty22\ sinfty12}$$
$$EQ6 := \frac{((-t11 - t31)\ sinfty22 - sinfty21)\ sinfty12 - sinfty22\ sinfty11}{sinfty12\ sinfty22}$$

```
> solve({EQ2,EQ6},{t11,t31});
  t11 := -s21/s22;
  t31 := -s11/s12;
  simplify(EQ2);
  simplify(EQ6);
  EQ3:=simplify(EQ3);
  EQ7:=simplify(EQ7);
```

$$\left\{t11 = -\frac{sinfty21}{sinfty22}, t31 = -\frac{sinfty11}{sinfty12}\right\} \quad \textbf{(2.3)}$$

$$t11 := -\frac{sinfty21}{sinfty22}$$

$$t31 := -\frac{sinfty11}{sinfty12}$$

$$0$$

$$0$$

$$EQ3 := \frac{-t20 - sinfty10 - sinfty20}{sinfty22\ sinfty12}$$

$$EQ7 := \frac{(-t20 - t30 - sinfty20)\ sinfty12 + sinfty22\ (t30 - sinfty10)}{sinfty12\ sinfty22}$$

```
> solve({EQ3,EQ7},{t20,t30});
  t20 := -s10-s20;
  t30 := s10;
  simplify(EQ3);
  simplify(EQ7);
  EQ4:=simplify(unapply(EQ4,Q,P)):
  EQ8:=simplify(unapply(EQ8,Q,P)):
  EQ9:=simplify(unapply(EQ9,Q,P)):
  EQ10:=simplify(unapply(EQ10,Q,P)):
```

$$\{t20 = -sinfty10 - sinfty20, t30 = sinfty10\} \quad \textbf{(2.4)}$$

$$t20 := -sinfty10 - sinfty20$$

$$t30 := sinfty10$$

$$0$$

$$0$$

```
> t21:=X1;
  simplify(EQ4(p,q+t22*p));
  series(simplify(EQ8(p,q+t22*p)),p=0);
```

$$t21 := X1 \quad \textbf{(2.5)}$$

$$0$$

$$\frac{sX10\ (sX10 + sinfty10 + sinfty20)}{sinfty22\ sinfty12\ (-q + X1)}$$

We find sX10*sX20=0

```
> sX10:=0;
  simplify(ToCancelShiftedfunction(p,q+t22*p));
```

$$sX10 := 0 \quad \textbf{(2.6)}$$

$$0$$

```
> t12:=t12;
  t32:=t32;
  t22:=t22;
  t11:=t11;
  t21:=t21;
  t31:=t31;
  t20:=t20;
```

```
t30:=t30;
sX10:=sX10;
```

$$t12 := \frac{t22\ sinfty22 - 1}{sinfty22} \tag{2.7}$$

$$t32 := \frac{t22\ sinfty12 - 1}{sinfty12}$$

$$t22 := t22$$

$$t11 := -\frac{sinfty21}{sinfty22}$$

$$t21 := X1$$

$$t31 := -\frac{sinfty11}{sinfty12}$$

$$t20 := -sinfty10 - sinfty20$$

$$t30 := sinfty10$$

$$sX10 := 0$$