In this Maple file, we check that the las for the Lax matrices are correct provided that the Hamiltonian are given as the linear combination of the spectral invariants (Theorem 8.1). We check the formulas both in the oper or in the geometric gauge using also the symmetric Darboux coordinates.

Loarding some procedures for the symmetric polynomials

```
> restart:
  with(LinearAlgebra):
  with(ListTools):
  with(combinat):
  with(PolynomialTools):
  with(Groebner):

  chk:=proc()
  local VV,AA,pp,LL,K,N,KK:
  VV:=[seq(args[i],i=2..nargs)];
  AA:=[seq(sigma[i],i=1..nargs-1)];
  pp:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
  LL := Reverse([seq((-1)^(r+nargs-1)*coeff(pp, x_, r), r = 0..
  nargs-2)])-AA;
  K:=Basis(LL,tdeg(VV[])):
  N:=NormalForm(args[1],K,tdeg(VV[]));
  KK:=Basis(AA,tdeg(AA[]));
  NormalForm(N,KK,tdeg(AA[]));
  if is(NormalForm(N,KK,tdeg(AA[]))=0) then print("symmetric")else
  print("not symmetric")fi;
  end proc:

  es:=proc()
  local VV, AA, pp, LL, K;
  VV:=[seq(args[i],i=2..nargs)];AA:=[seq(sigma[i],i=1..nargs-1)];
  pp:=simplify(expand(mul(x_-args[i],i=2..nargs)),x_);
  LL := Reverse([seq((-1)^(r+nargs-1)*coeff(pp, x_, r), r = 0..
  nargs-2)])-AA;
  K:=Basis(LL,tdeg(VV[]));
  NormalForm(args[1],K,tdeg(VV[]));
  end proc:

  ss:=proc() local LL, LLL, t, LLLL, H, K;
  LL:=[seq(args[i],i=2..nargs)];
  LLL:=[seq(map(x->x^r,LL),r=1..nargs-1)];
  t:=seq(s[i],i=1..nargs-1);
  LLLL:=[seq(add(i,i in LLL[u]),u=1..nops(LLL))];
  H:=LLLL-[t];
  K:=Basis(H,grlex(LL[]));
```

```
NormalForm(args[1],K,grlex(LL[]));
end proc:

ElementaryS:= proc(k)
local aux,i,Coeff:
aux:=0: for i from 1 to g do aux:=aux+q[i]^k od:
Coeff:=unapply(es(aux,q[1],q[2],q[3]),sigma[1],sigma[2],sigma[3])
:
return(Coeff(Q[1],Q[2],Q[3])):
end proc:

rinfty:=6:
g:=rinfty-3:
S[0]:=ElementaryS(0);
S[1]:=ElementaryS(1);
S[2]:=ElementaryS(2);
S[3]:=ElementaryS(3);
S[4]:=ElementaryS(4);
S[5]:=ElementaryS(5);
S[6]:=ElementaryS(6);
S[7]:=ElementaryS(7);
S[8]:=ElementaryS(8);
```

$$S_0 := 3$$
$$S_1 := Q_1$$
$$S_2 := Q_1^2 - 2\, Q_2$$
$$S_3 := Q_1^3 - 3\, Q_1\, Q_2 + 3\, Q_3$$
$$S_4 := Q_1^4 - 4\, Q_1^2\, Q_2 + 4\, Q_1\, Q_3 + 2\, Q_2^2$$
$$S_5 := Q_1^5 - 5\, Q_1^3\, Q_2 + 5\, Q_1^2\, Q_3 + 5\, Q_1\, Q_2^2 - 5\, Q_2\, Q_3$$
$$S_6 := Q_1^6 - 6\, Q_1^4\, Q_2 + 6\, Q_1^3\, Q_3 + 9\, Q_1^2\, Q_2^2 - 12\, Q_1\, Q_2\, Q_3 - 2\, Q_2^3 + 3\, Q_3^2$$
$$S_7 := Q_1^7 - 7\, Q_1^5\, Q_2 + 7\, Q_1^4\, Q_3 + 14\, Q_1^3\, Q_2^2 - 21\, Q_1^2\, Q_2\, Q_3 - 7\, Q_1\, Q_2^3 + 7\, Q_1\, Q_3^2 + 7\, Q_2^2\, Q_3$$
$$S_8 := Q_1^8 - 8\, Q_1^6\, Q_2 + 8\, Q_1^5\, Q_3 + 20\, Q_1^4\, Q_2^2 - 32\, Q_1^3\, Q_2\, Q_3 - 16\, Q_1^2\, Q_2^3 + 12\, Q_1^2\, Q_3^2 + 24\, Q_1\, Q_2^2\, Q_3 + 2\, Q_2^4 - 8\, Q_2\, Q_3^2 \tag{1}$$

```
> res:=-lambda^(2*rinfty-5):
for k from (rinfty-2) to (2*rinfty-7) do aux:=2*tau[2*rinfty-k-6]
:
for m from (k-rinfty+6) to (rinfty-3) do aux:=aux+tau[rinfty-m-2]
*tau[rinfty-k+m-5]: od:
res:=res-aux*lambda^k: od:
aux2:=2*tau[rinfty-3]:
```

```
for m from 3 to (rinfty-3) do aux2:=aux2+tau[rinfty-m-2]*tau[m-2]
: od:
res:=res-aux2*lambda^(rinfty-3):
res;
```

$$-\lambda^7 - 2\,\tau_1\,\lambda^5 - 2\,\tau_2\,\lambda^4 - \left(\tau_1^2 + 2\,\tau_3\right)\lambda^3 \tag{2}$$

```
> tdP2:=unapply(-lambda^7-2*tau[1]*lambda^5-2*tau[2]*lambda^4-(tau
[1]^2+2*tau[3])*lambda^3,lambda);
for k from rinfty-3 to 2*rinfty-5 do P2[k]:=-residue(tdP2(lambda)
/lambda^(k+1), lambda=infinity): od:
P2[3];
P2[4];
P2[5];
P2[6];
P2[7];
q1:=q[1]:
q2:=q[2]:
q3:=q[3]:
p1:=p[1]:
p2:=p[2]:
p3:=p[3]:

Elementaryh:= proc(k)
local aux,i,Coeff:
aux:=1: for i from 1 to g do aux:=aux/(1-t*q[i]): od:
Coeff:=unapply(es(residue(aux/t^(k+1),t=0),q[1],q[2],q[3]),sigma
[1],sigma[2],sigma[3]):
return(Coeff(Q[1],Q[2],Q[3])):
end proc:

h[0]:=simplify(Elementaryh(0));
h[1]:=simplify(Elementaryh(1));
h[2]:=simplify(Elementaryh(2));
h[3]:=simplify(Elementaryh(3));
h[4]:=simplify(Elementaryh(4));
h[5]:=simplify(Elementaryh(5));
h[6]:=simplify(Elementaryh(6));
h[7]:=simplify(Elementaryh(7));
h[8]:=simplify(Elementaryh(8));

Q[0]:=1:
Q[1]:=q[1]+q[2]+q[3]:
Q[2]:=q[1]*q[2]+q[1]*q[3]+q[2]*q[3]:
```

```
Q[3]:=q[1]*q[2]*q[3]:
SymMatrix:=Matrix(g,g,0):
for i from 1 to g do for j from 1 to g do SymMatrix[i,j]:=diff(Q
[j],q[i]): od: od:
SymMatrix:
Vectorp:=Matrix(g,1,0):
for i from 1 to g do Vectorp[i,1]:=p[i]: od:
Vectorp:
VectorP:=Multiply(SymMatrix^(-1),Vectorp);
for i from 1 to g do P[i]:=VectorP[i,1]: od:
```

$$tdP2 := \lambda \mapsto -\lambda^7 - 2\,\tau_1\,\lambda^5 - 2\,\tau_2\,\lambda^4 - \left(\tau_1^2 + 2\,\tau_3\right)\lambda^3$$

$$-\tau_1^2 - 2\,\tau_3$$

$$-2\,\tau_2$$

$$-2\,\tau_1$$

$$0$$

$$-1$$

$$h_0 := 1$$

$$h_1 := Q_1$$

$$h_2 := Q_1^2 - Q_2$$

$$h_3 := Q_1^3 - 2\,Q_1\,Q_2 + Q_3$$

$$h_4 := Q_1^4 - 3\,Q_1^2\,Q_2 + 2\,Q_1\,Q_3 + Q_2^2$$

$$h_5 := Q_1^5 - 4\,Q_1^3\,Q_2 + 3\,Q_1^2\,Q_3 + 3\,Q_1\,Q_2^2 - 2\,Q_2\,Q_3$$

$$h_6 := Q_1^6 - 5\,Q_1^4\,Q_2 + 4\,Q_1^3\,Q_3 + 6\,Q_1^2\,Q_2^2 - 6\,Q_1\,Q_2\,Q_3 - Q_2^3 + Q_3^2$$

$$h_7 := Q_1^7 - 6\,Q_1^5\,Q_2 + 5\,Q_1^4\,Q_3 + 10\,Q_1^3\,Q_2^2 - 12\,Q_1^2\,Q_2\,Q_3 - 4\,Q_1\,Q_2^3 + 3\,Q_1\,Q_3^2 + 3\,Q_2^2\,Q_3$$

$$h_8 := Q_1^8 - 7\,Q_1^6\,Q_2 + 6\,Q_1^5\,Q_3 + 15\,Q_1^4\,Q_2^2 - 20\,Q_1^3\,Q_2\,Q_3 - 10\,Q_1^2\,Q_2^3 + 6\,Q_1^2\,Q_3^2 + 12\,Q_1\,Q_2^2\,Q_3 + Q_2^4 - 3\,Q_2\,Q_3^2$$

$$VectorP := \left[\left[\frac{q_1^2\,p_1}{q_1^2 - q_1\,q_2 - q_3\,q_1 + q_3\,q_2} - \frac{q_2^2\,p_2}{q_1\,q_2 - q_3\,q_1 - q_2^2 + q_3\,q_2} + \frac{q_3^2\,p_3}{q_1\,q_2 - q_3\,q_1 - q_3\,q_2 + q_3^2}\right],\right.$$
$$\left[-\frac{q_1\,p_1}{q_1^2 - q_1\,q_2 - q_3\,q_1 + q_3\,q_2} + \frac{q_2\,p_2}{q_1\,q_2 - q_3\,q_1 - q_2^2 + q_3\,q_2} - \frac{q_3\,p_3}{q_1\,q_2 - q_3\,q_1 - q_3\,q_2 + q_3^2}\right], \tag{3}$$

$$\left[\left[\frac{p_1}{q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2} - \frac{p_2}{q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2} + \frac{p_3}{q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2}\right]\right]$$

Computing the spectral invariants using the theoretical formulas

```
> V:=Matrix(3,3,0):
V[1,1]:=1:
V[1,2]:=q1:
V[1,3]:=q1^2:
V[2,1]:=1:
V[2,2]:=q2:
V[2,3]:=q2^2:
V[3,1]:=1:
V[3,2]:=q3:
V[3,3]:=q3^2:
V;
HVector:=Matrix(3,1,0):
HVector[1,1]:=H0:
HVector[2,1]:=H1:
HVector[3,1]:=H2:
HVector;
RHSH:=Matrix(3,1,0):
RHSH[1,1]:=p1^2+tdP2(q1)+h*(p2-p1)/(q1-q2)+h*(p3-p1)/(q1-q3):
RHSH[2,1]:=p2^2+tdP2(q2)+h*(p1-p2)/(q2-q1)+h*(p3-p2)/(q2-q3):
RHSH[3,1]:=p3^2+tdP2(q3)+h*(p1-p3)/(q3-q1)+h*(p2-p3)/(q3-q2):
RHSH;
HVector:=simplify(Multiply(V^(-1),RHSH));
```

$$\begin{bmatrix} 1 & q_1 & q_1^2 \\ 1 & q_2 & q_2^2 \\ 1 & q_3 & q_3^2 \end{bmatrix}$$

$$\begin{bmatrix} H0 \\ H1 \\ H2 \end{bmatrix}$$

$$\begin{bmatrix} p_1^2 - q_1^7 - 2\,\tau_1\,q_1^5 - 2\,\tau_2\,q_1^4 - \left(\tau_1^2 + 2\,\tau_3\right) q_1^3 + \dfrac{h\,(p_2 - p_1)}{q_1 - q_2} + \dfrac{h\,(p_3 - p_1)}{q_1 - q_3} \\ p_2^2 - q_2^7 - 2\,\tau_1\,q_2^5 - 2\,\tau_2\,q_2^4 - \left(\tau_1^2 + 2\,\tau_3\right) q_2^3 + \dfrac{h\,(p_1 - p_2)}{q_2 - q_1} + \dfrac{h\,(p_3 - p_2)}{q_2 - q_3} \\ p_3^2 - q_3^7 - 2\,\tau_1\,q_3^5 - 2\,\tau_2\,q_3^4 - \left(\tau_1^2 + 2\,\tau_3\right) q_3^3 + \dfrac{h\,(p_1 - p_3)}{q_3 - q_1} + \dfrac{h\,(p_2 - p_3)}{q_3 - q_2} \end{bmatrix}$$

$$
\begin{aligned}
HVector := \Bigg[\bigg[& \frac{1}{(q_3-q_2)\,(q_3-q_1)\,(q_1-q_2)} \Big(\big((p_3-p_2)\, q_1^2 + \big((p_2-p_1)\, q_2 + q_3\,(p_1 \\
& - p_3) \big)\, q_1 - (q_3-q_2)\, \big((p_1-p_3)\, q_2 + q_3\,(p_1-p_2) \big) \big)\, h + q_3\, q_2\, (q_3-q_2)\, q_1^7 \\
& + 2\, q_3\, \tau_1\, q_2\, (q_3-q_2)\, q_1^5 + 2\, q_3\, \tau_2\, q_2\, (q_3-q_2)\, q_1^4 + q_3\, q_2\, \left(\tau_1^2 + 2\,\tau_3\right)\, (q_3-q_2)\, q_1^3 \\
& + \Big(q_3\, q_2^7 + 2\, q_2^5\, q_3\, \tau_1 + 2\, q_2^4\, q_3\, \tau_2 + q_3\, \left(\tau_1^2 + 2\,\tau_3\right) q_2^3 + \Big(-q_3^7 - 2\,\tau_1\, q_3^5 - 2\,\tau_2\, q_3^4 + \big(-\tau_1^2 \\
& - 2\,\tau_3 \big)\, q_3^3 + p_3^2 \Big)\, q_2 - p_2^2\, q_3 \Big)\, q_1^2 + \Big(-q_2^7\, q_3^2 - 2\, q_2^5\, q_3^2\, \tau_1 - 2\, q_2^4\, q_3^2\, \tau_2 - q_3^2\, \left(\tau_1^2 + 2\,\tau_3\right) q_2^3 \\
& + \left(q_3^7 + 2\,\tau_1\, q_3^5 + 2\,\tau_2\, q_3^4 + \left(\tau_1^2 + 2\,\tau_3\right) q_3^3 - p_3^2 \right) q_2^2 + p_2^2\, q_3^2 \Big)\, q_1 - q_3\, p_1^2\, q_2\, (q_3-q_2) \Big) \bigg], \\
\bigg[& \frac{1}{(q_3-q_2)\,(q_3-q_1)\,(q_1-q_2)} \Big(\big((p_2-p_3)\, q_1 + (p_3-p_1)\, q_2 + q_3\,(p_1-p_2) \big)\, h \\
& + \left(q_2^2 - q_3^2\right) q_1^7 + \left(2\, q_2^2\, \tau_1 - 2\, q_3^2\, \tau_1\right) q_1^5 + \left(2\, q_2^2\, \tau_2 - 2\, q_3^2\, \tau_2\right) q_1^4 - (q_3-q_2)\, (q_2+q_3)\, \big(\\
& \tau_1^2 + 2\,\tau_3 \big)\, q_1^3 + \Big(-q_2^7 - 2\,\tau_1\, q_2^5 - 2\,\tau_2\, q_2^4 + \left(-\tau_1^2 - 2\,\tau_3\right) q_2^3 + q_3^7 + 2\,\tau_1\, q_3^5 + 2\,\tau_2\, q_3^4 + \big(\tau_1^2 \\
& + 2\,\tau_3 \big)\, q_3^3 + p_2^2 - p_3^2 \Big)\, q_1^2 + q_2^7\, q_3^2 + 2\, q_2^5\, q_3^2\, \tau_1 + 2\, q_2^4\, q_3^2\, \tau_2 + q_3^2\, \left(\tau_1^2 + 2\,\tau_3\right) q_2^3 + \big(-q_3^7 \\
& - 2\,\tau_1\, q_3^5 - 2\,\tau_2\, q_3^4 + \left(-\tau_1^2 - 2\,\tau_3\right) q_3^3 - p_1^2 + p_3^2 \big)\, q_2^2 + q_3^2\, \left(p_1^2 - p_2^2\right) \Big) \bigg], \\
\bigg[& \frac{1}{(q_2-q_3)\,(q_1-q_3)\,(q_1-q_2)} \Big((q_3-q_2)\, q_1^7 - 2\,\tau_1\, (q_2-q_3)\, q_1^5 - 2\,\tau_2\, (q_2-q_3) \\
& q_1^4 - \left(\tau_1^2 + 2\,\tau_3\right) (q_2-q_3)\, q_1^3 + \Big(q_2^7 + 2\,\tau_1\, q_2^5 + 2\,\tau_2\, q_2^4 + \left(\tau_1^2 + 2\,\tau_3\right) q_2^3 - q_3^7 - 2\,\tau_1\, q_3^5 \\
& - 2\,\tau_2\, q_3^4 + \left(-\tau_1^2 - 2\,\tau_3\right) q_3^3 - p_2^2 + p_3^2 \Big)\, q_1 - q_3\, q_2^7 - 2\, q_2^5\, q_3\, \tau_1 - 2\, q_2^4\, q_3\, \tau_2 - q_3\, \big(\tau_1^2 \\
& + 2\,\tau_3 \big)\, q_2^3 + \left(q_3^7 + 2\,\tau_1\, q_3^5 + 2\,\tau_2\, q_3^4 + \left(\tau_1^2 + 2\,\tau_3\right) q_3^3 + p_1^2 - p_3^2 \right) q_2 + \left(-p_1^2 + p_2^2\right) q_3 \Big) \bigg]\Bigg]
\end{aligned} \tag{4}
$$

Defining the Hamiltonians as linear combination of the spectral invariants

```
> H0:=HVector[1,1]:
  H1:=HVector[2,1]:
  H2:=HVector[3,1]:

  TMatrix:=Matrix(3,3,0):
  TMatrix[1,1]:=1:
  TMatrix[2,2]:=1:
  TMatrix[3,3]:=1:
  TMatrix[3,1]:=tau[1]:
  TMatrix;
  RHStau1:=Matrix(3,1,0):
  RHStau1[1,1]:=1/(2*rinfty-2*1-5)*1:
  RHStau2:=Matrix(3,1,0):
  RHStau2[2,1]:=1/(2*rinfty-2*2-5)*1:
  RHStau3:=Matrix(3,1,0):
  RHStau3[3,1]:=1/(2*rinfty-2*3-5)*1:
  RHStau1;
  RHStau2;
```

```
RHStau3;
nutau1Vector:=Multiply(TMatrix^(-1),RHStau1);
nutau2Vector:=Multiply(TMatrix^(-1),RHStau2);
nutau3Vector:=Multiply(TMatrix^(-1),RHStau3);

nu1tau1:=nutau1Vector[1,1];
nu2tau1:=nutau1Vector[2,1];
nu3tau1:=nutau1Vector[3,1];

nu1tau2:=nutau2Vector[1,1];
nu2tau2:=nutau2Vector[2,1];
nu3tau2:=nutau2Vector[3,1];

nu1tau3:=nutau3Vector[1,1];
nu2tau3:=nutau3Vector[2,1];
nu3tau3:=nutau3Vector[3,1];


mutau1Vector:=Multiply((LinearAlgebra[Transpose](V))^
(-1),nutau1Vector);
mu1tau1:=mutau1Vector[1,1]:
mu2tau1:=mutau1Vector[2,1]:
mu3tau1:=mutau1Vector[3,1]:
mutau1Vector;

mutau2Vector:=Multiply((LinearAlgebra[Transpose](V))^(-1),
nutau2Vector);
mu1tau2:=mutau2Vector[1,1]:
mu2tau2:=mutau2Vector[2,1]:
mu3tau2:=mutau2Vector[3,1]:

mutau3Vector:=Multiply((LinearAlgebra[Transpose](V))^(-1),
nutau3Vector);
mu1tau3:=mutau3Vector[1,1]:
mu2tau3:=mutau3Vector[2,1]:
mu3tau3:=mutau3Vector[3,1]:

Hamtau1:= nu1tau1*H0+nu2tau1*H1+nu3tau1*H2;
Hamtau2:= nu1tau2*H0+nu2tau2*H1+nu3tau2*H2;
Hamtau3:= nu1tau3*H0+nu2tau3*H1+nu3tau3*H2;

QQ:=unapply(-p1*(lambda-q2)*(lambda-q3)/(q1-q2)/(q1-q3)-p2*
```

```
(lambda-q1)*(lambda-q3)/(q2-q1)/(q2-q3)
-p3*(lambda-q1)*(lambda-q2)/(q3-q1)/(q3-q2),lambda);;

J:=Matrix(2,2,0):
J[1,1]:=1:
J[1,2]:=0:
J[2,1]:=QQ(lambda)/(lambda-q1)/(lambda-q2)/(lambda-q3):
J[2,2]:=1/(lambda-q1)/(lambda-q2)/(lambda-q3):
J;
dJdlambda:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do
dJdlambda[i,j]:=diff(J[i,j],lambda): od: od:
dJdlambda;
```

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \tau_1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} \frac{1}{5} \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$nutau1Vector := \begin{bmatrix} \frac{1}{5} \\ 0 \\ -\frac{\tau_1}{5} \end{bmatrix}$$

$$nutau2Vector := \begin{bmatrix} 0 \\ \frac{1}{3} \\ 0 \end{bmatrix}$$

$$nutau3Vector := \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$nu1tau1 := \frac{1}{5}$$

$$nu2tau1 := 0$$

$$nu3tau1 := -\frac{\tau_1}{5}$$

$$nu1tau2 := 0$$

$$nu2tau2 := \frac{1}{3}$$

$$nu3tau2 := 0$$

$$nu1tau3 := 0$$

$$nu2tau3 := 0$$

$$nu3tau3 := 1$$

$$mutau1Vector := \begin{bmatrix} \frac{q_3 q_2}{5\left(q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2\right)} - \frac{\tau_1}{5\left(q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2\right)} \\ -\frac{q_3 q_1}{5\left(q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2\right)} + \frac{\tau_1}{5\left(q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2\right)} \\ \frac{q_1 q_2}{5\left(q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2\right)} - \frac{\tau_1}{5\left(q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2\right)} \end{bmatrix}$$

$$\begin{bmatrix} \frac{q_3 q_2}{5\left(q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2\right)} - \frac{\tau_1}{5\left(q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2\right)} \\ -\frac{q_3 q_1}{5\left(q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2\right)} + \frac{\tau_1}{5\left(q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2\right)} \\ \frac{q_1 q_2}{5\left(q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2\right)} - \frac{\tau_1}{5\left(q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2\right)} \end{bmatrix}$$

$$mutau2Vector := \begin{bmatrix} -\frac{q_2 + q_3}{3\left(q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2\right)} \\ \frac{q_1 + q_3}{3\left(q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2\right)} \\ -\frac{q_1 + q_2}{3\left(q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2\right)} \end{bmatrix}$$

$$
mutau3Vector := \begin{bmatrix} \dfrac{1}{q_1^2 - q_1 q_2 - q_3 q_1 + q_3 q_2} \\[2ex] -\dfrac{1}{q_1 q_2 - q_3 q_1 - q_2^2 + q_3 q_2} \\[2ex] \dfrac{1}{q_1 q_2 - q_3 q_1 - q_3 q_2 + q_3^2} \end{bmatrix}
$$

$$
\begin{aligned}
Hamtau1 :=\ & \frac{1}{5\,(q_3 - q_2)\,(q_3 - q_1)\,(q_1 - q_2)} \Big(\big(\big((p_3 - p_2)\, q_1^2 + \big((p_2 - p_1)\, q_2 + q_3\, (p_1 \\
& - p_3) \big)\, q_1 - (q_3 - q_2)\, \big((p_1 - p_3)\, q_2 + q_3\, (p_1 - p_2) \big) \big)\, h + q_3\, q_2\, (q_3 - q_2)\, q_1^7 \\
& + 2\, q_3\, \tau_1\, q_2\, (q_3 - q_2)\, q_1^5 + 2\, q_3\, \tau_2\, q_2\, (q_3 - q_2)\, q_1^4 + q_3\, q_2\, \big(\tau_1^2 + 2\,\tau_3\big)\, (q_3 - q_2)\, q_1^3 \\
& + \Big(q_3\, q_2^7 + 2\, q_2^5\, q_3\, \tau_1 + 2\, q_2^4\, q_3\, \tau_2 + q_3\, \big(\tau_1^2 + 2\,\tau_3\big)\, q_2^3 + \Big(-q_3^7 - 2\,\tau_1\, q_3^5 - 2\,\tau_2\, q_3^4 + \big(-\tau_1^2 \\
& - 2\,\tau_3 \big)\, q_3^3 + p_3^2 \Big)\, q_2 - p_2^2\, q_3 \Big)\, q_1^2 + \Big(-q_2^7\, q_3^2 - 2\, q_2^5\, q_3^2\, \tau_1 - 2\, q_2^4\, q_3^2\, \tau_2 - q_3^2\, \big(\tau_1^2 + 2\,\tau_3\big)\, q_2^3 \\
& + \Big(q_3^7 + 2\,\tau_1\, q_3^5 + 2\,\tau_2\, q_3^4 + \big(\tau_1^2 + 2\,\tau_3\big)\, q_3^3 - p_3^2 \Big)\, q_2^2 + p_2^2\, q_3^2 \Big)\, q_1 - q_3\, p_1^2\, q_2\, (q_3 \\
& - q_2) \Big) - \frac{1}{5\,(q_2 - q_3)\,(q_1 - q_3)\,(q_1 - q_2)} \Big(\Big((q_3 - q_2)\, q_1^7 - 2\,\tau_1\, (q_2 - q_3)\, q_1^5 \\
& - 2\,\tau_2\, (q_2 - q_3)\, q_1^4 - \big(\tau_1^2 + 2\,\tau_3\big)\, (q_2 - q_3)\, q_1^3 + \Big(q_2^7 + 2\,\tau_1\, q_2^5 + 2\,\tau_2\, q_2^4 + \big(\tau_1^2 + 2\,\tau_3\big) \\
& q_2^3 - q_3^7 - 2\,\tau_1\, q_3^5 - 2\,\tau_2\, q_3^4 + \big(-\tau_1^2 - 2\,\tau_3\big)\, q_3^3 - p_2^2 + p_3^2 \Big)\, q_1 - q_3\, q_2^7 - 2\, q_2^5\, q_3\, \tau_1 - 2 \\
& q_2^4\, q_3\, \tau_2 - q_3\, \big(\tau_1^2 + 2\,\tau_3\big)\, q_2^3 + \Big(q_3^7 + 2\,\tau_1\, q_3^5 + 2\,\tau_2\, q_3^4 + \big(\tau_1^2 + 2\,\tau_3\big)\, q_3^3 + p_1^2 - p_3^2 \Big)\, q_2 + \big(- \\
& p_1^2 + p_2^2 \big)\, q_3 \Big)\, \tau_1 \Big)
\end{aligned}
$$

$$
\begin{aligned}
Hamtau2 :=\ & \frac{1}{3\,(q_3 - q_2)\,(q_3 - q_1)\,(q_1 - q_2)} \Big(\big((p_2 - p_3)\, q_1 + (p_3 - p_1)\, q_2 + q_3\, (p_1 \\
& - p_2) \big)\, h + \big(q_2^2 - q_3^2\big)\, q_1^7 + \big(2\, q_2^2\, \tau_1 - 2\, q_3^2\, \tau_1\big)\, q_1^5 + \big(2\, q_2^2\, \tau_2 - 2\, q_3^2\, \tau_2\big)\, q_1^4 - (q_3 \\
& - q_2)\, (q_2 + q_3)\, \big(\tau_1^2 + 2\,\tau_3\big)\, q_1^3 + \Big(-q_2^7 - 2\,\tau_1\, q_2^5 - 2\,\tau_2\, q_2^4 + \big(-\tau_1^2 - 2\,\tau_3\big)\, q_2^3 + q_3^7 + 2\,\tau_1 \\
& q_3^5 + 2\,\tau_2\, q_3^4 + \big(\tau_1^2 + 2\,\tau_3\big)\, q_3^3 + p_2^2 - p_3^2 \Big)\, q_1^2 + q_2^7\, q_3^2 + 2\, q_2^5\, q_3^2\, \tau_1 + 2\, q_2^4\, q_3^2\, \tau_2 + q_3^2\, \big(\tau_1^2 \\
& + 2\,\tau_3\big)\, q_2^3 + \Big(-q_3^7 - 2\,\tau_1\, q_3^5 - 2\,\tau_2\, q_3^4 + \big(-\tau_1^2 - 2\,\tau_3\big)\, q_3^3 - p_1^2 + p_3^2 \Big)\, q_2^2 + q_3^2\, \big(p_1^2 - p_2^2\big) \Big)
\end{aligned}
$$

$$
\begin{aligned}
Hamtau3 :=\ & \frac{1}{(q_2 - q_3)\,(q_1 - q_3)\,(q_1 - q_2)} \Big((q_3 - q_2)\, q_1^7 - 2\,\tau_1\, (q_2 - q_3)\, q_1^5 - 2\,\tau_2\, (q_2 \\
& - q_3)\, q_1^4 - \big(\tau_1^2 + 2\,\tau_3\big)\, (q_2 - q_3)\, q_1^3 + \Big(q_2^7 + 2\,\tau_1\, q_2^5 + 2\,\tau_2\, q_2^4 + \big(\tau_1^2 + 2\,\tau_3\big)\, q_2^3 - q_3^7 \\
& - 2\,\tau_1\, q_3^5 - 2\,\tau_2\, q_3^4 + \big(-\tau_1^2 - 2\,\tau_3\big)\, q_3^3 - p_2^2 + p_3^2 \Big)\, q_1 - q_3\, q_2^7 - 2\, q_2^5\, q_3\, \tau_1 - 2\, q_2^4\, q_3\, \tau_2 \\
& - q_3\, \big(\tau_1^2 + 2\,\tau_3\big)\, q_2^3 + \Big(q_3^7 + 2\,\tau_1\, q_3^5 + 2\,\tau_2\, q_3^4 + \big(\tau_1^2 + 2\,\tau_3\big)\, q_3^3 + p_1^2 - p_3^2 \Big)\, q_2 + \big(-p_1^2 + \\
& p_2^2 \big)\, q_3 \Big)
\end{aligned}
$$

$$
QQ := \lambda \mapsto -\frac{p_1\, (\lambda - q_2)\, (\lambda - q_3)}{(q_1 - q_2)\, (q_1 - q_3)} - \frac{p_2\, (\lambda - q_1)\, (\lambda - q_3)}{(q_2 - q_1)\, (q_2 - q_3)} - \frac{p_3\, (\lambda - q_1)\, (\lambda - q_2)}{(q_3 - q_1)\, (q_3 - q_2)}
$$

$$
\Bigg[\bigg[\Big[1, 0\Big],
\left[\frac{-\frac{p_1(\lambda-q_2)(\lambda-q_3)}{(q_1-q_2)(q_1-q_3)} - \frac{p_2(\lambda-q_1)(\lambda-q_3)}{(q_2-q_1)(q_2-q_3)} - \frac{p_3(\lambda-q_1)(\lambda-q_2)}{(q_3-q_1)(q_3-q_2)}}{(\lambda-q_1)(\lambda-q_2)(\lambda-q_3)},
\frac{1}{(\lambda-q_1)(\lambda-q_2)(\lambda-q_3)}\right]\bigg],
$$

$$
\bigg[\Big[0, 0\Big],
\Bigg[\frac{1}{(\lambda-q_1)(\lambda-q_2)(\lambda-q_3)}\bigg(-\frac{p_1(\lambda-q_3)}{(q_1-q_2)(q_1-q_3)} - \frac{p_1(\lambda-q_2)}{(q_1-q_2)(q_1-q_3)}
- \frac{p_2(\lambda-q_3)}{(q_2-q_1)(q_2-q_3)} - \frac{p_2(\lambda-q_1)}{(q_2-q_1)(q_2-q_3)} - \frac{p_3(\lambda-q_2)}{(q_3-q_1)(q_3-q_2)}
- \frac{p_3(\lambda-q_1)}{(q_3-q_1)(q_3-q_2)}\bigg)
$$

$$
- \frac{-\frac{p_1(\lambda-q_2)(\lambda-q_3)}{(q_1-q_2)(q_1-q_3)} - \frac{p_2(\lambda-q_1)(\lambda-q_3)}{(q_2-q_1)(q_2-q_3)} - \frac{p_3(\lambda-q_1)(\lambda-q_2)}{(q_3-q_1)(q_3-q_2)}}{(\lambda-q_1)^2(\lambda-q_2)(\lambda-q_3)}
$$

$$
- \frac{-\frac{p_1(\lambda-q_2)(\lambda-q_3)}{(q_1-q_2)(q_1-q_3)} - \frac{p_2(\lambda-q_1)(\lambda-q_3)}{(q_2-q_1)(q_2-q_3)} - \frac{p_3(\lambda-q_1)(\lambda-q_2)}{(q_3-q_1)(q_3-q_2)}}{(\lambda-q_1)(\lambda-q_2)^2(\lambda-q_3)}
$$

$$
- \frac{-\frac{p_1(\lambda-q_2)(\lambda-q_3)}{(q_1-q_2)(q_1-q_3)} - \frac{p_2(\lambda-q_1)(\lambda-q_3)}{(q_2-q_1)(q_2-q_3)} - \frac{p_3(\lambda-q_1)(\lambda-q_2)}{(q_3-q_1)(q_3-q_2)}}{(\lambda-q_1)(\lambda-q_2)(\lambda-q_3)^2},
$$

$$
-\frac{1}{(\lambda-q_1)^2(\lambda-q_2)(\lambda-q_3)} - \frac{1}{(\lambda-q_1)(\lambda-q_2)^2(\lambda-q_3)}
\tag{5}
$$

$$-\frac{1}{(\lambda-q_1)(\lambda-q_2)(\lambda-q_3)^2}\Bigg]$$,

Defining the Lax matrix in the oper gauge using the general formula. Then defining the normalized geometric Lax matrix using the gauge transformation.

```
> L:=Matrix(2,2,0):
  L[1,1]:=0:
  L[1,2]:=1:
  L[2,1]:=-tdP2(lambda)+H2*lambda^2+H1*lambda+H0-h*p1/(lambda-q1)-
  h*p2/(lambda-q2)-h*p3/(lambda-q3):
  L[2,2]:=h/(lambda-q1)+h/(lambda-q2)+h/(lambda-q3):
  L:

  CheckL:=simplify(Multiply(Multiply(J,L),J^(-1))+h*Multiply
  (dJdlambda,J^(-1))):
```

Defining the Hamiltonian evolutions

```
> dq1dtau1:=factor(1/h*diff(Hamtau1,p1));
  dp1dtau1:=factor(-1/h*diff(Hamtau1,q1));
  dq2dtau1:=factor(1/h*diff(Hamtau1,p2));
  dp2dtau1:=factor(-1/h*diff(Hamtau1,q2));
  dq3dtau1:=factor(1/h*diff(Hamtau1,p3));
  dp3dtau1:=factor(-1/h*diff(Hamtau1,q3));

  dq1dtau2:=factor(1/h*diff(Hamtau2,p1));
  dp1dtau2:=factor(-1/h*diff(Hamtau2,q1));
  dq2dtau2:=factor(1/h*diff(Hamtau2,p2));
  dp2dtau2:=factor(-1/h*diff(Hamtau2,q2));
  dq3dtau2:=factor(1/h*diff(Hamtau2,p3));
  dp3dtau2:=factor(-1/h*diff(Hamtau2,q3));

  dq1dtau3:=factor(1/h*diff(Hamtau3,p1));
  dp1dtau3:=factor(-1/h*diff(Hamtau3,q1));
  dq2dtau3:=factor(1/h*diff(Hamtau3,p2));
  dp2dtau3:=factor(-1/h*diff(Hamtau3,q2));
  dq3dtau3:=factor(1/h*diff(Hamtau3,p3));
  dp3dtau3:=factor(-1/h*diff(Hamtau3,q3));

  Atau1:=Matrix(2,2,0):
  Atau1[1,1]:=-p1*mu1tau1/(lambda-q1)-p2*mu2tau1/(lambda-q2)-p3*
  mu3tau1/(lambda-q3):
  Atau1[1,2]:= mu1tau1/(lambda-q1)+mu2tau1/(lambda-q2)+mu3tau1/
  (lambda-q3):
  Atau1[2,1]:=h*diff(Atau1[1,1],lambda)+Atau1[1,2]*L[2,1]:
```

```
Atau1[2,2]:=h*diff(Atau1[1,2],lambda)+Atau1[1,1]+Atau1[1,2]*L[2,
2]:
Atau1:

Atau2:=Matrix(2,2,0):
Atau2[1,1]:=-p1*mu1tau2/(lambda-q1)-p2*mu2tau2/(lambda-q2)-p3*
mu3tau2/(lambda-q3):
Atau2[1,2]:=mu1tau2/(lambda-q1)+mu2tau2/(lambda-q2)+mu3tau2/
(lambda-q3):
Atau2[2,1]:=h*diff(Atau2[1,1],lambda)+Atau2[1,2]*L[2,1]:
Atau2[2,2]:=h*diff(Atau2[1,2],lambda)+Atau2[1,1]+Atau2[1,2]*L[2,
2]:
Atau2:

Atau3:=Matrix(2,2,0):
Atau3[1,1]:=-p1*mu1tau3/(lambda-q1)-p2*mu2tau3/(lambda-q2)-p3*
mu3tau3/(lambda-q3):
Atau3[1,2]:=mu1tau3/(lambda-q1)+mu2tau3/(lambda-q2)+mu3tau3/
(lambda-q3):
Atau3[2,1]:=h*diff(Atau3[1,1],lambda)+Atau3[1,2]*L[2,1]:
Atau3[2,2]:=h*diff(Atau3[1,2],lambda)+Atau3[1,1]+Atau3[1,2]*L[2,
2]:
Atau3:


dJdtau1:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do
dJdtau1[i,j]:=diff(J[i,j],tau1)+diff(J[i,j],q1)*dq1dtau1+diff(J
[i,j],p1)*dp1dtau1+diff(J[i,j],q2)*dq2dtau1+diff(J[i,j],p2)*
dp2dtau1+diff(J[i,j],q3)*dq3dtau1+diff(J[i,j],p3)*dp3dtau1: od:
od:
dJdtau1:

dJdtau2:=Matrix(2,2,0):
for i from 1 to 2 do for j from 1 to 2 do
dJdtau2[i,j]:=diff(J[i,j],tau2)+diff(J[i,j],q1)*dq1dtau2+diff(J
[i,j],p1)*dp1dtau2+diff(J[i,j],q2)*dq2dtau2+diff(J[i,j],p2)*
dp2dtau2+diff(J[i,j],q3)*dq3dtau2+diff(J[i,j],p3)*dp3dtau2: od:
od:
dJdtau2:

dJdtau3:=Matrix(2,2,0):
```

```
for i from 1 to 2 do for j from 1 to 2 do
dJdtau3[i,j]:=diff(J[i,j],tau3)+diff(J[i,j],q1)*dq1dtau3+diff(J
[i,j],p1)*dp1dtau3+diff(J[i,j],q2)*dq2dtau3+diff(J[i,j],p2)*
dp2dtau3+diff(J[i,j],q3)*dq3dtau3+diff(J[i,j],p3)*dp3dtau3: od:
od:
dJdtau3:

CheckAtau1:=simplify(Multiply(Multiply(J,Atau1),J^(-1))+h*
Multiply(dJdtau1,J^(-1)));
CheckAtau2:=simplify(Multiply(Multiply(J,Atau2),J^(-1))+h*
Multiply(dJdtau2,J^(-1)));
CheckAtau3:=simplify(Multiply(Multiply(J,Atau3),J^(-1))+h*
Multiply(dJdtau3,J^(-1)));
```

$$dq1dtau1 := -\frac{-2\,p_1\,q_2\,q_3 + h\,q_1 - h\,q_2 - h\,q_3 + 2\,p_1\,\tau_1}{5\,(q_1-q_3)\,(q_1-q_2)\,h}$$

$$\begin{aligned}
dp1dtau1 := & -\frac{1}{5\,h\,(q_1-q_2)^2\,(q_1-q_3)^2\,(q_2-q_3)}\Big(-5\,q_1^8\,q_2^2\,q_3 + 5\,q_1^8\,q_2\,q_3^2 + 6\,q_1^7\,q_2^3\,q_3 - 6 \\
& q_1^7\,q_2\,q_3^3 - 7\,q_1^6\,q_2^3\,q_3^2 + 7\,q_1^6\,q_2^2\,q_3^3 - q_1^2\,q_2^8\,q_3 + q_1^2\,q_2\,q_3^8 + 2\,q_1\,q_2^8\,q_3^2 - 2\,q_1\,q_2^2\,q_3^8 - q_2^8\,q_3^3 + q_2^3 \\
& q_3^8 + 5\,q_1^8\,q_2\,\tau_1 - 5\,q_1^8\,q_3\,\tau_1 - 6\,q_1^7\,q_2^2\,\tau_1 + 6\,q_1^7\,q_3^2\,\tau_1 + q_1^6\,q_2^2\,q_3\,\tau_1 - q_1^6\,q_2\,q_3^2\,\tau_1 + 8\,q_1^5\,q_2^3\,q_3\,\tau_1 \\
& - 8\,q_1^5\,q_2\,q_3^3\,\tau_1 - 10\,q_1^4\,q_2^3\,q_3^2\,\tau_1 + 10\,q_1^4\,q_2^2\,q_3^3\,\tau_1 + q_1^2\,q_2^7\,\tau_1 - 2\,q_1^2\,q_2^6\,q_3\,\tau_1 + 2\,q_1^2\,q_2\,q_3^6\,\tau_1 - \\
& q_1^2\,q_3^7\,\tau_1 - 2\,q_1\,q_2^7\,q_3\,\tau_1 + 4\,q_1\,q_2^6\,q_3^2\,\tau_1 - 4\,q_1\,q_2^2\,q_3^6\,\tau_1 + 2\,q_1\,q_2\,q_3^7\,\tau_1 + q_2^7\,q_3^2\,\tau_1 - 2\,q_2^6\,q_3^3\,\tau_1 \\
& + 2\,q_2^3\,q_3^6\,\tau_1 - q_2^2\,q_3^7\,\tau_1 + 6\,q_1^6\,q_2\,\tau_1^2 - 6\,q_1^6\,q_3\,\tau_1^2 - 4\,q_1^5\,q_2^2\,q_3\,\tau_2 - 8\,q_1^5\,q_2^2\,\tau_1^2 + 4\,q_1^5\,q_2\,q_3^2\,\tau_2 \\
& + 8\,q_1^5\,q_3^2\,\tau_1^2 + 6\,q_1^4\,q_2^3\,q_3\,\tau_2 + 9\,q_1^4\,q_2^2\,q_3\,\tau_1^2 - 6\,q_1^4\,q_2\,q_3^3\,\tau_2 - 9\,q_1^4\,q_2\,q_3^2\,\tau_1^2 - 8\,q_1^3\,q_2^3\,q_3^2\,\tau_2 + 2 \\
& q_1^3\,q_2^3\,q_3\,\tau_1^2 + 8\,q_1^3\,q_2^2\,q_3^3\,\tau_2 - 2\,q_1^3\,q_2\,q_3^3\,\tau_1^2 - 2\,q_1^2\,q_2^5\,q_3\,\tau_2 + 2\,q_1^2\,q_2^5\,\tau_1^2 - q_1^2\,q_2^4\,q_3\,\tau_1^2 - 3\,q_1^2\,q_2^3 \\
& q_3^2\,\tau_1^2 + 3\,q_1^2\,q_2^2\,q_3^3\,\tau_1^2 + 2\,q_1^2\,q_2\,q_3^5\,\tau_2 + q_1^2\,q_2\,q_3^4\,\tau_1^2 - 2\,q_1^2\,q_3^5\,\tau_1^2 + 4\,q_1\,q_2^5\,q_3^2\,\tau_2 - 4\,q_1\,q_2^5\,q_3\,\tau_1^2 \\
& + 2\,q_1\,q_2^4\,q_3^2\,\tau_1^2 - 4\,q_1\,q_2^2\,q_3^5\,\tau_2 - 2\,q_1\,q_2^2\,q_3^4\,\tau_1^2 + 4\,q_1\,q_2\,q_3^5\,\tau_1^2 - 2\,q_2^5\,q_3^3\,\tau_2 + 2\,q_2^5\,q_3^2\,\tau_1^2 - q_2^4 \\
& q_3^3\,\tau_1^2 + 2\,q_2^3\,q_3^5\,\tau_2 + q_2^3\,q_3^4\,\tau_1^2 - 2\,q_2^2\,q_3^5\,\tau_1^2 + 4\,q_1^5\,q_2\,\tau_1\,\tau_2 - 4\,q_1^5\,q_3\,\tau_1\,\tau_2 - 2\,q_1^4\,q_2^2\,q_3\,\tau_3 - 6\,q_1^4 \\
& q_2^2\,\tau_1\,\tau_2 + 2\,q_1^4\,q_2\,q_3^2\,\tau_3 + q_1^4\,q_2\,\tau_1^3 + 6\,q_1^4\,q_3^2\,\tau_1\,\tau_2 - q_1^4\,q_3\,\tau_1^3 + 4\,q_1^3\,q_2^3\,q_3\,\tau_3 + 8\,q_1^3\,q_2^2\,q_3\,\tau_1\,\tau_2 \\
& - 2\,q_1^3\,q_2^2\,\tau_1^3 - 4\,q_1^3\,q_2\,q_3^3\,\tau_3 - 8\,q_1^3\,q_2\,q_3^2\,\tau_1\,\tau_2 + 2\,q_1^3\,q_3^2\,\tau_1^3 - 2\,q_1^2\,q_2^4\,q_3\,\tau_3 + 2\,q_1^2\,q_2^4\,\tau_1\,\tau_2 - 6 \\
& q_1^2\,q_2^3\,q_3^2\,\tau_3 + q_1^2\,q_2^3\,\tau_1^3 + 6\,q_1^2\,q_2^2\,q_3^3\,\tau_3 + 3\,q_1^2\,q_2^2\,q_3\,\tau_1^3 + 2\,q_1^2\,q_2\,q_3^4\,\tau_3 - 3\,q_1^2\,q_2\,q_3^2\,\tau_1^3 - 2\,q_1^2 \\
& q_3^4\,\tau_1\,\tau_2 - q_1^2\,q_3^3\,\tau_1^3 + 4\,q_1\,q_2^4\,q_3^2\,\tau_3 - 4\,q_1\,q_2^4\,q_3\,\tau_1\,\tau_2 - 2\,q_1\,q_2^3\,q_3\,\tau_1^3 - 4\,q_1\,q_2^2\,q_3^4\,\tau_3 + 4\,q_1\,q_2 \\
& q_3^4\,\tau_1\,\tau_2 + 2\,q_1\,q_2\,q_3^3\,\tau_1^3 - 2\,q_2^4\,q_3^3\,\tau_3 + 2\,q_2^4\,q_3^2\,\tau_1\,\tau_2 + 2\,q_2^3\,q_3^4\,\tau_3 + q_2^3\,q_3^2\,\tau_1^3 - 2\,q_2^2\,q_3^4\,\tau_1\,\tau_2 - q_2^2 \\
& q_3^3\,\tau_1^3 + 2\,q_1^4\,q_2\,\tau_1\,\tau_3 - 2\,q_1^4\,q_3\,\tau_1\,\tau_3 - 4\,q_1^3\,q_2^2\,\tau_1\,\tau_3 + 4\,q_1^3\,q_3^2\,\tau_1\,\tau_3 + 2\,q_1^2\,q_2^3\,\tau_1\,\tau_3 + 6\,q_1^2 \\
& q_2^2\,q_3\,\tau_1\,\tau_3 - 6\,q_1^2\,q_2\,q_3^2\,\tau_1\,\tau_3 - 2\,q_1^2\,q_3^3\,\tau_1\,\tau_3 - 4\,q_1\,q_2^3\,q_3\,\tau_1\,\tau_3 + 4\,q_1\,q_2\,q_3^3\,\tau_1\,\tau_3 + 2\,q_2^3\,q_3^2\,\tau_1\,\tau_3
\end{aligned}$$

$$-2\,q_2^2\,q_3^3\,\tau_1\,\tau_3-2\,p_1^2\,q_1\,q_2^2\,q_3+2\,p_1^2\,q_1\,q_2\,q_3^2+p_1^2\,q_2^3\,q_3-p_1^2\,q_2\,q_3^3+p_2^2\,q_1^2\,q_2\,q_3-2\,p_2^2\,q_1\,q_2$$
$$q_3^2+p_2^2\,q_2\,q_3^3-p_3^2\,q_1^2\,q_2\,q_3+2\,p_3^2\,q_1\,q_2^2\,q_3-p_3^2\,q_2^3\,q_3+h\,p_1\,q_1^2\,q_2-h\,p_1\,q_1^2\,q_3-2\,h\,p_1\,q_1\,q_2^2$$
$$+2\,h\,p_1\,q_1\,q_3^2+h\,p_1\,q_2^3-h\,p_1\,q_3^3+h\,p_2\,q_1^2\,q_3-2\,h\,p_2\,q_1\,q_3^2+h\,p_2\,q_3^3-h\,p_3\,q_1^2\,q_2$$
$$+2\,h\,p_3\,q_1\,q_2^2-h\,p_3\,q_2^3+2\,p_1^2\,q_1\,q_2\,\tau_1-2\,p_1^2\,q_1\,q_3\,\tau_1-p_1^2\,q_2^2\,\tau_1+p_1^2\,q_3^2\,\tau_1-p_2^2\,q_1^2\,\tau_1+2$$
$$p_2^2\,q_1\,q_3\,\tau_1-p_2^2\,q_3^2\,\tau_1+p_3^2\,q_1^2\,\tau_1-2\,p_3^2\,q_1\,q_2\,\tau_1+p_3^2\,q_2^2\,\tau_1)$$

$$dq2dtau1 := -\frac{2\,p_2\,q_1\,q_3+h\,q_1-h\,q_2+h\,q_3-2\,p_2\,\tau_1}{5\,(q_2-q_3)\,(q_1-q_2)\,h}$$

$$dp2dtau1 := -\frac{1}{5\,h\,(q_1-q_2)^2\,(q_1-q_3)\,(q_2-q_3)^2}\,(-q_1^8\,q_2^2\,q_3+2\,q_1^8\,q_2\,q_3^2-q_1^8\,q_3^3+6\,q_1^3$$
$$q_2^7\,q_3-7\,q_1^3\,q_2^6\,q_3^2+q_1^3\,q_3^8-5\,q_1^2\,q_2^8\,q_3+7\,q_1^2\,q_2^6\,q_3^3-2\,q_1^2\,q_2\,q_3^8+5\,q_1\,q_2^8\,q_3^2-6\,q_1\,q_2^7\,q_3^3$$
$$+q_1\,q_2^2\,q_3^8+q_1^7\,q_2^2\,\tau_1-2\,q_1^7\,q_2\,q_3\,\tau_1+q_1^7\,q_3^2\,\tau_1-2\,q_1^6\,q_2^2\,q_3\,\tau_1+4\,q_1^6\,q_2\,q_3^2\,\tau_1-2\,q_1^6\,q_3^3\,\tau_1$$
$$+8\,q_1^3\,q_2^5\,q_3\,\tau_1-10\,q_1^3\,q_2^4\,q_3^2\,\tau_1+2\,q_1^3\,q_3^6\,\tau_1-6\,q_1^2\,q_2^7\,\tau_1+q_1^2\,q_2^6\,q_3\,\tau_1+10\,q_1^2\,q_2^4\,q_3^3\,\tau_1-4$$
$$q_1^2\,q_2\,q_3^6\,\tau_1-q_1^2\,q_3^7\,\tau_1+5\,q_1\,q_2^8\,\tau_1-q_1\,q_2^6\,q_3^2\,\tau_1-8\,q_1\,q_2^5\,q_3^3\,\tau_1+2\,q_1\,q_2^2\,q_3^6\,\tau_1+2\,q_1\,q_2\,q_3^7\,\tau_1$$
$$-5\,q_2^8\,q_3\,\tau_1+6\,q_2^7\,q_3^2\,\tau_1-q_2^2\,q_3^7\,\tau_1-2\,q_1^5\,q_2^2\,q_3\,\tau_2+2\,q_1^5\,q_2^2\,\tau_1^2+4\,q_1^5\,q_2\,q_3^2\,\tau_2-4\,q_1^5\,q_2\,q_3\,\tau_1^2$$
$$-2\,q_1^5\,q_3^3\,\tau_2+2\,q_1^5\,q_3^2\,\tau_1^2-q_1^4\,q_2^2\,q_3\,\tau_1^2+2\,q_1^4\,q_2\,q_3^2\,\tau_1^2-q_1^4\,q_3^3\,\tau_1^2+6\,q_1^3\,q_2^4\,q_3\,\tau_2-8\,q_1^3\,q_2^3$$
$$q_3^2\,\tau_2+2\,q_1^3\,q_2^3\,q_3\,\tau_1^2-3\,q_1^3\,q_2^2\,q_3^2\,\tau_1^2+2\,q_1^3\,q_3^5\,\tau_2+q_1^3\,q_3^4\,\tau_1^2-4\,q_1^2\,q_2^5\,q_3\,\tau_2-8\,q_1^2\,q_2^5\,\tau_1^2+9\,q_1^2$$
$$q_2^4\,q_3\,\tau_1^2+8\,q_1^2\,q_2^3\,q_3^3\,\tau_2+3\,q_1^2\,q_2^2\,q_3^3\,\tau_1^2-4\,q_1^2\,q_2\,q_3^5\,\tau_2-2\,q_1^2\,q_2\,q_3^4\,\tau_1^2-2\,q_1^2\,q_3^5\,\tau_1^2+6\,q_1\,q_2^6\,\tau_1^2$$
$$+4\,q_1\,q_2^5\,q_3^2\,\tau_2-6\,q_1\,q_2^4\,q_3^3\,\tau_2-9\,q_1\,q_2^4\,q_3^2\,\tau_1^2-2\,q_1\,q_2^3\,q_3^3\,\tau_1^2+2\,q_1\,q_2^2\,q_3^5\,\tau_2+q_1\,q_2^2\,q_3^4\,\tau_1^2$$
$$+4\,q_1\,q_2\,q_3^5\,\tau_1^2-6\,q_2^6\,q_3\,\tau_1^2+8\,q_2^5\,q_3^2\,\tau_1^2-2\,q_2^2\,q_3^5\,\tau_1^2-2\,q_1^4\,q_2^2\,q_3\,\tau_3+2\,q_1^4\,q_2^2\,\tau_1\,\tau_2+4\,q_1^4\,q_2$$
$$q_3^2\,\tau_3-4\,q_1^4\,q_2\,q_3\,\tau_1\,\tau_2-2\,q_1^4\,q_3^3\,\tau_3+2\,q_1^4\,q_3^2\,\tau_1\,\tau_2+4\,q_1^3\,q_2^3\,q_3\,\tau_3-6\,q_1^3\,q_2^2\,q_3^2\,\tau_3+q_1^3\,q_2^2\,\tau_1^3$$
$$-2\,q_1^3\,q_2\,q_3\,\tau_1^3+2\,q_1^3\,q_3^4\,\tau_3+q_1^3\,q_3^2\,\tau_1^3-2\,q_1^2\,q_2^4\,q_3\,\tau_3-6\,q_1^2\,q_2^4\,\tau_1\,\tau_2+8\,q_1^2\,q_2^3\,q_3\,\tau_1\,\tau_2-2\,q_1^2$$
$$q_2^3\,\tau_1^3+6\,q_1^2\,q_2^2\,q_3^3\,\tau_3+3\,q_1^2\,q_2^2\,q_3\,\tau_1^3-4\,q_1^2\,q_2\,q_3^4\,\tau_3-2\,q_1^2\,q_3^4\,\tau_1\,\tau_2-q_1^2\,q_3^3\,\tau_1^3+4\,q_1\,q_2^5\,\tau_1\,\tau_2$$
$$+2\,q_1\,q_2^4\,q_3^2\,\tau_3+q_1\,q_2^4\,\tau_1^3-4\,q_1\,q_2^3\,q_3^3\,\tau_3-8\,q_1\,q_2^3\,q_3^2\,\tau_1\,\tau_2+2\,q_1\,q_2^2\,q_3^4\,\tau_3-3\,q_1\,q_2^2\,q_3^2\,\tau_1^3$$
$$+4\,q_1\,q_2\,q_3^4\,\tau_1\,\tau_2+2\,q_1\,q_2\,q_3^3\,\tau_1^3-4\,q_2^5\,q_3\,\tau_1\,\tau_2+6\,q_2^4\,q_3^2\,\tau_1\,\tau_2-q_2^4\,q_3\,\tau_1^3+2\,q_2^3\,q_3^2\,\tau_1^3-2\,q_2^2$$
$$q_3^4\,\tau_1\,\tau_2-q_2^2\,q_3^3\,\tau_1^3+2\,q_1^3\,q_2^2\,\tau_1\,\tau_3-4\,q_1^3\,q_2\,q_3\,\tau_1\,\tau_3+2\,q_1^3\,q_3^2\,\tau_1\,\tau_3-4\,q_1^2\,q_2^3\,\tau_1\,\tau_3+6\,q_1^2$$
$$q_2^2\,q_3\,\tau_1\,\tau_3-2\,q_1^2\,q_3^3\,\tau_1\,\tau_3+2\,q_1\,q_2^4\,\tau_1\,\tau_3-6\,q_1\,q_2^2\,q_3^2\,\tau_1\,\tau_3+4\,q_1\,q_2\,q_3^3\,\tau_1\,\tau_3-2\,q_2^4\,q_3\,\tau_1\,\tau_3$$
$$+4\,q_2^3\,q_3^2\,\tau_1\,\tau_3-2\,q_2^2\,q_3^3\,\tau_1\,\tau_3+p_1^2\,q_1\,q_2^2\,q_3-2\,p_1^2\,q_1\,q_2\,q_3^2+p_1^2\,q_1\,q_3^3+p_2^2\,q_1^3\,q_3-2\,p_2^2$$
$$q_1^2\,q_2\,q_3+2\,p_2^2\,q_1\,q_2\,q_3^2-p_2^2\,q_1\,q_3^3-p_3^2\,q_1^3\,q_3+2\,p_3^2\,q_1^2\,q_2\,q_3-p_3^2\,q_1\,q_2^2\,q_3+h\,p_1\,q_2^2\,q_3$$
$$-2\,h\,p_1\,q_2\,q_3^2+h\,p_1\,q_3^3+h\,p_2\,q_1^3-2\,h\,p_2\,q_1^2\,q_2+h\,p_2\,q_1\,q_2^2-h\,p_2\,q_2^2\,q_3+2\,h\,p_2\,q_2\,q_3^2$$
$$-h\,p_2\,q_3^3-h\,p_3\,q_1^3+2\,h\,p_3\,q_1^2\,q_2-h\,p_3\,q_1\,q_2^2-p_1^2\,q_2^2\,\tau_1+2\,p_1^2\,q_2\,q_3\,\tau_1-p_1^2\,q_3^2\,\tau_1-p_2^2\,q_1^2\,\tau_1$$
$$+2\,p_2^2\,q_1\,q_2\,\tau_1-2\,p_2^2\,q_2\,q_3\,\tau_1+p_2^2\,q_3^2\,\tau_1+p_3^2\,q_1^2\,\tau_1-2\,p_3^2\,q_1\,q_2\,\tau_1+p_3^2\,q_2^2\,\tau_1)$$

$$\mathit{dq3dtau1} := \frac{2 p_3 q_1 q_2 + h q_1 + h q_2 - h q_3 - 2 p_3 \tau_1}{5 (q_2 - q_3) (q_1 - q_3) h}$$

$$\mathit{dp3dtau1} := -\frac{1}{5 h (q_1 - q_2) (q_1 - q_3)^2 (q_2 - q_3)^2} (-q_1^8 q_2^3 + 2 q_1^8 q_2^2 q_3 - q_1^8 q_2 q_3^2 + q_1^3 q_2^8$$
$$- 7 q_1^3 q_2^2 q_3^6 + 6 q_1^3 q_2 q_3^7 - 2 q_1^2 q_2^8 q_3 + 7 q_1^2 q_2^3 q_3^6 - 5 q_1^2 q_2 q_3^8 + q_1 q_2^8 q_3^2 - 6 q_1 q_2^3 q_3^7$$
$$+ 5 q_1 q_2^2 q_3^8 + q_1^7 q_2^2 \tau_1 - 2 q_1^7 q_2 q_3 \tau_1 + q_1^7 q_3^2 \tau_1 - 2 q_1^6 q_2^3 \tau_1 + 4 q_1^6 q_2^2 q_3 \tau_1 - 2 q_1^6 q_2 q_3^2 \tau_1$$
$$+ 2 q_1^3 q_2^6 \tau_1 - 10 q_1^3 q_2^2 q_3^4 \tau_1 + 8 q_1^3 q_2 q_3^5 \tau_1 - q_1^2 q_2^7 \tau_1 - 4 q_1^2 q_2^6 q_3 \tau_1 + 10 q_1^2 q_2^3 q_3^4 \tau_1 +$$
$$q_1^2 q_2 q_3^6 \tau_1 - 6 q_1^2 q_3^7 \tau_1 + 2 q_1 q_2^7 q_3 \tau_1 + 2 q_1 q_2^6 q_3^2 \tau_1 - 8 q_1 q_2^3 q_3^5 \tau_1 - q_1 q_2^2 q_3^6 \tau_1 + 5 q_1 q_3^8 \tau_1$$
$$- q_2^7 q_3^2 \tau_1 + 6 q_2^2 q_3^7 \tau_1 - 5 q_2 q_3^8 \tau_1 - 2 q_1^5 q_2^3 \tau_2 + 4 q_1^5 q_2^2 q_3 \tau_2 + 2 q_1^5 q_2^2 \tau_1^2 - 2 q_1^5 q_2 q_3^2 \tau_2$$
$$- 4 q_1^5 q_2 q_3 \tau_1^2 + 2 q_1^5 q_3^2 \tau_1^2 - q_1^4 q_2^3 \tau_1^2 + 2 q_1^4 q_2^2 q_3 \tau_1^2 - q_1^4 q_2 q_3^2 \tau_1^2 + 2 q_1^3 q_2^5 \tau_2 + q_1^3 q_2^4 \tau_1^2$$
$$- 8 q_1^3 q_2^2 q_3^3 \tau_2 - 3 q_1^3 q_2^2 q_3^2 \tau_1^2 + 6 q_1^3 q_2 q_3^4 \tau_2 + 2 q_1^3 q_2 q_3^3 \tau_1^2 - 4 q_1^2 q_2^5 q_3 \tau_2 - 2 q_1^2 q_2^5 \tau_1^2 - 2$$
$$q_1^2 q_2^4 q_3 \tau_1^2 + 8 q_1^2 q_2^3 q_3^3 \tau_2 + 3 q_1^2 q_2^3 q_3^2 \tau_1^2 - 4 q_1^2 q_2 q_3^5 \tau_2 + 9 q_1^2 q_2 q_3^4 \tau_1^2 - 8 q_1^2 q_3^5 \tau_1^2 + 2 q_1 q_2^5$$
$$q_3^2 \tau_2 + 4 q_1 q_2^5 q_3 \tau_1^2 + q_1 q_2^4 q_3^2 \tau_1^2 - 6 q_1 q_2^3 q_3^4 \tau_2 - 2 q_1 q_2^3 q_3^3 \tau_1^2 + 4 q_1 q_2^2 q_3^5 \tau_2 - 9 q_1 q_2^2 q_3^4 \tau_1^2$$
$$+ 6 q_1 q_3^6 \tau_1^2 - 2 q_2^5 q_3^2 \tau_1^2 + 8 q_2^2 q_3^5 \tau_1^2 - 6 q_2 q_3^6 \tau_1^2 - 2 q_1^4 q_2^3 \tau_3 + 4 q_1^4 q_2^2 q_3 \tau_3 + 2 q_1^4 q_2^2 \tau_1 \tau_2$$
$$- 2 q_1^4 q_2 q_3^2 \tau_3 - 4 q_1^4 q_2 q_3 \tau_1 \tau_2 + 2 q_1^4 q_3^2 \tau_1 \tau_2 + 2 q_1^3 q_2^4 \tau_3 - 6 q_1^3 q_2^2 q_3^2 \tau_3 + q_1^3 q_2^2 \tau_1^3 + 4$$
$$q_1^3 q_2 q_3^3 \tau_3 - 2 q_1^3 q_2 q_3 \tau_1^3 + q_1^3 q_3^2 \tau_1^3 - 4 q_1^2 q_2^4 q_3 \tau_3 - 2 q_1^2 q_2^4 \tau_1 \tau_2 + 6 q_1^2 q_2^3 q_3^2 \tau_3 - q_1^2 q_2^3 \tau_1^3$$
$$- 2 q_1^2 q_2 q_3^4 \tau_3 + 8 q_1^2 q_2 q_3^3 \tau_1 \tau_2 + 3 q_1^2 q_2 q_3^2 \tau_1^3 - 6 q_1^2 q_3^4 \tau_1 \tau_2 - 2 q_1^2 q_3^3 \tau_1^3 + 2 q_1 q_2^4 q_3^2 \tau_3$$
$$+ 4 q_1 q_2^4 q_3 \tau_1 \tau_2 - 4 q_1 q_2^3 q_3^3 \tau_3 + 2 q_1 q_2^3 q_3 \tau_1^3 + 2 q_1 q_2^2 q_3^4 \tau_3 - 8 q_1 q_2^2 q_3^3 \tau_1 \tau_2 - 3 q_1 q_2^2$$
$$q_3^2 \tau_1^3 + 4 q_1 q_3^5 \tau_1 \tau_2 + q_1 q_3^4 \tau_1^3 - 2 q_2^4 q_3^2 \tau_1 \tau_2 - q_2^3 q_3^2 \tau_1^3 + 6 q_2^2 q_3^4 \tau_1 \tau_2 + 2 q_2^2 q_3^3 \tau_1^3 - 4 q_2$$
$$q_3^5 \tau_1 \tau_2 - q_2 q_3^4 \tau_1^3 + 2 q_1^3 q_2^2 \tau_1 \tau_3 - 4 q_1^3 q_2 q_3 \tau_1 \tau_3 + 2 q_1^3 q_3^2 \tau_1 \tau_3 - 2 q_1^2 q_2^3 \tau_1 \tau_3 + 6 q_1^2 q_2$$
$$q_3^2 \tau_1 \tau_3 - 4 q_1^2 q_3^3 \tau_1 \tau_3 + 4 q_1 q_2^3 q_3 \tau_1 \tau_3 - 6 q_1 q_2^2 q_3^2 \tau_1 \tau_3 + 2 q_1 q_3^4 \tau_1 \tau_3 - 2 q_2^3 q_3^2 \tau_1 \tau_3 + 4$$
$$q_2^2 q_3^3 \tau_1 \tau_3 - 2 q_2 q_3^4 \tau_1 \tau_3 + p_1^2 q_1 q_2^3 - 2 p_1^2 q_1 q_2^2 q_3 + p_1^2 q_1 q_2 q_3^2 - p_2^2 q_1^3 q_2 + 2 p_2^2 q_1^2 q_2 q_3 -$$
$$p_2^2 q_1 q_2 q_3^2 + p_3^2 q_1^3 q_2 - 2 p_3^2 q_1^2 q_2 q_3 - p_3^2 q_1 q_2^3 + 2 p_3^2 q_1 q_2^2 q_3 + h p_1 q_2^3 - 2 h p_1 q_2^2 q_3$$
$$+ h p_1 q_2 q_3^2 - h p_2 q_1^3 + 2 h p_2 q_1^2 q_3 - h p_2 q_1 q_3^2 + h p_3 q_1^3 - 2 h p_3 q_1^2 q_3 + h p_3 q_1 q_3^2$$
$$- h p_3 q_2^3 + 2 h p_3 q_2^2 q_3 - h p_3 q_2 q_3^2 - p_1^2 q_2^2 \tau_1 + 2 p_1^2 q_2 q_3 \tau_1 - p_1^2 q_3^2 \tau_1 + p_2^2 q_1^2 \tau_1 - 2$$
$$p_2^2 q_1 q_3 \tau_1 + p_2^2 q_3^2 \tau_1 - p_3^2 q_1^2 \tau_1 + 2 p_3^2 q_1 q_3 \tau_1 + p_3^2 q_2^2 \tau_1 - 2 p_3^2 q_2 q_3 \tau_1)$$

$$\mathit{dq1dtau2} := -\frac{2 q_2 p_1 + 2 q_3 p_1 + h}{3 (q_1 - q_3) (q_1 - q_2) h}$$

$$\mathit{dp1dtau2} := -\frac{1}{3 h (q_1 - q_2)^2 (q_1 - q_3)^2 (q_2 - q_3)} (5 q_1^8 q_2^2 - 5 q_1^8 q_3^2 - 6 q_1^7 q_2^3 - 6 q_1^7 q_2^2 q_3$$
$$+ 6 q_1^7 q_2 q_3^2 + 6 q_1^7 q_3^3 + 7 q_1^6 q_2^3 q_3 - 7 q_1^6 q_2 q_3^3 + q_1^2 q_2^8 + q_1^2 q_2^7 q_3 - q_1^2 q_2 q_3^7 - q_1^2 q_3^8 - 2 q_1$$

$$q_2^8 q_3 - 2 q_1 q_2^7 q_3^2 + 2 q_1 q_2^2 q_3^7 + 2 q_1 q_2 q_3^8 + q_2^8 q_3^2 + q_2^7 q_3^3 - q_2^3 q_3^7 - q_2^2 q_3^8 + 6 q_1^6 q_2^2 \tau_1 - 6$$
$$q_1^6 q_3^2 \tau_1 - 8 q_1^5 q_2^3 \tau_1 - 8 q_1^5 q_2^2 q_3 \tau_1 + 8 q_1^5 q_2 q_3^2 \tau_1 + 8 q_1^5 q_3^3 \tau_1 + 10 q_1^4 q_2^3 q_3 \tau_1 - 10 q_1^4 q_2$$
$$q_3^3 \tau_1 + 2 q_1^2 q_2^6 \tau_1 + 2 q_1^2 q_2^5 q_3 \tau_1 - 2 q_1^2 q_2 q_3^5 \tau_1 - 2 q_1^2 q_3^6 \tau_1 - 4 q_1 q_2^6 q_3 \tau_1 - 4 q_1 q_2^5 q_3^2 \tau_1$$
$$+ 4 q_1 q_2^2 q_3^5 \tau_1 + 4 q_1 q_2 q_3^6 \tau_1 + 2 q_2^6 q_3^2 \tau_1 + 2 q_2^5 q_3^3 \tau_1 - 2 q_2^3 q_3^5 \tau_1 - 2 q_2^2 q_3^6 \tau_1 + 4 q_1^5 q_2^2 \tau_2$$
$$- 4 q_1^5 q_3^2 \tau_2 - 6 q_1^4 q_2^3 \tau_2 - 6 q_1^4 q_2^2 q_3 \tau_2 + q_1^4 q_2^2 \tau_1^2 + 6 q_1^4 q_2 q_3^2 \tau_2 + 6 q_1^4 q_3^3 \tau_2 - q_1^4 q_3^2 \tau_1^2 + 8$$
$$q_1^3 q_2^3 q_3 \tau_2 - 2 q_1^3 q_2^3 \tau_1^2 - 2 q_1^3 q_2^2 q_3 \tau_1^2 - 8 q_1^3 q_2 q_3^3 \tau_2 + 2 q_1^3 q_2 q_3^2 \tau_1^2 + 2 q_1^3 q_3^3 \tau_1^2 + 2 q_1^2 q_2^5 \tau_2$$
$$+ 2 q_1^2 q_2^4 q_3 \tau_2 + q_1^2 q_2^4 \tau_1^2 + 4 q_1^2 q_2^3 q_3 \tau_1^2 - 2 q_1^2 q_2 q_3^4 \tau_2 - 4 q_1^2 q_2 q_3^3 \tau_1^2 - 2 q_1^2 q_3^5 \tau_2 - q_1^2 q_3^4$$
$$\tau_1^2 - 4 q_1 q_2^5 q_3 \tau_2 - 4 q_1 q_2^4 q_3^2 \tau_2 - 2 q_1 q_2^4 q_3 \tau_1^2 - 2 q_1 q_2^3 q_3^2 \tau_1^2 + 4 q_1 q_2^2 q_3^4 \tau_2 + 2 q_1 q_2^2 q_3^3 \tau_1^2$$
$$+ 4 q_1 q_2 q_3^5 \tau_2 + 2 q_1 q_2 q_3^4 \tau_1^2 + 2 q_2^5 q_3^2 \tau_2 + 2 q_2^4 q_3^3 \tau_2 + q_2^4 q_3^2 \tau_1^2 - 2 q_2^3 q_3^4 \tau_2 - 2 q_2^2 q_3^5 \tau_2 -$$
$$q_2^2 q_3^4 \tau_1^2 + 2 q_1^4 q_2^2 \tau_3 - 2 q_1^4 q_3^2 \tau_3 - 4 q_1^3 q_2^3 \tau_3 - 4 q_1^3 q_2^2 q_3 \tau_3 + 4 q_1^3 q_2 q_3^2 \tau_3 + 4 q_1^3 q_3^3 \tau_3 + 2$$
$$q_1^2 q_2^4 \tau_3 + 8 q_1^2 q_2^3 q_3 \tau_3 - 8 q_1^2 q_2 q_3^3 \tau_3 - 2 q_1^2 q_3^4 \tau_3 - 4 q_1 q_2^4 q_3 \tau_3 - 4 q_1 q_2^3 q_3^2 \tau_3 + 4 q_1 q_2^2$$
$$q_3^3 \tau_3 + 4 q_1 q_2 q_3^4 \tau_3 + 2 q_2^4 q_3^2 \tau_3 - 2 q_2^2 q_3^4 \tau_3 + 2 p_1^2 q_1 q_2^2 - 2 p_1^2 q_1 q_3^2 - p_1^2 q_2^3 - p_1^2 q_2^2 q_3 +$$
$$p_1^2 q_2 q_3^2 + p_1^2 q_3^3 - p_2^2 q_1^2 q_2 - p_2^2 q_1^2 q_3 + 2 p_2^2 q_1 q_2 q_3 + 2 p_2^2 q_1 q_3^2 - p_2^2 q_2 q_3^2 - p_2^2 q_3^3 + p_3^2$$
$$q_1^2 q_2 + p_3^2 q_1^2 q_3 - 2 p_3^2 q_1 q_2^2 - 2 p_3^2 q_1 q_2 q_3 + p_3^2 q_2^3 + p_3^2 q_2^2 q_3 + 2 h p_1 q_1 q_2 - 2 h p_1 q_1 q_3$$
$$- h p_1 q_2^2 + h p_1 q_3^2 - h p_2 q_1^2 + 2 h p_2 q_1 q_3 - h p_2 q_3^2 + h p_3 q_1^2 - 2 h p_3 q_1 q_2 + h p_3 q_2^2 \Big)$$

$$dq2dtau2 := \frac{2 q_1 p_2 + 2 q_3 p_2 + h}{3 (q_2 - q_3) (q_1 - q_2) h}$$

$$dp2dtau2 := \frac{1}{3 h (q_1 - q_2)^2 (q_1 - q_3) (q_2 - q_3)^2} \Big(-q_1^8 q_2^2 + 2 q_1^8 q_2 q_3 - q_1^8 q_3^2 - q_1^7 q_2^2 q_3 + 2$$
$$q_1^7 q_2 q_3^2 - q_1^7 q_3^3 + 6 q_1^3 q_2^7 - 7 q_1^3 q_2^6 q_3 + q_1^3 q_3^7 - 5 q_1^2 q_2^8 + 6 q_1^2 q_2^7 q_3 - 2 q_1^2 q_2 q_3^7 + q_1^2 q_3^8$$
$$- 6 q_1 q_2^7 q_3^2 + 7 q_1 q_2^6 q_3^3 + q_1 q_2^2 q_3^7 - 2 q_1 q_2 q_3^8 + 5 q_2^8 q_3^2 - 6 q_2^7 q_3^3 + q_2^2 q_3^8 - 2 q_1^6 q_2^2 \tau_1 + 4$$
$$q_1^6 q_2 q_3 \tau_1 - 2 q_1^6 q_3^2 \tau_1 - 2 q_1^5 q_2^2 q_3 \tau_1 + 4 q_1^5 q_2 q_3^2 \tau_1 - 2 q_1^5 q_3^3 \tau_1 + 8 q_1^3 q_2^5 \tau_1 - 10 q_1^3$$
$$q_2^4 q_3 \tau_1 + 2 q_1^3 q_3^5 \tau_1 - 6 q_1^2 q_2^6 \tau_1 + 8 q_1^2 q_2^5 q_3 \tau_1 - 4 q_1^2 q_2 q_3^5 \tau_1 + 2 q_1^2 q_3^6 \tau_1 - 8 q_1 q_2^5 q_3^2 \tau_1$$
$$+ 10 q_1 q_2^4 q_3^3 \tau_1 + 2 q_1 q_2^2 q_3^5 \tau_1 - 4 q_1 q_2 q_3^6 \tau_1 + 6 q_2^6 q_3^2 \tau_1 - 8 q_2^5 q_3^3 \tau_1 + 2 q_2^2 q_3^6 \tau_1 - 2 q_1^5$$
$$q_2^2 \tau_2 + 4 q_1^5 q_2 q_3 \tau_2 - 2 q_1^5 q_3^2 \tau_2 - 2 q_1^4 q_2^2 q_3 \tau_2 - q_1^4 q_2^2 \tau_1^2 + 4 q_1^4 q_2 q_3^2 \tau_2 + 2 q_1^4 q_2 q_3 \tau_1^2 - 2$$
$$q_1^4 q_3^3 \tau_2 - q_1^4 q_3^2 \tau_1^2 + 6 q_1^3 q_2^4 \tau_2 - 8 q_1^3 q_2^3 q_3 \tau_2 + 2 q_1^3 q_2^3 \tau_1^2 - 4 q_1^3 q_2^2 q_3 \tau_1^2 + 2 q_1^3 q_2 q_3^2 \tau_1^2 + 2$$
$$q_1^3 q_3^4 \tau_2 - 4 q_1^2 q_2^5 \tau_2 + 6 q_1^2 q_2^4 q_3 \tau_2 - q_1^2 q_2^4 \tau_1^2 + 2 q_1^2 q_2^3 q_3 \tau_1^2 - 4 q_1^2 q_2 q_3^4 \tau_2 - 2 q_1^2 q_2 q_3^3 \tau_1^2$$
$$+ 2 q_1^2 q_3^5 \tau_2 + q_1^2 q_3^4 \tau_1^2 - 6 q_1 q_2^4 q_3^2 \tau_2 + 8 q_1 q_2^3 q_3^3 \tau_2 - 2 q_1 q_2^3 q_3^2 \tau_1^2 + 2 q_1 q_2^2 q_3^4 \tau_2 + 4 q_1$$
$$q_2^2 q_3^3 \tau_1^2 - 4 q_1 q_2 q_3^5 \tau_2 - 2 q_1 q_2 q_3^4 \tau_1^2 + 4 q_2^5 q_3^2 \tau_2 - 6 q_2^4 q_3^3 \tau_2 + q_2^4 q_3^2 \tau_1^2 - 2 q_2^3 q_3^3 \tau_1^2 + 2 q_2^2$$
$$q_3^5 \tau_2 + q_2^2 q_3^4 \tau_1^2 - 2 q_1^4 q_2^2 \tau_3 + 4 q_1^4 q_2 q_3 \tau_3 - 2 q_1^4 q_3^2 \tau_3 + 4 q_1^3 q_2^3 \tau_3 - 8 q_1^3 q_2^2 q_3 \tau_3 + 4 q_1^3 q_2$$
$$q_3^2 \tau_3 - 2 q_1^2 q_2^4 \tau_3 + 4 q_1^2 q_2^3 q_3 \tau_3 - 4 q_1^2 q_2 q_3^3 \tau_3 + 2 q_1^2 q_3^4 \tau_3 - 4 q_1 q_2^3 q_3^2 \tau_3 + 8 q_1 q_2^2 q_3^3 \tau_3$$

$$-4 q_1 q_2 q_3^4 \tau_3+2 q_2^4 q_3^2 \tau_3-4 q_2^3 q_3^3 \tau_3+2 q_2^2 q_3^4 \tau_3+p_1^2 q_1 q_2^2-2 p_1^2 q_1 q_2 q_3+p_1^2 q_1 q_3^2+p_1^2 q_2^2 q_3-2 p_1^2 q_2 q_3^2+p_1^2 q_3^3+p_2^2 q_1^3-2 p_2^2 q_1^2 q_2+p_2^2 q_1^2 q_3-p_2^2 q_1 q_3^2+2 p_2^2 q_2 q_3^2-p_2^2 q_3^3-p_3^2 q_1^3+2 p_3^2 q_1^2 q_2-p_3^2 q_1^2 q_3-p_3^2 q_1 q_2^2+2 p_3^2 q_1 q_2 q_3-p_3^2 q_2^2 q_3+h p_1 q_2^2-2 h p_1 q_2 q_3+h p_1 q_3^2+h p_2 q_1^2-2 h p_2 q_1 q_2+2 h p_2 q_2 q_3-h p_2 q_3^2-h p_3 q_1^2+2 h p_3 q_1 q_2-h p_3 q_2^2\Big)$$

$$dq3dtau2 := -\frac{2 q_1 p_3+2 q_2 p_3+h}{3 (q_2-q_3) (q_1-q_3) h}$$

$$dp3dtau2 := \frac{1}{3 h (q_1-q_2) (q_1-q_3)^2 (q_2-q_3)^2}\Big(-q_1^8 q_2^2+2 q_1^8 q_2 q_3-q_1^8 q_3^2-q_1^7 q_2^3+2 q_1^7 q_2^2 q_3-q_1^7 q_2 q_3^2+q_1^3 q_2^7-7 q_1^3 q_2 q_3^6+6 q_1^3 q_3^7+q_1^2 q_2^8-2 q_1^2 q_2^7 q_3+6 q_1^2 q_2 q_3^7-5 q_1^2 q_3^8-2 q_1 q_2^8 q_3+q_1 q_2^7 q_3^2+7 q_1 q_2^3 q_3^6-6 q_1 q_2^2 q_3^7+q_2^8 q_3^2-6 q_2^3 q_3^7+5 q_2^2 q_3^8-2 q_1^6 q_2^2 \tau_1+4 q_1^6 q_2 q_3 \tau_1-2 q_1^6 q_3^2 \tau_1-2 q_1^5 q_2^3 \tau_1+4 q_1^5 q_2^2 q_3 \tau_1-2 q_1^5 q_2 q_3^2 \tau_1+2 q_1^3 q_2^5 \tau_1-10 q_1^3 q_2 q_3^4 \tau_1+8 q_1^3 q_3^5 \tau_1+2 q_1^2 q_2^6 \tau_1-4 q_1^2 q_2^5 q_3 \tau_1+8 q_1^2 q_2 q_3^5 \tau_1-6 q_1^2 q_3^6 \tau_1-4 q_1 q_2^6 q_3 \tau_1+2 q_1 q_2^5 q_3^2 \tau_1+10 q_1 q_2^3 q_3^4 \tau_1-8 q_1 q_2^2 q_3^5 \tau_1+2 q_2^6 q_3^2 \tau_1-8 q_2^3 q_3^5 \tau_1+6 q_2^2 q_3^6 \tau_1-2 q_1^5 q_2^2 \tau_2+4 q_1^5 q_2 q_3 \tau_2-2 q_1^5 q_3^2 \tau_2-2 q_1^4 q_2^3 \tau_2+4 q_1^4 q_2^2 q_3 \tau_2-q_1^4 q_2^2 \tau_1^2-2 q_1^4 q_2 q_3^2 \tau_2+2 q_1^4 q_2 q_3 \tau_1^2-q_1^4 q_3^2 \tau_1^2+2 q_1^3 q_2^4 \tau_2+2 q_1^3 q_2^2 q_3 \tau_1^2-8 q_1^3 q_2 q_3^3 \tau_2-4 q_1^3 q_2 q_3^2 \tau_1^2+6 q_1^3 q_3^4 \tau_2+2 q_1^3 q_3^3 \tau_1^2+2 q_1^2 q_2^5 \tau_2-4 q_1^2 q_2^4 q_3 \tau_2+q_1^2 q_2^4 \tau_1^2-2 q_1^2 q_2^3 q_3 \tau_1^2+6 q_1^2 q_2 q_3^4 \tau_2+2 q_1^2 q_2 q_3^3 \tau_1^2-4 q_1^2 q_3^5 \tau_2-q_1^2 q_3^4 \tau_1^2-4 q_1 q_2^5 q_3 \tau_2+2 q_1 q_2^4 q_3^2 \tau_2-2 q_1 q_2^4 q_3 \tau_1^2+8 q_1 q_2^3 q_3^3 \tau_2+4 q_1 q_2^3 q_3^2 \tau_1^2-6 q_1 q_2^2 q_3^4 \tau_2-2 q_1 q_2^2 q_3^3 \tau_1^2+2 q_2^5 q_3^2 \tau_2+q_2^4 q_3^2 \tau_1^2-6 q_2^3 q_3^4 \tau_2-2 q_2^3 q_3^3 \tau_1^2+4 q_2^2 q_3^5 \tau_2+q_2^2 q_3^4 \tau_1^2-2 q_1^4 q_2^2 \tau_3+4 q_1^4 q_2 q_3 \tau_3-2 q_1^4 q_3^2 \tau_3+4 q_1^3 q_2^2 q_3 \tau_3-8 q_1^3 q_2 q_3^2 \tau_3+4 q_1^3 q_3^3 \tau_3+2 q_1^2 q_2^4 \tau_3-4 q_1^2 q_2^3 q_3 \tau_3+4 q_1^2 q_2 q_3^3 \tau_3-2 q_1^2 q_3^4 \tau_3-4 q_1 q_2^4 q_3 \tau_3+8 q_1 q_2^3 q_3^2 \tau_3-4 q_1 q_2^2 q_3^3 \tau_3+2 q_2^4 q_3^2 \tau_3-4 q_2^3 q_3^3 \tau_3+2 q_2^2 q_3^4 \tau_3+p_1^2 q_1 q_2^2-2 p_1^2 q_1 q_2 q_3+p_1^2 q_1 q_3^2+p_1^2 q_2^3-2 p_1^2 q_2^2 q_3+p_1^2 q_2 q_3^2-p_2^2 q_1^3-p_2^2 q_1^2 q_2+2 p_2^2 q_1^2 q_3+2 p_2^2 q_1 q_2 q_3-p_2^2 q_1 q_3^2-p_2^2 q_2 q_3^2+p_3^2 q_1^3+p_3^2 q_1^2 q_2-2 p_3^2 q_1^2 q_3-p_3^2 q_1 q_2^2-p_3^2 q_2^3+2 p_3^2 q_2^2 q_3+h p_1 q_2^2-2 h p_1 q_2 q_3+h p_1 q_3^2-h p_2 q_1^2+2 h p_2 q_1 q_3-h p_2 q_3^2+h p_3 q_1^2-2 h p_3 q_1 q_3-h p_3 q_2^2+2 h p_3 q_2 q_3\Big)$$

$$dq1dtau3 := \frac{2 p_1}{(q_1-q_3) (q_1-q_2) h}$$

$$dp1dtau3 := \frac{1}{h (q_1-q_2)^2 (q_1-q_3)^2 (q_2-q_3)}\Big(5 q_1^8 q_2-5 q_1^8 q_3-6 q_1^7 q_2^2+6 q_1^7 q_3^2+7 q_1^6 q_2^2 q_3-7 q_1^6 q_2 q_3^2+q_1^2 q_2^7-q_1^2 q_3^7-2 q_1 q_2^7 q_3+2 q_1 q_2 q_3^7+q_2^7 q_3^2-q_2^2 q_3^7+6 q_1^6 q_2 \tau_1-6 q_1^6 q_3 \tau_1-8 q_1^5 q_2^2 \tau_1+8 q_1^5 q_3^2 \tau_1+10 q_1^4 q_2^2 q_3 \tau_1-10 q_1^4 q_2 q_3^2 \tau_1+2 q_1^2 q_2^5 \tau_1-2 q_1^2 q_3^5 \tau_1$$

$$-4 q_1 q_2^5 q_3 \tau_1 + 4 q_1 q_2 q_3^5 \tau_1 + 2 q_2^5 q_3^2 \tau_1 - 2 q_2^2 q_3^5 \tau_1 + 4 q_1^5 q_2 \tau_2 - 4 q_1^5 q_3 \tau_2 - 6 q_1^4 q_2^2 \tau_2$$
$$+ q_1^4 q_2 \tau_1^2 + 6 q_1^4 q_3^2 \tau_2 - q_1^4 q_3 \tau_1^2 + 8 q_1^3 q_2^2 q_3 \tau_2 - 2 q_1^3 q_2^2 \tau_1^2 - 8 q_1^3 q_2 q_3^2 \tau_2 + 2 q_1^3 q_3^2 \tau_1^2 + 2$$
$$q_1^2 q_2^4 \tau_2 + q_1^2 q_2^3 \tau_1^2 + 3 q_1^2 q_2^2 q_3 \tau_1^2 - 3 q_1^2 q_2 q_3^2 \tau_1^2 - 2 q_1^2 q_3^4 \tau_2 - q_1^2 q_3^3 \tau_1^2 - 4 q_1 q_2^4 q_3 \tau_2$$
$$-2 q_1 q_2^3 q_3 \tau_1^2 + 4 q_1 q_2 q_3^4 \tau_2 + 2 q_1 q_2 q_3^3 \tau_1^2 + 2 q_2^4 q_3^2 \tau_2 + q_2^3 q_3^2 \tau_1^2 - 2 q_2^2 q_3^4 \tau_2 - q_2^2 q_3^3 \tau_1^2$$
$$+ 2 q_1^4 q_2 \tau_3 - 2 q_1^4 q_3 \tau_3 - 4 q_1^3 q_2^2 \tau_3 + 4 q_1^3 q_3^2 \tau_3 + 2 q_1^2 q_2^3 \tau_3 + 6 q_1^2 q_2^2 q_3 \tau_3 - 6 q_1^2 q_2 q_3^2 \tau_3$$
$$-2 q_1^2 q_3^3 \tau_3 - 4 q_1 q_2^3 q_3 \tau_3 + 4 q_1 q_2 q_3^3 \tau_3 + 2 q_2^3 q_3^2 \tau_3 - 2 q_2^2 q_3^3 \tau_3 + 2 p_1^2 q_1 q_2 - 2 p_1^2 q_1 q_3$$
$$- p_1^2 q_2^2 + p_1^2 q_3^2 - q_1^2 p_2^2 + 2 p_2^2 q_1 q_3 - p_2^2 q_3^2 + q_1^2 p_3^2 - 2 p_3^2 q_1 q_2 + p_3^2 q_2^2 \Big)$$

$$dq2dtau3 := - \frac{2 p_2}{(q_2 - q_3)\,(q_1 - q_2)\, h}$$

$$dp2dtau3 := - \frac{1}{h\,(q_1 - q_2)^2\,(q_1 - q_3)\,(q_2 - q_3)^2} \Big(-q_1^7 q_2^2 + 2 q_3 q_2 q_1^7 - q_1^7 q_3^2 + 6 q_1^2 q_2^7 - 7$$
$$q_1^2 q_2^6 q_3 + q_1^2 q_3^7 - 5 q_1 q_2^8 + 7 q_1 q_2^6 q_3^2 - 2 q_1 q_2 q_3^7 + 5 q_2^8 q_3 - 6 q_2^7 q_3^2 + q_2^2 q_3^7 - 2 q_1^5 q_2^2 \tau_1$$
$$+ 4 q_3 \tau_1 q_2 q_1^5 - 2 q_1^5 q_3^2 \tau_1 + 8 q_1^2 q_2^5 \tau_1 - 10 q_1^2 q_2^4 q_3 \tau_1 + 2 q_1^2 q_3^5 \tau_1 - 6 q_1 q_2^6 \tau_1 + 10 q_1 q_2^4$$
$$q_3^2 \tau_1 - 4 q_1 q_2 q_3^5 \tau_1 + 6 q_2^6 q_3 \tau_1 - 8 q_2^5 q_3^2 \tau_1 + 2 q_2^2 q_3^5 \tau_1 - 2 q_1^4 q_2^2 \tau_2 + 4 q_3 \tau_2 q_2 q_1^4 - 2 q_1^4$$
$$q_3^2 \tau_2 - q_1^3 q_2^2 \tau_1^2 + 2 q_1^3 q_2 q_3 \tau_1^2 - q_1^3 q_3^2 \tau_1^2 + 6 q_1^2 q_2^4 \tau_2 - 8 q_1^2 q_2^3 q_3 \tau_2 + 2 q_1^2 q_2^3 \tau_1^2 - 3 q_1^2$$
$$q_2^2 q_3 \tau_1^2 + 2 q_1^2 q_3^4 \tau_2 + q_1^2 q_3^3 \tau_1^2 - 4 q_1 q_2^5 \tau_2 - q_1 q_2^4 \tau_1^2 + 8 q_1 q_2^3 q_3^2 \tau_2 + 3 q_1 q_2^2 q_3^2 \tau_1^2$$
$$-4 q_1 q_2 q_3^4 \tau_2 - 2 q_1 q_2 q_3^3 \tau_1^2 + 4 q_2^5 q_3 \tau_2 - 6 q_2^4 q_3^2 \tau_2 + q_2^4 q_3 \tau_1^2 - 2 q_2^3 q_3^2 \tau_1^2 + 2 q_2^2 q_3^4 \tau_2 +$$
$$q_2^2 q_3^3 \tau_1^2 - 2 q_1^3 q_2^2 \tau_3 + 4 q_1^3 q_2 q_3 \tau_3 - 2 q_1^3 q_3^2 \tau_3 + 4 q_1^2 q_2^3 \tau_3 - 6 q_1^2 q_2^2 q_3 \tau_3 + 2 q_1^2 q_3^3 \tau_3$$
$$-2 q_1 q_2^4 \tau_3 + 6 q_1 q_2^2 q_3^2 \tau_3 - 4 q_1 q_2 q_3^3 \tau_3 + 2 q_2^4 q_3 \tau_3 - 4 q_2^3 q_3^2 \tau_3 + 2 q_2^2 q_3^3 \tau_3 + p_1^2 q_2^2$$
$$-2 q_3 p_1^2 q_2 + p_1^2 q_3^2 + q_1^2 p_2^2 - 2 p_2^2 q_1 q_2 + 2 p_2^2 q_2 q_3 - p_2^2 q_3^2 - q_1^2 p_3^2 + 2 p_3^2 q_1 q_2 - p_3^2 q_2^2 \Big)$$

$$dq3dtau3 := \frac{2 p_3}{(q_2 - q_3)\,(q_1 - q_3)\, h}$$

$$dp3dtau3 := - \frac{1}{h\,(q_1 - q_2)\,(q_1 - q_3)^2\,(q_2 - q_3)^2} \Big(-q_1^7 q_2^2 + 2 q_3 q_2 q_1^7 - q_1^7 q_3^2 + q_1^2 q_2^7 - 7$$
$$q_1^2 q_2 q_3^6 + 6 q_1^2 q_3^7 - 2 q_1 q_2^7 q_3 + 7 q_1 q_2^2 q_3^6 - 5 q_1 q_3^8 + q_2^7 q_3^2 - 6 q_2^2 q_3^7 + 5 q_2 q_3^8 - 2 q_1^5 q_2^2 \tau_1$$
$$+ 4 q_3 \tau_1 q_2 q_1^5 - 2 q_1^5 q_3^2 \tau_1 + 2 q_1^2 q_2^5 \tau_1 - 10 q_1^2 q_2 q_3^4 \tau_1 + 8 q_1^2 q_3^5 \tau_1 - 4 q_1 q_2^5 q_3 \tau_1 + 10 q_1$$
$$q_2^2 q_3^4 \tau_1 - 6 q_1 q_3^6 \tau_1 + 2 q_2^5 q_3^2 \tau_1 - 8 q_2^2 q_3^5 \tau_1 + 6 q_2 q_3^6 \tau_1 - 2 q_1^4 q_2^2 \tau_2 + 4 q_3 \tau_2 q_2 q_1^4 - 2 q_1^4$$
$$q_3^2 \tau_2 - q_1^3 q_2^2 \tau_1^2 + 2 q_1^3 q_2 q_3 \tau_1^2 - q_1^3 q_3^2 \tau_1^2 + 2 q_1^2 q_2^4 \tau_2 + q_1^2 q_2^3 \tau_1^2 - 8 q_1^2 q_2 q_3^3 \tau_2 - 3 q_1^2 q_2 q_3^2$$
$$\tau_1^2 + 6 q_1^2 q_3^4 \tau_2 + 2 q_1^2 q_3^3 \tau_1^2 - 4 q_1 q_2^4 q_3 \tau_2 - 2 q_1 q_2^3 q_3 \tau_1^2 + 8 q_1 q_2^2 q_3^3 \tau_2 + 3 q_1 q_2^2 q_3^2 \tau_1^2$$
$$-4 q_1 q_3^5 \tau_2 - q_1 q_3^4 \tau_1^2 + 2 q_2^4 q_3^2 \tau_2 + q_2^3 q_3^2 \tau_1^2 - 6 q_2^2 q_3^4 \tau_2 - 2 q_2^2 q_3^3 \tau_1^2 + 4 q_2 q_3^5 \tau_2 + q_2 q_3^4 \tau_1^2$$
$$-2 q_1^3 q_2^2 \tau_3 + 4 q_1^3 q_2 q_3 \tau_3 - 2 q_1^3 q_3^2 \tau_3 + 2 q_1^2 q_2^3 \tau_3 - 6 q_1^2 q_2 q_3^2 \tau_3 + 4 q_1^2 q_3^3 \tau_3 - 4 q_1$$

$$q_2^3 q_3 \tau_3 + 6 q_1 q_2^2 q_3^2 \tau_3 - 2 q_1 q_3^4 \tau_3 + 2 q_2^3 q_3^2 \tau_3 - 4 q_2^2 q_3^3 \tau_3 + 2 q_2 q_3^4 \tau_3 + p_1^2 q_2^2 - 2 q_3 p_1^2 q_2 + p_1^2 q_3^2 - q_1^2 p_2^2 + 2 p_2^2 q_1 q_3 - p_2^2 q_3^2 + q_1^2 p_3^2 - 2 p_3^2 q_1 q_3 - p_3^2 q_2^2 + 2 p_3^2 q_2 q_3\Big)$$

$$CheckAtau1 := \Bigg[$$

$$\Bigg[\frac{1}{5\,(q_1-q_2)\,(q_1-q_3)\,(q_2-q_3)}\Big(\big((p_2-p_3)\,q_1^2 - \lambda\,(p_2-p_3)\,q_1 + (p_3-p_1)\,q_2^2 + \lambda\,(p_1-p_3)\,q_2 - q_3\,(p_1-p_2)\,(\lambda-q_3)\big)\Big), \frac{\lambda^2}{5} + \frac{(-q_1-q_2-q_3)\,\lambda}{5} + \frac{(q_2+q_3)\,q_1}{5} + \frac{q_3 q_2}{5} - \frac{\tau_1}{5}\Bigg],$$

$$\Bigg[\frac{1}{5\,(q_2-q_3)^2\,(q_1-q_3)^2\,(q_1-q_2)^2}\Bigg((q_2-q_3)^2 q_1^7 + (q_2-q_3)^2\,(\lambda-q_2-q_3)\,q_1^6 + (q_2-q_3)^2\,\big((-\lambda+2 q_3)\,q_2 + \lambda\,(\lambda-q_3)\big)\,q_1^5 + (q_2-q_3)^2\Big(-2 q_3 q_2^2 + \big(-\lambda^2 + \lambda q_3 - 2 q_3^2\big)\,q_2 + \lambda^3 - \lambda^2 q_3 + \tau_1 \lambda + 2 \tau_2\Big)\,q_1^4 - 2\,(q_2-q_3)^2\Bigg(\frac{q_2^4}{2} + \frac{\lambda q_2^3}{2} + \Big(\frac{1}{2}\lambda^2 + \frac{1}{2}\lambda q_3 - \frac{3}{2} q_3^2\Big)\,q_2^2 + \Big(\lambda^3 + \frac{1}{2}\lambda q_3^2 + \tau_1\lambda + 2\tau_2\Big)\,q_2 + q_3\Big(\lambda^3 + \frac{1}{2}\lambda^2 q_3 + \frac{1}{2}\lambda q_3^2 + \frac{1}{2} q_3^3 + \tau_1 \lambda + 2 \tau_2\Big)\Bigg)\,q_1^3 + \Big(q_2^7 + (\lambda+q_3)\,q_2^6 + \big(\lambda^2 + \lambda q_3 - 4 q_3^2\big)\,q_2^5 + \big(\lambda^3 + \lambda^2 q_3 - 2 \lambda q_3^2 + 2 q_3^3 + \tau_1 \lambda + 2 \tau_2\big)\,q_2^4 + 2 q_3\,\big(\lambda^3 - \lambda^2 q_3 + q_3^3 + \tau_1 \lambda + 2 \tau_2\big)\,q_2^3 - 6 q_3^2\Big(\lambda^3 + \frac{1}{3}\lambda^2 q_3 + \frac{1}{3}\lambda q_3^2 + \frac{2}{3} q_3^3 + \tau_1\lambda + 2\tau_2\Big)\,q_2^2 + 2 q_3^3\Big(\lambda^3 + \frac{1}{2}\lambda^2 q_3 + \frac{1}{2}\lambda q_3^2 + \frac{1}{2} q_3^3 + \tau_1\lambda + 2\tau_2\Big)\,q_2 + q_3^7 + \lambda q_3^6 + \lambda^2 q_3^5 + \big(\lambda^3 + \tau_1 \lambda + 2 \tau_2\big)\,q_3^4 - (p_2-p_3)^2\Big)\,q_1^2 + \Big(-2 q_3 q_2^7 + \big(-2 \lambda q_3 + q_3^2\big)\,q_2^6 + \big(-2 \lambda^2 q_3 + \lambda q_3^2 + 2 q_3^3\big)\,q_2^5 - 2\Big(\lambda^3 - \frac{1}{2}\lambda^2 q_3 - \frac{1}{2}\lambda q_3^2 + q_3^3 + \tau_1\lambda + 2\tau_2\Big)\,q_3 q_2^4 + 2\Big(\lambda^3 + \lambda^2 q_3 + \frac{1}{2}\lambda q_3^2 + q_3^3 + \tau_1\lambda + 2\tau_2\Big)\,q_3^2 q_2^3 + 2 q_3^3\Big(\lambda^3 + \frac{1}{2}\lambda^2 q_3 + \frac{1}{2}\lambda q_3^2 + \frac{1}{2} q_3^3 + \tau_1\lambda + 2\tau_2\Big)\,q_2^2 + \Big(-2 q_3^7 - 2 \lambda q_3^6 - 2 \lambda^2 q_3^5 + \Big(-2 \lambda^3$$

$$-2\tau_1\lambda-4\tau_2\Big)q_3^4+2(p_2-p_3)(p_1-p_3)\Big)q_2-2q_3(p_2-p_3)(p_1-p_2)\Big)q_1+q_2^7q_3^2$$

$$+q_3^2(\lambda-q_3)q_2^6+\lambda q_3^2(\lambda-q_3)q_2^5+q_3^2\left(\lambda^3-\lambda^2q_3+\tau_1\lambda+2\tau_2\right)q_2^4-2q_3^3\Big(\lambda^3$$

$$+\frac{1}{2}\lambda^2q_3+\frac{1}{2}\lambda q_3^2+\frac{1}{2}q_3^3+\tau_1\lambda+2\tau_2\Big)q_2^3+\Big(q_3^7+\lambda q_3^6+\lambda^2q_3^5+\Big(\lambda^3+\tau_1\lambda$$

$$+2\tau_2\Big)q_3^4-(p_1-p_3)^2\Big)q_2^2+2q_3(p_1-p_3)(p_1-p_2)q_2-q_3^2(p_1-p_2)^2\Big),$$

$$\frac{1}{5(q_1-q_2)(q_1-q_3)(q_2-q_3)}\big((p_3-p_2)q_1^2+\lambda(p_2-p_3)q_1+(p_1-p_3)q_2^2$$

$$-\lambda(p_1-p_3)q_2+q_3(p_1-p_2)(\lambda-q_3)\big)\Big]\Big]$$

$$CheckAtau2 := \left[\left[\frac{(p_3-p_2)q_1+(p_1-p_3)q_2-q_3(p_1-p_2)}{3(q_2-q_3)(q_1-q_3)(q_1-q_2)}, -\frac{q_2}{3}+\frac{\lambda}{3}-\frac{q_1}{3}-\frac{q_3}{3}\right],\right.$$

$$\left[\frac{\lambda^2}{3}+\frac{(q_1+q_2+q_3)\lambda}{3}+\frac{q_1^2}{3}+\frac{q_2^2}{3}+\frac{q_3^2}{3}+\frac{2\tau_1}{3},\right.$$

$$\left.\left.\frac{(p_2-p_3)q_1+(p_3-p_1)q_2+q_3(p_1-p_2)}{3(q_2-q_3)(q_1-q_3)(q_1-q_2)}\right]\right]$$

$$CheckAtau3 := \begin{bmatrix} 0 & 1 \\ 2q_2+\lambda+2q_1+2q_3 & 0 \end{bmatrix} \tag{6}$$

Check of the theoretical formulas for the Lax matrix L

```
> tdL11theo:=0:
  for j from 0 to g-1 do aux:=0: for i from j+1 to g do aux:=aux+P
  [i]*Q[i-j-1]: od: tdL11theo:=tdL11theo-(-1)^(j-1)*aux*lambda^j:
  od:
  tdL11theo:=simplify(tdL11theo);
  simplify(tdL11theo-CheckL[1,1]);
```

$$tdL11theo := \frac{1}{(q_1-q_2)(q_1-q_3)(q_2-q_3)}\Big(\big((p_3-p_2)q_1+(p_1-p_3)q_2-q_3(p_1$$

$$-p_2)\big)\lambda^2+\big((p_2-p_3)q_1^2+(p_3-p_1)q_2^2+q_3^2(p_1-p_2)\big)\lambda+(-q_3p_2+q_2p_3)q_1^2$$

$$+\left(p_2q_3^2-p_3q_2^2\right)q_1+p_1q_2q_3(q_2-q_3)\Big)$$

$$0 \tag{7}$$

```
> tdL12theo:=0:
  for m from 0 to g do tdL12theo:=tdL12theo+(-1)^(g-m)*Q[g-m]*
  lambda^m: od:
  tdL12theo:=simplify(tdL12theo);
  simplify(tdL12theo-CheckL[1,2]);
```

$$tdL12theo := (\lambda - q_1)(\lambda - q_2)(\lambda - q_3)$$
$$0 \quad (8)$$

```
> Term1:=0:
  for i from 0 to rinfty-2 do for j from g+i to 2*rinfty-5 do
  Term1:=Term1- P2[j]*h[j-g-i]*lambda^i: od: od:
  Term1:
  Term2:=0:
  for i from 0 to g-2 do for j1 from i+1 to g-1 do for j2 from g+i-
  j1 to g-1 do for i1 from j1+1 to g do for i2 from j2+1 to g do
  Term2:=Term2- (-1)^(j1+j2)*P[i1]*Q[i1-j1-1]*P[i2]*Q[i2-j2-1]*h
  [j1+j2-g-i]*lambda^i:
  od: od: od: od: od:
  Term2:
  tdL21theo:=simplify(Term1+Term2):
  simplify(tdL21theo-CheckL[2,1]);
```

$$0 \quad (9)$$

Inverting the symmetric Darboux coordinates

```
> factor(P[1] -( q[1]^2*(q[2]-q[3])*p[1]- q[2]^2*(q[1]-q[3])*p[2]+q
  [3]^2*(q[1]-q[2])*p[3])/(q[1]-q[2])/(q[1]-q[3])/(q[2]-q[3]));
  factor(P[2]-(-(q[1]*(q[2]-q[3])*p[1]- q[2]*(q[1]-q[3])*p[2]+q[3]*
  (q[1]-q[2])*p[3])/(q[1]-q[2])/(q[1]-q[3])/(q[2]-q[3]) ));
  factor(P[3]-(((q[2]-q[3])*p[1]- (q[1]-q[3])*p[2]+(q[1]-q[2])*p[3]
  )/(q[1]-q[2])/(q[1]-q[3])/(q[2]-q[3]) ));

  solve({PP1 -( q[1]^2*(q[2]-q[3])*p[1]- q[2]^2*(q[1]-q[3])*p[2]+q
  [3]^2*(q[1]-q[2])*p[3])/(q[1]-q[2])/(q[1]-q[3])/(q[2]-q[3])=0,
  PP2-(-(q[1]*(q[2]-q[3])*p[1]- q[2]*(q[1]-q[3])*p[2]+q[3]*(q[1]-q
  [2])*p[3])/(q[1]-q[2])/(q[1]-q[3])/(q[2]-q[3]) ), PP3-(((q[2]-q
  [3])*p[1]- (q[1]-q[3])*p[2]+(q[1]-q[2])*p[3])/(q[1]-q[2])/(q[1]-q
  [3])/(q[2]-q[3]) )},{p[1],p[2],p[3]});

  QQfunction:=unapply(QQ(lambda),p[1],p[2],p[3]):
  simplify(series(es(simplify(QQfunction(PP3*q[2]*q[3]+PP2*q[2]+
  PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q[2]+
  PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
```

$$0$$
$$0$$
$$0$$
$$\{p_1 = PP3\, q_2\, q_3 + PP2\, q_2 + PP2\, q_3 + PP1,\ p_2 = PP3\, q_1\, q_3 + PP2\, q_1 + PP2\, q_3 + PP1,\ p_3 = PP3\, q_1\, q_2 + PP2\, q_1 + PP2\, q_2 + PP1\}$$
$$-PP2\,\sigma_1 - PP3\,\sigma_2 - PP1 + (PP3\,\sigma_1 + PP2)\,\lambda - PP3\,\lambda^2 \quad (10)$$

Expressing the spectral invariants in terms of symmetric Darboux coordinates

```
> H0function:=unapply(H0,p[1],p[2],p[3]):
  H1function:=unapply(H1,p[1],p[2],p[3]):
  H2function:=unapply(H2,p[1],p[2],p[3]):
  simplify(series(es(simplify(H0function(PP3*q[2]*q[3]+PP2*q[2]+
  PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q[2]+
  PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),h=0));
  simplify(series(es(simplify(H1function(PP3*q[2]*q[3]+PP2*q[2]+
  PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q[2]+
  PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),h=0));
  simplify(series(es(simplify(H2function(PP3*q[2]*q[3]+PP2*q[2]+
  PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q[2]+
  PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),h=0));
```

$$-\sigma_1^4\sigma_3+PP2^2\sigma_1^2+2\,PP2\,PP3\,\sigma_1\sigma_2-PP3^2\sigma_1\sigma_3+PP3^2\sigma_2^2+3\sigma_1^2\sigma_2\sigma_3-2\sigma_1^2\sigma_3\tau_1+2\,PP1\,PP2\,\sigma_1+2\,PP1\,PP3\,\sigma_2-2\,PP2\,PP3\,\sigma_3-2\sigma_1\sigma_3^2-2\sigma_1\sigma_3\tau_2-\sigma_2^2\sigma_3+2\sigma_2\sigma_3\tau_1-\sigma_3\tau_1^2+PP1^2-2\sigma_3\tau_3+\left(PP3\,\sigma_1+2\,PP2\right)h$$

$$\sigma_1^4\sigma_2-2\,PP2\,PP3\,\sigma_1^2-PP3^2\sigma_1\sigma_2-\sigma_1^3\sigma_3-3\sigma_1^2\sigma_2^2+2\sigma_1^2\sigma_2\tau_1-2\,PP1\,PP3\,\sigma_1-2\,PP2^2\sigma_1+PP3^2\sigma_3+4\sigma_1\sigma_2\sigma_3+2\sigma_1\sigma_2\tau_2-2\sigma_1\sigma_3\tau_1+\sigma_2^3-2\sigma_2^2\tau_1+\sigma_2\tau_1^2-2\,PP1\,PP2+2\sigma_2\tau_3-\sigma_3^2-2\sigma_3\tau_2-PP3\,h$$

$$-\sigma_1^5+4\sigma_1^3\sigma_2-2\sigma_1^3\tau_1+2\,PP2\,PP3\,\sigma_1+PP3^2\sigma_2-3\sigma_1^2\sigma_3-2\sigma_1^2\tau_2-3\sigma_1\sigma_2^2+4\sigma_1\sigma_2\tau_1-\sigma_1\tau_1^2+2\,PP1\,PP3+PP2^2-2\sigma_1\tau_3+2\sigma_2\sigma_3+2\sigma_2\tau_2-2\sigma_3\tau_1 \qquad \textbf{(11)}$$

Expressing \td{L} using the symmetric Draboux coordinates

```
> CheckL11function:= unapply(CheckL[1,1],p[1],p[2],p[3]):
  CheckL12function:= unapply(CheckL[1,2],p[1],p[2],p[3]):
  CheckL21function:= unapply(CheckL[2,1],p[1],p[2],p[3]):
  simplify(series(es(simplify(CheckL11function(PP3*q[2]*q[3]+PP2*q
  [2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q
  [2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
  simplify(series(es(simplify(CheckL12function(PP3*q[2]*q[3]+PP2*q
  [2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q
  [2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
  simplify(series(es(simplify(CheckL21function(PP3*q[2]*q[3]+PP2*q
  [2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q[1]*q
  [2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
```

$$PP2\,\sigma_1+PP3\,\sigma_2+PP1+\left(-PP3\,\sigma_1-PP2\right)\lambda+PP3\,\lambda^2$$

$$-\sigma_3+\sigma_2\lambda-\sigma_1\lambda^2+\lambda^3$$

$$\sigma_1^4+\left(2\tau_1-3\sigma_2\right)\sigma_1^2+\left(PP3^2+2\sigma_3+2\tau_2\right)\sigma_1+2\,PP2\,PP3+\tau_1^2-2\sigma_2\tau_1+\sigma_2^2+2\tau_3+\left(\sigma_1^3+\left(2\tau_1-2\sigma_2\right)\sigma_1-PP3^2+2\tau_2+\sigma_3\right)\lambda+\left(\sigma_1^2-\sigma_2+2\tau_1\right)\lambda^2+\sigma_1\lambda^3+\lambda^4 \qquad \textbf{(12)}$$

Expressing the Hamiltonians in terms of symmetric Darboux coordinates

```
> Hamtau1function:=unapply(Hamtau1,p[1],p[2],p[3]):
Hamtau2function:=unapply(Hamtau2,p[1],p[2],p[3]):
Hamtau3function:=unapply(Hamtau3,p[1],p[2],p[3]):
Hamtau1R:=simplify(series(es(simplify(Hamtau1function(PP3*q[2]*q
[3]+PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,
PP3*q[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),h=0));
Hamtau2R:=simplify(series(es(simplify(Hamtau2function(PP3*q[2]*q
[3]+PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,
PP3*q[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),h=0));
Hamtau3R:=simplify(series(es(simplify(Hamtau3function(PP3*q[2]*q
[3]+PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,
PP3*q[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),h=0));
```

$$Hamtau1R := -\frac{1}{5} PP3^2 \sigma_1 \sigma_3 + \frac{3}{5} \sigma_1^2 \sigma_2 \sigma_3 + \frac{1}{5} \sigma_1^2 \sigma_3 \tau_1 - \frac{2}{5} PP2\, PP3\, \sigma_3 - \frac{2}{5} \sigma_1 \sigma_3 \tau_2 + \frac{2}{5} PP1\, PP3\, \sigma_2 + \frac{2}{5} PP1\, PP2\, \sigma_1 + \frac{1}{5} PP2^2 \sigma_1^2 - \frac{1}{5} \sigma_1^4 \sigma_3 - \frac{2}{5} \sigma_1 \sigma_3^2 - \frac{1}{5} \sigma_2^2 \sigma_3 + \frac{1}{5} \sigma_3 \tau_1^2 - \frac{2}{5} \sigma_3 \tau_3 + \frac{1}{5} PP3^2 \sigma_2^2 - \frac{2}{5} PP2\, PP3\, \sigma_1 \tau_1 + \frac{2}{5} PP2\, PP3\, \sigma_1 \sigma_2 + \frac{1}{5} PP1^2 - \frac{2}{5} PP1\, PP3\, \tau_1 - \frac{1}{5} PP2^2 \tau_1 - \frac{4}{5} \sigma_1^3 \sigma_2 \tau_1 - \frac{1}{5} PP3^2 \sigma_2 \tau_1 + \frac{3}{5} \sigma_1 \sigma_2^2 \tau_1 - \frac{4}{5} \sigma_1 \sigma_2 \tau_1^2 - \frac{2}{5} \sigma_2 \tau_1 \tau_2 + \frac{2}{5} \sigma_1^2 \tau_1 \tau_2 + \frac{2}{5} \sigma_1 \tau_1 \tau_3 + \frac{1}{5} \sigma_1^5 \tau_1 + \frac{2}{5} \sigma_1^3 \tau_1^2 + \frac{1}{5} \sigma_1 \tau_1^3 + \left(\frac{2\, PP2}{5} + \frac{PP3\, \sigma_1}{5} \right) h$$

$$Hamtau2R := \frac{1}{3} \sigma_1^4 \sigma_2 - \frac{2}{3} PP2\, PP3\, \sigma_1^2 - \frac{1}{3} PP3^2 \sigma_1 \sigma_2 - \frac{1}{3} \sigma_1^3 \sigma_3 - \sigma_1^2 \sigma_2^2 + \frac{2}{3} \sigma_1^2 \sigma_2 \tau_1 - \frac{2}{3} PP1\, PP3\, \sigma_1 - \frac{2}{3} PP2^2 \sigma_1 + \frac{1}{3} PP3^2 \sigma_3 + \frac{4}{3} \sigma_1 \sigma_2 \sigma_3 + \frac{2}{3} \sigma_1 \sigma_2 \tau_2 - \frac{2}{3} \sigma_1 \sigma_3 \tau_1 + \frac{1}{3} \sigma_2^3 - \frac{2}{3} \sigma_2^2 \tau_1 + \frac{1}{3} \sigma_2 \tau_1^2 - \frac{2}{3} PP1\, PP2 + \frac{2}{3} \sigma_2 \tau_3 - \frac{1}{3} \sigma_3^2 - \frac{2}{3} \sigma_3 \tau_2 - \frac{1}{3} PP3\, h$$

$$Hamtau3R := -\sigma_1^5 + 4 \sigma_1^3 \sigma_2 - 2 \sigma_1^3 \tau_1 + 2\, PP2\, PP3\, \sigma_1 + PP3^2 \sigma_2 - 3 \sigma_1^2 \sigma_3 - 2 \sigma_1^2 \tau_2 - 3 \sigma_1 \sigma_2^2 + 4 \sigma_1 \sigma_2 \tau_1 - \sigma_1 \tau_1^2 + 2\, PP1\, PP3 + PP2^2 - 2 \sigma_1 \tau_3 + 2 \sigma_2 \sigma_3 + 2 \sigma_2 \tau_2 - 2 \sigma_3 \tau_1 \tag{13}$$

```
> Hamtau1bis:=((1/5)*PP3*sigma[1]+2*PP2*(1/5))*h-(1/5)*PP3^2*sigma
[2]*tau[1]-(1/5)*PP3^2*sigma[1]*sigma[3]+(1/5)*sigma[1]^2*sigma
[3]*tau[1]+(1/5)*sigma[1]^5*tau[1]+(1/5)*PP2^2*sigma[1]^2+(1/5)*
sigma[1]*tau[1]^3-(1/5)*sigma[1]^4*sigma[3]-(1/5)*sigma[2]^2*
sigma[3]+(1/5)*sigma[3]*tau[1]^2+(1/5)*PP3^2*sigma[2]^2-(2/5*PP1)
*PP3*tau[1]-(2/5*sigma[3])*tau[3]-(2/5*sigma[1])*sigma[3]^2+(2/5*
(sigma[1]^3))*tau[1]^2+(1/5)*PP1^2-(2/5*sigma[1])*sigma[3]*tau[2]
-(2/5*PP2)*PP3*sigma[3]+(2/5*(sigma[1]^2))*tau[1]*tau[2]+(2/5*
sigma[1])*tau[1]*tau[3]+(2/5*PP1)*PP2*sigma[1]+(3/5*sigma[1])*
sigma[2]^2*tau[1]-(4/5*sigma[1])*sigma[2]*tau[1]^2+(3/5*(sigma[1]
```

```
^2))*sigma[2]*sigma[3]+(2/5*PP1)*PP3*sigma[2]-(2/5*sigma[2])*tau
[1]*tau[2]-(1/5)*PP2^2*tau[1]-(4/5*(sigma[1]^3))*sigma[2]*tau[1]+
(2/5*PP2)*PP3*sigma[1]*sigma[2]-(2/5*PP2)*PP3*sigma[1]*tau[1]:
simplify(Hamtau1R-Hamtau1bis);
Hamtau1ter:=1/5*((sigma[2]^2-sigma[1]*sigma[3])*PP3^2+2*(sigma[1]
+sigma[2])*PP1*PP3 +sigma[1]^2*PP2^2+2*sigma[1]*PP1*PP2+PP1^2+(2*
sigma[1]*sigma[2]-2*sigma[3])*PP2*PP3 -2*sigma[1]*PP1*PP3-sigma
[3]*(sigma[1]^4-3*sigma[1]^2*sigma[2]+2*sigma[1]*sigma[3]+sigma
[2]^2)
+ 2*sigma[1]^2*tau[1]*tau[2]-2*sigma[2]*tau[1]*tau[2]+ 2*sigma[1]
*tau[1]*tau[3]+sigma[1]*tau[1]^3+(sigma[3]+2*sigma[1]^3-4*sigma
[1]*sigma[2])*tau[1]^2
-(2*PP1*PP3+2*sigma[1]*PP2*PP3+PP2^2+sigma[2]*PP3^2-sigma[1]^5-
sigma[1]^2*sigma[3]-3*sigma[1]*sigma[2]^2+4*sigma[1]^3*sigma[2])*
tau[1]-2*sigma[1]*sigma[3]*tau[2]- 2*sigma[3]*tau[3]+ (PP3*sigma
[1]+2*PP2)*h  ):
factor(simplify(series(Hamtau1bis-Hamtau1ter,PP3=0)));
Hamtau1ter;
```

$$0$$

$$0$$

$$\frac{\left(-\sigma_1\sigma_3+\sigma_2^2\right)PP3^2}{5}+\frac{2\left(\sigma_1+\sigma_2\right)PP1\,PP3}{5}+\frac{PP2^2\sigma_1^2}{5}+\frac{2\,PP1\,PP2\,\sigma_1}{5}+\frac{PP1^2}{5}$$
$$+\frac{\left(2\sigma_1\sigma_2-2\sigma_3\right)PP2\,PP3}{5}-\frac{2\,PP1\,PP3\,\sigma_1}{5}-\frac{\sigma_3\left(\sigma_1^4-3\sigma_1^2\sigma_2+2\sigma_1\sigma_3+\sigma_2^2\right)}{5}$$
$$+\frac{2\sigma_1^2\tau_1\tau_2}{5}-\frac{2\sigma_2\tau_1\tau_2}{5}+\frac{2\sigma_1\tau_1\tau_3}{5}+\frac{\sigma_1\tau_1^3}{5}+\frac{\left(2\sigma_1^3-4\sigma_1\sigma_2+\sigma_3\right)\tau_1^2}{5}$$
$$-\frac{1}{5}\Big(\Big(-\sigma_1^5+4\sigma_1^3\sigma_2+2\,PP2\,PP3\,\sigma_1+PP3^2\sigma_2-\sigma_1^2\sigma_3-3\sigma_1\sigma_2^2+2\,PP1\,PP3$$
$$+PP2^2\Big)\tau_1\Big)-\frac{2\sigma_1\sigma_3\tau_2}{5}-\frac{2\sigma_3\tau_3}{5}+\frac{\left(PP3\,\sigma_1+2\,PP2\right)h}{5} \tag{14}$$

```
> Hamtau2bis:=(1/3)*sigma[1]^4*sigma[2]-(2/3*PP2)*PP3*sigma[1]^2-
(1/3)*PP3^2*sigma[1]*sigma[2]-(1/3)*sigma[1]^3*sigma[3]-sigma[1]
^2*sigma[2]^2+(2/3*(sigma[1]^2))*sigma[2]*tau[1]-(2/3*PP1)*PP3*
sigma[1]-(2/3*(PP2^2))*sigma[1]+(1/3)*PP3^2*sigma[3]+(4/3*sigma
[1])*sigma[2]*sigma[3]+(2/3*sigma[1])*sigma[2]*tau[2]-(2/3*sigma
[1])*sigma[3]*tau[1]+(1/3)*sigma[2]^3-(2/3*(sigma[2]^2))*tau[1]+
(1/3)*sigma[2]*tau[1]^2-(2/3*PP1)*PP2+(2/3*sigma[2])*tau[3]-(1/3)
*sigma[3]^2-(2/3*sigma[3])*tau[2]-h*(1/3)*PP3:
simplify(Hamtau2R-Hamtau2bis);
Hamtau2ter:=1/3*(-2*sigma[1]^2*PP2*PP3-2*sigma[1]*PP1*PP3-2*PP1*
PP2+(sigma[3]-sigma[1]*sigma[2])*PP3^2-2*sigma[1]*PP2^2+4*sigma
```

```
[1]*sigma[2]*sigma[3]+sigma[1]^4*sigma[2]-3*sigma[1]^2*sigma[2]^2
-sigma[1]^3*sigma[3]+sigma[2]^3-sigma[3]^2+   sigma[2]*tau[1]^2+2*
(sigma[1]^2*sigma[2]-sigma[1]*sigma[3]-sigma[2]^2)*tau[1]+    2*
(sigma[1]*sigma[2]-sigma[3])*tau[2]+2*sigma[2]*tau[3]-PP3*h):
factor(simplify(series(Hamtau2bis-Hamtau2ter,tau[1]=0)));
Hamtau2ter;
```

$$0$$

$$0$$

$$-\frac{2\,PP2\,PP3\,\sigma_1^2}{3}-\frac{2\,PP1\,PP3\,\sigma_1}{3}-\frac{2\,PP1\,PP2}{3}+\frac{\left(-\sigma_1\,\sigma_2+\sigma_3\right)PP3^2}{3}-\frac{2\,PP2^2\,\sigma_1}{3}+\frac{4\,\sigma_1\,\sigma_2\,\sigma_3}{3}+\frac{\sigma_1^4\,\sigma_2}{3}-\sigma_1^2\,\sigma_2^2-\frac{\sigma_1^3\,\sigma_3}{3}+\frac{\sigma_2^3}{3}-\frac{\sigma_3^2}{3}+\frac{\sigma_2\,\tau_1^2}{3}+\frac{2\left(\sigma_1^2\,\sigma_2-\sigma_1\,\sigma_3-\sigma_2^2\right)\tau_1}{3}+\frac{2\left(\sigma_1\,\sigma_2-\sigma_3\right)\tau_2}{3}+\frac{2\,\sigma_2\,\tau_3}{3}-\frac{PP3\,h}{3} \tag{15}$$

```
> Hamtau3bis:=-sigma[1]^5+4*sigma[1]^3*sigma[2]-2*sigma[1]^3*tau[1]
+2*PP2*PP3*sigma[1]+PP3^2*sigma[2]-3*sigma[1]^2*sigma[3]-2*sigma
[1]^2*tau[2]-3*sigma[1]*sigma[2]^2+4*sigma[1]*sigma[2]*tau[1]-
sigma[1]*tau[1]^2+2*PP1*PP3+PP2^2-2*sigma[1]*tau[3]+2*sigma[2]*
sigma[3]+2*sigma[2]*tau[2]-2*sigma[3]*tau[1]:
simplify(Hamtau3R-Hamtau3bis);
Hamtau3ter:=2*PP1*PP3+PP2^2+sigma[2]*PP3^2+2*sigma[1]*PP2*PP3-
sigma[1]^5+4*sigma[1]^3*sigma[2]-3*sigma[1]^2*sigma[3]-3*sigma[1]
*sigma[2]^2+2*sigma[2]*sigma[3]-sigma[1]*tau[1]^2+2*(2*sigma[1]*
sigma[2]-sigma[1]^3-sigma[3])*tau[1]+ 2*(sigma[2]-sigma[1]^2)*tau
[2]-2*sigma[1]*tau[3]:
factor(simplify(series(Hamtau3bis-Hamtau3ter,tau[1]=0)));
Hamtau3ter;
```

$$0$$

$$0$$

$$2\,PP1\,PP3+PP2^2+PP3^2\,\sigma_2+2\,PP2\,PP3\,\sigma_1-\sigma_1^5+4\,\sigma_1^3\,\sigma_2-3\,\sigma_1^2\,\sigma_3-3\,\sigma_1\,\sigma_2^2+2\,\sigma_2\,\sigma_3-\sigma_1\,\tau_1^2+2\left(-\sigma_1^3+2\,\sigma_1\,\sigma_2-\sigma_3\right)\tau_1+2\left(-\sigma_1^2+\sigma_2\right)\tau_2-2\,\sigma_1\,\tau_3 \tag{16}$$

Verification of the formulas for the matrix A^{tau1}.

```
> nu[1]:=nu1tau1;
nu[2]:=nu2tau1;
nu[3]:=nu3tau1;

nu[rinfty-2]:=0:
for k from 1 to g do nu[rinfty-2]:=nu[rinfty-2]+(-1)^(g-k)*nu[k]*
Q[g+1-k]: od:
nu[rinfty-2];
```

$$\nu_1 := \frac{1}{5}$$
$$\nu_2 := 0$$
$$\nu_3 := -\frac{\tau_1}{5}$$
$$\frac{q_1 q_2 q_3}{5} - \frac{\tau_1 (q_1 + q_2 + q_3)}{5} \tag{17}$$

```
> tdA11theo:=0:
  for i from 0 to g-2 do for m from 1 to g-1-i do for r from i+m+1
  to g do tdA11theo:=tdA11theo-(-1)^(i+m-1)*nu[m]*P[r]*Q[r-i-m-1]*
  lambda^i od: od: od:
  tdA11theo:=simplify(tdA11theo);
  factor(simplify(tdA11theo-CheckAtau1[1,1]));
```

$$tdA11theo := \frac{1}{5 (q_1 - q_2)(q_1 - q_3)(q_2 - q_3)} \big((p_2 - p_3) q_1^2 - \lambda (p_2 - p_3) q_1 + (p_3 - p_1) q_2^2 + \lambda (p_1 - p_3) q_2 - q_3 (p_1 - p_2)(\lambda - q_3) \big)$$
$$0 \tag{18}$$

```
> tdA12theo:=0:
  for j from 0 to g-1 do for m from 1 to g-j do tdA12theo:=
  tdA12theo+(-1)^(g-j-m)*nu[m]*Q[g-j-m]*lambda^j: od: od:
  tdA12theo:=simplify(tdA12theo);
  factor(simplify(tdA12theo-CheckAtau1[1,2]));
```

$$tdA12theo := \frac{\lambda^2}{5} + \frac{(-q_1 - q_2 - q_3)\lambda}{5} + \frac{(q_2 + q_3) q_1}{5} + \frac{q_3 q_2}{5} - \frac{\tau_1}{5}$$
$$0 \tag{19}$$

```
> tdA21theoTerm1:=0:
  for i from 0 to g do lowerpoint:=max(g,g+i-1): for j from
  lowerpoint to 2*rinfty-5 do for m from 1 to j-g-i do
   tdA21theoTerm1:=tdA21theoTerm1-nu[m]*h[j-g-m-i]*P2[j]*lambda^i:
  od: od: od:
  tdA21theoTerm1;
  tdA21theoTerm2:=0:
  for i from 0 to g do for j1 from 0 to g-1 do for j2 from 1 to g-1
  do for m from 1 to j1+j2-g-i do for r1 from j1+1 to g do for r2
  from j2+1 to g do
  tdA21theoTerm2:=tdA21theoTerm2-(-1)^(j1+j2)*nu[m]*h[j1+j2-g-i-m]*
  P[r1]*P[r2]*Q[r1-j1-1]*Q[r2-j2-1]*lambda^i:
  od: od: od: od: od: od:
  tdA21theoTerm2:
  tdA21theo:=simplify(tdA21theoTerm1+tdA21theoTerm2):
  factor(simplify(tdA21theo-CheckAtau1[2,1]));
```

$$\frac{2\,\tau_2}{5} + \frac{(q_1+q_2+q_3)^3}{5} - \frac{2\,(q_1+q_2+q_3)\,(q_1 q_2 + q_3 q_1 + q_3 q_2)}{5} + \frac{2\,q_1 q_2 q_3}{5} + \frac{\tau_1 \lambda}{5} + \frac{\left(-q_1 q_2 - q_3 q_1 - q_3 q_2 + (q_1+q_2+q_3)^2\right)\lambda}{5} + \frac{(q_1+q_2+q_3)\,\lambda^2}{5} + \frac{\lambda^3}{5}$$

$$0 \qquad \textbf{(20)}$$

Verification of the formulas for the Hamiltonians

```
> Term0Hamtheo:=0:
  for i from 1 to g do for k from i+1 to g do Term0Hamtheo:=
  Term0Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:

  Term1Hamtheo:=0:
  for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
  do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
  : od: od: od:

  Term2Hamtheo:=0:
  for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
  for r1 from max(0,i-k2) to min(k1-1,i-1) do
  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(i-1)*Q[k1-1-
  r1]*Q[k2-i+r1]: od: od: od: od:

  for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
  for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
  g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
  [k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
  r2+m-i-g+1]: fi: od: od: od: od: od: od:

  Term3Hamtheo:=0:
  for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
  to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
  h[r+m-i-g+1]: od: od: od:

  Hamilton:=simplify(Term0Hamtheo+Term1Hamtheo+Term2Hamtheo+
  Term3Hamtheo):
  simplify(Hamilton-Hamtau1);
```

$$0 \qquad \textbf{(21)}$$

```
> Term1Hamtheo:=0:
  for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
  Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
  for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
  do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
  : od: od: od:
```

```
Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from max(0,i-k2) to min(k1-1,i-1) do
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(i-1)*Q[k1-1-
r1]*Q[k2-i+r1]: od: od: od: od:

for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau1);
```

$$0 \tag{22}$$

Expression of \td{A} in terms of the symmetric Darboux coordinates

```
> CheckA11tau1function:= unapply(CheckAtau1[1,1],p[1],p[2],p[3]):
CheckA12tau1function:= unapply(CheckAtau1[1,2],p[1],p[2],p[3]):
CheckA21tau1function:= unapply(CheckAtau1[2,1],p[1],p[2],p[3]):
simplify(series(es(simplify(CheckA11tau1function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
simplify(series(es(simplify(CheckA12tau1function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
simplify(series(es(simplify(CheckA21tau1function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
```

$$-\frac{PP3\,\sigma_1}{5}-\frac{PP2}{5}+\frac{1}{5}\,PP3\,\lambda$$

$$\frac{\sigma_2}{5}-\frac{\tau_1}{5}-\frac{1}{5}\,\sigma_1\,\lambda+\frac{1}{5}\,\lambda^2$$

$$\frac{1}{5}\,\sigma_1^3-\frac{1}{5}\,PP3^2-\frac{2}{5}\,\sigma_1\,\sigma_2+\frac{2}{5}\,\sigma_3+\frac{2}{5}\,\tau_2+\left(\frac{\sigma_1^2}{5}-\frac{\sigma_2}{5}+\frac{\tau_1}{5}\right)\lambda+\frac{1}{5}\,\sigma_1\,\lambda^2+\frac{1}{5}\,\lambda^3 \tag{23}$$

We have verified the \td{A}^1 formula. Let us do A^{tau2}.

```
> nu[1]:=nu1tau2;
nu[2]:=nu2tau2;
```

```
nu[3]:=nu3tau2;

nu[rinfty-2]:=0:
for k from 1 to g do nu[rinfty-2]:=nu[rinfty-2]+(-1)^(g-k)*nu[k]*
Q[g+1-k]: od:
nu[rinfty-2];
```

$$\nu_1 := 0$$

$$\nu_2 := \frac{1}{3}$$

$$\nu_3 := 0$$

$$-\frac{1}{3}\, q_1\, q_2 - \frac{1}{3}\, q_3\, q_1 - \frac{1}{3}\, q_3\, q_2 \tag{24}$$

```
> tdA11theo:=0:
for i from 0 to g-2 do for m from 1 to g-1-i do for r from i+m+1
to g do tdA11theo:=tdA11theo-(-1)^(i+m-1)*nu[m]*P[r]*Q[r-i-m-1]*
lambda^i od: od: od:
tdA11theo:=simplify(tdA11theo);
factor(simplify(tdA11theo-CheckAtau2[1,1]));
```

$$tdA11theo := \frac{(p_3 - p_2)\, q_1 + (p_1 - p_3)\, q_2 - q_3\, (p_1 - p_2)}{3\, (q_2 - q_3)\, (q_1 - q_3)\, (q_1 - q_2)}$$

$$0 \tag{25}$$

```
> tdA12theo:=0:
for j from 0 to g-1 do for m from 1 to g-j do tdA12theo:=
tdA12theo+(-1)^(g-j-m)*nu[m]*Q[g-j-m]*lambda^j: od: od:
tdA12theo:=simplify(tdA12theo);
factor(simplify(tdA12theo-CheckAtau2[1,2]));
```

$$tdA12theo := -\frac{q_2}{3} + \frac{\lambda}{3} - \frac{q_1}{3} - \frac{q_3}{3}$$

$$0 \tag{26}$$

```
> tdA21theoTerm1:=0:
for i from 0 to g do lowerpoint:=max(g,g+i-1): for j from
lowerpoint to 2*rinfty-5 do for m from 1 to j-g-i do
 tdA21theoTerm1:=tdA21theoTerm1-nu[m]*h[j-g-m-i]*P2[j]*lambda^i:
od: od: od:
tdA21theoTerm1;
tdA21theoTerm2:=0:
for i from 0 to g do for j1 from 0 to g-1 do for j2 from 1 to g-1
do for m from 1 to j1+j2-g-i do for r1 from j1+1 to g do for r2
from j2+1 to g do
tdA21theoTerm2:=tdA21theoTerm2-(-1)^(j1+j2)*nu[m]*h[j1+j2-g-i-m]*
P[r1]*P[r2]*Q[r1-j1-1]*Q[r2-j2-1]*lambda^i:
od: od: od: od: od: od:
```

```
tdA21theoTerm2:
tdA21theo:=simplify(tdA21theoTerm1+tdA21theoTerm2):
factor(simplify(tdA21theo-CheckAtau2[2,1]));
```

$$\frac{2\,\tau_1}{3}-\frac{2\,q_1\,q_2}{3}-\frac{2\,q_3\,q_1}{3}-\frac{2\,q_3\,q_2}{3}+\frac{(q_1+q_2+q_3)^2}{3}+\frac{(q_1+q_2+q_3)\,\lambda}{3}+\frac{\lambda^2}{3}$$

$$0 \tag{27}$$

```
> Term1Hamtheo:=0:
for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
: od: od: od:

Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from max(0,i-k2) to min(k1-1,i-1) do
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(i-1)*Q[k1-1-
r1]*Q[k2-i+r1]: od: od: od: od:

for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau2);
```

$$0 \tag{28}$$

```
> CheckA11tau2function:= unapply(CheckAtau2[1,1],p[1],p[2],p[3]):
CheckA12tau2function:= unapply(CheckAtau2[1,2],p[1],p[2],p[3]):
CheckA21tau2function:= unapply(CheckAtau2[2,1],p[1],p[2],p[3]):
simplify(series(es(simplify(CheckA11tau2function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
simplify(series(es(simplify(CheckA12tau2function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
```

```
simplify(series(es(simplify(CheckA21tau2function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
```

$$\frac{PP3}{3}$$

$$-\frac{\sigma_1}{3}+\frac{1}{3}\lambda$$

$$\frac{\sigma_1^2}{3}-\frac{2\sigma_2}{3}+\frac{2\tau_1}{3}+\frac{1}{3}\sigma_1\lambda+\frac{1}{3}\lambda^2 \tag{29}$$

We have verified the \td{A}^2 formula. Let us do A^{tau3}.

```
> nu[1]:=nu1tau3;
nu[2]:=nu2tau3;
nu[3]:=nu3tau3;

nu[rinfty-2]:=0:
for k from 1 to g do nu[rinfty-2]:=nu[rinfty-2]+(-1)^(g-k)*nu[k]*
Q[g+1-k]: od:
nu[rinfty-2];
```

$$\nu_1 := 0$$

$$\nu_2 := 0$$

$$\nu_3 := 1$$

$$q_1+q_2+q_3 \tag{30}$$

```
> tdA11theo:=0:
for i from 0 to g-2 do for m from 1 to g-1-i do for r from i+m+1
to g do tdA11theo:=tdA11theo-(-1)^(i+m-1)*nu[m]*P[r]*Q[r-i-m-1]*
lambda^i od: od: od:
tdA11theo:=simplify(tdA11theo);
factor(simplify(tdA11theo-CheckAtau3[1,1]));
```

$$tdA11theo := 0$$

$$0 \tag{31}$$

```
> tdA12theo:=0:
for j from 0 to g-1 do for m from 1 to g-j do tdA12theo:=
tdA12theo+(-1)^(g-j-m)*nu[m]*Q[g-j-m]*lambda^j: od: od:
tdA12theo:=simplify(tdA12theo);
factor(simplify(tdA12theo-CheckAtau3[1,2]));
```

$$tdA12theo := 1$$

$$0 \tag{32}$$

```
> tdA21theoTerm1:=0:
for i from 0 to g do lowerpoint:=max(g,g+i-1): for j from
lowerpoint to 2*rinfty-5 do for m from 1 to j-g-i do
 tdA21theoTerm1:=tdA21theoTerm1-nu[m]*h[j-g-m-i]*P2[j]*lambda^i:
```

```
od: od: od:
tdA21theoTerm1;
tdA21theoTerm2:=0:
for i from 0 to g do for j1 from 0 to g-1 do for j2 from 1 to g-1
do for m from 1 to j1+j2-g-i do for r1 from j1+1 to g do for r2
from j2+1 to g do
tdA21theoTerm2:=tdA21theoTerm2-(-1)^(j1+j2)*nu[m]*h[j1+j2-g-i-m]*
P[r1]*P[r2]*Q[r1-j1-1]*Q[r2-j2-1]*lambda^i:
od: od: od: od: od: od:
tdA21theoTerm2:
tdA21theo:=simplify(tdA21theoTerm1+tdA21theoTerm2):
factor(simplify(tdA21theo-CheckAtau3[2,1]));
```

$$2\,q_2+\lambda+2\,q_1+2\,q_3$$

$$0 \qquad (33)$$

```
> Term1Hamtheo:=0:
for i from 1 to g do for k from i+1 to g do Term1Hamtheo:=
Term1Hamtheo-h*nu[i]*(-1)^i*(g-i)*P[k]*Q[k-1-i]: od: od:
for i from 1 to g do for k from i+1 to g do for m from i+1 to k-1
do Term1Hamtheo:=Term1Hamtheo-h*nu[i]*(-1)^m*P[k]*Q[k-1-m]*S[m-i]
: od: od: od:

Term2Hamtheo:=0:
for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from max(0,i-k2) to min(k1-1,i-1) do
Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P[k2]*(-1)^(i-1)*Q[k1-1-
r1]*Q[k2-i+r1]: od: od: od: od:

for i from 1 to g do for k1 from 1 to g do for k2 from 1 to g do
for r1 from 0 to k1-1 do for r2 from 0 to k2-1 do for m from i to
g do if r1+r2>=g then  Term2Hamtheo:=Term2Hamtheo+nu[i]*P[k1]*P
[k2]*(-1)^(r1+r2)*Q[k1-1-r1]*Q[k2-1-r2]*(-1)^(g-m)*Q[g-m]*h[r1+
r2+m-i-g+1]: fi: od: od: od: od: od: od:

Term3Hamtheo:=0:
for i from 1 to g do for r from g to 2*rinfty-5 do for m from i
to g do Term3Hamtheo:=Term3Hamtheo+nu[i]*(-1)^(g-m)*P2[r]*Q[g-m]*
h[r+m-i-g+1]: od: od: od:

Hamilton:=simplify(Term1Hamtheo+Term2Hamtheo+Term3Hamtheo):
simplify(Hamilton-Hamtau3);
```

$$0 \qquad (34)$$

```
> CheckA11tau3function:= unapply(CheckAtau3[1,1],p[1],p[2],p[3]):
```

```
CheckA12tau3function:= unapply(CheckAtau3[1,2],p[1],p[2],p[3]):
CheckA21tau3function:= unapply(CheckAtau3[2,1],p[1],p[2],p[3]):
simplify(series(es(simplify(CheckA11tau3function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
simplify(series(es(simplify(CheckA12tau3function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
simplify(series(es(simplify(CheckA21tau3function(PP3*q[2]*q[3]+
PP2*q[2]+PP2*q[3]+PP1,PP3*q[1]*q[3]+PP2*q[1]+PP2*q[3]+PP1,PP3*q
[1]*q[2]+PP2*q[1]+PP2*q[2]+PP1)),q[1],q[2],q[3]),lambda=0));
```

$$0$$

$$1$$

$$2\sigma_1 + \lambda \tag{35}$$