In this Maple file, we perform the gauge transformation to obtain the geometric Lax pairs for the second element of the Painlevé 1 hierarchy. This Lax pair is then expressed in terms of the symmetric Darboux coordinates and formulas are checked with the theoretical ones.

We load the results on the Lax pair expressed in the oper gauge as well as some procedures to obtain the elementary symmetric polynomials. We define \td{L} and \td{A} using the gauge transformation from the oper gauge.

```
> restart:
  with(LinearAlgebra):
  with(ListTools):
  with(combinat):
  with(PolynomialTools):
  with(Groebner):

  chk:=proc()
  local V,A,p,L,K,N,KK:
  V:=[seq(args[i],i=2..nargs)];
  A:=[seq(sigma[i],i=1..nargs-1)];
  p:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
  L := Reverse([seq((-1)^(r+nargs-1)*coeff(p, x_, r), r = 0..
  nargs-2)])-A;
  K:=Basis(L,tdeg(V[])):
  N:=NormalForm(args[1],K,tdeg(V[]));
  KK:=Basis(A,tdeg(A[]));
  NormalForm(N,KK,tdeg(A[]));
  if is(NormalForm(N,KK,tdeg(A[]))=0) then print("symmetric")else
  print("not symmetric")fi;
  end proc:

  es:=proc()
  local V, A, p, L, K;
  V:=[seq(args[i],i=2..nargs)];A:=[seq(sigma[i],i=1..nargs-1)];
  p:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
  L := Reverse([seq((-1)^(r+nargs-1)*coeff(p, x_, r), r = 0..
  nargs-2)])-A;
  K:=Basis(L,tdeg(V[]));
  NormalForm(args[1],K,tdeg(V[]));
  end proc:

  ss:=proc() local L, LL, t, LLL, H, K;
  L:=[seq(args[i],i=2..nargs)];
  LL:=[seq(map(x->x^r,L),r=1..nargs-1)];
  t:=seq(s[i],i=1..nargs-1);
  LLL:=[seq(add(i,i in LL[u]),u=1..nops(LL))];
```

```
H:=LLL-[t];
K:=Basis(H,grlex(L[]));
NormalForm(args[1],K,grlex(L[]));
end proc:

rinfty:=4:
g:=rinfty-2:
tinfty27:=-tinfty17:
tinfty25:=-tinfty15:
tinfty23:=-tinfty13:
tinfty21:=-tinfty11:
tinfty20:=-tinfty10:
tinfty26:=tinfty16:
tinfty24:=tinfty14:
tinfty22:=tinfty12:
tinfty10:=0:
Pinfty01 := -tinfty12;
Pinfty11 := -tinfty14;
Pinfty21 := -tinfty16;
Pinfty22 := -(1/2)*tinfty11*tinfty17+(1/2)*tinfty12*tinfty16-
(1/2)*tinfty13*tinfty15+(1/4)*tinfty14^2;
Pinfty32 := -(1/2)*tinfty13*tinfty17+(1/2)*tinfty14*tinfty16-
(1/4)*tinfty15^2;
Pinfty42 := -(1/2)*tinfty15*tinfty17+(1/4)*tinfty16^2;
Pinfty52 := -(1/4)*tinfty17^2;

Pinfty2[1]:=Pinfty21:
Pinfty2[2]:=Pinfty22:
Pinfty2[3]:=Pinfty32:
Pinfty2[4]:=Pinfty42:
Pinfty2[5]:=Pinfty52:

P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2;
P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
x^4+Pinfty52*x^5;
tdP2:=unapply(Pinfty22*x^2+Pinfty32*x^3+Pinfty42*x^4+Pinfty52*
x^5,x);

dP1dlambda:=unapply(diff(P1(lambda),lambda),lambda):
dP2dlambda:=unapply(diff(P2(lambda),lambda),lambda):
dtdP2dlambda:=unapply(diff(tdP2(lambda),lambda),lambda):
```

```
c3:=-(7*alpha6*tinfty17-6*alpha7*tinfty16)/(42*tinfty17):
c2:=-(35*alpha4*tinfty17^2-28*alpha5*tinfty16*tinfty17-20*alpha7*
tinfty14*tinfty17+20*alpha7*tinfty15*tinfty16)/(140*tinfty17^2):
c1:=-(105*alpha2*tinfty17^3-70*alpha3*tinfty16*tinfty17^2-42*
alpha5*tinfty14*tinfty17^2+42*alpha5*tinfty15*tinfty16*tinfty17
-30*alpha7*tinfty12*tinfty17^2+30*alpha7*tinfty13*tinfty16*
tinfty17+30*alpha7*tinfty14*tinfty15*tinfty17-30*alpha7*
tinfty15^2*tinfty16)/(210*tinfty17^3):

c[0]:=c0:
c[1]:=c1:
c[2]:=c2:
c[3]:=c3:

alpha[1]:=alpha1:
alpha[2]:=alpha2:
alpha[3]:=alpha3:
alpha[4]:=alpha4:
alpha[5]:=alpha5:
alpha[6]:=alpha6:
alpha[7]:=alpha7:

tinfty[1]:=tinfty11:
tinfty[2]:=tinfty12:
tinfty[3]:=tinfty13:
tinfty[4]:=tinfty14:
tinfty[5]:=tinfty15:
tinfty[6]:=tinfty16:
tinfty[7]:=tinfty17:

nuMoins1:=2*alpha7/(7*tinfty17):
nu0:=(2*(7*alpha5*tinfty17-5*alpha7*tinfty15))/(35*tinfty17^2):
nu1:=-(2*(tinfty13*tinfty17-tinfty15^2))*alpha7/(7*tinfty17^3)-2*
tinfty15*alpha5/(5*tinfty17^2)+2*alpha3/(3*tinfty17):
nu2:=-(2*(tinfty11*tinfty17^2-2*tinfty13*tinfty15*tinfty17+
tinfty15^3))*alpha7/(7*tinfty17^4)-(2*(tinfty13*tinfty17-
tinfty15^2))*alpha5/(5*tinfty17^3)-2*tinfty15*alpha3/(3*
tinfty17^2)+2*alpha1/tinfty17:

nu[-1]:=nuMoins1:
nu[0]:=nu0:
nu[1]:=nu1:
```

```
nu[2]:=nu2:

mu1:=2*(-35*alpha3*q2*tinfty17^3+21*alpha5*q2*tinfty15*
tinfty17^2+15*alpha7*q2*tinfty13*tinfty17^2-15*alpha7*q2*
tinfty15^2*tinfty17+105*alpha1*tinfty17^3-35*alpha3*tinfty15*
tinfty17^2-21*alpha5*tinfty13*tinfty17^2+21*alpha5*tinfty15^2*
tinfty17-15*alpha7*tinfty11*tinfty17^2+30*alpha7*tinfty13*
tinfty15*tinfty17-15*alpha7*tinfty15^3)/(105*tinfty17^4*(q1-q2)):
mu2:=-1/(105*tinfty17^4*(q1-q2))* 2*(-35*alpha3*q1*tinfty17^3+21*
alpha5*q1*tinfty15*tinfty17^2+15*alpha7*q1*tinfty13*tinfty17^2
-15*alpha7*q1*tinfty15^2*tinfty17+105*alpha1*tinfty17^3-35*
alpha3*tinfty15*tinfty17^2-21*alpha5*tinfty13*tinfty17^2+21*
alpha5*tinfty15^2*tinfty17-15*alpha7*tinfty11*tinfty17^2+30*
alpha7*tinfty13*tinfty15*tinfty17-15*alpha7*tinfty15^3):

rho1:=-mu1*p1:
rho2:=-mu2*p2:

C0:=-(-q1^5*q2*tinfty17^2+q1*q2^5*tinfty17^2-2*q1^4*q2*tinfty15*
tinfty17+q1^4*q2*tinfty16^2+2*q1*q2^4*tinfty15*tinfty17-q1*q2^4*
tinfty16^2-2*q1^3*q2*tinfty13*tinfty17+2*q1^3*q2*tinfty14*
tinfty16-q1^3*q2*tinfty15^2+2*q1*q2^3*tinfty13*tinfty17-2*q1*
q2^3*tinfty14*tinfty16+q1*q2^3*tinfty15^2+4*p1*q1^2*q2*tinfty16
-4*p2*q1*q2^2*tinfty16-2*q1^2*q2*tinfty11*tinfty17+2*q1^2*q2*
tinfty12*tinfty16-2*q1^2*q2*tinfty13*tinfty15+q1^2*q2*
tinfty14^2+2*q1*q2^2*tinfty11*tinfty17-2*q1*q2^2*tinfty12*
tinfty16+2*q1*q2^2*tinfty13*tinfty15-q1*q2^2*tinfty14^2+4*p1*q1*
q2*tinfty14-4*p2*q1*q2*tinfty14+4*p1^2*q2+4*p1*q2*tinfty12-4*
p2^2*q1-4*p2*q1*tinfty12+4*h*p1-4*h*p2)/(4*(q1-q2)):
C1:=1/4/(q1-q2)*(-q1^5*tinfty17^2+q2^5*tinfty17^2-2*q1^4*
tinfty15*tinfty17+q1^4*tinfty16^2+2*q2^4*tinfty15*tinfty17-q2^4*
tinfty16^2-2*q1^3*tinfty13*tinfty17+2*q1^3*tinfty14*tinfty16-
q1^3*tinfty15^2+2*q2^3*tinfty13*tinfty17-2*q2^3*tinfty14*
tinfty16+q2^3*tinfty15^2+4*p1*q1^2*tinfty16-4*p2*q2^2*tinfty16-2*
q1^2*tinfty11*tinfty17+2*q1^2*tinfty12*tinfty16-2*q1^2*tinfty13*
tinfty15+q1^2*tinfty14^2+2*q2^2*tinfty11*tinfty17-2*q2^2*
tinfty12*tinfty16+2*q2^2*tinfty13*tinfty15-q2^2*tinfty14^2+4*p1*
q1*tinfty14-4*p2*q2*tinfty14+4*p1^2+4*p1*tinfty12-4*p2^2-4*p2*
tinfty12):

CurlyLq1:=2*mu1*(p1-1/2*P1(q1))-h*nu0-h*nuMoins1*q1-h*(mu1+mu2)/
(q1-q2):
```

```
CurlyLq2:=2*mu2*(p2-1/2*P1(q2))-h*nu0-h*nuMoins1*q2-h*(mu2+mu1)/
(q2-q1):
CurlyLp1:=h*(mu2+mu1)*(p2-p1)/(q1-q2)^2+mu1*(p1*dP1dlambda(q1)-
dtdP2dlambda(q1)+1*C1*q1^0) +h*nuMoins1*p1+ h*(1*c1*q1^0+2*c2*
q1^1+3*c3*q1^2):
CurlyLp2:=h*(mu1+mu2)*(p1-p2)/(q2-q1)^2+mu2*(p2*dP1dlambda(q2)-
dtdP2dlambda(q2)+1*C1*q2^0) +h*nuMoins1*p2+ h*(1*c1*q2^0+2*c2*
q2^1+3*c3*q2^2):

CurlyLQ1:=CurlyLq1+CurlyLq2:
CurlyLQ2:=CurlyLq1*q2+q1*CurlyLq2:

dP1dlambda:=unapply(diff(P1(lambda),lambda),lambda):
dP2dlambda:=unapply(diff(P2(lambda),lambda),lambda):
L:=Matrix(2,2,0):
L[1,1]:=0:
L[1,2]:=1:
L[2,1]:=-tdP2(lambda)+C1*lambda+C0 - h*p1/(lambda-q1)-h*p2/
(lambda-q2):
L[2,2]:= P1(lambda)  +h/(lambda-q1)+h/(lambda-q2) :

A:=Matrix(2,2,0):
A[1,1]:=c3*lambda^3+c2*lambda^2+c1*lambda+c0+ rho1/(lambda-q1)+
rho2/(lambda-q2):
A[1,2]:=nuMoins1*lambda+nu0+ mu1/(lambda-q1)+ mu2/(lambda-q2):
A[2,1]:=AA21(lambda):
A[2,2]:=AA22(lambda):
dAdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dAdlambda[i,j]:=diff(A
[i,j],lambda):  od: od:

L:
A:

Q2Poly:=unapply( -p1*(lambda-q2)/(q1-q2)-p2*(lambda-q1)/(q2-q1),
lambda);
simplify(Q2Poly(q1));
simplify(Q2Poly(q2));

J:=Matrix(2,2,0):
J[1,1]:=1:
J[1,2]:=0:
```

```
J[2,1]:=Q2Poly(lambda)/(lambda-q1)/(lambda-q2)+1/2*tinfty16:
J[2,2]:=1/(lambda-q1)/(lambda-q2):
dJdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do dJdlambda[i,j]:=diff(J
[i,j],lambda): od: od:
J:

Ltilde:=simplify(Multiply(Multiply(J,L),J^(-1))+h*Multiply
(dJdlambda,J^(-1))):
```

$$Pinfty01 := -tinfty12$$
$$Pinfty11 := -tinfty14$$
$$Pinfty21 := -tinfty16$$
$$Pinfty22 := -\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}$$
$$Pinfty32 := -\frac{tinfty13\ tinfty17}{2} + \frac{tinfty14\ tinfty16}{2} - \frac{tinfty15^2}{4}$$
$$Pinfty42 := -\frac{tinfty15\ tinfty17}{2} + \frac{tinfty16^2}{4}$$
$$Pinfty52 := -\frac{tinfty17^2}{4}$$
$$P1 := x \mapsto Pinfty01 + Pinfty11\, x + Pinfty21\, x^2$$
$$P2 := x \mapsto Pinfty02 + Pinfty12\, x + Pinfty22\, x^2 + Pinfty32\, x^3 + Pinfty42\, x^4 + Pinfty52\, x^5$$
$$tdP2 := x \mapsto -\frac{x^5\ tinfty17^2}{4} + x^4\left(-\frac{tinfty15\ tinfty17}{2} + \frac{tinfty16^2}{4}\right) + x^3\left(-\frac{tinfty13\ tinfty17}{2} + \frac{tinfty14\ tinfty16}{2} - \frac{tinfty15^2}{4}\right) + x^2\left(-\frac{tinfty11\ tinfty17}{2} + \frac{tinfty12\ tinfty16}{2} - \frac{tinfty13\ tinfty15}{2} + \frac{tinfty14^2}{4}\right)$$
$$Q2Poly := \lambda \mapsto -\frac{p1\ (\lambda - q2)}{q1 - q2} - \frac{p2\ (\lambda - q1)}{q2 - q1}$$
$$-p1$$
$$-p2 \qquad (1)$$

```
> Elementaryh:= proc(k)
local aux,i,Coeff:
aux:=1: for i from 1 to 2 do aux:=aux/(1-t*q[i]): od:
Coeff:=unapply(es(residue(aux/t^(k+1),t=0),q[1],q[2]),sigma[1],
sigma[2]):
return(Coeff(Q1,Q2)):
end proc:

hh[0]:=Elementaryh(0);
hh[1]:=Elementaryh(1);
hh[2]:=Elementaryh(2);
hh[3]:=Elementaryh(3);
```

```
hh[4]:=Elementaryh(4);
hh[5]:=Elementaryh(5);

e[0]:=1:
e[1]:=q[1]+q[2]:
e[2]:=q[1]*q[2]:
PolyP:=unapply(lambda^2-Q1*lambda+Q2,lambda);
```

$$hh_0 := 1$$

$$hh_1 := Q1$$

$$hh_2 := Q1^2 - Q2$$

$$hh_3 := Q1^3 - 2\,Q2\,Q1$$

$$hh_4 := Q1^4 - 3\,Q1^2\,Q2 + Q2^2$$

$$hh_5 := Q1^5 - 4\,Q1^3\,Q2 + 3\,Q1\,Q2^2$$

$$PolyP := \lambda \mapsto -Q1\,\lambda + \lambda^2 + Q2 \tag{2}$$

The compatibility equation reads \mathcal{L}L=h\partial_x A+[A,L]
Because the first line of L is trivial (0,1) we may obtain A[2,1] and A[2,2] to complete the knowledge of A

```
> CurlyL:=h*dAdlambda+(Multiply(A,L)-Multiply(L,A)):

  Entree11:=CurlyL[1,1]:
  Entree12:=CurlyL[1,2]:

  AA21:=unapply(solve(Entree11=0,AA21(lambda)),lambda):
  AA21bis:=h*dAdlambda[1,1]+A[1,2]*L[2,1]:

  simplify(AA21(lambda)-AA21bis);
  AA22:=unapply(solve(Entree12=0,AA22(lambda)),lambda):
  AA22bis:=h*dAdlambda[1,2]+A[1,1]+A[1,2]*L[2,2]:

  simplify(AA22(lambda)-AA22bis);
  simplify(Entree11);
  simplify(Entree12);
  CurlyL:=h*dAdlambda+(Multiply(A,L)-Multiply(L,A)):
```

$$0$$

$$0$$

$$0$$

$$0 \tag{3}$$

```
> e1:=q1+q2:
  e2:=q1*q2:
  p1:=P1*diff(e1,q1)+P2*diff(e2,q1);
  p2:=P1*diff(e1,q2)+P2*diff(e2,q2);
  Q2Polyfunction:=unapply(es(simplify(-p1*(lambda-q2)/(q1-q2)-p2*
```

```
(lambda-q1)/(q2-q1)),q1,q2),sigma[1],sigma[2]):
Q2Poly:=unapply( Q2Polyfunction(Q[1],Q[2]),lambda);

Q[0]:=1:
Q[1]:=Q1:
Q[2]:=Q2:
P[1]:=P1:
P[2]:=P2:
q[1]:=q1:
q[2]:=q2:
p[1]:=p1:
p[2]:=p2:
tinfty[1]:=tinfty11:
tinfty[2]:=tinfty12:
tinfty[3]:=tinfty13:
tinfty[4]:=tinfty14:
tinfty[5]:=tinfty15:
tinfty[6]:=tinfty16:
tinfty[7]:=tinfty17:

C0function:=unapply(es(simplify(C0),q1,q2),sigma[1],sigma[2]):
C1function:=unapply(es(simplify(C1),q1,q2),sigma[1],sigma[2]):
C[0]:=C0function(Q[1],Q[2]):
C[1]:=C1function(Q[1],Q[2]):
```

$$p1 := P2\, q2 + P1$$
$$p2 := P2\, q1 + P1$$
$$Q2Poly := \lambda \mapsto P2\, \lambda - P2\, Q_1 - P1 \quad \textbf{(4)}$$

```
> solve({pp1=PP1*diff(e1,q1)+PP2*diff(e2,q1),pp2=PP1*diff(e1,q2)+
PP2*diff(e2,q2)},{PP1,PP2}):
PP1 := (pp1*q1-pp2*q2)/(q1-q2):
PP2 := -(pp1-pp2)/(q1-q2):
CurlyLP1fonction:=unapply(diff(PP1,pp1)*CurlyLp1+diff(PP1,pp2)*
CurlyLp2+diff(PP1,q1)*CurlyLq1+diff(PP1,q2)*CurlyLq2,pp1,pp2):
CurlyLP2fonction:=unapply(diff(PP2,pp1)*CurlyLp1+diff(PP2,pp2)*
CurlyLp2+diff(PP2,q1)*CurlyLq1+diff(PP2,q2)*CurlyLq2,pp1,pp2):
CurlyLP1:=simplify(CurlyLP1fonction(p1,p2)):
CurlyLP2:=simplify(CurlyLP2fonction(p1,p2)):
simplify(CurlyLp1-CurlyLP2*q2-P2*CurlyLq2-CurlyLP1);
simplify(CurlyLp2-CurlyLP2*q1-P2*CurlyLq1-CurlyLP1);
```

$$0$$
$$0 \quad \textbf{(5)}$$

Expression of \td{L} in the Darboux coordinates (q_1,q_2,p_1,p_2)

```
> Ltilde11:=simplify(Ltilde[1,1]);
```

```
Ltilde12:=simplify(Ltilde[1,2]);
Ltilde21:=simplify(Ltilde[2,1]):
Ltilde22:=simplify(Ltilde[2,2]);
```

$$Ltilde11 := -\frac{(-\lambda+q2)\,(-\lambda+q1)\,tinfty16}{2} - P2\,\lambda + \frac{(2\,q1+2\,q2)\,P2}{2} + P1$$

$$Ltilde12 := (-\lambda+q1)\,(-\lambda+q2)$$

$$Ltilde22 := -\frac{tinfty16\,\lambda^2}{2} + \frac{((-q1-q2)\,tinfty16+2\,P2-2\,tinfty14)\,\lambda}{2} + \frac{q1\,q2\,tinfty16}{2} + \frac{(-2\,q1-2\,q2)\,P2}{2} - P1 - tinfty12 \tag{6}$$

Expression of \td{L} in the Darboux coordinates (Q_1,Q_2,P_1,P_2)

```
> Ltilde11NewVar:=unapply( es(simplify(Ltilde11), q1, q2),sigma[1],
  sigma[2]):
  Ltilde12NewVar:=unapply( es(simplify(Ltilde12), q1, q2),sigma[1],
  sigma[2]):
  Ltilde22NewVar:=unapply( es(simplify(Ltilde22), q1, q2),sigma[1],
  sigma[2]):
  Ltilde11NewVar(Q1,Q2);
  Ltilde12NewVar(Q1,Q2);
  Ltilde22NewVar(Q1,Q2);
```

$$-\frac{1}{2}\,tinfty16\,\lambda^2 + \frac{1}{2}\,\lambda\,tinfty16\,Q1 - P2\,\lambda + P2\,Q1 - \frac{1}{2}\,tinfty16\,Q2 + P1$$

$$-\lambda\,Q1 + \lambda^2 + Q2$$

$$-\frac{1}{2}\,tinfty16\,\lambda^2 - \frac{1}{2}\,\lambda\,tinfty16\,Q1 + P2\,\lambda - P2\,Q1 - tinfty14\,\lambda + \frac{1}{2}\,tinfty16\,Q2 - P1 - tinfty12 \tag{7}$$

Let us now check that the matrix \td{L} matches with the theoretical formulas.

```
> Ltilde11Theo:=proc()
  local res,j,aux,i:
  res:=0:
  for j from 0 to g-1 do aux:=0:  if j+1<=g then for i from j+1 to
  g do aux:=aux+P[i]*Q[i-j-1]: od: fi:  res:=res-(-1)^(j-1)*aux*
  lambda^j: od:
  for j from 0 to g do res:=res-1/2*tinfty16*(-1)^(g-j)*Q[g-j]*
  lambda^j: od:
  return(res):
  end proc:

  Ltilde11Theo();
  simplify(Ltilde11NewVar(Q1,Q2)-Ltilde11Theo());
```

$$-\frac{1}{2}\,tinfty16\,\lambda^2 + \frac{1}{2}\,\lambda\,tinfty16\,Q1 - P2\,\lambda + P2\,Q1 - \frac{1}{2}\,tinfty16\,Q2 + P1$$

$$0 \tag{8}$$

```
> Ltilde12Theo:=proc()
  local res,m:
  res:=0:
  for m from 0 to g do  res:=res+(-1)^(g-m)*Q[g-m]*lambda^m: od:
  return(res):
  end proc:

  Ltilde12Theo();
  simplify(Ltilde12NewVar(Q1,Q2)-Ltilde12Theo());
```

$$-\lambda Q1 + \lambda^2 + Q2$$

$$0 \qquad \textbf{(9)}$$

```
> Ltilde22Theo:=proc()
  local res,j,aux,i,s:
  res:=0:
  for j from 0 to g-1 do aux:=0:  if j+1<=g then for i from j+1 to
  g do aux:=aux+P[i]*Q[i-j-1]: od: fi:  res:=res+(-1)^(j-1)*aux*
  lambda^j: od:
  for j from 0 to g do res:=res+1/2*tinfty16*(-1)^(g-j)*Q[g-j]*
  lambda^j: od:
  for s from 0 to rinfty-2 do res:=res- tinfty[2*s+2]*lambda^s: od:
  return(res):
  end proc:

  Ltilde22Theo();
  simplify(Ltilde22NewVar(Q1,Q2)-Ltilde22Theo());
```

$$-\frac{1}{2}\, tinfty16\, \lambda^2 - \frac{1}{2}\, \lambda\, tinfty16\, Q1 + P2\, \lambda - P2\, Q1 - tinfty14\, \lambda + \frac{1}{2}\, tinfty16\, Q2 - P1 - tinfty12$$

$$0 \qquad \textbf{(10)}$$

```
> Ltilde21Theo:=proc()
  local res,j,i,j1,j2,i1,i2,r,s,m:
  res:=0:
  for i from 0 to rinfty-1 do for j from g+i to 2*rinfty-3 do res:=
  res-Pinfty2[j]*hh[j-g-i]*lambda^i: od: od:

  for i from 0 to g-1 do for j from i to g do for s from g+i-j to g
  do for r from j+1 to g do
  res:=res+ (-1)^(j-1)*tinfty[2*s+2]*P[r]*Q[r-j-1]*hh[s+j-i-g]*
  lambda^(i): od: od: od: od:

  for i from 0 to g-2 do for j1 from i+1 to g-1 do for j2 from g+i-
  j1 to g-1 do for i1 from j1+1 to g do for i2 from j2+1 to g do
  res:=res- (-1)^(j1+j2)*P[i1]*Q[i1-j1-1]*P[i2]*Q[i2-j2-1]*hh[j1+j2
```

```
-g-i]*lambda^i:
od: od: od: od: od:

for m from 0 to g do res:=res-1/4*tinfty[2*rinfty-2]^2*(-1)^(g-m)
*Q[g-m]*lambda^m: od:

for s from 0 to g do res:=res+1/2*tinfty[2*rinfty-2]*tinfty[2*
s+2]*lambda^s: od:

for j from 0 to g-1 do for i from j+1 to g do res:=res-tinfty[2*
rinfty-2]*(-1)^(j-1)*P[i]*Q[i-j-1]*lambda^j: od: od:

return(res):
end proc:
```

```
> Ltilde21NewVar:=unapply( es(simplify(Ltilde21), q1, q2),sigma[1],
sigma[2]):
Ltilde21NewVar(Q1,Q2);
factor(series(simplify(Ltilde21NewVar(Q1,Q2)-Ltilde21Theo()),
lambda=0));
```

$$\frac{1}{4}\, tinfty17^2\, \lambda^3 + \frac{1}{4}\, \lambda^2\, tinfty17^2\, Q1 + \frac{1}{4}\, \lambda\, tinfty17^2\, Q1^2 + \frac{1}{4}\, tinfty17^2\, Q1^3 + \frac{1}{2}\, \lambda^2\, tinfty15\, tinfty17 + \frac{1}{2}\, \lambda\, tinfty15\, tinfty17\, Q1 - \frac{1}{4}\, \lambda\, tinfty17^2\, Q2 + \frac{1}{2}\, tinfty15\, tinfty17\, Q1^2 - \frac{1}{4}\, tinfty16^2\, Q1^2 - \frac{1}{2}\, tinfty17^2\, Q1\, Q2 + P2\, tinfty16\, Q1 + \frac{1}{2}\, \lambda\, tinfty13\, tinfty17 + \frac{1}{4}\, \lambda\, tinfty15^2 + \frac{1}{2}\, tinfty13\, tinfty17\, Q1 - \frac{1}{2}\, tinfty14\, tinfty16\, Q1 + \frac{1}{4}\, tinfty15^2\, Q1 - \frac{1}{2}\, tinfty15\, tinfty17\, Q2 - P2^2 + tinfty14\, P2 + \frac{1}{2}\, tinfty11\, tinfty17 + \frac{1}{2}\, tinfty13\, tinfty15 - \frac{1}{4}\, tinfty14^2$$

$$0 \qquad (11)$$

Let us now compute the auxiliary matrix \td{A} and check that it corresponds to the theoretical formulas.

```
> J:=simplify(J):
CurlyLJ:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do CurlyLJ[i,j]:=
h*(alpha7*diff(J[i,j],tinfty17)+alpha6*diff(J[i,j],tinfty16)+
alpha5*diff(J[i,j],tinfty15)+alpha4*diff(J[i,j],tinfty14)+alpha3*
diff(J[i,j],tinfty13)+alpha2*diff(J[i,j],tinfty12)+alpha1*diff(J
[i,j],tinfty11))
+CurlyLq1*diff(J[i,j],q1)+CurlyLq2*diff(J[i,j],q2)+CurlyLP1*diff
(J[i,j],P1)+CurlyLP2*diff(J[i,j],P2): od: od:
```

```
Atilde:=simplify(Multiply(Multiply(J,A),J^(-1))+Multiply(CurlyLJ,
J^(-1))):
```

Let us now check that the formula for \td{A} matches with the theoretical formulas for each of its entries.

```
> Atilde12Theo:=proc()
  local res,j,m:
  res:=0:
  for j from 0 to g+1 do for m from max(-1,-j) to g-j do res:=res+
  (-1)^(g-j-m)*nu[m]*Q[g-j-m]*lambda^j: od: od:
  return(res):
  end proc:
> Atilde12fonction:=unapply(es(Atilde[1,2],q[1],q[2]),sigma[1],
  sigma[2]):
  Atilde12NewCoordinates:=Atilde12fonction(Q1,Q2):
  simplify(Atilde12NewCoordinates-Atilde12Theo());
```

$$0 \tag{12}$$

```
> Atilde11Theo:=proc()
  local res,i,r,j,m:
  res:=0:
  for i from 0 to rinfty-1 do  res:=res+c[i]*lambda^i: od:
  for i from 0 to g do for m from (max(-1,-i)) to g-1-i do for r
  from i+m+1 to g do res:=res+(-1)^(i+m)*nu[m]*P[r]*Q[r-i-m-1]*
  lambda^i: od: od: od:

  for i from 0 to g+1 do for m from (max(-1,-i)) to g-i do res:=
  res-1/2*tinfty[2*rinfty-2]*(-1)^(g-i-m)*Q[g-i-m]*nu[m]*lambda^i:
  od: od:
  return(res):
  end proc:

> Atilde11fonction:=unapply(es(simplify(Atilde[1,1]),q[1],q[2]),
  sigma[1],sigma[2]):
  Atilde11NewCoordinates:=Atilde11fonction(Q1,Q2):
  simplify(Atilde11NewCoordinates-Atilde11Theo());
```

$$0 \tag{13}$$

```
> Atilde22Theo:=proc()
  local res,s,j:
  res:=-Atilde11Theo():
  res:=res+2*c0+h*(g+1)*nu[-1]:
  for s from 1 to rinfty-1 do res:=res-1/s*alpha[2*s]*lambda^s: od:
  for j from 0 to rinfty-2 do res:=res-tinfty[2*j+2]*nu[j]: od:
  return(res):
```

```
end proc:
```

```
> Atilde22fonction:=unapply(es(simplify(Atilde[2,2]),q[1],q[2]),
sigma[1],sigma[2]):
Atilde22NewCoordinates:=Atilde22fonction(Q1,Q2):
simplify(Atilde22NewCoordinates-Atilde22Theo());
```

$$0 \tag{14}$$

```
> Atilde21Theo:=proc()
local res,i,j,s,m,r,j1,j2,r1,r2:
res:=-1/2*h*(g+1)*nu[-1]*tinfty[2*rinfty-2]+h/2*alpha[2*rinfty-2]
+h*(rinfty-1)*c[rinfty-1]+nu[-1]*C[rinfty-3]:

for i from 0 to g do for j from max(0,i-1) to g-1 do for s from
g+i-j-1 to g do for r from j+1 to g do for m from -1 to s+j-g-i
do
res:=res-(-1)^j*tinfty[2*s+2]*nu[m]*hh[s+j-g-m-i]*P[r]*Q[r-j-1]*
lambda^i:
od: od: od: od: od:

for i from 0 to rinfty do for j from (max(g,g+i-1)) to 2*rinfty-3
do for m from -1 to j-g-i do
res:=res-nu[m]*hh[j-g-m-i]*Pinfty2[j]*lambda^i: od: od: od:

for i from 0 to g do for j1 from 0 to g-1 do for j2 from 0 to g-1
do
if j1+j2-g-i>=-1 then for m from -1 to j1+j2-g-i do
for r1 from j1+1 to g do for r2 from j2+1 to g do
res:=res-(-1)^(j1+j2)*nu[m]*hh[j1+j2-g-i-m]*P[r1]*P[r2]*Q[r1-j1
-1]*Q[r2-j2-1]*lambda^i:
 od:  od: od: fi: od: od: od:

for i from 0 to rinfty-1 do for s from max(0,i-1) to rinfty-2 do
res:=res+1/2*tinfty[2*rinfty-2]*tinfty[2*s+2]*nu[s-i]*lambda^i:
od: od:

for i from 0 to g do for j from max(0,i-1) to g-1 do for r from
j+1 to g do res:=res-tinfty[2*rinfty-2]*(-1)^(j-1)*nu[j-i]*P[r]*Q
[r-j-1]*lambda^i: od: od: od:

for i from 0 to g+1 do for j from max(0,i-1) to g do res:=
res-1/4*(tinfty[2*rinfty-2])^2*(-1)^(g-j)*Q[g-j]*nu[j-i]*
lambda^i: od: od:
```

```
  return(res):
  end proc:
> Atilde21fonction:=unapply(es(simplify(Atilde[2,1]),q[1],q[2]),
  sigma[1],sigma[2]):
  Atilde21NewCoordinates:=Atilde21fonction(Q1,Q2):
  Atilde21NewCoordinates:
  simplify(factor(Atilde21NewCoordinates-Atilde21Theo())):

  nu3fonction:=unapply(es(simplify(q1^g*mu1+q2^g*mu2),q1,q2),sigma
  [1],sigma[2]):
  nu3fonction(Q1,Q2):
  nu[3]:=nu3fonction(Q1,Q2):
  factor(Atilde21NewCoordinates-Atilde21Theo());
```

$$0 \tag{15}$$

```
> VerifSomme:=proc()
  local res,k:
  res:=0:
  for k from 0 to g-1 do res:=res+(g-k)*Q[k]*P[k+1]: od:
  return(res):
  end proc:

  VerifSomme();
  es(simplify(p1+p2),q1,q2);
```

$$P2\,Q1 + 2\,P1$$

$$P2\,\sigma_1 + 2\,P1 \tag{16}$$