

In this Maple sheet, we compute the Lax matrices using the asymptotics of the wave functions and the local diagonalization for the Painlevé 2 equation.

We first use the expression of the coefficients of the spectral curve in terms of the irregular times and monodromies.

```
> restart;
CoherenceEquation1:=tinfty10+tinfty20;
tinfty20:=-tinfty10;
Pinfty42 := tinfty13*tinfty23;
Pinfty32 := tinfty12*tinfty23+tinfty13*tinfty22;
Pinfty22 := tinfty12*tinfty22+tinfty13*tinfty21+tinfty11*
tinfty23;
Pinfty12 := tinfty20*tinfty13+tinfty12*tinfty21+tinfty10*
tinfty23+tinfty11*tinfty22;
Pinfty01 := -tinfty11-tinfty21;
Pinfty11 := -tinfty12-tinfty22;
Pinfty21 := -tinfty13-tinfty23;
P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2;
P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
x^4;
```

$$\begin{aligned}
 \text{CoherenceEquation1} &:= \text{tinfty10} + \text{tinfty20} \\
 \text{Pinfty42} &:= \text{tinfty13} \text{ tinfty23} \\
 \text{Pinfty32} &:= \text{tinfty12} \text{ tinfty23} + \text{tinfty13} \text{ tinfty22} \\
 \text{Pinfty22} &:= \text{tinfty11} \text{ tinfty23} + \text{tinfty12} \text{ tinfty22} + \text{tinfty13} \text{ tinfty21} \\
 \text{Pinfty12} &:= -\text{tinfty10} \text{ tinfty13} + \text{tinfty10} \text{ tinfty23} + \text{tinfty11} \text{ tinfty22} + \text{tinfty12} \text{ tinfty21} \\
 \text{Pinfty01} &:= -\text{tinfty11} - \text{tinfty21} \\
 \text{Pinfty11} &:= -\text{tinfty12} - \text{tinfty22} \\
 \text{Pinfty21} &:= -\text{tinfty13} - \text{tinfty23}
 \end{aligned}
 \tag{1}$$

### Study of the asymptotics at infinity

```
> logPsi1Infty:=-tinfty13/3/h*lambda^3-tinfty12/2/h*lambda^2-
tinfty11/h*lambda-tinfty10/h*ln(lambda)+A10-A12/(2-1)/lambda^(2
-1)-A13/(3-1)/lambda^(3-1)-A14/(4-1)/lambda^(4-1)-A15/(5-1)
/lambda^(5-1)-A16/(6-1)/lambda^(6-1)-A17/(7-1)/lambda^(7-1) ;
logPsi2Infty:=-tinfty23/3/h*lambda^3-tinfty22/2/h*lambda^2-
tinfty21/h*lambda-tinfty20/h*ln(lambda)-1*ln(lambda)+A20-A22/(2
-1)/lambda^(2-1)-A23/(3-1)/lambda^(3-1)-A24/(4-1)/lambda^(4-1)-
A25/(5-1)/lambda^(5-1)-A26/(6-1)/lambda^(6-1)-A27/(7-1)/lambda^(7
-1) ;
LlogPsi1Infty:=-Ltinfty13/3/h*lambda^3-Ltinfty12/2/h*lambda^2-
Ltinfty11/h*lambda-Ltinfty10/h*ln(lambda)+LA10-LA12/(2-1)/lambda^(
2-1)-LA13/(3-1)/lambda^(3-1)-LA14/(4-1)/lambda^(4-1)-LA15/(5-1)
/lambda^(5-1)-LA16/(6-1)/lambda^(6-1)-LA17/(7-1)/lambda^(7-1) ;
LlogPsi2Infty:=-Ltinfty23/3/h*lambda^3-Ltinfty22/2/h*lambda^2-
```

```

Ltinfty21/h*lambda-Ltinfty20/h*ln(lambda)+LA20-LA22/(2-1)/lambda^
(2-1)-LA23/(3-1)/lambda^(3-1)-LA24/(4-1)/lambda^(4-1)-LA25/(5-1)
/lambda^(5-1)-LA26/(6-1)/lambda^(6-1)-LA27/(7-1)/lambda^(7-1) ;
Lpsi1Infty := exp(1/h*(-tinfty13/3*lambda^3-tinfty12/2*lambda^2-
tinfty11*lambda-tinfty10*ln(lambda)+h*A10-h*A12/lambda-1/2*h*
A13/lambda^2-1/3*h*A14/lambda^3-1/4*h*A15/lambda^4-1/5*h*
A16/lambda^5-1/6*h*A17/lambda^6))*1/h*(-Ltinfty13/3*lambda^3-
Ltinfty12/2*lambda^2-Ltinfty11*lambda-Ltinfty10*ln(lambda)+h*LA10
-h*LA12/lambda-1/2*h*LA13/lambda^2-1/3*h*LA14/lambda^3-1/4*h*
LA15/lambda^4-1/5*h*LA16/lambda^5-1/6*h*LA17/lambda^6) ;
Lpsi2Infty := exp(1/h*(-tinfty23/3*lambda^3-tinfty22/2*lambda^2-
tinfty21*lambda-tinfty20*ln(lambda)-h*ln(lambda)+h*A20-h*
A22/lambda-1/2*h*A23/lambda^2-1/3*h*A24/lambda^3-1/4*h*
A25/lambda^4-1/5*h*A26/lambda^5-1/6*h*A27/lambda^6))*1/h*(-
Ltinfty23/3*lambda^3-Ltinfty22/2*lambda^2-Ltinfty21*lambda-
Ltinfty20*ln(lambda)+h*LA20-h*LA22/lambda-1/2*h*LA23/lambda^2
-1/3*h*LA24/lambda^3-1/4*h*LA25/lambda^4-1/5*h*LA26/lambda^5-1/6*
h*LA27/lambda^6) ;
psi1Infty:=exp(logPsi1Infty) ;
psi2Infty:=exp(logPsi2Infty) ;
dpsi1dlambdaInfty:=diff(psi1Infty,lambda) :
dpsi2dlambdaInfty:=diff(psi2Infty,lambda) :
d2psi1dlambda2Infty:=diff(psi1Infty,lambda$2) :
d2psi2dlambda2Infty:=diff(psi2Infty,lambda$2) :

```

```

WronskianLambdaInfty:=h*factor(psi1Infty*dpsi2dlambdaInfty-
psi2Infty*dpsi1dlambdaInfty) :
WronskianLambdabisInfty:=h*simplify(factor(diff(logPsi2Infty,
lambda)-diff(logPsi1Infty,lambda))*exp(logPsi1Infty+logPsi2Infty)
)) :

```

```

WronskianTildeLambdaInfty:=h^3*factor(dpsi2dlambdaInfty*
d2psi1dlambda2Infty-dpsi1dlambdaInfty*d2psi2dlambda2Infty) :

```

$$\begin{aligned}
\log\Psi_1\text{Infty} &:= -\frac{1}{3} \frac{\text{tinfty13} \lambda^3}{h} - \frac{1}{2} \frac{\text{tinfty12} \lambda^2}{h} - \frac{\text{tinfty11} \lambda}{h} - \frac{\text{tinfty10} \ln(\lambda)}{h} + A_{10} \\
&\quad - \frac{A_{12}}{\lambda} - \frac{1}{2} \frac{A_{13}}{\lambda^2} - \frac{1}{3} \frac{A_{14}}{\lambda^3} - \frac{1}{4} \frac{A_{15}}{\lambda^4} - \frac{1}{5} \frac{A_{16}}{\lambda^5} - \frac{1}{6} \frac{A_{17}}{\lambda^6} \\
\log\Psi_2\text{Infty} &:= -\frac{1}{3} \frac{\text{tinfty23} \lambda^3}{h} - \frac{1}{2} \frac{\text{tinfty22} \lambda^2}{h} - \frac{\text{tinfty21} \lambda}{h} + \frac{\text{tinfty10} \ln(\lambda)}{h} - \ln(\lambda)
\end{aligned} \tag{2}$$

$$+ A20 - \frac{A22}{\lambda} - \frac{1}{2} \frac{A23}{\lambda^2} - \frac{1}{3} \frac{A24}{\lambda^3} - \frac{1}{4} \frac{A25}{\lambda^4} - \frac{1}{5} \frac{A26}{\lambda^5} - \frac{1}{6} \frac{A27}{\lambda^6}$$

$$Llogpsi1Infty := -\frac{1}{3} \frac{Ltiny13 \lambda^3}{h} - \frac{1}{2} \frac{Ltiny12 \lambda^2}{h} - \frac{Ltiny11 \lambda}{h} - \frac{Ltiny10 \ln(\lambda)}{h}$$

$$+ LA10 - \frac{LA12}{\lambda} - \frac{1}{2} \frac{LA13}{\lambda^2} - \frac{1}{3} \frac{LA14}{\lambda^3} - \frac{1}{4} \frac{LA15}{\lambda^4} - \frac{1}{5} \frac{LA16}{\lambda^5} - \frac{1}{6} \frac{LA17}{\lambda^6}$$

$$Llogpsi2Infty := -\frac{1}{3} \frac{Ltiny23 \lambda^3}{h} - \frac{1}{2} \frac{Ltiny22 \lambda^2}{h} - \frac{Ltiny21 \lambda}{h} - \frac{Ltiny20 \ln(\lambda)}{h}$$

$$+ LA20 - \frac{LA22}{\lambda} - \frac{1}{2} \frac{LA23}{\lambda^2} - \frac{1}{3} \frac{LA24}{\lambda^3} - \frac{1}{4} \frac{LA25}{\lambda^4} - \frac{1}{5} \frac{LA26}{\lambda^5} - \frac{1}{6} \frac{LA27}{\lambda^6}$$

$$Lpsi1Infty := 1 /$$

$$h \left( e^{\frac{1}{h} \left( -\frac{1}{3} tiny13 \lambda^3 - \frac{1}{2} tiny12 \lambda^2 - tiny11 \lambda - tiny10 \ln(\lambda) + h A10 - \frac{h A12}{\lambda} - \frac{1}{2} \frac{h A13}{\lambda^2} \right.} \right.$$

$$\left. - \frac{1}{3} \frac{h A14}{\lambda^3} - \frac{1}{4} \frac{h A15}{\lambda^4} - \frac{1}{5} \frac{h A16}{\lambda^5} - \frac{1}{6} \frac{h A17}{\lambda^6} \right) \left( -\frac{1}{3} Ltiny13 \lambda^3 - \frac{1}{2} Ltiny12 \lambda^2 \right. \\ \left. - Ltiny11 \lambda - Ltiny10 \ln(\lambda) + h LA10 - \frac{h LA12}{\lambda} - \frac{1}{2} \frac{h LA13}{\lambda^2} - \frac{1}{3} \frac{h LA14}{\lambda^3} \right. \\ \left. - \frac{1}{4} \frac{h LA15}{\lambda^4} - \frac{1}{5} \frac{h LA16}{\lambda^5} - \frac{1}{6} \frac{h LA17}{\lambda^6} \right) \right)$$

$$Lpsi2Infty := 1 /$$

$$h \left( e^{\frac{1}{h} \left( -\frac{1}{3} tiny23 \lambda^3 - \frac{1}{2} tiny22 \lambda^2 - tiny21 \lambda + tiny10 \ln(\lambda) - h \ln(\lambda) + h A20 - \frac{h A22}{\lambda} \right.} \right.$$

$$\left. - \frac{1}{2} \frac{h A23}{\lambda^2} - \frac{1}{3} \frac{h A24}{\lambda^3} - \frac{1}{4} \frac{h A25}{\lambda^4} - \frac{1}{5} \frac{h A26}{\lambda^5} - \frac{1}{6} \frac{h A27}{\lambda^6} \right) \left( -\frac{1}{3} Ltiny23 \lambda^3 - \frac{1}{2} Ltiny22 \lambda^2 \right. \\ \left. - Ltiny21 \lambda - Ltiny20 \ln(\lambda) + h LA20 - \frac{h LA22}{\lambda} - \frac{1}{2} \frac{h LA23}{\lambda^2} - \frac{1}{3} \frac{h LA24}{\lambda^3} \right. \\ \left. - \frac{1}{4} \frac{h LA25}{\lambda^4} - \frac{1}{5} \frac{h LA26}{\lambda^5} - \frac{1}{6} \frac{h LA27}{\lambda^6} \right) \right)$$

$$psi1Infty :=$$

$$e^{-\frac{1}{3} \frac{tiny13 \lambda^3}{h} - \frac{1}{2} \frac{tiny12 \lambda^2}{h} - \frac{tiny11 \lambda}{h} - \frac{tiny10 \ln(\lambda)}{h} + A10 - \frac{A12}{\lambda} - \frac{1}{2} \frac{A13}{\lambda^2} - \frac{1}{3} \frac{A14}{\lambda^3}}$$

$$- \frac{1}{4} \frac{A15}{\lambda^4} - \frac{1}{5} \frac{A16}{\lambda^5} - \frac{1}{6} \frac{A17}{\lambda^6}$$

$$psi2Infty :=$$

$$e^{-\frac{1}{3} \frac{tiny23 \lambda^3}{h} - \frac{1}{2} \frac{tiny22 \lambda^2}{h} - \frac{tiny21 \lambda}{h} + \frac{tiny10 \ln(\lambda)}{h} - \ln(\lambda) + A20 - \frac{A22}{\lambda} - \frac{1}{2} \frac{A23}{\lambda^2}}$$

$$-\frac{1}{3} \frac{A24}{\lambda^3} - \frac{1}{4} \frac{A25}{\lambda^4} - \frac{1}{5} \frac{A26}{\lambda^5} - \frac{1}{6} \frac{A27}{\lambda^6}$$

## Expression of the Lax matrix L

Shape of  $L_{\{2,1\}}$  at infinity.

```
> L21Infty:=factor(simplify
(WronskianTildeLambdaInfty/WronskianLambdabisInfty)):
L21InftyOrderlambda5:=factor(-residue(L21Infty/lambda^6,lambda=
infinity));
L21InftyOrderlambda4:=factor(-residue(L21Infty/lambda^5,lambda=
infinity));
L21InftyOrderlambda3:=factor(-residue(L21Infty/lambda^4,lambda=
infinity));
L21InftyOrderlambda2:=factor(-residue(L21Infty/lambda^3,lambda=
infinity));
L21InftyOrderlambda1:=factor(-residue(L21Infty/lambda^2,lambda=
infinity));
L21InftyOrderlambda0:=factor(-residue(L21Infty/lambda^1,lambda=
infinity));
```

$$\begin{aligned} L21InftyOrderlambda5 &:= 0 \\ L21InftyOrderlambda4 &:= -\text{tinfy}13 \text{tinfy}23 \\ L21InftyOrderlambda3 &:= -\text{tinfy}12 \text{tinfy}23 - \text{tinfy}13 \text{tinfy}22 \\ L21InftyOrderlambda2 &:= -\text{tinfy}11 \text{tinfy}23 - \text{tinfy}12 \text{tinfy}22 - \text{tinfy}13 \text{tinfy}21 \\ L21InftyOrderlambda1 &:= -h \text{tinfy}13 + \text{tinfy}10 \text{tinfy}13 - \text{tinfy}10 \text{tinfy}23 \\ &\quad - \text{tinfy}11 \text{tinfy}22 - \text{tinfy}12 \text{tinfy}21 \\ L21InftyOrderlambda0 &:= \frac{1}{-\text{tinfy}23 + \text{tinfy}13} (A12 h \text{tinfy}13 \text{tinfy}23 - A12 h \text{tinfy}23^2 \\ &\quad + A22 h \text{tinfy}13^2 - A22 h \text{tinfy}13 \text{tinfy}23 - h \text{tinfy}12 \text{tinfy}13 + h \text{tinfy}13 \text{tinfy}22 \\ &\quad + \text{tinfy}10 \text{tinfy}12 \text{tinfy}13 - \text{tinfy}10 \text{tinfy}12 \text{tinfy}23 - \text{tinfy}10 \text{tinfy}13 \text{tinfy}22 \\ &\quad + \text{tinfy}10 \text{tinfy}22 \text{tinfy}23 - \text{tinfy}11 \text{tinfy}13 \text{tinfy}21 + \text{tinfy}11 \text{tinfy}21 \text{tinfy}23) \end{aligned} \quad (1.1)$$

We conclude that  $L_{\{2,1\}}$  behaves at infinity like

$$-\text{tinfy}13 \text{tinfy}23 \lambda^4 - (\text{tinfy}22 \text{tinfy}13 + \text{tinfy}23 \text{tinfy}12) \lambda^3 - (\text{tinfy}21 \text{tinfy}13 + \text{tinfy}23 \text{tinfy}11 + \text{tinfy}22 \text{tinfy}12) \lambda^2 - (\text{tinfy}13 \text{tinfy}20 + \text{tinfy}21 \text{tinfy}12 + \text{tinfy}23 \text{tinfy}10 + \text{tinfy}22 \text{tinfy}11 + \text{tinfy}13 h) \lambda + O(1)$$

In other words:

$$L_{\{2,1\}} = -P_2(\lambda) + P_{\text{infty}02} + C - h \lambda \text{tinfy}13 + O(1)$$

```
> factor(simplify(L21InftyOrderlambda4*lambda^4+
L21InftyOrderlambda3*lambda^3+L21InftyOrderlambda2*lambda^2+
L21InftyOrderlambda1*lambda- (-P2(lambda)+Pinfy02-h*lambda*
tinfy13)));
```

$$0 \quad (1.2)$$

Shape of  $L_{\{2,2\}}$  at infinity.

```
> L22Infty:=factor(h*simplify(diff(WronskianLambdabisInfty,
lambda)/WronskianLambdabisInfty));
```

```

L22InftyOrderlambda5:=factor(-residue(L22Infty/lambda^6,lambda=
infinity));
L22InftyOrderlambda4:=factor(-residue(L22Infty/lambda^5,lambda=
infinity));
L22InftyOrderlambda3:=factor(-residue(L22Infty/lambda^4,lambda=
infinity));
L22InftyOrderlambda2:=factor(-residue(L22Infty/lambda^3,lambda=
infinity));
L22InftyOrderlambda1:=factor(-residue(L22Infty/lambda^2,lambda=
infinity));
L22InftyOrderlambda0:=factor(-residue(L22Infty/lambda^1,lambda=
infinity));
L22InftyOrderlambdaMinus1:=factor(-residue(L22Infty/lambda^0,
lambda=infinity));
L22InftyOrderlambdaMinus2:=factor(-residue(L22Infty/lambda^
(-1),lambda=infinity));

```

$$\begin{aligned}
L22InftyOrderlambda5 &:= 0 \\
L22InftyOrderlambda4 &:= 0 \\
L22InftyOrderlambda3 &:= 0 \\
L22InftyOrderlambda2 &:= -tinfty13 - tinfty23 \\
L22InftyOrderlambda1 &:= -tinfty12 - tinfty22 \\
L22InftyOrderlambda0 &:= -tinfty11 - tinfty21 \\
L22InftyOrderlambdaMinus1 &:= h
\end{aligned} \tag{1.3}$$

$$L22InftyOrderlambdaMinus2 := \frac{1}{-tinfty23 + tinfty13} (h (A12 tinfty13 - A12 tinfty23 + A22 tinfty13 - A22 tinfty23 - tinfty12 + tinfty22))$$

We deduce that  $L_{\{2,2\}}$  behaves at infinity like  $-(tinfty13+tinfty23)*lambda^2 -(tinfty12+tinfty22)*lambda +h/lambda +O(1/lambda^2) =P_1(lambda) +h/lambda+O(1/lambda^2)$

**Conclusion: Using the additional apparent singularities and the definition of the Darboux coordinates, we conclude that  $L_{\{2,2\}}= P_1(lambda) +h/(lambda-q)$ ,  $L_{\{2,1\}}=-P_2(lambda)+Pinfty02+C -h*lambda*tinfty13 -p*h/(lambda-q)$**

```

> L21Form:=-P2(lambda)+Pinfty02 -h*lambda*tinfty13- p*h/(lambda-
q);
L22Form:=P1(lambda) +h/(lambda-q);

```

$$L21Form := -(-tinfty10 tinfty13 + tinfty10 tinfty23 + tinfty11 tinfty22 + tinfty12 tinfty21) \lambda - (tinfty11 tinfty23 + tinfty12 tinfty22 + tinfty13 tinfty21) \lambda^2 - (tinfty12 tinfty23 + tinfty13 tinfty22) \lambda^3 - tinfty13 tinfty23 \lambda^4 - h \lambda tinfty13 - \frac{p h}{\lambda - q} \tag{1.4}$$

$$L22Form := -tinfty11 - tinfty21 + (-tinfty12 - tinfty22) \lambda + (-tinfty13 - tinfty23) \lambda^2 + \frac{h}{\lambda - q}$$

## Auxiliary matrix A

We define the operator  $\mathcal{L} = \hbar (\alpha_{13} \partial_{t^{\infty}\{1\},3} + \alpha_{23} \partial_{t^{\infty}\{2\},3} + \alpha_{12} \partial_{t^{\infty}\{1\},2} + \alpha_{22} \partial_{t^{\infty}\{2\},2} + \alpha_{11} \partial_{t^{\infty}\{1\},1} + \alpha_{21} \partial_{t^{\infty}\{2\},1})$

```
> WronskianLInfty:=factor(psi1Infty*Lpsi2Infty-psi2Infty*
Lpsi1Infty):
A12Infty:=factor(simplify(WronskianLInfty/WronskianLambdaInfty)
):
```

We test the alternative formulas for  $A_{\{1,2\}}$  and  $A_{\{1,1\}}$

```
> Y1Infty:=h*factor(dpsildlambdaInfty/psi1Infty):
Y2Infty:=h*factor(dpsi2dlambdaInfty/psi2Infty):
Z1Infty:=factor(Lpsi1Infty/psi1Infty):
Z2Infty:=factor(Lpsi2Infty/psi2Infty):
A12bisInfty:=factor(simplify((Z2Infty-Z1Infty)/(Y2Infty-
Y1Infty))):
A11Infty:=factor(simplify((Y2Infty*Z1Infty-Y1Infty*Z2Infty)/
(Y2Infty-Y1Infty))):
factor(simplify(A12bisInfty-A12Infty));
```

0

(2.1)

We define the coefficients of the operator  $\mathcal{L}$ .

```
> Ltinfty13:=h*alpha13:
Ltinfty23:=h*alpha23:
Ltinfty12:=h*alpha12:
Ltinfty22:=h*alpha22:
Ltinfty11:=h*alpha11:
Ltinfty21:=h*alpha21:
Ltinfty10:=0:
Ltinfty20:=0:
```

Asymptotic of  $A_{\{1,2\}}$  at infinity.

```
> A12InftyLambda3:=factor(-residue(A12Infty/lambda^4,lambda=
infinity));
A12InftyLambda2:=factor(-residue(A12Infty/lambda^3,lambda=
infinity));
A12InftyLambda1:=factor(-residue(A12Infty/lambda^2,lambda=
infinity));
A12InftyLambda0:=factor(-residue(A12Infty/lambda^1,lambda=
```

```

infinity) );
A12InftyLambdaMoins1:=factor(-residue(A12Infty/lambda^0,lambda=
infinity) );
A12InftyLambdaMoins2:=factor(-residue(A12Infty/lambda^(-1),
lambda=infinity) ) :

```

$$\begin{aligned} A12InftyLambda3 &:= 0 \\ A12InftyLambda2 &:= 0 \end{aligned} \quad (2.2)$$

$$\begin{aligned} A12InftyLambda1 &:= \frac{1}{3} \frac{\alpha3 - \alpha23}{-tinfy23 + tinfy13} \\ A12InftyLambda0 &:= \frac{1}{6} \frac{1}{(-tinfy23 + tinfy13)^2} (3 \alpha2 tinfy13 - 3 \alpha2 tinfy23 \\ &\quad - 2 \alpha3 tinfy12 + 2 \alpha3 tinfy22 - 3 \alpha22 tinfy13 + 3 \alpha22 tinfy23 + 2 \alpha23 tinfy12 \\ &\quad - 2 \alpha23 tinfy22) \\ A12InftyLambdaMoins1 &:= \frac{1}{6} \frac{1}{(-tinfy23 + tinfy13)^3} (6 \alpha1 tinfy13^2 \\ &\quad - 12 \alpha1 tinfy13 tinfy23 + 6 \alpha1 tinfy23^2 - 3 \alpha2 tinfy12 tinfy13 \\ &\quad + 3 \alpha2 tinfy12 tinfy23 + 3 \alpha2 tinfy13 tinfy22 - 3 \alpha2 tinfy22 tinfy23 \\ &\quad - 2 \alpha3 tinfy11 tinfy13 + 2 \alpha3 tinfy11 tinfy23 + 2 \alpha3 tinfy12^2 \\ &\quad - 4 \alpha3 tinfy12 tinfy22 + 2 \alpha3 tinfy13 tinfy21 - 2 \alpha3 tinfy21 tinfy23 \\ &\quad + 2 \alpha3 tinfy22^2 - 6 \alpha21 tinfy13^2 + 12 \alpha21 tinfy13 tinfy23 - 6 \alpha21 tinfy23^2 \\ &\quad + 3 \alpha22 tinfy12 tinfy13 - 3 \alpha22 tinfy12 tinfy23 - 3 \alpha22 tinfy13 tinfy22 \\ &\quad + 3 \alpha22 tinfy22 tinfy23 + 2 \alpha23 tinfy11 tinfy13 - 2 \alpha23 tinfy11 tinfy23 \\ &\quad - 2 \alpha23 tinfy12^2 + 4 \alpha23 tinfy12 tinfy22 - 2 \alpha23 tinfy13 tinfy21 \\ &\quad + 2 \alpha23 tinfy21 tinfy23 - 2 \alpha23 tinfy22^2) \end{aligned}$$

We thus deduce that  $A_{\{1,2\}} = (\alpha_{13} - \alpha_{23})/3 / (tinfy_{13} - tinfy_{23}) * \lambda + \nu + \mu / (\lambda - q)$ . Expressions of  $(\mu, \nu)$  are obtained below.

```

> A12Form := (alpha13 - alpha23) / 3 / (-tinfy23 + tinfy13) * lambda + nu +
mu / (lambda - q) ;
simplify(-residue(A12Form/lambda^2, lambda=infinity) -
A12InftyLambda1) ;
solve({factor(-residue(A12Form/lambda, lambda=infinity)) =
A12InftyLambda0, factor(-residue(A12Form, lambda=infinity)) =
factor(A12InftyLambdaMoins1)}, {mu, nu}) ;

```

$$\begin{aligned} \mu &:= -1/6 * (-2 * \alpha_{23} * tinfy_{21} * tinfy_{23} + 3 * \alpha_{22} * tinfy_{13} * \\ &\quad tinfy_{22} + 3 * \alpha_{22} * tinfy_{23} * tinfy_{12} - 3 * \alpha_{22} * tinfy_{23} * \\ &\quad tinfy_{22} - 12 * \alpha_{21} * tinfy_{13} * tinfy_{23} + 12 * \alpha_{11} * tinfy_{13} * \\ &\quad tinfy_{23} + 2 * \alpha_{13} * tinfy_{11} * tinfy_{13} - 2 * \alpha_{13} * tinfy_{11} * \\ &\quad tinfy_{23} - 2 * \alpha_{13} * tinfy_{21} * tinfy_{13} + 2 * \alpha_{13} * tinfy_{21} * \\ &\quad tinfy_{23} - 2 * \alpha_{23} * tinfy_{11} * tinfy_{13} + 2 * \alpha_{23} * tinfy_{11} * \\ &\quad tinfy_{23} + 2 * \alpha_{23} * tinfy_{21} * tinfy_{13} + 4 * \alpha_{13} * tinfy_{22} * \end{aligned}$$

```

tinfty12-4*alpha23*tinfty22*tinfty12+3*alpha12*tinfty13*
tinfty12-3*alpha12*tinfty13*tinfty22-3*alpha12*tinfty23*
tinfty12+3*alpha12*tinfty23*tinfty22-3*alpha22*tinfty13*
tinfty12-6*alpha11*tinfty23^2-2*alpha13*tinfty12^2-2*alpha13*
tinfty22^2+2*alpha23*tinfty12^2+2*alpha23*tinfty22^2+6*alpha21*
tinfty13^2+6*alpha21*tinfty23^2-6*alpha11*tinfty13^2) / (-
tinfty23^3+3*tinfty13*tinfty23^2-3*tinfty23*tinfty13^2+
tinfty13^3) ;
nu := 1/6*(3*alpha22*tinfty23-3*alpha22*tinfty13-3*alpha12*
tinfty23+3*alpha12*tinfty13-2*alpha23*tinfty22+2*alpha23*
tinfty12+2*alpha13*tinfty22-2*alpha13*tinfty12) / (tinfty23^2-2*
tinfty13*tinfty23+tinfty13^2) ;

```

$$A12Form := \frac{1}{3} \frac{(\alpha_{13} - \alpha_{23}) \lambda}{-tinfty23 + tinfty13} + v + \frac{\mu}{\lambda - q} \quad (2.3)$$

$$\left\{ \begin{aligned} \mu = & \frac{1}{6} (6 \alpha_{11} tinfty13^2 - 12 \alpha_{11} tinfty13 tinfty23 + 6 \alpha_{11} tinfty23^2 \\ & - 3 \alpha_{12} tinfty12 tinfty13 + 3 \alpha_{12} tinfty12 tinfty23 + 3 \alpha_{12} tinfty13 tinfty22 \\ & - 3 \alpha_{12} tinfty22 tinfty23 - 2 \alpha_{13} tinfty11 tinfty13 + 2 \alpha_{13} tinfty11 tinfty23 \\ & + 2 \alpha_{13} tinfty12^2 - 4 \alpha_{13} tinfty12 tinfty22 + 2 \alpha_{13} tinfty13 tinfty21 \\ & - 2 \alpha_{13} tinfty21 tinfty23 + 2 \alpha_{13} tinfty22^2 - 6 \alpha_{21} tinfty13^2 + 12 \alpha_{21} tinfty13 tinfty23 \\ & - 6 \alpha_{21} tinfty23^2 + 3 \alpha_{22} tinfty12 tinfty13 - 3 \alpha_{22} tinfty12 tinfty23 \\ & - 3 \alpha_{22} tinfty13 tinfty22 + 3 \alpha_{22} tinfty22 tinfty23 + 2 \alpha_{23} tinfty11 tinfty13 \\ & - 2 \alpha_{23} tinfty11 tinfty23 - 2 \alpha_{23} tinfty12^2 + 4 \alpha_{23} tinfty12 tinfty22 \\ & - 2 \alpha_{23} tinfty13 tinfty21 + 2 \alpha_{23} tinfty21 tinfty23 - 2 \alpha_{23} tinfty22^2) / (tinfty13^3 \\ & - 3 tinfty13^2 tinfty23 + 3 tinfty13 tinfty23^2 - tinfty23^3), v \\ = & \frac{1}{6} \frac{1}{tinfty13^2 - 2 tinfty13 tinfty23 + tinfty23^2} (3 \alpha_{12} tinfty13 - 3 \alpha_{12} tinfty23 \\ & - 2 \alpha_{13} tinfty12 + 2 \alpha_{13} tinfty22 - 3 \alpha_{22} tinfty13 + 3 \alpha_{22} tinfty23 + 2 \alpha_{23} tinfty12 \\ & - 2 \alpha_{23} tinfty22) \end{aligned} \right\}$$



$$\mu := -\frac{1}{6} \left( -6 \alpha_1 t_{13}^2 + 12 \alpha_1 t_{13} t_{23} - 6 \alpha_1 t_{23}^2 \right. \\
+ 3 \alpha_2 t_{12} t_{13} - 3 \alpha_2 t_{12} t_{23} - 3 \alpha_2 t_{13} t_{22} \\
+ 3 \alpha_2 t_{22} t_{23} + 2 \alpha_3 t_{11} t_{13} - 2 \alpha_3 t_{11} t_{23} \\
- 2 \alpha_3 t_{12}^2 + 4 \alpha_3 t_{12} t_{22} - 2 \alpha_3 t_{13} t_{21} \\
+ 2 \alpha_3 t_{21} t_{23} - 2 \alpha_3 t_{22}^2 + 6 \alpha_2 t_{13}^2 - 12 \alpha_2 t_{13} t_{23} \\
+ 6 \alpha_2 t_{23}^2 - 3 \alpha_2 t_{12} t_{13} + 3 \alpha_2 t_{12} t_{23} \\
+ 3 \alpha_2 t_{13} t_{22} - 3 \alpha_2 t_{22} t_{23} - 2 \alpha_3 t_{11} t_{13} \\
+ 2 \alpha_3 t_{11} t_{23} + 2 \alpha_3 t_{12}^2 - 4 \alpha_3 t_{12} t_{22} \\
+ 2 \alpha_3 t_{13} t_{21} - 2 \alpha_3 t_{21} t_{23} + 2 \alpha_3 t_{22}^2 \left. \right) / (t_{13}^3 \\
- 3 t_{13}^2 t_{23} + 3 t_{13} t_{23}^2 - t_{23}^3) \\
\nu := \frac{1}{6} \frac{1}{t_{13}^2 - 2 t_{13} t_{23} + t_{23}^2} (3 \alpha_2 t_{13} - 3 \alpha_2 t_{23} \\
- 2 \alpha_3 t_{12} + 2 \alpha_3 t_{22} - 3 \alpha_2 t_{13} + 3 \alpha_2 t_{23} + 2 \alpha_3 t_{12} \\
- 2 \alpha_3 t_{22})$$

Study of  $A_{\{1,1\}}$  at infinity

```
> AllInftyLambda4:=factor(-residue(AllInfty/lambda^5,lambda=
infinity));
AllInftyLambda3:=factor(-residue(AllInfty/lambda^4,lambda=
infinity));
AllInftyLambda2:=factor(-residue(AllInfty/lambda^3,lambda=
infinity));
AllInftyLambda1:=factor(-residue(AllInfty/lambda^2,lambda=
infinity));
AllInftyLambda0:=factor(-residue(AllInfty/lambda^1,lambda=
infinity));
AllInftyLambdaMoins1:=factor(-residue(AllInfty/lambda^0,lambda=
infinity));
```

$$AllInftyLambda4 := 0 \tag{2.4}$$

$$AllInftyLambda3 := \frac{1}{3} \frac{\alpha_3 t_{23} - \alpha_3 t_{13}}{-t_{23} + t_{13}}$$

$$AllInftyLambda2 := \frac{1}{6} \frac{1}{(-t_{23} + t_{13})^2} (3 \alpha_2 t_{13} t_{23} - 3 \alpha_2 t_{23}^2 \\
- 2 \alpha_3 t_{12} t_{23} + 2 \alpha_3 t_{13} t_{22} - 3 \alpha_2 t_{13}^2 \\
+ 3 \alpha_2 t_{13} t_{23} + 2 \alpha_3 t_{12} t_{23} - 2 \alpha_3 t_{13} t_{22})$$

$$AllInftyLambda1 := \frac{1}{6} \frac{1}{(-t_{23} + t_{13})^3} (6 \alpha_1 t_{13}^2 t_{23} \\
- 12 \alpha_1 t_{13} t_{23}^2 + 6 \alpha_1 t_{23}^3 - 3 \alpha_2 t_{12} t_{13} t_{23} \\
+ 3 \alpha_2 t_{12} t_{23}^2 + 3 \alpha_2 t_{13}^2 t_{22} - 3 \alpha_2 t_{13} t_{22} t_{23} \\
- 2 \alpha_3 t_{11} t_{13} t_{23} + 2 \alpha_3 t_{11} t_{23}^2 + 2 \alpha_3 t_{12}^2 t_{23} \\
- 2 \alpha_3 t_{12} t_{13} t_{22} - 2 \alpha_3 t_{12} t_{22} t_{23})$$

$$\begin{aligned}
& + 2 \alpha_{13} \tau_{13}^2 \tau_{21} - 2 \alpha_{13} \tau_{13} \tau_{21} \tau_{23} + 2 \alpha_{13} \tau_{13} \tau_{22}^2 \\
& - 6 \alpha_{21} \tau_{13}^3 + 12 \alpha_{21} \tau_{13}^2 \tau_{23} - 6 \alpha_{21} \tau_{13} \tau_{23}^2 \\
& + 3 \alpha_{22} \tau_{12} \tau_{13} \tau_{23} - 3 \alpha_{22} \tau_{12} \tau_{23}^2 - 3 \alpha_{22} \tau_{13}^2 \tau_{22} \\
& + 3 \alpha_{22} \tau_{13} \tau_{22} \tau_{23} + 2 \alpha_{23} \tau_{11} \tau_{13} \tau_{23} \\
& - 2 \alpha_{23} \tau_{11} \tau_{23}^2 - 2 \alpha_{23} \tau_{12}^2 \tau_{23} + 2 \alpha_{23} \tau_{12} \tau_{13} \tau_{22} \\
& + 2 \alpha_{23} \tau_{12} \tau_{22} \tau_{23} - 2 \alpha_{23} \tau_{13}^2 \tau_{21} \\
& + 2 \alpha_{23} \tau_{13} \tau_{21} \tau_{23} - 2 \alpha_{23} \tau_{13} \tau_{22}^2)
\end{aligned}$$

We deduce that  $A_{\{1,1\}} = 1/3 * (\alpha_{13} * \tau_{23} - \alpha_{23} * \tau_{13}) / (\tau_{13} - \tau_{23}) * \lambda^3 + c_2 * \lambda^2 + c_1 * \lambda + c_0 + \rho / (\lambda - q)$ . Expressions of  $(c_1, c_2)$  are obtained below.

**> AllForm:=1/3\*(alpha13\*tau23-alpha23\*tau13)/(tau13-tau23)\*lambda^3+c2\*lambda^2+c1\*lambda+c0+rho/(lambda-q);**

**simplify(-residue(AllForm/lambda^4,lambda=infinity)-AllInftyLambda3);**

**solve({factor(-residue(AllForm/lambda^3,lambda=infinity))=AllInftyLambda2,factor(-residue(AllForm/lambda^2,lambda=infinity))=AllInftyLambda1},{c2,c1});**

$$AllForm := \frac{1}{3} \frac{(\alpha_{13} \tau_{23} - \alpha_{23} \tau_{13}) \lambda^3}{-\tau_{23} + \tau_{13}} + c_2 \lambda^2 + c_1 \lambda + c_0 + \frac{\rho}{\lambda - q} \quad (2.5)$$

$$\left\{ c_1 = \frac{1}{6} (6 \alpha_{11} \tau_{13}^2 \tau_{23} - 12 \alpha_{11} \tau_{13} \tau_{23}^2 + 6 \alpha_{11} \tau_{23}^3 \right.$$

$$- 3 \alpha_{12} \tau_{12} \tau_{13} \tau_{23} + 3 \alpha_{12} \tau_{12} \tau_{23}^2 + 3 \alpha_{12} \tau_{13}^2 \tau_{22}$$

$$- 3 \alpha_{12} \tau_{13} \tau_{22} \tau_{23} - 2 \alpha_{13} \tau_{11} \tau_{13} \tau_{23}$$

$$+ 2 \alpha_{13} \tau_{11} \tau_{23}^2 + 2 \alpha_{13} \tau_{12}^2 \tau_{23} - 2 \alpha_{13} \tau_{12} \tau_{13} \tau_{22}$$

$$- 2 \alpha_{13} \tau_{12} \tau_{22} \tau_{23} + 2 \alpha_{13} \tau_{13}^2 \tau_{21}$$

$$- 2 \alpha_{13} \tau_{13} \tau_{21} \tau_{23} + 2 \alpha_{13} \tau_{13} \tau_{22}^2 - 6 \alpha_{21} \tau_{13}^3$$

$$+ 12 \alpha_{21} \tau_{13}^2 \tau_{23} - 6 \alpha_{21} \tau_{13} \tau_{23}^2 + 3 \alpha_{22} \tau_{12} \tau_{13} \tau_{23}$$

$$- 3 \alpha_{22} \tau_{12} \tau_{23}^2 - 3 \alpha_{22} \tau_{13}^2 \tau_{22} + 3 \alpha_{22} \tau_{13} \tau_{22} \tau_{23}$$

$$+ 2 \alpha_{23} \tau_{11} \tau_{13} \tau_{23} - 2 \alpha_{23} \tau_{11} \tau_{23}^2 - 2 \alpha_{23} \tau_{12}^2 \tau_{23}$$

$$+ 2 \alpha_{23} \tau_{12} \tau_{13} \tau_{22} + 2 \alpha_{23} \tau_{12} \tau_{22} \tau_{23}$$

$$\begin{aligned}
& -2 \alpha_3 \text{tinfty}13^2 \text{tinfty}21 + 2 \alpha_3 \text{tinfty}13 \text{tinfty}21 \text{tinfty}23 - 2 \alpha_3 \text{tinfty}13 \text{tinfty}22^2) / \\
& (\text{tinfty}13^3 - 3 \text{tinfty}13^2 \text{tinfty}23 + 3 \text{tinfty}13 \text{tinfty}23^2 - \text{tinfty}23^3), c2 \\
& = \frac{1}{6} \frac{1}{\text{tinfty}13^2 - 2 \text{tinfty}13 \text{tinfty}23 + \text{tinfty}23^2} (3 \alpha_2 \text{tinfty}13 \text{tinfty}23 - 3 \alpha_2 \text{tinfty}23^2 \\
& - 2 \alpha_3 \text{tinfty}12 \text{tinfty}23 + 2 \alpha_3 \text{tinfty}13 \text{tinfty}22 - 3 \alpha_2 \text{tinfty}13^2 \\
& + 3 \alpha_2 \text{tinfty}13 \text{tinfty}23 + 2 \alpha_3 \text{tinfty}12 \text{tinfty}23 - 2 \alpha_3 \text{tinfty}13 \text{tinfty}22) \}
\end{aligned}$$

```

> c1 := factor(-(2*alpha13*tinfty11*tinfty13*tinfty23-2*alpha13*
tinfty11*tinfty23^2-2*alpha13*tinfty12^2*tinfty23+2*alpha13*
tinfty12*tinfty13*tinfty22+2*alpha13*tinfty12*tinfty22*tinfty23
-2*alpha13*tinfty13^2*tinfty21+2*alpha13*tinfty13*tinfty21*
tinfty23-2*alpha13*tinfty13*tinfty22^2-2*alpha23*tinfty11*
tinfty13*tinfty23+2*alpha23*tinfty11*tinfty23^2+2*alpha23*
tinfty12^2*tinfty23-2*alpha23*tinfty12*tinfty13*tinfty22-2*
alpha23*tinfty12*tinfty22*tinfty23+2*alpha23*tinfty13^2*
tinfty21-2*alpha23*tinfty13*tinfty21*tinfty23+2*alpha23*
tinfty13*tinfty22^2+3*alpha12*tinfty12*tinfty13*tinfty23-3*
alpha12*tinfty12*tinfty23^2-3*alpha12*tinfty13^2*tinfty22+3*
alpha12*tinfty13*tinfty22*tinfty23-3*alpha22*tinfty12*tinfty13*
tinfty23+3*alpha22*tinfty12*tinfty23^2+3*alpha22*tinfty13^2*
tinfty22-3*alpha22*tinfty13*tinfty22*tinfty23-6*alpha11*
tinfty13^2*tinfty23+12*alpha11*tinfty13*tinfty23^2-6*alpha11*
tinfty23^3+6*alpha21*tinfty13^3-12*alpha21*tinfty13^2*
tinfty23+6*alpha21*tinfty13*tinfty23^2) / (6*(tinfty13^3-3*
tinfty13^2*tinfty23+3*tinfty13*tinfty23^2-tinfty23^3)));
c2 := factor(-(2*alpha13*tinfty12*tinfty23-2*alpha13*tinfty13*
tinfty22-2*alpha23*tinfty12*tinfty23+2*alpha23*tinfty13*
tinfty22-3*alpha12*tinfty13*tinfty23+3*alpha12*tinfty23^2+3*
alpha22*tinfty13^2-3*alpha22*tinfty13*tinfty23) / (6*(tinfty13^2
-2*tinfty13*tinfty23+tinfty23^2)));

```

$$\begin{aligned}
c1 := \frac{1}{6} \frac{1}{(-\text{tinfty}23 + \text{tinfty}13)^3} & (6 \alpha_1 \text{tinfty}13^2 \text{tinfty}23 - 12 \alpha_1 \text{tinfty}13 \text{tinfty}23^2 & (2.6) \\
& + 6 \alpha_1 \text{tinfty}23^3 - 3 \alpha_2 \text{tinfty}12 \text{tinfty}13 \text{tinfty}23 + 3 \alpha_2 \text{tinfty}12 \text{tinfty}23^2 \\
& + 3 \alpha_2 \text{tinfty}13^2 \text{tinfty}22 - 3 \alpha_2 \text{tinfty}13 \text{tinfty}22 \text{tinfty}23 \\
& - 2 \alpha_3 \text{tinfty}11 \text{tinfty}13 \text{tinfty}23 + 2 \alpha_3 \text{tinfty}11 \text{tinfty}23^2 + 2 \alpha_3 \text{tinfty}12^2 \text{tinfty}23 \\
& - 2 \alpha_3 \text{tinfty}12 \text{tinfty}13 \text{tinfty}22 - 2 \alpha_3 \text{tinfty}12 \text{tinfty}22 \text{tinfty}23 \\
& + 2 \alpha_3 \text{tinfty}13^2 \text{tinfty}21 - 2 \alpha_3 \text{tinfty}13 \text{tinfty}21 \text{tinfty}23 + 2 \alpha_3 \text{tinfty}13 \text{tinfty}22^2 \\
& - 6 \alpha_2 \text{tinfty}13^3 + 12 \alpha_2 \text{tinfty}13^2 \text{tinfty}23 - 6 \alpha_2 \text{tinfty}13 \text{tinfty}23^2 \\
& + 3 \alpha_2 \text{tinfty}12 \text{tinfty}13 \text{tinfty}23 - 3 \alpha_2 \text{tinfty}12 \text{tinfty}23^2 - 3 \alpha_2 \text{tinfty}13^2 \text{tinfty}22 \\
& + 3 \alpha_2 \text{tinfty}13 \text{tinfty}22 \text{tinfty}23 + 2 \alpha_3 \text{tinfty}11 \text{tinfty}13 \text{tinfty}23
\end{aligned}$$

$$\begin{aligned}
& -2 \alpha_{23} \text{tiny}11 \text{tiny}23^2 - 2 \alpha_{23} \text{tiny}12^2 \text{tiny}23 + 2 \alpha_{23} \text{tiny}12 \text{tiny}13 \text{tiny}22 \\
& + 2 \alpha_{23} \text{tiny}12 \text{tiny}22 \text{tiny}23 - 2 \alpha_{23} \text{tiny}13^2 \text{tiny}21 \\
& + 2 \alpha_{23} \text{tiny}13 \text{tiny}21 \text{tiny}23 - 2 \alpha_{23} \text{tiny}13 \text{tiny}22^2) \\
c2 := & \frac{1}{6} \frac{1}{(-\text{tiny}23 + \text{tiny}13)^2} (3 \alpha_{12} \text{tiny}13 \text{tiny}23 - 3 \alpha_{12} \text{tiny}23^2 \\
& - 2 \alpha_{13} \text{tiny}12 \text{tiny}23 + 2 \alpha_{13} \text{tiny}13 \text{tiny}22 - 3 \alpha_{22} \text{tiny}13^2 \\
& + 3 \alpha_{22} \text{tiny}13 \text{tiny}23 + 2 \alpha_{23} \text{tiny}12 \text{tiny}23 - 2 \alpha_{23} \text{tiny}13 \text{tiny}22)
\end{aligned}$$

> ?