In this Maple sheet, we compute the Lax matrices using the asymptotics of the wave functions and the local diagonalization for the Painlevé 3 equation.

We first use the expression of the coefficients of the spectral curve in terms of the irregular times and monodromies.

```
> restart:
tinfty20:=-tinfty10:
Pinfty42 := tinfty13*tinfty23;
Pinfty32 := tinfty12*tinfty23+tinfty13*tinfty22;
Pinfty22 := tinfty12*tinfty22+tinfty13*tinfty21+tinfty11*
tinfty23;
Pinfty12 := tinfty20*tinfty13+tinfty12*tinfty21+tinfty10*
tinfty23+tinfty11*tinfty22;
Pinfty01 := -tinfty11-tinfty21;
Pinfty11 := -tinfty12-tinfty22;
Pinfty21 := -tinfty13-tinfty23;
P1:=x-> Pinfty01+Pinfty11*x+Pinfty21*x^2:
P2:=x-> Pinfty02+Pinfty12*x+Pinfty22*x^2+Pinfty32*x^3+Pinfty42*
x^4:
CoherenceEquation1 :=tinfty10+tinfty20+t010+t020;
```

$$\begin{gathered} Pinfty42 := tinfty13\, tinfty23 \\ Pinfty32 := tinfty12\, tinfty23 + tinfty13\, tinfty22 \\ Pinfty22 := tinfty11\, tinfty23 + tinfty12\, tinfty22 + tinfty13\, tinfty21 \\ Pinfty12 := -tinfty10\, tinfty13 + tinfty10\, tinfty23 + tinfty11\, tinfty22 + tinfty12\, tinfty21 \\ Pinfty01 := -tinfty11 - tinfty21 \\ Pinfty11 := -tinfty12 - tinfty22 \\ Pinfty21 := -tinfty13 - tinfty23 \\ CoherenceEquation1 := t010 + t020 \end{gathered} \tag{1}$$

Expression of the Lax matrix L

Study of the asymptotics at infinity

```
> logPsi1Infty:=-tinfty11/h*lambda-tinfty10/h*ln(lambda)+A10-A12/
(2-1)/lambda^(2-1)-A13/(3-1)/lambda^(3-1)-A14/(4-1)/lambda^(4
-1)-A15/(5-1)/lambda^(5-1)-A16/(6-1)/lambda^(6-1)-A17/(7-1)
/lambda^(7-1)   ;
logPsi2Infty:=-tinfty21/h*lambda-tinfty20/h*ln(lambda)+A20-A22/
(2-1)/lambda^(2-1)-A23/(3-1)/lambda^(3-1)-A24/(4-1)/lambda^(4
-1)-A25/(5-1)/lambda^(5-1)-A26/(6-1)/lambda^(6-1)-A27/(7-1)
/lambda^(7-1)   ;
Llogpsi1Infty:=-Ltinfty11/h*lambda-Ltinfty10/h*ln(lambda)+LA10-
LA12/(2-1)/lambda^(2-1)-LA13/(3-1)/lambda^(3-1)-LA14/(4-1)
/lambda^(4-1)-LA15/(5-1)/lambda^(5-1)-LA16/(6-1)/lambda^(6-1)-
LA17/(7-1)/lambda^(7-1)   ;
Llogpsi2Infty:=-Ltinfty21/h*lambda-Ltinfty20/h*ln(lambda)+LA20-
```

```
LA22/(2-1)/lambda^(2-1)-LA23/(3-1)/lambda^(3-1)-LA24/(4-1)
/lambda^(4-1)-LA25/(5-1)/lambda^(5-1)-LA26/(6-1)/lambda^(6-1)-
LA27/(7-1)/lambda^(7-1)   ;
Lpsi1Infty := exp(1/h*(-tinfty11*lambda-tinfty10*ln(lambda)+h*
A10-h*A12/lambda-1/2*h*A13/lambda^2-1/3*h*A14/lambda^3-1/4*h*
A15/lambda^4-1/5*h*A16/lambda^5-1/6*h*A17/lambda^6))*1/h*(-
Ltinfty11*lambda-Ltinfty10*ln(lambda)+h*LA10-h*LA12/lambda-1/2*
h*LA13/lambda^2-1/3*h*LA14/lambda^3-1/4*h*LA15/lambda^4-1/5*h*
LA16/lambda^5-1/6*h*LA17/lambda^6);
Lpsi2Infty := exp(1/h*(-tinfty21*lambda-tinfty20*ln(lambda)+h*
A20-h*A22/lambda-1/2*h*A23/lambda^2-1/3*h*A24/lambda^3-1/4*h*
A25/lambda^4-1/5*h*A26/lambda^5-1/6*h*A27/lambda^6))*1/h*(-
Ltinfty21*lambda-Ltinfty20*ln(lambda)+h*LA20-h*LA22/lambda-1/2*
h*LA23/lambda^2-1/3*h*LA24/lambda^3-1/4*h*LA25/lambda^4-1/5*h*
LA26/lambda^5-1/6*h*LA27/lambda^6);
psi1Infty:=exp(logPsi1Infty);
psi2Infty:=exp(logPsi2Infty);
dpsi1dlambdaInfty:=diff(psi1Infty,lambda):
dpsi2dlambdaInfty:=diff(psi2Infty,lambda):
d2psi1dlambda2Infty:=diff(psi1Infty,lambda$2):
d2psi2dlambda2Infty:=diff(psi2Infty,lambda$2):
Vinfty1:=tinfty11*lambda+tinfty10*ln(lambda);
Vinfty2:=tinfty21*lambda+tinfty20*ln(lambda);


WronskianLambdaInfty:=h*factor(psi1Infty*dpsi2dlambdaInfty-
psi2Infty*dpsi1dlambdaInfty):
WronskianLambdabisInfty:=h*simplify(factor( (diff(logPsi2Infty,
lambda)-diff(logPsi1Infty,lambda))*exp(logPsi1Infty+
logPsi2Infty))):

WronskianTildeLambdaInfty:=h^3*factor(dpsi2dlambdaInfty*
d2psi1dlambda2Infty-dpsi1dlambdaInfty*d2psi2dlambda2Infty):
```

$$logPsi1Infty := -\frac{tinfty11\,\lambda}{h} - \frac{tinfty10\ln(\lambda)}{h} + A10 - \frac{A12}{\lambda} - \frac{1}{2}\frac{A13}{\lambda^2} - \frac{1}{3}\frac{A14}{\lambda^3} - \frac{1}{4}\frac{A15}{\lambda^4} - \frac{1}{5}\frac{A16}{\lambda^5} - \frac{1}{6}\frac{A17}{\lambda^6} \qquad (1.1)$$

$$logPsi2Infty := -\frac{tinfty21\,\lambda}{h} + \frac{tinfty10\ln(\lambda)}{h} + A20 - \frac{A22}{\lambda} - \frac{1}{2}\frac{A23}{\lambda^2} - \frac{1}{3}\frac{A24}{\lambda^3}$$

$$-\frac{1}{4}\frac{A25}{\lambda^4}-\frac{1}{5}\frac{A26}{\lambda^5}-\frac{1}{6}\frac{A27}{\lambda^6}$$

$$Llogpsi1Infty := -\frac{Ltinfty11\,\lambda}{h}-\frac{Ltinfty10\ln(\lambda)}{h}+LA10-\frac{LA12}{\lambda}-\frac{1}{2}\frac{LA13}{\lambda^2}-\frac{1}{3}\frac{LA14}{\lambda^3}-\frac{1}{4}\frac{LA15}{\lambda^4}-\frac{1}{5}\frac{LA16}{\lambda^5}-\frac{1}{6}\frac{LA17}{\lambda^6}$$

$$Llogpsi2Infty := -\frac{Ltinfty21\,\lambda}{h}-\frac{Ltinfty20\ln(\lambda)}{h}+LA20-\frac{LA22}{\lambda}-\frac{1}{2}\frac{LA23}{\lambda^2}-\frac{1}{3}\frac{LA24}{\lambda^3}-\frac{1}{4}\frac{LA25}{\lambda^4}-\frac{1}{5}\frac{LA26}{\lambda^5}-\frac{1}{6}\frac{LA27}{\lambda^6}$$

$$Lpsi1Infty := 1\Bigg/\left(h\,\mathrm{e}^{\frac{-tinfty11\,\lambda-tinfty10\ln(\lambda)+h\,A10-\frac{h\,A12}{\lambda}-\frac{1}{2}\frac{h\,A13}{\lambda^2}-\frac{1}{3}\frac{h\,A14}{\lambda^3}-\frac{1}{4}\frac{h\,A15}{\lambda^4}-\frac{1}{5}\frac{h\,A16}{\lambda^5}-\frac{1}{6}\frac{h\,A17}{\lambda^6}}{h}}\left(-Ltinfty11\,\lambda-Ltinfty10\ln(\lambda)+h\,LA10-\frac{h\,LA12}{\lambda}-\frac{1}{2}\frac{h\,LA13}{\lambda^2}-\frac{1}{3}\frac{h\,LA14}{\lambda^3}-\frac{1}{4}\frac{h\,LA15}{\lambda^4}-\frac{1}{5}\frac{h\,LA16}{\lambda^5}-\frac{1}{6}\frac{h\,LA17}{\lambda^6}\right)\right)$$

$$Lpsi2Infty := 1\Bigg/\left(h\,\mathrm{e}^{\frac{-tinfty21\,\lambda+tinfty10\ln(\lambda)+h\,A20-\frac{h\,A22}{\lambda}-\frac{1}{2}\frac{h\,A23}{\lambda^2}-\frac{1}{3}\frac{h\,A24}{\lambda^3}-\frac{1}{4}\frac{h\,A25}{\lambda^4}-\frac{1}{5}\frac{h\,A26}{\lambda^5}-\frac{1}{6}\frac{h\,A27}{\lambda^6}}{h}}\left(-Ltinfty21\,\lambda-Ltinfty20\ln(\lambda)+h\,LA20-\frac{h\,LA22}{\lambda}-\frac{1}{2}\frac{h\,LA23}{\lambda^2}-\frac{1}{3}\frac{h\,LA24}{\lambda^3}-\frac{1}{4}\frac{h\,LA25}{\lambda^4}-\frac{1}{5}\frac{h\,LA26}{\lambda^5}-\frac{1}{6}\frac{h\,LA27}{\lambda^6}\right)\right)$$

$$psi1Infty := \mathrm{e}^{-\frac{tinfty11\,\lambda}{h}-\frac{tinfty10\ln(\lambda)}{h}+A10-\frac{A12}{\lambda}-\frac{1}{2}\frac{A13}{\lambda^2}-\frac{1}{3}\frac{A14}{\lambda^3}-\frac{1}{4}\frac{A15}{\lambda^4}-\frac{1}{5}\frac{A16}{\lambda^5}-\frac{1}{6}\frac{A17}{\lambda^6}}$$

$$psi2Infty := \mathrm{e}^{-\frac{tinfty21\,\lambda}{h}+\frac{tinfty10\ln(\lambda)}{h}+A20-\frac{A22}{\lambda}-\frac{1}{2}\frac{A23}{\lambda^2}-\frac{1}{3}\frac{A24}{\lambda^3}-\frac{1}{4}\frac{A25}{\lambda^4}-\frac{1}{5}\frac{A26}{\lambda^5}-\frac{1}{6}\frac{A27}{\lambda^6}}$$

$$Vinfty1 := tinfty11\,\lambda+tinfty10\ln(\lambda)$$

$$Vinfty2 := tinfty21\,\lambda - tinfty10\ln(\lambda)$$

```
> L21Infty:=factor(simplify
  (WronskianTildeLambdaInfty/WronskianLambdabisInfty)):
  L21InftyOrderlambda3:=factor(-residue(L21Infty/lambda^4,lambda=
  infinity));
  L21InftyOrderlambda2:=factor(-residue(L21Infty/lambda^3,lambda=
  infinity));
  L21InftyOrderlambda1:=factor(-residue(L21Infty/lambda^2,lambda=
  infinity));
  L21InftyOrderlambda0:=factor(-residue(L21Infty/lambda^1,lambda=
  infinity));
  L21InftyOrderlambdaMinus1:=factor(-residue(L21Infty/lambda^0,
  lambda=infinity));
  L21InftyOrderlambdaMinus2:=factor(-residue(series
  (L21InftyNumer/L21InftyDenom,lambda=infinity,12)/lambda^(-1),
  lambda=infinity));
  L21InftyOrderlambdaMinus3:=factor(-residue(series
  (L21InftyNumer/L21InftyDenom,lambda=infinity,12)/lambda^(-2),
  lambda=infinity));
```

$$L21InftyOrderlambda3 := 0 \qquad \textbf{(1.2)}$$
$$L21InftyOrderlambda2 := 0$$
$$L21InftyOrderlambda1 := 0$$
$$L21InftyOrderlambda0 := -tinfty11\,tinfty21$$
$$L21InftyOrderlambdaMinus1 := (tinfty11 - tinfty21)\,tinfty10$$
$$L21InftyOrderlambdaMinus2 := 0$$
$$L21InftyOrderlambdaMinus3 := 0$$

We deduce that L_{2,1} behaves at infinity like -tinfty11*tinfty21-(tinfty11*tinfty20+tinfty21*tinfty10)/lambda +O(1/lambda^2)

```
> L22Infty:=factor(h*simplify(diff(WronskianLambdabisInfty,
  lambda)/WronskianLambdabisInfty)):
  L22InftyOrderlambda3:=factor(-residue(L22Infty/lambda^4,lambda=
  infinity));
  L22InftyOrderlambda2:=factor(-residue(L22Infty/lambda^3,lambda=
  infinity));
  L22InftyOrderlambda1:=factor(-residue(L22Infty/lambda^2,lambda=
  infinity));
  L22InftyOrderlambda0:=factor(-residue(L22Infty/lambda^1,lambda=
  infinity));
  L22InftyOrderlambdaMinus1:=factor(-residue(L22Infty/lambda^0,
  lambda=infinity));
  L22InftyOrderlambdaMinus2:=factor(-residue(L22Infty/lambda^
  (-1),lambda=infinity));
  L22InftyOrderlambdaMinus3:=factor(-residue(L22Infty/lambda^
```

```
(-2),lambda=infinity)):
```

$$L22InftyOrderlambda3 := 0 \quad (1.3)$$
$$L22InftyOrderlambda2 := 0$$
$$L22InftyOrderlambda1 := 0$$
$$L22InftyOrderlambda0 := -tinfty11 - tinfty21$$
$$L22InftyOrderlambdaMinus1 := 0$$
$$L22InftyOrderlambdaMinus2 := \frac{1}{tinfty11 - tinfty21} \left(h \left(A12\ tinfty11 - A12\ tinfty21 + A22\ tinfty11 - A22\ tinfty21 - 2\ tinfty10\right)\right)$$

We deduce that L_{2,2} behaves at infinity like -(tinfty11+tinfty21)-(tinfty10+tinfty20)/lambda+h*O(1/lambda^2)

Study at lambda=0

```
> logPsi1Zero:=-t011/h/lambda+t010/h*ln(lambda)+B10+B12/(2-1)*lambda^(2-1)+B13/(3-1)*lambda^(3-1)+B14/(4-1)*lambda^(4-1)+B15/(5-1)*lambda^(5-1)+B16/(6-1)*lambda^(6-1)+B17/(7-1)*lambda^(7-1)  ;
logPsi2Zero:=-t021/h/lambda+t020/h*ln(lambda)+B20+B22/(2-1)*lambda^(2-1)+B23/(3-1)*lambda^(3-1)+B24/(4-1)*lambda^(4-1)+B25/(5-1)*lambda^(5-1)+B26/(6-1)*lambda^(6-1)+B27/(7-1)*lambda^(7-1)  ;
Llogpsi1Zero:=-Lt011/h/lambda+Lt010/h*ln(lambda)+LB10+LB12/(2-1)*lambda^(2-1)+LB13/(3-1)*lambda^(3-1)+LB14/(4-1)*lambda^(4-1)+LB15/(5-1)*lambda^(5-1)+LB16/(6-1)*lambda^(6-1)+LB17/(7-1)*lambda^(7-1)   ;
Llogpsi2Zero:=-Lt021/h/lambda+Lt020/h*ln(lambda)+LB20+LB22/(2-1)*lambda^(2-1)+LB23/(3-1)*lambda^(3-1)+LB24/(4-1)*lambda^(4-1)+LB25/(5-1)*lambda^(5-1)+LB26/(6-1)*lambda^(6-1)+LB27/(7-1)*lambda^(7-1)   ;
Lpsi1Zero  := exp((-t011/h/lambda+t010/h*ln(lambda)+B10+B12/(2-1)*lambda^(2-1)+B13/(3-1)*lambda^(3-1)+B14/(4-1)*lambda^(4-1)+B15/(5-1)*lambda^(5-1)+B16/(6-1)*lambda^(6-1)+B17/(7-1)*lambda^(7-1)))
*(-Lt011/h/lambda+Lt010/h*ln(lambda)+LB10+LB12/(2-1)*lambda^(2-1)+LB13/(3-1)*lambda^(3-1)+LB14/(4-1)*lambda^(4-1)+LB15/(5-1)*lambda^(5-1)+LB16/(6-1)*lambda^(6-1)+LB17/(7-1)*lambda^(7-1));
Lpsi2Zero  := exp((-t021/h/lambda+t020/h*ln(lambda)+B20+B22/(2-1)*lambda^(2-1)+B23/(3-1)*lambda^(3-1)+B24/(4-1)*lambda^(4-1)+B25/(5-1)*lambda^(5-1)+B26*(6-1)*lambda^(6-1)+B27*(7-1)*lambda^(7-1)))
*(-Lt021/h/lambda+Lt020/h*ln(lambda)+LB20+LB22/(2-1)*lambda^(2-1)+LB23/(3-1)*lambda^(3-1)+LB24/(4-1)*lambda^(4-1)+LB25/(5-1)*lambda^(5-1)+LB26/(6-1)*lambda^(6-1)+LB27/(7-1)*lambda^(7-1));
psi1Zero:=exp(logPsi1Zero);
```

```
psi2Zero:=exp(logPsi2Zero);
dpsi1dlambdaZero:=diff(psi1Zero,lambda):
dpsi2dlambdaZero:=diff(psi2Zero,lambda):
d2psi1dlambda2Zero:=diff(psi1Zero,lambda$2):
d2psi2dlambda2Zero:=diff(psi2Zero,lambda$2):
VZero1:=t010*ln(lambda);
VZero2:=t020*ln(lambda);


WronskianLambdaZero:=h*factor(psi1Zero*dpsi2dlambdaZero-
psi2Zero*dpsi1dlambdaZero):
WronskianLambdabisZero:=h*simplify(factor( (diff(logPsi2Zero,
lambda)-diff(logPsi1Zero,lambda))*exp(logPsi1Zero+logPsi2Zero))
):

WronskianTildeLambdaZero:=h^3*factor(dpsi2dlambdaZero*
d2psi1dlambda2Zero-dpsi1dlambdaZero*d2psi2dlambda2Zero):
```

$$logPsi1Zero := -\frac{t011}{h\lambda} + \frac{t010\ln(\lambda)}{h} + B10 + B12\lambda + \frac{1}{2}B13\lambda^2 + \frac{1}{3}B14\lambda^3 + \frac{1}{4}B15\lambda^4 + \frac{1}{5}B16\lambda^5 + \frac{1}{6}B17\lambda^6 \quad \textbf{(1.4)}$$

$$logPsi2Zero := -\frac{t021}{h\lambda} + \frac{t020\ln(\lambda)}{h} + B20 + B22\lambda + \frac{1}{2}B23\lambda^2 + \frac{1}{3}B24\lambda^3 + \frac{1}{4}B25\lambda^4 + \frac{1}{5}B26\lambda^5 + \frac{1}{6}B27\lambda^6$$

$$Llogpsi1Zero := -\frac{Lt011}{h\lambda} + \frac{Lt010\ln(\lambda)}{h} + LB10 + LB12\lambda + \frac{1}{2}LB13\lambda^2 + \frac{1}{3}LB14\lambda^3 + \frac{1}{4}LB15\lambda^4 + \frac{1}{5}LB16\lambda^5 + \frac{1}{6}LB17\lambda^6$$

$$Llogpsi2Zero := -\frac{Lt021}{h\lambda} + \frac{Lt020\ln(\lambda)}{h} + LB20 + LB22\lambda + \frac{1}{2}LB23\lambda^2 + \frac{1}{3}LB24\lambda^3 + \frac{1}{4}LB25\lambda^4 + \frac{1}{5}LB26\lambda^5 + \frac{1}{6}LB27\lambda^6$$

$$Lpsi1Zero := e^{-\frac{t011}{h\lambda} + \frac{t010\ln(\lambda)}{h} + B10 + B12\lambda + \frac{1}{2}B13\lambda^2 + \frac{1}{3}B14\lambda^3 + \frac{1}{4}B15\lambda^4 + \frac{1}{5}B16\lambda^5 + \frac{1}{6}B17\lambda^6}\left(-\frac{Lt011}{h\lambda} + \frac{Lt010\ln(\lambda)}{h} + LB10 + LB12\lambda + \frac{1}{2}LB13\lambda^2 + \frac{1}{3}LB14\lambda^3 + \frac{1}{4}LB15\lambda^4 + \frac{1}{5}LB16\lambda^5 + \frac{1}{6}LB17\lambda^6\right)$$

$$Lpsi2Zero := e^{-\frac{t021}{h\lambda}+\frac{t020\ln(\lambda)}{h}+B20+B22\lambda+\frac{1}{2}B23\lambda^2+\frac{1}{3}B24\lambda^3+\frac{1}{4}B25\lambda^4+5B26\lambda^5+6B27\lambda^6}\left(-\frac{Lt021}{h\lambda}+\frac{Lt020\ln(\lambda)}{h}+LB20+LB22\lambda+\frac{1}{2}LB23\lambda^2+\frac{1}{3}LB24\lambda^3+\frac{1}{4}LB25\lambda^4+\frac{1}{5}LB26\lambda^5+\frac{1}{6}LB27\lambda^6\right)$$

$$psi1Zero := e^{-\frac{t011}{h\lambda}+\frac{t010\ln(\lambda)}{h}+B10+B12\lambda+\frac{1}{2}B13\lambda^2+\frac{1}{3}B14\lambda^3+\frac{1}{4}B15\lambda^4+\frac{1}{5}B16\lambda^5+\frac{1}{6}B17\lambda^6}$$

$$psi2Zero := e^{-\frac{t021}{h\lambda}+\frac{t020\ln(\lambda)}{h}+B20+B22\lambda+\frac{1}{2}B23\lambda^2+\frac{1}{3}B24\lambda^3+\frac{1}{4}B25\lambda^4+\frac{1}{5}B26\lambda^5+\frac{1}{6}B27\lambda^6}$$

$$VZero1 := t010\ln(\lambda)$$

$$VZero2 := t020\ln(\lambda)$$

```
> L22Zero:=factor(h*simplify(diff(WronskianLambdabisZero,lambda)
  /WronskianLambdabisZero)):
  L22ZeroOrderlambdaMinus3:=factor(residue(L22Zero*lambda^2,
  lambda=0));
  L22ZeroOrderlambdaMinus2:=factor(residue(L22Zero*lambda^1,
  lambda=0));
  L22ZeroOrderlambdaMinus1:=factor(residue(L22Zero*lambda^0,
  lambda=0));
  L22ZeroOrderlambda0:=factor(residue(L22Zero*lambda^(-1),lambda=
  0));
  L22ZeroOrderlambda1:=factor(residue(L22Zero*lambda^(-2),lambda=
  0)):
  L22ZeroOrderlambda2:=factor(residue(L22Zero*lambda^(-3),lambda=
  0)):
```

$$L22ZeroOrderlambdaMinus3 := 0 \quad \textbf{(1.5)}$$

$$L22ZeroOrderlambdaMinus2 := t011 + t021$$

$$L22ZeroOrderlambdaMinus1 := -2h + t010 + t020$$

$$L22ZeroOrderlambda0 := \frac{1}{t011-t021}\left(h\left(B12\,t011 - B12\,t021 + B22\,t011 - B22\,t021 + t010 - t020\right)\right)$$

We deduce that L_{2,2} behaves at lambda=0 like (t011+t021)/lambda^2+(t010+t020-2h)/lambda + O(1)

```
> L21Zero:=factor(simplify
  (WronskianTildeLambdaZero/WronskianLambdabisZero)):
  L21ZeroOrderlambdaMinus5:=factor(residue(L21Zero*lambda^4,
  lambda=0));
  L21ZeroOrderlambdaMinus4:=factor(residue(L21Zero*lambda^3,
  lambda=0));
  L21ZeroOrderlambdaMinus3:=factor(residue(L21Zero*lambda^2,
  lambda=0));
```

```
L21ZeroOrderlambdaMinus2:=factor(residue(L21Zero*lambda^1,
lambda=0)):
L21ZeroOrderlambdaMinus1:=factor(residue(L21Zero*lambda^0,
lambda=0)):
L21ZeroOrderlambda0:=factor(residue(L21Zero*lambda^(-1),lambda=
0)):
L21ZeroOrderlambda1:=factor(residue(L21Zero*lambda^(-2),lambda=
0)):
L21ZeroOrderlambda2:=factor(residue(L21Zero*lambda^(-3),lambda=
0)):
```

$$L21ZeroOrderlambdaMinus5 := 0 \quad (1.6)$$

$$L21ZeroOrderlambdaMinus4 := -t011\ t021$$

$$L21ZeroOrderlambdaMinus3 := -t010\ t021 - t011\ t020$$

Formulas for L_{2,2} et L_{2,1}

We have L_{2,2} behaves at infinity like -(tinfty11+tinfty21)-(tinfty10+tinfty20)/lambda+h*O(1/lambda^2)

L_{2,2} behaves at lambda=0 like (t011+t021)/lambda^2+(t010+t020-2h)/lambda +O(1)

Thus, **L_{2,2}= (t011+t021)/lambda^2+ (t010+t020-2*h)/lambda -(tinfty11+tinfty21) +h/(lambda-q)**

with the condition t010+t020+tinfty10+tinfty20=h

We have L_{2,1} behaves at infinity like -tinfty11*tinfty21-(tinfty11*tinfty20+tinfty21*tinfty10 + h*tinfty11)/lambda +O(1/lambda^2)

and L_{2,1} behaves at lambda=0 like -t011*t021/lambda^4-(t010*t021+t020*t011)/lambda^3+ O(1/lambda^2)

Thus, **L_{2,1}=-t011*t021/lambda^4 -(t010*t021+t020*t011)/lambda^3- tinfty11*tinfty21- H/lambda^2 - (tinfty11*tinfty20+tinfty21*tinfty10+h*tinfty11-h*p)/lambda- p*h/(lambda-q)**

```
> L21Form:=-t011*t021/lambda^4  -(t010*t021+t020*t011)/lambda^3-
  H/lambda^2- tinfty11*tinfty21 - (tinfty11*tinfty20+tinfty21*
  tinfty10+h*tinfty11-h*p)/lambda- p*h/(lambda-q);
  L21FormOrderLambdaMinus1:=factor(-residue(L21Form,lambda=
  infinity));
  L21FormOrderLambdaMinus2:=factor(-residue(L21Form*lambda,lambda=
  infinity)):
```

$$L21Form := -\frac{t011\ t021}{\lambda^4} - \frac{t010\ t021 + t011\ t020}{\lambda^3} - \frac{H}{\lambda^2} - tinfty11\ tinfty21 - \frac{-h\ p + h\ tinfty11 - tinfty10\ tinfty11 + tinfty10\ tinfty21}{\lambda} - \frac{p\ h}{\lambda - q} \quad (2)$$

$$L21FormOrderLambdaMinus1 := -h\ tinfty11 + tinfty10\ tinfty11 - tinfty10\ tinfty21$$

Expression of the auxiliary matrix A

The deformation operator is \mathcal{L}=\hbar (alphainf11\partial_{t_{\infty^{(1)},1} +

alphainf21\partial_{t_{\infty^{(2)},1}+alpha011\partial_{t_{0^{(1)},1} +alpha021\partial_{t_{0^{(2)},1})))

```
> WronskianLInfty:=factor(psi1Infty*Lpsi2Infty-psi2Infty*
  Lpsi1Infty):
  WronskianLZero:=factor(psi1Zero*Lpsi2Zero-psi2Zero*Lpsi1Zero):
  A12Infty:=factor(simplify(WronskianLInfty/WronskianLambdaInfty)
  ):
  A12Zero:=factor(simplify(WronskianLZero/WronskianLambdaZero)):
  Y1Infty:=h*factor(dpsi1dlambdaInfty/psi1Infty):
  Y2Infty:=h*factor(dpsi2dlambdaInfty/psi2Infty):
  Y1Zero:=h*factor(dpsi1dlambdaZero/psi1Zero):
  Y2Zero:=h*factor(dpsi2dlambdaZero/psi2Zero):
  Z1Infty:=factor(Lpsi1Infty/psi1Infty):
  Z2Infty:=factor(Lpsi2Infty/psi2Infty):
  Z1Zero:=factor(Lpsi1Zero/psi1Zero):
  Z2Zero:=factor(Lpsi2Zero/psi2Zero):
  A12bisInfty:=factor(simplify((Z2Infty-Z1Infty)/(Y2Infty-
  Y1Infty))):
  A12bisZero:=factor(simplify((Z2Zero-Z1Zero)/(Y2Zero-Y1Zero))):
  A11Infty:=factor(simplify( (Y2Infty*Z1Infty-Y1Infty*Z2Infty)/
  (Y2Infty-Y1Infty) )):
  A11Zero:=factor(simplify( (Y2Zero*Z1Zero-Y1Zero*Z2Zero)/
  (Y2Zero-Y1Zero) )):
  factor(simplify(A12bisInfty-A12Infty));
  factor(simplify(A12bisZero-A12Zero));
```

$$0 \quad (2.1)$$
$$0$$

```
> Lt011:=h*alpha011:
  Lt021:=h*alpha021:
  Lt010:=0:
  Lt020:=0:
  Ltinfty11:=h*alphainf11:
  Ltinfty21:=h*alphainf21:
  Ltinfty10:=0:
  Ltinfty20:=0:
```

```
> A12InftyLambda3:=factor(-residue(A12Infty/lambda^4,lambda=
  infinity));
```

```
A12InftyLambda2:=factor(-residue(A12Infty/lambda^3,lambda=
infinity));
A12InftyLambda1:=factor(-residue(A12Infty/lambda^2,lambda=
infinity));
A12InftyLambda0:=factor(-residue(A12Infty/lambda^1,lambda=
infinity));
A12InftyLambdaMinus1:=factor(-residue(A12Infty/lambda^0,lambda=
infinity)):
```

$$A12InftyLambda3 := 0 \tag{2.2}$$
$$A12InftyLambda2 := 0$$
$$A12InftyLambda1 := \frac{alphainf11 - alphainf21}{tinfty11 - tinfty21}$$
$$A12InftyLambda0 := -\frac{1}{(tinfty11 - tinfty21)^2}(LA10\ tinfty11 - LA10\ tinfty21 - LA20\ tinfty11 + LA20\ tinfty21 + 2\ alphainf11\ tinfty10 - 2\ alphainf21\ tinfty10)$$

```
> A12ZeroLambdaMinus3:=factor(residue(A12bisZero*lambda^2,lambda=
0));
A12ZeroLambdaMinus2:=factor(residue(A12bisZero*lambda^1,lambda=
0));
A12ZeroLambdaMinus1:=factor(residue(A12bisZero*lambda^0,lambda=
0));
A12ZeroLambda0:=factor(residue(A12bisZero*lambda^(-1),lambda=0)
);
A12ZeroLambda1:=factor(residue(A12bisZero*lambda^(-2),lambda=0)
);
A12ZeroLambda2:=factor(residue(A12bisZero*lambda^(-3),lambda=0)
);
```

$$A12ZeroLambdaMinus3 := 0 \tag{2.3}$$
$$A12ZeroLambdaMinus2 := 0$$
$$A12ZeroLambdaMinus1 := 0$$
$$A12ZeroLambda0 := 0$$
$$A12ZeroLambda1 := -\frac{-\alpha021 + \alpha011}{t011 - t021}$$
$$A12ZeroLambda2 := \frac{1}{(t011 - t021)^2}(LB10\ t011 - LB10\ t021 - LB20\ t011 + LB20\ t021 + t010\ \alpha011 - t010\ \alpha021 - t020\ \alpha011 + t020\ \alpha021)$$

We get that A_{1,2}=(alphainf11-alphainf21)/(tinfty11-tinfty21)*lambda+nu+mu/(lambda-q)

```
> A12Form:=(alphainf11-alphainf21)/(-tinfty21+tinfty11)*lambda+
nu+ mu/(lambda-q);
factor(series(A12Form,lambda=0));
solve({factor(residue(A12Form/lambda,lambda=0))=A12ZeroLambda0,
```

```
factor(residue(A12Form/lambda^2,lambda=0))=A12ZeroLambda1},{mu,
nu});
```

$$A12Form := \frac{(alphainf11 - alphainf21)\,\lambda}{tinfty11 - tinfty21} + \nu + \frac{\mu}{\lambda - q} \tag{2.4}$$

$$-\frac{-\nu q + \mu}{q} - \frac{-q^2\, alphainf11 + q^2\, alphainf21 + \mu\, tinfty11 - \mu\, tinfty21}{(tinfty11 - tinfty21)\, q^2}\,\lambda - \frac{\mu}{q^3}\,\lambda^2 - \frac{\mu}{q^4}\,\lambda^3 - \frac{\mu}{q^5}\,\lambda^4 - \frac{\mu}{q^6}\,\lambda^5 + \mathrm{O}(\lambda^6)$$

$$\{\mu = \big(q^2\,(t011\, alphainf11 - t011\, alphainf21 - t021\, alphainf11 + t021\, alphainf21 + \alpha011\, tinfty11 - \alpha011\, tinfty21 - \alpha021\, tinfty11 + \alpha021\, tinfty21)\big) \big/ (t011\, tinfty11 - t011\, tinfty21 - t021\, tinfty11 + t021\, tinfty21), \nu = \big(q\,(t011\, alphainf11 - t011\, alphainf21 - t021\, alphainf11 + t021\, alphainf21 + \alpha011\, tinfty11 - \alpha011\, tinfty21 - \alpha021\, tinfty11 + \alpha021\, tinfty21)\big) \big/ (t011\, tinfty11 - t011\, tinfty21 - t021\, tinfty11 + t021\, tinfty21)\}$$

```
> mu := q^2*(t011*alphainf11-t011*alphainf21-t021*alphainf11+
  t021*alphainf21+alpha011*tinfty11-alpha011*tinfty21-alpha021*
  tinfty11+alpha021*tinfty21)/(t011*tinfty11-t011*tinfty21-t021*
  tinfty11+t021*tinfty21);

  nu := q*(t011*alphainf11-t011*alphainf21-t021*alphainf11+t021*
  alphainf21+alpha011*tinfty11-alpha011*tinfty21-alpha021*
  tinfty11+alpha021*tinfty21)/(t011*tinfty11-t011*tinfty21-t021*
  tinfty11+t021*tinfty21);
```

$$\mu := \big(q^2\,(t011\, alphainf11 - t011\, alphainf21 - t021\, alphainf11 + t021\, alphainf21 + \alpha011\, tinfty11 - \alpha011\, tinfty21 - \alpha021\, tinfty11 + \alpha021\, tinfty21)\big) \big/ (t011\, tinfty11 - t011\, tinfty21 - t021\, tinfty11 + t021\, tinfty21) \tag{2.5}$$

$$\nu := \big(q\,(t011\, alphainf11 - t011\, alphainf21 - t021\, alphainf11 + t021\, alphainf21 + \alpha011\, tinfty11 - \alpha011\, tinfty21 - \alpha021\, tinfty11 + \alpha021\, tinfty21)\big) \big/ (t011\, tinfty11 - t011\, tinfty21 - t021\, tinfty11 + t021\, tinfty21)$$

```
> A11InftyLambda3:=factor(-residue(A11Infty/lambda^4,lambda=
  infinity));
  A11InftyLambda2:=factor(-residue(A11Infty/lambda^3,lambda=
  infinity));
  A11InftyLambda1:=factor(-residue(A11Infty/lambda^2,lambda=
  infinity));
  A11InftyLambda0:=factor(-residue(A11Infty/lambda^1,lambda=
  infinity));
  A11InftyLambdaMinus1:=factor(-residue(A11Infty/lambda^0,lambda=
```

```
infinity)):

A11ZeroLambdaMinus3:=factor(residue(A11Zero*lambda^2,lambda=0))
;
A11ZeroLambdaMinus2:=factor(residue(A11Zero*lambda^1,lambda=0))
;
A11ZeroLambdaMinus1:=factor(residue(A11Zero*lambda^0,lambda=0))
;
A11ZeroLambda0:=factor(residue(A11Zero*lambda^(-1),lambda=0));
A11ZeroLambda1:=factor(residue(A11Zero*lambda^(-2),lambda=0)):
```

$$A11InftyLambda3 := 0 \tag{2.6}$$

$$A11InftyLambda2 := 0$$

$$A11InftyLambda1 := \frac{alphainf11\ tinfty21 - alphainf21\ tinfty11}{tinfty11 - tinfty21}$$

$$A11InftyLambda0 := -\frac{1}{(tinfty11 - tinfty21)^2} \big(LA10\ tinfty11\ tinfty21 - LA10\ tinfty21^2 - LA20\ tinfty11^2 + LA20\ tinfty11\ tinfty21 + alphainf11\ tinfty10\ tinfty11 + alphainf11\ tinfty10\ tinfty21 - alphainf21\ tinfty10\ tinfty11 - alphainf21\ tinfty10\ tinfty21\big)$$

$$A11ZeroLambdaMinus3 := 0$$

$$A11ZeroLambdaMinus2 := 0$$

$$A11ZeroLambdaMinus1 := -\frac{t011\ \alpha021 - t021\ \alpha011}{t011 - t021}$$

$$A11ZeroLambda0 := -\frac{1}{(t011 - t021)^2} \big(LB10\ t011\ t021 - LB10\ t021^2 - LB20\ t011^2 + LB20\ t011\ t021 + t010\ t021\ \alpha011 - t010\ t021\ \alpha021 - t011\ t020\ \alpha011 + t011\ t020\ \alpha021\big)$$

A_{1,1}=(alphainf11*tinfty21-alphainf21*tinfty11)/(tinfty11-tinfty21)*lambda+C-rho/(lambda-q)-(t011*alpha021-t021*alpha011)/(t011-t021)/lambda