In this Maple file, we compute the Casimir coefficients of the Lax matrix L associated to the Painlevé 4 equation in relation with the spectral curve

For convenience the irregular times and monodromies are denoted t_{i,j} at lambda=infinity and s_{i,j} at lambda=t

```
> restart:
  P1:=x-> P021/(x-t)^2+P011/(x-t)+Pinfty01+Pinfty11*x;
  P2:=x-> P042/(x-t)^4+P032/(x-t)^3+P022/(x-t)^2+P012/(x-t)+
  Pinfty02+Pinfty12*x+Pinfty22*x^2;
  SpectralCurve:=unapply(y^2-P1(x)*y+P2(x),y);
```

$$P1 := x \to \frac{P021}{(x-t)^2} + \frac{P011}{x-t} + Pinfty01 + Pinfty11\,x \tag{1}$$

$$P2 := x \to \frac{P042}{(x-t)^4} + \frac{P032}{(x-t)^3} + \frac{P022}{(x-t)^2} + \frac{P012}{x-t} + Pinfty02 + Pinfty12\,x + Pinfty22\,x^2$$

$$SpectralCurve := y \to y^2 - \left(\frac{P021}{(x-t)^2} + \frac{P011}{x-t} + Pinfty01 + Pinfty11\,x \right) y + \frac{P042}{(x-t)^4} + \frac{P032}{(x-t)^3} + \frac{P022}{(x-t)^2} + \frac{P012}{x-t} + Pinfty02 + Pinfty12\,x + Pinfty22\,x^2$$

```
> DiaginftySheet1:=-t12*x-t11-t10/x+Unknown/x^2;
  DiaginftySheet2:=-t22*x-t21-t20/x+Unknown2/x^2;
  DiagtSheet1:=s10/(x-t)+u1+v1*x+Unknown3*x^2;
  DiagtSheet2:=s20/(x-t)+u2+v2*x+Unknown4*x^2;
```

$$DiaginftySheet1 := -t12\,x - t11 - \frac{t10}{x} + \frac{Unknown}{x^2} \tag{2}$$

$$DiaginftySheet2 := -t22\,x - t21 - \frac{t20}{x} + \frac{Unknown2}{x^2}$$

$$DiagtSheet1 := \frac{s10}{x-t} + u1 + v1\,x + Unknown3\,x^2$$

$$DiagtSheet2 := \frac{s20}{x-t} + u2 + v2\,x + Unknown4\,x^2$$

```
> series(DiaginftySheet1+DiaginftySheet2-P1(x),x=infinity);
  series(DiagtSheet1+DiagtSheet2-P1(x),x=t,10);
```

$$(-t12 - t22 - Pinfty11)\,x - t11 - t21 - Pinfty01 + \frac{-t10 - t20 - P011}{x} + \frac{-P011\,t - P021 + Unknown + Unknown2}{x^2} + \frac{-P011\,t^2 - 2\,P021\,t}{x^3} + \frac{-P011\,t^3 - 3\,P021\,t^2}{x^4} + \frac{-P011\,t^4 - 4\,P021\,t^3}{x^5} + O\left(\frac{1}{x^6}\right) \tag{3}$$

$$-\frac{P021}{(x-t)^2} + \frac{s10 + s20 - P011}{x-t} + t^2\,Unknown3 + t^2\,Unknown4 + t\,v1 + t\,v2 - t\,Pinfty11 + u1 + u2 - Pinfty01 + (2\,t\,Unknown3 + 2\,t\,Unknown4 + v1 + v2 - Pinfty11)\,(x-t) + (Unknown3 + Unknown4)\,(x-t)^2$$

```
> Pinfty11:=-t12-t22;
  Pinfty01:=-t11-t21;
```

```
P021:=s11+s21;
P011:=s10+s20;
```

$$Pinfty11 := -t12 - t22$$
$$Pinfty01 := -t11 - t21$$
$$P021 := s11 + s21$$
$$P011 := s10 + s20 \tag{4}$$

```
> series(factor(series(SpectralCurve(DiaginftySheet1),x=infinity)),
  x=infinity);
  series(factor(series(SpectralCurve(DiaginftySheet2),x=infinity)),
  x=infinity);
```

$$(-t12\,t22 + Pinfty22)\,x^2 + (-t11\,t22 - t12\,t21 + Pinfty12)\,x + t12\,s10 + t12\,s20 + t12\,t10 \tag{5}$$
$$- t10\,t22 - t11\,t21 + Pinfty02 + \frac{1}{x}(s10\,t\,t12 + s20\,t\,t12 - Unknown\,t12$$
$$+ Unknown\,t22 + s10\,t11 + s11\,t12 + s20\,t11 + s21\,t12 + t10\,t11 - t10\,t21 + P012)$$
$$+ \frac{1}{x^2}(s10\,t^2\,t12 + s20\,t^2\,t12 + s10\,t\,t11 + 2\,s11\,t\,t12 + s20\,t\,t11 + 2\,s21\,t\,t12 + P012\,t$$
$$- Unknown\,t11 + Unknown\,t21 + s10\,t10 + s11\,t11 + s20\,t10 + s21\,t11 + t10^2 + P022)$$
$$+ \frac{1}{x^3}(s10\,t^3\,t12 + s20\,t^3\,t12 + s10\,t^2\,t11 + 3\,s11\,t^2\,t12 + s20\,t^2\,t11 + 3\,s21\,t^2\,t12$$
$$+ P012\,t^2 + s10\,t\,t10 + 2\,s11\,t\,t11 + s20\,t\,t10 + 2\,s21\,t\,t11 + 2\,P022\,t - Unknown\,s10$$
$$- Unknown\,s20 - 2\,Unknown\,t10 + s11\,t10 + s21\,t10 + P032) + \frac{1}{x^4}(s10\,t^4\,t12$$
$$+ s20\,t^4\,t12 + s10\,t^3\,t11 + 4\,s11\,t^3\,t12 + s20\,t^3\,t11 + 4\,s21\,t^3\,t12 + P012\,t^3 + s10\,t^2\,t10$$
$$+ 3\,s11\,t^2\,t11 + s20\,t^2\,t10 + 3\,s21\,t^2\,t11 + 3\,P022\,t^2 - Unknown\,s10\,t - Unknown\,s20\,t$$
$$+ 2\,s11\,t\,t10 + 2\,s21\,t\,t10 + 3\,P032\,t + Unknown^2 - Unknown\,s11 - Unknown\,s21$$
$$+ P042) + O\left(\frac{1}{x^5}\right)$$
$$(-t12\,t22 + Pinfty22)\,x^2 + (-t11\,t22 - t12\,t21 + Pinfty12)\,x + t22\,s10 + t22\,s20 - t11\,t21$$
$$- t20\,t12 + t22\,t20 + Pinfty02 + \frac{1}{x}(s10\,t\,t22 + s20\,t\,t22 + s10\,t21 + s11\,t22$$
$$+ s20\,t21 + s21\,t22 - t11\,t20 + t12\,Unknown2 + t20\,t21 - t22\,Unknown2 + P012)$$
$$+ \frac{1}{x^2}(s10\,t^2\,t22 + s20\,t^2\,t22 + s10\,t\,t21 + 2\,s11\,t\,t22 + s20\,t\,t21 + 2\,s21\,t\,t22 + P012\,t$$
$$+ s10\,t20 + s11\,t21 + s20\,t20 + s21\,t21 + t11\,Unknown2 + t20^2 - t21\,Unknown2 + P022)$$
$$+ \frac{1}{x^3}(s10\,t^3\,t22 + s20\,t^3\,t22 + s10\,t^2\,t21 + 3\,s11\,t^2\,t22 + s20\,t^2\,t21 + 3\,s21\,t^2\,t22$$
$$+ P012\,t^2 + s10\,t\,t20 + 2\,s11\,t\,t21 + s20\,t\,t20 + 2\,s21\,t\,t21 + 2\,P022\,t - s10\,Unknown2$$
$$+ s11\,t20 - s20\,Unknown2 + s21\,t20 - 2\,t20\,Unknown2 + P032) + \frac{1}{x^4}(s10\,t^4\,t22$$
$$+ s20\,t^4\,t22 + s10\,t^3\,t21 + 4\,s11\,t^3\,t22 + s20\,t^3\,t21 + 4\,s21\,t^3\,t22 + P012\,t^3 + s10\,t^2\,t20$$
$$+ 3\,s11\,t^2\,t21 + s20\,t^2\,t20 + 3\,s21\,t^2\,t21 + 3\,P022\,t^2 - s10\,t\,Unknown2 + 2\,s11\,t\,t20$$
$$- s20\,t\,Unknown2 + 2\,s21\,t\,t20 + 3\,P032\,t - s11\,Unknown2 - s21\,Unknown2$$
$$+ Unknown2^2 + P042) + O\left(\frac{1}{x^5}\right)$$

```
> Pinfty22:=t12*t22;
  Pinfty12:=t11*t22+t12*t21;
  Pinfty02:=-s10*t12-s20*t12-t10*t12+t10*t22+t11*t21;
  factor(residue(SpectralCurve(DiaginftySheet2)/x,x=infinity));
  CoherenceEquation:=s10+s20+t10+t20;
```

$$Pinfty22 := t22\,t12$$
$$Pinfty12 := t11\,t22 + t12\,t21$$
$$Pinfty02 := -s10\,t12 - s20\,t12 - t10\,t12 + t10\,t22 + t11\,t21$$
$$(t12 - t22)\,(s10 + s20 + t10 + t20)$$
$$CoherenceEquation := s10 + s20 + t10 + t20$$

(6)

```
> factor(series(SpectralCurve(DiagtSheet1),x=t));
  factor(series(SpectralCurve(DiagtSheet2),x=t));
  EQ1:=residue((x-t)^3*SpectralCurve(DiagtSheet1),x=t);
  EQ2:=residue((x-t)^3*SpectralCurve(DiagtSheet2),x=t);
  EQ3:=residue((x-t)^2*SpectralCurve(DiagtSheet1),x=t);
  EQ4:=residue((x-t)^2*SpectralCurve(DiagtSheet2),x=t);
  EQ5:=residue((x-t)*SpectralCurve(DiagtSheet1),x=t);
  EQ6:=residue((x-t)*SpectralCurve(DiagtSheet2),x=t);
  EQ7:=residue((x-t)^0*SpectralCurve(DiagtSheet1),x=t);
  EQ8:=residue((x-t)^0*SpectralCurve(DiagtSheet2),x=t);

  P001 := s11+s21;
  P042 := s21*s11;
  simplify(EQ1);
  simplify(EQ2);
```

$$\frac{P042}{(x-t)^4} + \frac{-s10\,s11 - s10\,s21 + P032}{(x-t)^3} + \frac{1}{(x-t)^2}\left(-s11\,t^2\,Unknown3 - s21\,t^2\,Unknown3\right. \qquad \textbf{(7)}$$
$$\left. - s11\,t\,v1 - s21\,t\,v1 - s10\,s20 - s11\,u1 - s21\,u1 + P022\right) + \frac{1}{x-t}\left(s10\,t^2\,Unknown3\right.$$
$$\left. - s20\,t^2\,Unknown3 + s10\,t\,t12 + s10\,t\,t22 + s10\,t\,v1 - 2\,s11\,t\,Unknown3 - s20\,t\,v1\right.$$
$$\left. - 2\,s21\,t\,Unknown3 + s10\,t11 + s10\,t21 + s10\,u1 - s11\,v1 - s20\,u1 - s21\,v1 + P012\right)$$
$$+ t^4\,Unknown3^2 + t^3\,t12\,Unknown3 + t^3\,t22\,Unknown3 + 2\,t^3\,v1\,Unknown3$$
$$+ t^2\,t11\,Unknown3 + t^2\,t12\,t22 + t^2\,t12\,v1 + t^2\,t21\,Unknown3 + t^2\,t22\,v1$$
$$+ 2\,t^2\,u1\,Unknown3 + t^2\,v1^2 + 2\,s10\,t\,Unknown3 - 2\,s20\,t\,Unknown3 + t\,t11\,t22 + t\,t11\,v1$$
$$+ t\,t12\,t21 + t\,t12\,u1 + t\,t21\,v1 + t\,t22\,u1 + 2\,t\,u1\,v1 + s10\,t22 + s10\,v1 - s11\,Unknown3$$
$$- s20\,t12 - s20\,v1 - s21\,Unknown3 - t10\,t12 + t10\,t22 + t11\,t21 + t11\,u1 + t21\,u1 + u1^2$$
$$+ \left(4\,t^3\,Unknown3^2 + 3\,t^2\,t12\,Unknown3 + 3\,t^2\,t22\,Unknown3 + 6\,t^2\,v1\,Unknown3\right.$$
$$+ 2\,t\,t11\,Unknown3 + 2\,t\,t12\,t22 + 2\,t\,t12\,v1 + 2\,t\,t21\,Unknown3 + 2\,t\,t22\,v1$$
$$\left. + 4\,t\,u1\,Unknown3 + 2\,t\,v1^2 + s10\,Unknown3 - s20\,Unknown3 + t11\,t22 + t11\,v1\right.$$
$$\left. + t12\,t21 + t12\,u1 + t21\,v1 + t22\,u1 + 2\,u1\,v1\right)\,(x-t) + \left(6\,t^2\,Unknown3^2\right.$$
$$\left. + 3\,t\,t12\,Unknown3 + 3\,t\,t22\,Unknown3 + 6\,t\,v1\,Unknown3 + t11\,Unknown3 + t12\,t22\right.$$
$$\left. + t12\,v1 + t21\,Unknown3 + t22\,v1 + 2\,u1\,Unknown3 + v1^2\right)\,(x-t)^2$$
$$+ Unknown3\,(4\,t\,Unknown3 + t12 + t22 + 2\,v1)\,(x-t)^3 + Unknown3^2\,(x-t)^4$$

$$\frac{P042}{(x-t)^4}+\frac{-s11\,s20-s20\,s21+P032}{(x-t)^3}+\frac{1}{(x-t)^2}\left(-s11\,t^2\,Unknown4-s21\,t^2\,Unknown4\right.$$
$$\left.-s11\,t\,v2-s21\,t\,v2-s10\,s20-s11\,u2-s21\,u2+P022\right)+\frac{1}{x-t}\left(\right.$$
$$-s10\,t^2\,Unknown4+s20\,t^2\,Unknown4-s10\,t\,v2-2\,s11\,t\,Unknown4+s20\,t\,t12+s20\,t\,t22$$
$$+s20\,t\,v2-2\,s21\,t\,Unknown4-s10\,u2-s11\,v2+s20\,t11+s20\,t21+s20\,u2-s21\,v2$$
$$\left.+P012\right)+t^4\,Unknown4^2+t^3\,t12\,Unknown4+t^3\,t22\,Unknown4+2\,t^3\,v2\,Unknown4$$
$$+t^2\,t11\,Unknown4+t^2\,t12\,t22+t^2\,t12\,v2+t^2\,t21\,Unknown4+t^2\,t22\,v2$$
$$+2\,t^2\,u2\,Unknown4+t^2\,v2^2-2\,s10\,t\,Unknown4+2\,s20\,t\,Unknown4+t\,t11\,t22+t\,t11\,v2$$
$$+t\,t12\,t21+t\,t12\,u2+t\,t21\,v2+t\,t22\,u2+2\,t\,u2\,v2-s10\,t12-s10\,v2-s11\,Unknown4$$
$$+s20\,t22+s20\,v2-s21\,Unknown4-t10\,t12+t10\,t22+t11\,t21+t11\,u2+t21\,u2+u2^2$$
$$+\left(4\,t^3\,Unknown4^2+3\,t^2\,t12\,Unknown4+3\,t^2\,t22\,Unknown4+6\,t^2\,v2\,Unknown4\right.$$
$$+2\,t\,t11\,Unknown4+2\,t\,t12\,t22+2\,t\,t12\,v2+2\,t\,t21\,Unknown4+2\,t\,t22\,v2$$
$$\left.+4\,t\,u2\,Unknown4+2\,t\,v2^2-s10\,Unknown4+s20\,Unknown4+t11\,t22+t11\,v2\right.$$
$$\left.+t12\,t21+t12\,u2+t21\,v2+t22\,u2+2\,u2\,v2\right)(x-t)+\left(6\,t^2\,Unknown4^2\right.$$
$$+3\,t\,t12\,Unknown4+3\,t\,t22\,Unknown4+6\,t\,v2\,Unknown4+t11\,Unknown4+t12\,t22$$
$$\left.+t12\,v2+t21\,Unknown4+t22\,v2+2\,u2\,Unknown4+v2^2\right)(x-t)^2$$
$$+Unknown4\,(4\,t\,Unknown4+t12+t22+2\,v2)\,(x-t)^3+Unknown4^2\,(x-t)^4$$

$$EQ1 := P042$$
$$EQ2 := P042$$
$$EQ3 := -s10\,s11-s10\,s21+P032$$
$$EQ4 := -s11\,s20-s20\,s21+P032$$

$$EQ5 := -s11\,t^2\,Unknown3-s21\,t^2\,Unknown3-s11\,t\,v1-s21\,t\,v1-s10\,s20-s11\,u1$$
$$-s21\,u1+P022$$

$$EQ6 := -s11\,t^2\,Unknown4-s21\,t^2\,Unknown4-s11\,t\,v2-s21\,t\,v2-s10\,s20-s11\,u2$$
$$-s21\,u2+P022$$

$$EQ7 := s10\,t^2\,Unknown3-s20\,t^2\,Unknown3+s10\,t\,t12+s10\,t\,t22+s10\,t\,v1$$
$$-2\,s11\,t\,Unknown3-s20\,t\,v1-2\,s21\,t\,Unknown3+s10\,t11+s10\,t21+s10\,u1-s11\,v1$$
$$-s20\,u1-s21\,v1+P012$$

$$EQ8 := -s10\,t^2\,Unknown4+s20\,t^2\,Unknown4-s10\,t\,v2-2\,s11\,t\,Unknown4+s20\,t\,t12$$
$$+s20\,t\,t22+s20\,t\,v2-2\,s21\,t\,Unknown4-s10\,u2-s11\,v2+s20\,t11+s20\,t21+s20\,u2$$
$$-s21\,v2+P012$$

$$P001 := s11+s21$$
$$P042 := s21\,s11$$
$$s21\,s11$$
$$s21\,s11$$

```
> solve(EQ3,P032):
  P032 := s21*s10+s20*s11;
  simplify(EQ3);
  simplify(EQ4);
```

$$P032 := s10\,s21+s11\,s20 \qquad \textbf{(8)}$$
$$-s11\,(s10-s20)$$
$$s21\,(s10-s20)$$

```
> s11:=0:
  s21:=0:
```

```
solve(EQ5,P022);
P022:=s10*s20;
simplify(EQ5);
```

$$s10\,s20 \qquad (9)$$
$$P022 := s10\,s20$$
$$0$$

Summary of the coefficients:

```
> P1:=unapply(P1(lambda),lambda);
  P2:=unapply(P2(lambda),lambda);
  tdP2:=unapply(s10*s20/(lambda-t)^2-t12*s10-t12*s20-t12*t10+t10*
  t22+t11*t21+(t11*t22+t12*t21)*lambda+t12*t22*lambda^2,lambda);
```

$$P1 := \lambda \rightarrow \frac{s10 + s20}{\lambda - t} - t11 - t21 + (-t12 - t22)\,\lambda \qquad (10)$$

$$P2 := \lambda \rightarrow \frac{s10\,s20}{(\lambda - t)^2} + \frac{P012}{\lambda - t} - t12\,s10 - t12\,s20 - t12\,t10 + t10\,t22 + t11\,t21 + (t11\,t22 + t12\,t21)\,\lambda + t22\,t12\,\lambda^2$$

$$tdP2 := \lambda \rightarrow \frac{s10\,s20}{(\lambda - t)^2} - t12\,s10 - t12\,s20 - t12\,t10 + t10\,t22 + t11\,t21 + (t11\,t22 + t12\,t21)\,\lambda + t22\,t12\,\lambda^2$$