In this Maple file we compute the Lax pair in the oper gauge in terms of the irregular times at infinity. We check the results with the theoretical ones.

```
> restart:
  with(LinearAlgebra):
  rinfty:=4:
  g:=rinfty-2:
  tinfty27:=-tinfty17:
  tinfty25:=-tinfty15:
  tinfty23:=-tinfty13:
  tinfty21:=-tinfty11:
  tinfty20:=-tinfty10:
  tinfty26:=tinfty16:
  tinfty24:=tinfty14:
  tinfty22:=tinfty12:
  tinfty10:=0:
  Pinfty01  := -tinfty12;
  Pinfty11  := -tinfty14;
  Pinfty21  := -tinfty16;
  Pinfty22  := -(1/2)*tinfty11*tinfty17+(1/2)*tinfty12*tinfty16-
  (1/2)*tinfty13*tinfty15+(1/4)*tinfty14^2;
  Pinfty32  := -(1/2)*tinfty13*tinfty17+(1/2)*tinfty14*tinfty16-
  (1/4)*tinfty15^2;
  Pinfty42  := -(1/2)*tinfty15*tinfty17+(1/4)*tinfty16^2;
  Pinfty52  := -(1/4)*tinfty17^2;

  P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2;
  P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
  x^4+Pinfty52*x^5;
  tdP2:=unapply(Pinfty22*x^2+Pinfty32*x^3+Pinfty42*x^4+Pinfty52*
  x^5,x);

  Unknownn:=-Unknown:
  LUnknownn:=-LUnknown:
  Unknownn2:=Unknown2:
  LUnknownn2:=LUnknown2:

  Ltinfty27:=-Ltinfty17;
  Ltinfty25:=-Ltinfty15;
  Ltinfty23:=-Ltinfty13;
  Ltinfty21:=-Ltinfty11;
  Ltinfty20:=-Ltinfty10;
  Ltinfty26:=Ltinfty16;
```

```
Ltinfty24:=Ltinfty14;
Ltinfty22:=Ltinfty12;
Ltinfty10:=0:
```

$$Pinfty01 := -tinfty12$$
$$Pinfty11 := -tinfty14$$
$$Pinfty21 := -tinfty16$$
$$Pinfty22 := -\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}$$
$$Pinfty32 := -\frac{tinfty13\ tinfty17}{2} + \frac{tinfty14\ tinfty16}{2} - \frac{tinfty15^2}{4}$$
$$Pinfty42 := -\frac{tinfty15\ tinfty17}{2} + \frac{tinfty16^2}{4}$$
$$Pinfty52 := -\frac{tinfty17^2}{4}$$
$$P1 := x \mapsto Pinfty01 + Pinfty11\ x + Pinfty21\ x^2$$
$$P2 := x \mapsto Pinfty02 + Pinfty12\ x + Pinfty22\ x^2 + Pinfty32\ x^3 + Pinfty42\ x^4 + Pinfty52\ x^5$$
$$tdP2 := x \mapsto -\frac{x^5\ tinfty17^2}{4} + x^4 \left(-\frac{tinfty15\ tinfty17}{2} + \frac{tinfty16^2}{4}\right) + x^3 \left(-\frac{tinfty13\ tinfty17}{2} + \frac{tinfty14\ tinfty16}{2} - \frac{tinfty15^2}{4}\right) + x^2 \left(-\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}\right)$$
$$Ltinfty27 := -Ltinfty17$$
$$Ltinfty25 := -Ltinfty15$$
$$Ltinfty23 := -Ltinfty13$$
$$Ltinfty21 := -Ltinfty11$$
$$Ltinfty20 := -Ltinfty10$$
$$Ltinfty26 := Ltinfty16$$
$$Ltinfty24 := Ltinfty14$$
$$Ltinfty22 := Ltinfty12 \quad \textbf{(1)}$$

We have recopied the results regarding the coefficients of the classical spectral curve and adapted the deformation parameters according to the symmetries.

Study at infinity

```
> logPsi1Infty:=-1/7*tinfty17/h*lambda^(7/2)-1/6*tinfty16/h*
lambda^3-1/5*tinfty15/h*lambda^(5/2)-1/4*tinfty14/h*lambda^2-1/3*
tinfty13/h*lambda^(3/2)-1/2*tinfty12/h*lambda^1-
tinfty11/h*lambda^(1/2)-1/4*ln(lambda)+A10+ Unknown/lambda^(1/2)+
Unknown2/lambda;
logPsi2Infty:=-1/7*tinfty27/h*lambda^(7/2)-1/6*tinfty26/h*
lambda^3-1/5*tinfty25/h*lambda^(5/2)-1/4*tinfty24/h*lambda^2-1/3*
tinfty23/h*lambda^(3/2)-1/2*tinfty22/h*lambda^1-tinfty21/h*
lambda^(1/2)-1/4*ln(lambda)+A20+ Unknownn/lambda^(1/2)+
Unknownn2/lambda;
```

```
CurlyLlogpsi1Infty:=-1/7*Ltinfty17/h*lambda^(7/2)-1/6*
Ltinfty16/h*lambda^3-1/5*Ltinfty15/h*lambda^(5/2)-1/4*
Ltinfty14/h*lambda^2-1/3*Ltinfty13/h*lambda^(3/2)-1/2*
Ltinfty12/h*lambda^1-Ltinfty11/h*lambda^(1/2)+LA10+
LUnknown/lambda^(1/2)+ LUnknown2/lambda  ;
CurlyLlogpsi2Infty:=-1/7*Ltinfty27/h*lambda^(7/2)-1/6*
Ltinfty26/h*lambda^3-1/5*Ltinfty25/h*lambda^(5/2)-1/4*
Ltinfty24/h*lambda^2-1/3*Ltinfty23/h*lambda^(3/2)-1/2*
Ltinfty22/h*lambda^1-Ltinfty21/h*lambda^(1/2)+LA20+
LUnknownn/lambda^(1/2)+ LUnknownn2/lambda;

CurlyLpsi1Infty := exp(-1/7*tinfty17/h*lambda^(7/2)-1/6*
tinfty16/h*lambda^3-1/5*tinfty15/h*lambda^(5/2)-1/4*tinfty14/h*
lambda^2-1/3*tinfty13/h*lambda^(3/2)-1/2*tinfty12/h*lambda^1-
tinfty11/h*lambda^(1/2)-1/4*ln(lambda)+A10+ Unknown/lambda^(1/2)+
Unknown2/lambda)*(-1/7*Ltinfty17/h*lambda^(7/2)-1/6*Ltinfty16/h*
lambda^3-1/5*Ltinfty15/h*lambda^(5/2)-1/4*Ltinfty14/h*lambda^2
-1/3*Ltinfty13/h*lambda^(3/2)-1/2*Ltinfty12/h*lambda^1-
Ltinfty11/h*lambda^(1/2)+LA10+ LUnknown/lambda^(1/2)+
LUnknown2/lambda);
CurlyLpsi2Infty := exp(-1/7*tinfty27/h*lambda^(7/2)-1/6*
tinfty26/h*lambda^3-1/5*tinfty25/h*lambda^(5/2)-1/4*tinfty24/h*
lambda^2-1/3*tinfty23/h*lambda^(3/2)-1/2*tinfty22/h*lambda^1-
tinfty21/h*lambda^(1/2)-1/4*ln(lambda)+A20+ Unknownn/lambda^(1/2)
+ Unknownn2/lambda)*(-1/7*Ltinfty27/h*lambda^(7/2)-1/6*
Ltinfty26/h*lambda^3-1/5*Ltinfty25/h*lambda^(5/2)-1/4*
Ltinfty24/h*lambda^2-1/3*Ltinfty23/h*lambda^(3/2)-1/2*
Ltinfty22/h*lambda^1-Ltinfty21/h*lambda^(1/2)+LA20+
LUnknownn/lambda^(1/2)+ LUnknownn2/lambda);

psi1Infty:=exp(logPsi1Infty);
psi2Infty:=exp(logPsi2Infty);
dpsi1dlambdaInfty:=diff(psi1Infty,lambda):
dpsi2dlambdaInfty:=diff(psi2Infty,lambda):
d2psi1dlambda2Infty:=diff(psi1Infty,lambda$2):
d2psi2dlambda2Infty:=diff(psi2Infty,lambda$2):


WronskianLambdaInfty:=h*factor(psi1Infty*dpsi2dlambdaInfty-
psi2Infty*dpsi1dlambdaInfty):
WronskianLambdabisInfty:=h*simplify(factor( (diff(logPsi2Infty,
lambda)-diff(logPsi1Infty,lambda))*exp(logPsi1Infty+logPsi2Infty)
```

```
)):

WronskianTildeLambdaInfty:=h^3*factor(dpsi2dlambdaInfty*
d2psi1dlambda2Infty-dpsi1dlambdaInfty*d2psi2dlambda2Infty):
```

$$logPsi1Infty := -\frac{tinfty17\,\lambda^{7/2}}{7\,h} - \frac{tinfty16\,\lambda^3}{6\,h} - \frac{tinfty15\,\lambda^{5/2}}{5\,h} - \frac{tinfty14\,\lambda^2}{4\,h} - \frac{tinfty13\,\lambda^{3/2}}{3\,h} - \frac{tinfty12\,\lambda}{2\,h} - \frac{tinfty11\,\sqrt{\lambda}}{h} - \frac{\ln(\lambda)}{4} + A10 + \frac{Unknown}{\sqrt{\lambda}} + \frac{Unknown2}{\lambda}$$

$$logPsi2Infty := \frac{tinfty17\,\lambda^{7/2}}{7\,h} - \frac{tinfty16\,\lambda^3}{6\,h} + \frac{tinfty15\,\lambda^{5/2}}{5\,h} - \frac{tinfty14\,\lambda^2}{4\,h} + \frac{tinfty13\,\lambda^{3/2}}{3\,h} - \frac{tinfty12\,\lambda}{2\,h} + \frac{tinfty11\,\sqrt{\lambda}}{h} - \frac{\ln(\lambda)}{4} + A20 - \frac{Unknown}{\sqrt{\lambda}} + \frac{Unknown2}{\lambda}$$

$$CurlyLlogpsi1Infty := -\frac{Ltinfty17\,\lambda^{7/2}}{7\,h} - \frac{Ltinfty16\,\lambda^3}{6\,h} - \frac{Ltinfty15\,\lambda^{5/2}}{5\,h} - \frac{Ltinfty14\,\lambda^2}{4\,h} - \frac{Ltinfty13\,\lambda^{3/2}}{3\,h} - \frac{Ltinfty12\,\lambda}{2\,h} - \frac{Ltinfty11\,\sqrt{\lambda}}{h} + LA10 + \frac{LUnknown}{\sqrt{\lambda}} + \frac{LUnknown2}{\lambda}$$

$$CurlyLlogpsi2Infty := \frac{Ltinfty17\,\lambda^{7/2}}{7\,h} - \frac{Ltinfty16\,\lambda^3}{6\,h} + \frac{Ltinfty15\,\lambda^{5/2}}{5\,h} - \frac{Ltinfty14\,\lambda^2}{4\,h} + \frac{Ltinfty13\,\lambda^{3/2}}{3\,h} - \frac{Ltinfty12\,\lambda}{2\,h} + \frac{Ltinfty11\,\sqrt{\lambda}}{h} + LA20 - \frac{LUnknown}{\sqrt{\lambda}} + \frac{LUnknown2}{\lambda}$$

$$CurlyLpsi1Infty := \mathrm{e}^{-\frac{tinfty17\,\lambda^{7/2}}{7\,h} - \frac{tinfty16\,\lambda^3}{6\,h} - \frac{tinfty15\,\lambda^{5/2}}{5\,h} - \frac{tinfty14\,\lambda^2}{4\,h} - \frac{tinfty13\,\lambda^{3/2}}{3\,h} - \frac{tinfty12\,\lambda}{2\,h} - \frac{tinfty11\,\sqrt{\lambda}}{h} - \frac{\ln(\lambda)}{4} + A10 + \frac{Unknown}{\sqrt{\lambda}} + \frac{Unknown2}{\lambda}} \left(-\frac{Ltinfty17\,\lambda^{7/2}}{7\,h} - \frac{Ltinfty16\,\lambda^3}{6\,h} - \frac{Ltinfty15\,\lambda^{5/2}}{5\,h} - \frac{Ltinfty14\,\lambda^2}{4\,h} - \frac{Ltinfty13\,\lambda^{3/2}}{3\,h} - \frac{Ltinfty12\,\lambda}{2\,h} - \frac{Ltinfty11\,\sqrt{\lambda}}{h} + LA10 + \frac{LUnknown}{\sqrt{\lambda}} + \frac{LUnknown2}{\lambda} \right)$$

$$CurlyLpsi2Infty :=$$

$$e^{\frac{tinfty17\lambda^{7/2}}{7h}-\frac{tinfty16\lambda^{3}}{6h}+\frac{tinfty15\lambda^{5/2}}{5h}-\frac{tinfty14\lambda^{2}}{4h}+\frac{tinfty13\lambda^{3/2}}{3h}-\frac{tinfty12\lambda}{2h}+\frac{tinfty11\sqrt{\lambda}}{h}-\frac{\ln(\lambda)}{4}+A20-\frac{Unknown}{\sqrt{\lambda}}+\frac{Unknown2}{\lambda}}\left(\frac{Ltinfty17\,\lambda^{7/2}}{7\,h}-\frac{Ltinfty16\,\lambda^{3}}{6\,h}+\frac{Ltinfty15\,\lambda^{5/2}}{5\,h}-\frac{Ltinfty14\,\lambda^{2}}{4\,h}+\frac{Ltinfty13\,\lambda^{3/2}}{3\,h}-\frac{Ltinfty12\,\lambda}{2\,h}+\frac{Ltinfty11\,\sqrt{\lambda}}{h}+LA20-\frac{LUnknown}{\sqrt{\lambda}}+\frac{LUnknown2}{\lambda}\right)$$

$$psi1Infty := e^{-\frac{tinfty17\lambda^{7/2}}{7h}-\frac{tinfty16\lambda^{3}}{6h}-\frac{tinfty15\lambda^{5/2}}{5h}-\frac{tinfty14\lambda^{2}}{4h}-\frac{tinfty13\lambda^{3/2}}{3h}-\frac{tinfty12\lambda}{2h}-\frac{tinfty11\sqrt{\lambda}}{h}-\frac{\ln(\lambda)}{4}+A10+\frac{Unknown}{\sqrt{\lambda}}+\frac{Unknown2}{\lambda}}$$

$$psi2Infty := e^{\frac{tinfty17\lambda^{7/2}}{7h}-\frac{tinfty16\lambda^{3}}{6h}+\frac{tinfty15\lambda^{5/2}}{5h}-\frac{tinfty14\lambda^{2}}{4h}+\frac{tinfty13\lambda^{3/2}}{3h}-\frac{tinfty12\lambda}{2h}+\frac{tinfty11\sqrt{\lambda}}{h}-\frac{\ln(\lambda)}{4}+A20-\frac{Unknown}{\sqrt{\lambda}}+\frac{Unknown2}{\lambda}} \tag{2}$$

```
> L21Infty:=factor(simplify
  (WronskianTildeLambdaInfty/WronskianLambdabisInfty)):
  L21InftyOrdrelambda6:=factor(-residue(L21Infty/lambda^7,lambda=
  infinity));
  L21InftyOrdrelambda5:=factor(-residue(L21Infty/lambda^6,lambda=
  infinity));
  L21InftyOrdrelambda4:=factor(-residue(L21Infty/lambda^5,lambda=
  infinity));
  L21InftyOrdrelambda3:=factor(-residue(L21Infty/lambda^4,lambda=
  infinity));
  L21InftyOrdrelambda2:=factor(-residue(L21Infty/lambda^3,lambda=
  infinity));
  L21InftyOrdrelambda1:=factor(-residue(L21Infty/lambda^2,lambda=
  infinity));
  L21InftyOrdrelambda0:=factor(-residue(L21Infty/lambda^1,lambda=
  infinity));
```

$$L21InftyOrdrelambda6 := 0$$

$$L21InftyOrdrelambda5 := \frac{tinfty17^2}{4}$$

$$L21InftyOrdrelambda4 := \frac{tinfty15\ tinfty17}{2} - \frac{tinfty16^2}{4}$$

$$L21InftyOrdrelambda3 := \frac{tinfty13\ tinfty17}{2} - \frac{tinfty14\ tinfty16}{2} + \frac{tinfty15^2}{4}$$

$$L21InftyOrdrelambda2 := \frac{tinfty11\ tinfty17}{2} - \frac{tinfty12\ tinfty16}{2} + \frac{tinfty13\ tinfty15}{2} - \frac{tinfty14^2}{4}$$

$$L21InftyOrdrelambda1 := \frac{Unknown\ h\ tinfty17}{2} + \frac{tinfty11\ tinfty15}{2} - \frac{tinfty12\ tinfty14}{2} + \frac{tinfty13^2}{4}$$

$$L21InftyOrdrelambda0 := \frac{1}{4\ tinfty17}\big(2\ Unknown\ h\ tinfty15\ tinfty17 - 4\ h\ Unknown2\ tinfty16\ tinfty17 + 2\ h\ tinfty14\ tinfty17 - 2\ h\ tinfty15\ tinfty16 + 2\ tinfty11\ tinfty13\ tinfty17 - tinfty17\ tinfty12^2\big) \tag{3}$$

```
> factor(simplify(L21InftyOrdrelambda5*
  lambda^5+L21InftyOrdrelambda4*lambda^4+L21InftyOrdrelambda3*
  lambda^3+L21InftyOrdrelambda2*lambda^2+L21InftyOrdrelambda1*
  lambda- (-tdP2(lambda))));
```

$$\frac{\left(2\ Unknown\ h\ tinfty17 + 2\ tinfty11\ tinfty15 - 2\ tinfty12\ tinfty14 + tinfty13^2\right)\lambda}{4} \tag{4}$$

We deduce that L_{2,1} behaves at infinity like -tdP2(\lambda)+C_1*lambda+C_0+ O(1/lambda)

```
> L22Infty:=factor(h*simplify(diff(WronskianLambdabisInfty,lambda)
  /WronskianLambdabisInfty)):
  L22InftyOrdrelambda6:=factor(-residue(L22Infty/lambda^7,lambda=
  infinity));
  L22InftyOrdrelambda5:=factor(-residue(L22Infty/lambda^6,lambda=
  infinity));
  L22InftyOrdrelambda4:=factor(-residue(L22Infty/lambda^5,lambda=
  infinity));
  L22InftyOrdrelambda3:=factor(-residue(L22Infty/lambda^4,lambda=
  infinity));
  L22InftyOrdrelambda2:=factor(-residue(L22Infty/lambda^3,lambda=
  infinity));
  L22InftyOrdrelambda1:=factor(-residue(L22Infty/lambda^2,lambda=
  infinity));
  L22InftyOrdrelambda0:=factor(-residue(L22Infty/lambda^1,lambda=
  infinity));
  L22InftyOrdrelambdaMinus1:=factor(-residue(L22Infty/lambda^0,
  lambda=infinity));
```

```
L22InftyOrdrelambdaMinus2:=factor(-residue(L22Infty/lambda^(-1),
lambda=infinity));
```

$$L22InftyOrdrelambda6 := 0$$
$$L22InftyOrdrelambda5 := 0$$
$$L22InftyOrdrelambda4 := 0$$
$$L22InftyOrdrelambda3 := 0$$
$$L22InftyOrdrelambda2 := -tinfty16$$
$$L22InftyOrdrelambda1 := -tinfty14$$
$$L22InftyOrdrelambda0 := -tinfty12$$
$$L22InftyOrdrelambdaMinus1 := 2\,h$$
$$L22InftyOrdrelambdaMinus2 := -\frac{h\,(2\,Unknown2\,tinfty17 + tinfty15)}{tinfty17} \quad (5)$$

```
> simplify(L22InftyOrdrelambda2*lambda^2+L22InftyOrdrelambda1*
lambda+L22InftyOrdrelambda0-P1(lambda));
```

$$0 \quad (6)$$

We deduce that L_{2,2} behaves at infinity like -tinfty16*lambda^2-tinfty14*lambda -tinfty12+2*h/lambda +O(1/lambda^2) =P1(lambda) +h/lambda+O(1/lambda^2)

Thus we get the form of the matrix L:

L_{2,2}= P_1(lambda) +h/(lambda-q1) +h/(lambda-q2)
L_{2,1}= -tdP_2(lambda)+C1*lambda+C0 -h*p1/(lambda-q1)-h*p2/(lambda-q2)

```
> L21Form:=-tdP2(lambda)- h*p1/(lambda-q1)-h*p2/(lambda-q2);
  L22Form:=P1(lambda)   +h/(lambda-q1)+h/(lambda-q2);
```

$$L21Form := \frac{\lambda^5\,tinfty17^2}{4} - \lambda^4\left(-\frac{tinfty15\,tinfty17}{2} + \frac{tinfty16^2}{4}\right) - \lambda^3\left(-\frac{tinfty13\,tinfty17}{2} + \frac{tinfty14\,tinfty16}{2} - \frac{tinfty15^2}{4}\right) - \lambda^2\left(-\frac{tinfty11\,tinfty17}{2} + \frac{tinfty12\,tinfty16}{2} - \frac{tinfty13\,tinfty15}{2} + \frac{tinfty14^2}{4}\right) - \frac{h\,p1}{\lambda - q1} - \frac{h\,p2}{\lambda - q2}$$

$$L22Form := -tinfty16\,\lambda^2 - tinfty14\,\lambda - tinfty12 + \frac{h}{\lambda - q1} + \frac{h}{\lambda - q2} \quad (7)$$

Computation of the Auxiliary matrix A

We define the deformation operator \mathcal{L}=\hbar (alpha7*\partial_{t_{\infty^{(1)},7} +alpha6*\partial_{t_{\infty^{(1)},6}+... alpha1*\partial_{t_{\infty^{(1)},1})

```
> WronskianCurlyLInfty:=factor(psi1Infty*CurlyLpsi2Infty-psi2Infty*
CurlyLpsi1Infty):
A12Infty:=factor(simplify
(WronskianCurlyLInfty/WronskianLambdaInfty)):
Y1Infty:=h*factor(dpsi1dlambdaInfty/psi1Infty):
Y2Infty:=h*factor(dpsi2dlambdaInfty/psi2Infty):
Z1Infty:=factor(CurlyLpsi1Infty/psi1Infty):
Z2Infty:=factor(CurlyLpsi2Infty/psi2Infty):
A12bisInfty:=factor(simplify((Z2Infty-Z1Infty)/(Y2Infty-Y1Infty))
):
```

```
A11Infty:=factor(simplify( (Y2Infty*Z1Infty-Y1Infty*Z2Infty)/
(Y2Infty-Y1Infty) )):
factor(simplify(A12bisInfty-A12Infty));
```

$$0 \tag{8}$$

```
> Ltinfty17:=h*alpha7:
Ltinfty16:=h*alpha6:
Ltinfty15:=h*alpha5:
Ltinfty14:=h*alpha4:
Ltinfty13:=h*alpha3:
Ltinfty12:=h*alpha2:
Ltinfty11:=h*alpha1:
Ltinfty10:=0:
Ltinfty20:=0:
LA20:=LA10:
```

```
> A12InftyLambda4:=factor(-residue(A12Infty/lambda^5,lambda=
infinity));
A12InftyLambda3:=factor(-residue(A12Infty/lambda^4,lambda=
infinity));
A12InftyLambda2:=factor(-residue(A12Infty/lambda^3,lambda=
infinity));
A12InftyLambda1:=factor(-residue(A12Infty/lambda^2,lambda=
infinity));
A12InftyLambda0:=factor(-residue(A12Infty/lambda^1,lambda=
infinity));
A12InftyLambdaMinus1:=factor(-residue(A12Infty/lambda^0,lambda=
infinity));
A12InftyLambdaMinus2:=factor(-residue(A12Infty/lambda^(-1),
lambda=infinity));
```

$$A12InftyLambda4 := 0$$
$$A12InftyLambda3 := 0$$
$$A12InftyLambda2 := 0$$
$$A12InftyLambda1 := \frac{2\,\alpha 7}{7\,tinfty17}$$
$$A12InftyLambda0 := \frac{2\,(7\,\alpha 5\,tinfty17 - 5\,\alpha 7\,tinfty15)}{35\,tinfty17^2}$$
$$A12InftyLambdaMinus1 := \frac{2\,(35\,\alpha 3\,tinfty17^2 - 21\,\alpha 5\,tinfty15\,tinfty17 - 15\,\alpha 7\,tinfty13\,tinfty17 + 15\,\alpha 7\,tinfty15^2)}{105\,tinfty17^3}$$
$$A12InftyLambdaMinus2 := \frac{1}{105\,tinfty17^4}\,(2\,(105\,\alpha 1\,tinfty17^3 - 35\,\alpha 3\,tinfty15\,tinfty17^2 - 21\,\alpha 5\,tinfty13\,tinfty17^2 + 21\,\alpha 5\,tinfty15^2\,tinfty17 - 15\,\alpha 7\,tinfty11\,tinfty17^2 + 30\,\alpha 7\,tinfty13\,tinfty15\,tinfty17 - 15\,\alpha 7\,tinfty15^3)) \tag{9}$$

We thus get **A_{1,2}=nuMinus1*lambda+nu0 +mu1/(lambda-q1)+mu2/(lambda-q2)**

```
> A12Form:=2*nuMinus1*lambda+nu0+ mu1/(lambda-q1)+ mu2/(lambda-q2);
  nuMinus1:=A12InftyLambda1;
  nu0:=A12InftyLambda0;
  Equation1:=A12InftyLambdaMinus1-(-residue(A12Form,lambda=
  infinity));
  Equation2:=A12InftyLambdaMinus2-(-residue(A12Form*lambda,lambda=
  infinity));
```

$$A12Form := 2\ nuMinus1\ \lambda + \nu 0 + \frac{\mu 1}{\lambda - q1} + \frac{\mu 2}{\lambda - q2}$$

$$nuMinus1 := \frac{2\ \alpha 7}{7\ tinfty17}$$

$$\nu 0 := \frac{2\ (7\ \alpha 5\ tinfty17 - 5\ \alpha 7\ tinfty15)}{35\ tinfty17^2}$$

$$Equation1 := \frac{2\ (35\ \alpha 3\ tinfty17^2 - 21\ \alpha 5\ tinfty15\ tinfty17 - 15\ \alpha 7\ tinfty13\ tinfty17 + 15\ \alpha 7\ tinfty15^2)}{105\ tinfty17^3} - \mu 1 - \mu 2$$

$$Equation2 := \frac{1}{105\ tinfty17^4} (2\ (105\ \alpha 1\ tinfty17^3 - 35\ \alpha 3\ tinfty15\ tinfty17^2 - 21\ \alpha 5\ tinfty13\ tinfty17^2 + 21\ \alpha 5\ tinfty15^2\ tinfty17 - 15\ \alpha 7\ tinfty11\ tinfty17^2 + 30\ \alpha 7\ tinfty13\ tinfty15\ tinfty17 - 15\ \alpha 7\ tinfty15^3)) - \mu 1\ q1 - \mu 2\ q2 \quad \textbf{(10)}$$

```
> mu1:=2*(-35*alpha3*q2*tinfty17^3+21*alpha5*q2*tinfty15*
  tinfty17^2+15*alpha7*q2*tinfty13*tinfty17^2-15*alpha7*q2*
  tinfty15^2*tinfty17+105*alpha1*tinfty17^3-35*alpha3*tinfty15*
  tinfty17^2-21*alpha5*tinfty13*tinfty17^2+21*alpha5*tinfty15^2*
  tinfty17-15*alpha7*tinfty11*tinfty17^2+30*alpha7*tinfty13*
  tinfty15*tinfty17-15*alpha7*tinfty15^3)/(105*tinfty17^4*(q1-q2));
  mu2:=-1/(105*tinfty17^4*(q1-q2))* 2*(-35*alpha3*q1*tinfty17^3+21*
  alpha5*q1*tinfty15*tinfty17^2+15*alpha7*q1*tinfty13*tinfty17^2
  -15*alpha7*q1*tinfty15^2*tinfty17+105*alpha1*tinfty17^3-35*
  alpha3*tinfty15*tinfty17^2-21*alpha5*tinfty13*tinfty17^2+21*
  alpha5*tinfty15^2*tinfty17-15*alpha7*tinfty11*tinfty17^2+30*
  alpha7*tinfty13*tinfty15*tinfty17-15*alpha7*tinfty15^3);
  simplify(Equation1);
  simplify(Equation2);
```

$$\mu 1 := \frac{1}{105\ tinfty17^4\ (q1 - q2)} (2\ (-35\ \alpha 3\ q2\ tinfty17^3 + 21\ \alpha 5\ q2\ tinfty15\ tinfty17^2 + 15\ \alpha 7\ q2\ tinfty13\ tinfty17^2 - 15\ \alpha 7\ q2\ tinfty15^2\ tinfty17 + 105\ \alpha 1\ tinfty17^3 - 35\ \alpha 3\ tinfty15\ tinfty17^2 - 21\ \alpha 5\ tinfty13\ tinfty17^2 + 21\ \alpha 5\ tinfty15^2\ tinfty17 - 15\ \alpha 7\ tinfty11\ tinfty17^2 + 30\ \alpha 7\ tinfty13\ tinfty15\ tinfty17 - 15\ \alpha 7\ tinfty15^3))$$

$$\mu 2 := -\frac{1}{105\, tinfty17^4\, (q1 - q2)} \big(2 \big(-35\, \alpha 3\, q1\, tinfty17^3 + 21\, \alpha 5\, q1\, tinfty15\, tinfty17^2 + 15\, \alpha 7\, q1\, tinfty13\, tinfty17^2 - 15\, \alpha 7\, q1\, tinfty15^2\, tinfty17 + 105\, \alpha 1\, tinfty17^3 - 35\, \alpha 3\, tinfty15\, tinfty17^2 - 21\, \alpha 5\, tinfty13\, tinfty17^2 + 21\, \alpha 5\, tinfty15^2\, tinfty17 - 15\, \alpha 7\, tinfty11\, tinfty17^2 + 30\, \alpha 7\, tinfty13\, tinfty15\, tinfty17 - 15\, \alpha 7\, tinfty15^3\big)\big)$$

0

0 **(11)**

We compute the values of nu_{\infty,k} using the theoretical results

```
> Minfty:=Matrix(rinfty,rinfty,0):
  2*rinfty-1;
  Minfty[1,1]:=tinfty17:
  Minfty[2,2]:=tinfty17:
  Minfty[3,3]:=tinfty17:
  Minfty[4,4]:=tinfty17:
  Minfty[2,1]:=tinfty15:
  Minfty[3,2]:=tinfty15:
  Minfty[4,3]:=tinfty15:
  Minfty[3,1]:=tinfty13:
  Minfty[4,2]:=tinfty13:
  Minfty[4,1]:=tinfty11:
  Minfty;
  RHSMatrix:=Matrix(rinfty,1,0):
  RHSMatrix[1,1]:=2*alpha7/7:
  RHSMatrix[2,1]:=2*alpha5/5:
  RHSMatrix[3,1]:=2*alpha3/3:
  RHSMatrix[4,1]:=2*alpha1/1:
  RHSMatrix;
  NuVector:=Multiply(Minfty^(-1),RHSMatrix):
  nuMinus1Theo:=NuVector[1,1];
  nu0Theo:=NuVector[2,1];
  nu1Theo:=NuVector[3,1];
  nu2Theo:=NuVector[4,1];
```

7

$$\begin{bmatrix} tinfty17 & 0 & 0 & 0 \\ tinfty15 & tinfty17 & 0 & 0 \\ tinfty13 & tinfty15 & tinfty17 & 0 \\ tinfty11 & tinfty13 & tinfty15 & tinfty17 \end{bmatrix}$$

$$\begin{bmatrix} \frac{2\,\alpha 7}{7} \\ \frac{2\,\alpha 5}{5} \\ \frac{2\,\alpha 3}{3} \\ 2\,\alpha 1 \end{bmatrix}$$

$$nuMinus1Theo := \frac{2\,\alpha 7}{7\,tinfty17}$$

$$nu0Theo := -\frac{2\,tinfty15\,\alpha 7}{7\,tinfty17^2} + \frac{2\,\alpha 5}{5\,tinfty17}$$

$$nu1Theo := -\frac{2\,(tinfty13\,tinfty17 - tinfty15^2)\,\alpha 7}{7\,tinfty17^3} - \frac{2\,tinfty15\,\alpha 5}{5\,tinfty17^2} + \frac{2\,\alpha 3}{3\,tinfty17}$$

$$nu2Theo := -\frac{2\,(tinfty17^2\,tinfty11 - 2\,tinfty13\,tinfty15\,tinfty17 + tinfty15^3)\,\alpha 7}{7\,tinfty17^4} - \frac{2\,(tinfty13\,tinfty17 - tinfty15^2)\,\alpha 5}{5\,tinfty17^3} - \frac{2\,tinfty15\,\alpha 3}{3\,tinfty17^2} + \frac{2\,\alpha 1}{tinfty17} \qquad \textbf{(12)}$$

We check agreement between theoretical formulas and practical one for nu_{infty,-1} and \nu_{\infty, 0}

```
> simplify(nuMinus1-nuMinus1Theo);
  simplify(nu0-nu0Theo);
```

$$0$$
$$0 \qquad \textbf{(13)}$$

We check theoretical formulas giving the relations between the mu's and the nu's and with the deformation parameters alpha's.

```
> Vinfty:=Matrix(g,g,0):
  Vinfty[1,1]:=1:
  Vinfty[1,2]:=1:
  Vinfty[2,1]:=q1:
  Vinfty[2,2]:=q2:
  Vinfty;
  muVector:=Matrix(g,1,0):
  muVector[1,1]:=mu1:
  muVector[2,1]:=mu2:
  nuVectorReducedTheo:=Matrix(g,1,0):
  nuVectorReducedTheo[1,1]:=nu1Theo:
  nuVectorReducedTheo[2,1]:=nu2Theo:
  simplify(Multiply(Vinfty,muVector)-nuVectorReducedTheo);

  BigMatrix:=Matrix(g+2,g+2,0):
  BigMatrix[1,1]:=1:
```

```
BigMatrix[2,2]:=1:
for i from 3 to g+2 do for j from 3 to g+2 do
BigMatrix[i,j]:=Vinfty[i-2,j-2]: od: od:
BigMatrix;

BigVector:=Matrix(g+2,1,0):
BigVector[1,1]:=nuMinus1:
BigVector[2,1]:=nu0:
BigVector[3,1]:=mu1:
BigVector[4,1]:=mu2:
BigVector:

simplify(Multiply(Minfty,Multiply(BigMatrix,BigVector))
-RHSMatrix);
nu1:=nu1Theo:
nu2:=nu2Theo:
```

$$\begin{bmatrix} 1 & 1 \\ q1 & q2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & q1 & q2 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad (14)$$

We now deal with A_{1,1}

```
> A11InftyLambda5:=factor(-residue(A11Infty/lambda^6,lambda=
infinity));
A11InftyLambda4:=factor(-residue(A11Infty/lambda^5,lambda=
infinity));
A11InftyLambda3:=factor(-residue(A11Infty/lambda^4,lambda=
infinity));
A11InftyLambda2:=factor(-residue(A11Infty/lambda^3,lambda=
infinity));
A11InftyLambda1:=factor(-residue(A11Infty/lambda^2,lambda=
```

```
infinity));
A11InftyLambda0:=factor(-residue(A11Infty/lambda^1,lambda=
infinity));
A11InftyLambdaMinus1:=factor(-residue(A11Infty/lambda^0,lambda=
infinity)):
```

$$A11InftyLambda5 := 0$$
$$A11InftyLambda4 := 0$$
$$A11InftyLambda3 := -\frac{7\ \alpha6\ tinfty17 - 6\ \alpha7\ tinfty16}{42\ tinfty17}$$

$$A11InftyLambda2 := -\frac{1}{140\ tinfty17^2}\left(35\ \alpha4\ tinfty17^2 - 28\ \alpha5\ tinfty16\ tinfty17 - 20\ \alpha7\ tinfty14\ tinfty17 + 20\ \alpha7\ tinfty15\ tinfty16\right)$$

$$A11InftyLambda1 := -\frac{1}{210\ tinfty17^3}\left(105\ \alpha2\ tinfty17^3 - 70\ \alpha3\ tinfty16\ tinfty17^2 - 42\ \alpha5\ tinfty14\ tinfty17^2 + 42\ \alpha5\ tinfty15\ tinfty16\ tinfty17 - 30\ \alpha7\ tinfty12\ tinfty17^2 + 30\ \alpha7\ tinfty13\ tinfty16\ tinfty17 + 30\ \alpha7\ tinfty14\ tinfty15\ tinfty17 - 30\ \alpha7\ tinfty15^2\ tinfty16\right)$$

$$A11InftyLambda0 := \frac{1}{210\ tinfty17^4}\left(210\ LA10\ tinfty17^4 + 210\ \alpha1\ tinfty16\ tinfty17^3 + 70\ \alpha3\ tinfty14\ tinfty17^3 - 70\ \alpha3\ tinfty15\ tinfty16\ tinfty17^2 + 42\ \alpha5\ tinfty12\ tinfty17^3 - 42\ \alpha5\ tinfty13\ tinfty16\ tinfty17^2 - 42\ \alpha5\ tinfty14\ tinfty15\ tinfty17^2 + 42\ \alpha5\ tinfty15^2\ tinfty16\ tinfty17 + 15\ \alpha7\ h\ tinfty17^3 - 30\ \alpha7\ tinfty11\ tinfty16\ tinfty17^2 - 30\ \alpha7\ tinfty12\ tinfty15\ tinfty17^2 - 30\ \alpha7\ tinfty13\ tinfty14\ tinfty17^2 + 60\ \alpha7\ tinfty13\ tinfty15\ tinfty16\ tinfty17 + 30\ \alpha7\ tinfty14\ tinfty15^2\ tinfty17 - 30\ \alpha7\ tinfty15^3\ tinfty16\right) \quad (15)$$

A_{1,1}=c3*lambda^3+c2*lambda^2+c1*lambda+c0+ rho1/(lambda-q1)+ rho2/(lambda-q2)

```
> A11Form:=c3*lambda^3+c2*lambda^2+c1*lambda+c0+ rho1/(lambda-q1)+
rho2/(lambda-q2);

simplify(-residue(A11Form/lambda^4,lambda=infinity)-
A11InftyLambda3);
solve({
factor(-residue(A11Form/lambda^4,lambda=infinity))=
A11InftyLambda3,
factor(-residue(A11Form/lambda^3,lambda=infinity))=
A11InftyLambda2,
factor(-residue(A11Form/lambda^2,lambda=infinity))=
A11InftyLambda1},{c1,c2,c3}):
```

$$A11Form := c3\ \lambda^3 + c2\ \lambda^2 + c1\ \lambda + c0 + \frac{\rho1}{\lambda - q1} + \frac{\rho2}{\lambda - q2}$$

$$\frac{(42\, c3 + 7\, \alpha 6)\, tinfty17 - 6\, \alpha 7\, tinfty16}{42\, tinfty17} \tag{16}$$

```
> c1 := -(105*alpha2*tinfty17^3-70*alpha3*tinfty16*tinfty17^2-42*
  alpha5*tinfty14*tinfty17^2+42*alpha5*tinfty15*tinfty16*tinfty17
  -30*alpha7*tinfty12*tinfty17^2+30*alpha7*tinfty13*tinfty16*
  tinfty17+30*alpha7*tinfty14*tinfty15*tinfty17-30*alpha7*
  tinfty15^2*tinfty16)/(210*tinfty17^3):
  c2 := -(35*alpha4*tinfty17^2-28*alpha5*tinfty16*tinfty17-20*
  alpha7*tinfty14*tinfty17+20*alpha7*tinfty15*tinfty16)/(140*
  tinfty17^2):
  c3 := -(7*alpha6*tinfty17-6*alpha7*tinfty16)/(42*tinfty17):
  factor(-residue(A11Form/lambda^4,lambda=infinity)-
  A11InftyLambda3);
  factor(-residue(A11Form/lambda^3,lambda=infinity)-
  A11InftyLambda2);
  factor(-residue(A11Form/lambda^2,lambda=infinity)-
  A11InftyLambda1);
```

$$0$$
$$0$$
$$0 \tag{17}$$

We check the result for (c_{\infty,k})_{1\leq k\leq r_\infty-1} with the theoretical formulas

```
> cVectorTheo:=Matrix(rinfty-1,1,0):
  cVectorTheo[1,1]:=c3Theo:
  cVectorTheo[2,1]:=c2Theo:
  cVectorTheo[3,1]:=c1Theo:
  cVectorTheo;
  MinftyReduced:=Matrix(rinfty-1,rinfty-1,0):
  MinftyReduced[1,1]:=tinfty17:
  MinftyReduced[2,2]:=tinfty17:
  MinftyReduced[3,3]:=tinfty17:
  MinftyReduced[2,1]:=tinfty15:
  MinftyReduced[3,2]:=tinfty15:
  MinftyReduced[3,1]:=tinfty13:
  MinftyReduced;

  RHSTheo:=Matrix(rinfty-1,1,0):
  RHSTheo[1,1]:=alpha7/7*tinfty16-alpha6/6*tinfty17:
  RHSTheo[2,1]:=alpha7/7*tinfty14-alpha6/6*tinfty15+alpha5/5*
  tinfty16-alpha4/4*tinfty17:
  RHSTheo[3,1]:=alpha7/7*tinfty12-alpha6/6*tinfty13+alpha5/5*
  tinfty14-alpha4/4*tinfty15+alpha3/3*tinfty16-alpha2/2*tinfty17:
```

$$\begin{bmatrix} c3Theo \\ c2Theo \\ c1Theo \end{bmatrix}$$

$$\begin{bmatrix} tinfty17 & 0 & 0 \\ tinfty15 & tinfty17 & 0 \\ tinfty13 & tinfty15 & tinfty17 \end{bmatrix} \quad (18)$$

```
> cVectorTheo:=Multiply(MinftyReduced^(-1),RHSTheo):
  simplify(cVectorTheo[1,1]-c3);
  simplify(cVectorTheo[2,1]-c2);
  simplify(cVectorTheo[3,1]-c1);
```

$$0$$
$$0$$
$$0 \quad (19)$$

Summary of all coefficients

```
> c3;
  c2;
  c1;
```

$$-\frac{7\ \alpha 6\ tinfty17 - 6\ \alpha 7\ tinfty16}{42\ tinfty17}$$

$$-\frac{35\ \alpha 4\ tinfty17^2 - 28\ \alpha 5\ tinfty16\ tinfty17 - 20\ \alpha 7\ tinfty14\ tinfty17 + 20\ \alpha 7\ tinfty15\ tinfty16}{140\ tinfty17^2}$$

$$-\frac{1}{210\ tinfty17^3}\left(105\ \alpha 2\ tinfty17^3 - 70\ \alpha 3\ tinfty16\ tinfty17^2 - 42\ \alpha 5\ tinfty14\ tinfty17^2 + 42\ \alpha 5\ tinfty15\ tinfty16\ tinfty17 - 30\ \alpha 7\ tinfty12\ tinfty17^2 + 30\ \alpha 7\ tinfty13\ tinfty16\ tinfty17 + 30\ \alpha 7\ tinfty14\ tinfty15\ tinfty17 - 30\ \alpha 7\ tinfty15^2\ tinfty16\right) \quad (20)$$

```
> nuMinus1;
  nu0;
  nu1;
  nu2;
```

$$\frac{2\ \alpha 7}{7\ tinfty17}$$

$$\frac{2\left(7\ \alpha 5\ tinfty17 - 5\ \alpha 7\ tinfty15\right)}{35\ tinfty17^2}$$

$$-\frac{2\left(tinfty13\ tinfty17 - tinfty15^2\right)\ \alpha 7}{7\ tinfty17^3} - \frac{2\ tinfty15\ \alpha 5}{5\ tinfty17^2} + \frac{2\ \alpha 3}{3\ tinfty17}$$

$$-\frac{2\left(tinfty17^2\ tinfty11 - 2\ tinfty13\ tinfty15\ tinfty17 + tinfty15^3\right)\ \alpha 7}{7\ tinfty17^4} - \frac{2\left(tinfty13\ tinfty17 - tinfty15^2\right)\ \alpha 5}{5\ tinfty17^3} - \frac{2\ tinfty15\ \alpha 3}{3\ tinfty17^2} + \frac{2\ \alpha 1}{tinfty17} \quad (21)$$

```
> mu1;
```

```
mu2;
```

$$\frac{1}{105\,tinfty17^4\,(q1-q2)}\Big(2\,\big(-35\,\alpha3\,q2\,tinfty17^3+21\,\alpha5\,q2\,tinfty15\,tinfty17^2+15\,\alpha7\,q2\,tinfty13\,tinfty17^2-15\,\alpha7\,q2\,tinfty15^2\,tinfty17+105\,\alpha1\,tinfty17^3-35\,\alpha3\,tinfty15\,tinfty17^2-21\,\alpha5\,tinfty13\,tinfty17^2+21\,\alpha5\,tinfty15^2\,tinfty17-15\,\alpha7\,tinfty11\,tinfty17^2+30\,\alpha7\,tinfty13\,tinfty15\,tinfty17-15\,\alpha7\,tinfty15^3\big)\Big)$$

$$-\frac{1}{105\,tinfty17^4\,(q1-q2)}\Big(2\,\big(-35\,\alpha3\,q1\,tinfty17^3+21\,\alpha5\,q1\,tinfty15\,tinfty17^2+15\,\alpha7\,q1\,tinfty13\,tinfty17^2-15\,\alpha7\,q1\,tinfty15^2\,tinfty17+105\,\alpha1\,tinfty17^3-35\,\alpha3\,tinfty15\,tinfty17^2-21\,\alpha5\,tinfty13\,tinfty17^2+21\,\alpha5\,tinfty15^2\,tinfty17-15\,\alpha7\,tinfty11\,tinfty17^2+30\,\alpha7\,tinfty13\,tinfty15\,tinfty17-15\,\alpha7\,tinfty15^3\big)\Big) \quad \textbf{(22)}$$