In this Maple sheet, we compute the Lax matrices using the asymptotics of the wave functions and the local diagonalization for the Painlevé 6 equation.

We first use the expression of the coefficients of the spectral curve in terms of the irregular times and monodromies.

```
> restart:
  CoherenceEquation1 := tinfty10+tt10+tt20+t010+t020+tinfty20+t110+
  t120;
  CoherenceEquation2 := Pt12+P012+P112;
  CoherenceEquation3 := -tinfty10*tinfty20+tt20*tt10+Pt12*t+t010*
  t020+P112+t120*t110;
```

$$CoherenceEquation1 := tinfty10 + tt10 + tt20 + t010 + t020 + tinfty20 + t110 + t120 \quad (1)$$

$$CoherenceEquation2 := Pt12 + P012 + P112$$

$$CoherenceEquation3 := Pt12\, t + t010\, t020 + t110\, t120 + tt10\, tt20 - tinfty10\, tinfty20 + P112$$

Computation of the Lax matrix L using the asymptotics of the wave functions

Study of the asymptotics at infinity

```
> logPsi1Infty:=-tinfty10/h*ln(lambda)+A10-A12/(2-1)/lambda^(2-1)
  -A13/(3-1)/lambda^(3-1)-A14/(4-1)/lambda^(4-1)-A15/(5-1)
  /lambda^(5-1)-A16/(6-1)/lambda^(6-1)-A17/(7-1)/lambda^(7-1)   ;
  logPsi2Infty:=-tinfty20/h*ln(lambda)-1*ln(lambda)+A20-A22/(2-1)
  /lambda^(2-1)-A23/(3-1)/lambda^(3-1)-A24/(4-1)/lambda^(4-1)-
  A25/(5-1)/lambda^(5-1)-A26/(6-1)/lambda^(6-1)-A27/(7-1)/lambda^
  (7-1)   ;
  Llogpsi1Infty:=-Ltinfty10/h*ln(lambda)+LA10-LA12/(2-1)/lambda^
  (2-1)-LA13/(3-1)/lambda^(3-1)-LA14/(4-1)/lambda^(4-1)-LA15/(5
  -1)/lambda^(5-1)-LA16/(6-1)/lambda^(6-1)-LA17/(7-1)/lambda^(7
  -1)   ;
  Llogpsi2Infty:=-Ltinfty20/h*ln(lambda)+LA20-LA22/(2-1)/lambda^
  (2-1)-LA23/(3-1)/lambda^(3-1)-LA24/(4-1)/lambda^(4-1)-LA25/(5
  -1)/lambda^(5-1)-LA26/(6-1)/lambda^(6-1)-LA27/(7-1)/lambda^(7
  -1)   ;
  Lpsi1Infty  := exp(1/h*(-tinfty10*ln(lambda)+h*A10-h*
  A12/lambda-1/2*h*A13/lambda^2-1/3*h*A14/lambda^3-1/4*h*
  A15/lambda^4-1/5*h*A16/lambda^5-1/6*h*A17/lambda^6))*1/h*(-
  Ltinfty10*ln(lambda)+h*LA10-h*LA12/lambda-1/2*h*LA13/lambda^2
  -1/3*h*LA14/lambda^3-1/4*h*LA15/lambda^4-1/5*h*LA16/lambda^5
  -1/6*h*LA17/lambda^6);
  Lpsi2Infty  := exp(1/h*(-tinfty20*ln(lambda)-h*ln(lambda)+h*A20-
  h*A22/lambda-1/2*h*A23/lambda^2-1/3*h*A24/lambda^3-1/4*h*
  A25/lambda^4-1/5*h*A26/lambda^5-1/6*h*A27/lambda^6))*1/h*(-
  Ltinfty20*ln(lambda)+h*LA20-h*LA22/lambda-1/2*h*LA23/lambda^2
```

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-1/3*h*LA24/lambda^3-1/4*h*LA25/lambda^4-1/5*h*LA26/lambda^5
-1/6*h*LA27/lambda^6);
psi1Infty:=exp(logPsi1Infty);
psi2Infty:=exp(logPsi2Infty);
dpsi1dlambdaInfty:=diff(psi1Infty,lambda):
dpsi2dlambdaInfty:=diff(psi2Infty,lambda):
d2psi1dlambda2Infty:=diff(psi1Infty,lambda$2):
d2psi2dlambda2Infty:=diff(psi2Infty,lambda$2):
Vinfty1:=+tinfty10*ln(lambda);
Vinfty2:=+tinfty20*ln(lambda);


WronskianLambdaInfty:=h*factor(psi1Infty*dpsi2dlambdaInfty-
psi2Infty*dpsi1dlambdaInfty):
WronskianLambdabisInfty:=h*simplify(factor( (diff(logPsi2Infty,
lambda)-diff(logPsi1Infty,lambda))*exp(logPsi1Infty+
logPsi2Infty))):

WronskianTildeLambdaInfty:=h^3*factor(dpsi2dlambdaInfty*
d2psi1dlambda2Infty-dpsi1dlambdaInfty*d2psi2dlambda2Infty):
```

$$logPsi1Infty := -\frac{tinfty10\ln(\lambda)}{h} + A10 - \frac{A12}{\lambda} - \frac{1}{2}\frac{A13}{\lambda^2} - \frac{1}{3}\frac{A14}{\lambda^3} - \frac{1}{4}\frac{A15}{\lambda^4} - \frac{1}{5}\frac{A16}{\lambda^5} - \frac{1}{6}\frac{A17}{\lambda^6} \quad \textbf{(1.1)}$$

$$logPsi2Infty := -\frac{tinfty20\ln(\lambda)}{h} - \ln(\lambda) + A20 - \frac{A22}{\lambda} - \frac{1}{2}\frac{A23}{\lambda^2} - \frac{1}{3}\frac{A24}{\lambda^3} - \frac{1}{4}\frac{A25}{\lambda^4} - \frac{1}{5}\frac{A26}{\lambda^5} - \frac{1}{6}\frac{A27}{\lambda^6}$$

$$Llogpsi1Infty := -\frac{Ltinfty10\ln(\lambda)}{h} + LA10 - \frac{LA12}{\lambda} - \frac{1}{2}\frac{LA13}{\lambda^2} - \frac{1}{3}\frac{LA14}{\lambda^3} - \frac{1}{4}\frac{LA15}{\lambda^4} - \frac{1}{5}\frac{LA16}{\lambda^5} - \frac{1}{6}\frac{LA17}{\lambda^6}$$

$$Llogpsi2Infty := -\frac{Ltinfty20\ln(\lambda)}{h} + LA20 - \frac{LA22}{\lambda} - \frac{1}{2}\frac{LA23}{\lambda^2} - \frac{1}{3}\frac{LA24}{\lambda^3} - \frac{1}{4}\frac{LA25}{\lambda^4} - \frac{1}{5}\frac{LA26}{\lambda^5} - \frac{1}{6}\frac{LA27}{\lambda^6}$$

$$Lpsi1Infty := 1$$

$$h\left(e^{\frac{-tinfty10\ln(\lambda)+hA10-\frac{hA12}{\lambda}-\frac{1}{2}\frac{hA13}{\lambda^2}-\frac{1}{3}\frac{hA14}{\lambda^3}-\frac{1}{4}\frac{hA15}{\lambda^4}-\frac{1}{5}\frac{hA16}{\lambda^5}-\frac{1}{6}\frac{hA17}{\lambda^6}}{h}}\left(-Ltinfty10\ln(\lambda)+h\,LA10-\frac{h\,LA12}{\lambda}-\frac{1}{2}\frac{h\,LA13}{\lambda^2}-\frac{1}{3}\frac{h\,LA14}{\lambda^3}-\frac{1}{4}\frac{h\,LA15}{\lambda^4}-\frac{1}{5}\frac{h\,LA16}{\lambda^5}-\frac{1}{6}\frac{h\,LA17}{\lambda^6}\right)\right)$$

$$Lpsi2Infty := 1\Big/\left(h\left(e^{\frac{-tinfty20\ln(\lambda)-h\ln(\lambda)+hA20-\frac{hA22}{\lambda}-\frac{1}{2}\frac{hA23}{\lambda^2}-\frac{1}{3}\frac{hA24}{\lambda^3}-\frac{1}{4}\frac{hA25}{\lambda^4}-\frac{1}{5}\frac{hA26}{\lambda^5}-\frac{1}{6}\frac{hA27}{\lambda^6}}{h}}\left(-Ltinfty20\ln(\lambda)+h\,LA20-\frac{h\,LA22}{\lambda}-\frac{1}{2}\frac{h\,LA23}{\lambda^2}-\frac{1}{3}\frac{h\,LA24}{\lambda^3}-\frac{1}{4}\frac{h\,LA25}{\lambda^4}-\frac{1}{5}\frac{h\,LA26}{\lambda^5}-\frac{1}{6}\frac{h\,LA27}{\lambda^6}\right)\right)\right)$$

$$psi1Infty := e^{-\frac{tinfty10\ln(\lambda)}{h}+A10-\frac{A12}{\lambda}-\frac{1}{2}\frac{A13}{\lambda^2}-\frac{1}{3}\frac{A14}{\lambda^3}-\frac{1}{4}\frac{A15}{\lambda^4}-\frac{1}{5}\frac{A16}{\lambda^5}-\frac{1}{6}\frac{A17}{\lambda^6}}$$

$$psi2Infty := e^{-\frac{tinfty20\ln(\lambda)}{h}-\ln(\lambda)+A20-\frac{A22}{\lambda}-\frac{1}{2}\frac{A23}{\lambda^2}-\frac{1}{3}\frac{A24}{\lambda^3}-\frac{1}{4}\frac{A25}{\lambda^4}-\frac{1}{5}\frac{A26}{\lambda^5}-\frac{1}{6}\frac{A27}{\lambda^6}}$$

$$Vinfty1 := tinfty10\ln(\lambda)$$

$$Vinfty2 := tinfty20\ln(\lambda)$$

```
> L21Infty:=factor(simplify
  (WronskianTildeLambdaInfty/WronskianLambdabisInfty)):
  L21InftyOrderlambda3:=factor(-residue(L21Infty/lambda^4,lambda=
  infinity));
  L21InftyOrderlambda2:=factor(-residue(L21Infty/lambda^3,lambda=
  infinity));
  L21InftyOrderlambda1:=factor(-residue(L21Infty/lambda^2,lambda=
  infinity));
  L21InftyOrderlambda0:=factor(-residue(L21Infty/lambda^1,lambda=
  infinity));
  L21InftyOrderlambdaMinus1:=factor(-residue(L21Infty/lambda^0,
  lambda=infinity));
  L21InftyNumer:=series(numer(L21Infty),lambda=infinity,30):
  L21InftyDenom:=series(denom(L21Infty),lambda=infinity,30):
  series(L21InftyNumer/L21InftyDenom,lambda=infinity,12):
  L21InftyOrderlambdaMinus2:=factor(-residue(series
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```
(L21InftyNumer/L21InftyDenom,lambda=infinity,12)/lambda^(-1),
lambda=infinity));
L21InftyOrderlambdaMinus3:=factor(-residue(series
(L21InftyNumer/L21InftyDenom,lambda=infinity,12)/lambda^(-2),
lambda=infinity));
L21InftyNumer:=series(numer(L21Infty),lambda=infinity,30):
L21InftyDenom:=series(denom(L21Infty),lambda=infinity,30):
series(L21InftyNumer/L21InftyDenom,lambda=infinity,10):
L21InftyOrderlambdaMinus2:=factor(-residue(series
(L21InftyNumer/L21InftyDenom,lambda=infinity,10)/lambda^(-1),
lambda=infinity)):
```

$$\begin{gathered} L21InftyOrderlambda3 := 0 \\ L21InftyOrderlambda2 := 0 \\ L21InftyOrderlambda1 := 0 \\ L21InftyOrderlambda0 := 0 \\ L21InftyOrderlambdaMinus1 := 0 \\ L21InftyOrderlambdaMinus2 := -(h + tinfty20)\, tinfty10 \\ L21InftyOrderlambdaMinus3 := -\frac{1}{h - tinfty10 + tinfty20}\big(h\,\big(A12\, h\, tinfty10 - A12\, h\, tinfty20 + A12\, tinfty10\, tinfty20 - A12\, tinfty20^2 - 2\, A22\, h\, tinfty10 + A22\, tinfty10^2 - A22\, tinfty10\, tinfty20\big)\big) \end{gathered} \tag{1.2}$$

We get that L21 behaves at infinity like -(h+tinfty20)tinfty10/lambda^2= (-tinfty10tinfty20 -h* tinfty10)/lambda^2 +O(lambda^{-1})

```
> L22Infty:=factor(h*simplify(diff(WronskianLambdabisInfty,
lambda)/WronskianLambdabisInfty)):
L22InftyOrderlambda3:=factor(-residue(L22Infty/lambda^4,lambda=
infinity));
L22InftyOrderlambda2:=factor(-residue(L22Infty/lambda^3,lambda=
infinity));
L22InftyOrderlambda1:=factor(-residue(L22Infty/lambda^2,lambda=
infinity));
L22InftyOrderlambda0:=factor(-residue(L22Infty/lambda^1,lambda=
infinity));
L22InftyOrderlambdaMinus1:=factor(-residue(L22Infty/lambda^0,
lambda=infinity));
L22InftyOrderlambdaMinus2:=factor(-residue(L22Infty/lambda^
(-1),lambda=infinity));
L22InftyOrderlambdaMinus3:=factor(-residue(L22Infty/lambda^
(-2),lambda=infinity)):
```

$$\begin{gathered} L22InftyOrderlambda3 := 0 \\ L22InftyOrderlambda2 := 0 \\ L22InftyOrderlambda1 := 0 \\ L22InftyOrderlambda0 := 0 \\ L22InftyOrderlambdaMinus1 := -2\, h - tinfty10 - tinfty20 \end{gathered} \tag{1.3}$$

$$L22InftyOrderlambdaMinus2 := -\frac{h\,(A12\,tinfty10 - A12\,tinfty20 - 2\,A22\,h + A22\,tinfty10 - A22\,tinfty20)}{h - tinfty10 + tinfty20}$$

We get that L_{2,2} behaves at infinity like -(tinfty10+tinfty20+2\hbar)/lambda+h*O(1/lambda^2)

Study of the asymptotics at lambda=0

```
> logPsi1Zero:=t010/h*ln(lambda)+B10+B12/(2-1)*lambda^(2-1)+B13/
(3-1)*lambda^(3-1)+B14/(4-1)*lambda^(4-1)+B15/(5-1)*lambda^(5
-1)+B16/(6-1)*lambda^(6-1)+B17/(7-1)*lambda^(7-1)   ;
logPsi2Zero:=t020/h*ln(lambda)+B20+B22/(2-1)*lambda^(2-1)+B23/
(3-1)*lambda^(3-1)+B24/(4-1)*lambda^(4-1)+B25/(5-1)*lambda^(5
-1)+B26/(6-1)*lambda^(6-1)+B27/(7-1)*lambda^(7-1)   ;
Llogpsi1Zero:=Lt010/h*ln(lambda)+LB10+LB12/(2-1)*lambda^(2-1)+
LB13/(3-1)*lambda^(3-1)+LB14/(4-1)*lambda^(4-1)+LB15/(5-1)*
lambda^(5-1)+LB16/(6-1)*lambda^(6-1)+LB17/(7-1)*lambda^(7-1)   ;
Llogpsi2Zero:=Lt020/h*ln(lambda)+LB20+LB22/(2-1)*lambda^(2-1)+
LB23/(3-1)*lambda^(3-1)+LB24/(4-1)*lambda^(4-1)+LB25/(5-1)*
lambda^(5-1)+LB26/(6-1)*lambda^(6-1)+LB27/(7-1)*lambda^(7-1)   ;
Lpsi1Zero := exp((t010/h*ln(lambda)+B10+B12/(2-1)*lambda^(2-1)+
B13/(3-1)*lambda^(3-1)+B14/(4-1)*lambda^(4-1)+B15/(5-1)*lambda^
(5-1)+B16/(6-1)*lambda^(6-1)+B17/(7-1)*lambda^(7-1)))
*(Lt010/h*ln(lambda)+LB10+LB12/(2-1)*lambda^(2-1)+LB13/(3-1)*
lambda^(3-1)+LB14/(4-1)*lambda^(4-1)+LB15/(5-1)*lambda^(5-1)+
LB16/(6-1)*lambda^(6-1)+LB17/(7-1)*lambda^(7-1));
Lpsi2Zero := exp((t020/h*ln(lambda)+B20+B22/(2-1)*lambda^(2-1)+
B23/(3-1)*lambda^(3-1)+B24/(4-1)*lambda^(4-1)+B25/(5-1)*lambda^
(5-1)+B26*(6-1)*lambda^(6-1)+B27*(7-1)*lambda^(7-1)))
*(Lt020/h*ln(lambda)+LB20+LB22/(2-1)*lambda^(2-1)+LB23/(3-1)*
lambda^(3-1)+LB24/(4-1)*lambda^(4-1)+LB25/(5-1)*lambda^(5-1)+
LB26/(6-1)*lambda^(6-1)+LB27/(7-1)*lambda^(7-1));
psi1Zero:=exp(logPsi1Zero);
psi2Zero:=exp(logPsi2Zero);
dpsi1dlambdaZero:=diff(psi1Zero,lambda):
dpsi2dlambdaZero:=diff(psi2Zero,lambda):
d2psi1dlambda2Zero:=diff(psi1Zero,lambda$2):
d2psi2dlambda2Zero:=diff(psi2Zero,lambda$2):
VZero1:=t010*ln(lambda);
VZero2:=t020*ln(lambda);


WronskianLambdaZero:=h*factor(psi1Zero*dpsi2dlambdaZero-
psi2Zero*dpsi1dlambdaZero):
WronskianLambdabisZero:=h*simplify(factor( (diff(logPsi2Zero,
```

```
lambda)-diff(logPsi1Zero,lambda))*exp(logPsi1Zero+logPsi2Zero))
):

WronskianTildeLambdaZero:=h^3*factor(dpsi2dlambdaZero*
d2psi1dlambda2Zero-dpsi1dlambdaZero*d2psi2dlambda2Zero):
```

$$logPsi1Zero := \frac{t010\ln(\lambda)}{h} + B10 + B12\lambda + \frac{1}{2}B13\lambda^2 + \frac{1}{3}B14\lambda^3 + \frac{1}{4}B15\lambda^4 + \frac{1}{5}B16\lambda^5 + \frac{1}{6}B17\lambda^6 \quad \textbf{(1.4)}$$

$$logPsi2Zero := \frac{t020\ln(\lambda)}{h} + B20 + B22\lambda + \frac{1}{2}B23\lambda^2 + \frac{1}{3}B24\lambda^3 + \frac{1}{4}B25\lambda^4 + \frac{1}{5}B26\lambda^5 + \frac{1}{6}B27\lambda^6$$

$$Llogpsi1Zero := \frac{Lt010\ln(\lambda)}{h} + LB10 + LB12\lambda + \frac{1}{2}LB13\lambda^2 + \frac{1}{3}LB14\lambda^3 + \frac{1}{4}LB15\lambda^4 + \frac{1}{5}LB16\lambda^5 + \frac{1}{6}LB17\lambda^6$$

$$Llogpsi2Zero := \frac{Lt020\ln(\lambda)}{h} + LB20 + LB22\lambda + \frac{1}{2}LB23\lambda^2 + \frac{1}{3}LB24\lambda^3 + \frac{1}{4}LB25\lambda^4 + \frac{1}{5}LB26\lambda^5 + \frac{1}{6}LB27\lambda^6$$

$$Lpsi1Zero := e^{\frac{t010\ln(\lambda)}{h} + B10 + B12\lambda + \frac{1}{2}B13\lambda^2 + \frac{1}{3}B14\lambda^3 + \frac{1}{4}B15\lambda^4 + \frac{1}{5}B16\lambda^5 + \frac{1}{6}B17\lambda^6} \left(1 / h\left(Lt010\ln(\lambda)\right) + LB10 + LB12\lambda + \frac{1}{2}LB13\lambda^2 + \frac{1}{3}LB14\lambda^3 + \frac{1}{4}LB15\lambda^4 + \frac{1}{5}LB16\lambda^5 + \frac{1}{6}LB17\lambda^6\right)$$

$$Lpsi2Zero := e^{\frac{t020\ln(\lambda)}{h} + B20 + B22\lambda + \frac{1}{2}B23\lambda^2 + \frac{1}{3}B24\lambda^3 + \frac{1}{4}B25\lambda^4 + 5B26\lambda^5 + 6B27\lambda^6} \left(1 / h\left(Lt020\ln(\lambda)\right) + LB20 + LB22\lambda + \frac{1}{2}LB23\lambda^2 + \frac{1}{3}LB24\lambda^3 + \frac{1}{4}LB25\lambda^4 + \frac{1}{5}LB26\lambda^5 + \frac{1}{6}LB27\lambda^6\right)$$

$$psi1Zero := e^{\frac{t010\ln(\lambda)}{h} + B10 + B12\lambda + \frac{1}{2}B13\lambda^2 + \frac{1}{3}B14\lambda^3 + \frac{1}{4}B15\lambda^4 + \frac{1}{5}B16\lambda^5 + \frac{1}{6}B17\lambda^6}$$

$$psi2Zero := e^{\frac{t020\ln(\lambda)}{h} + B20 + B22\lambda + \frac{1}{2}B23\lambda^2 + \frac{1}{3}B24\lambda^3 + \frac{1}{4}B25\lambda^4 + \frac{1}{5}B26\lambda^5 + \frac{1}{6}B27\lambda^6}$$

$$VZero1 := t010\ln(\lambda)$$

$$VZero2 := t020\ln(\lambda)$$

```
> L22Zero:=factor(h*simplify(diff(WronskianLambdabisZero,lambda)
/WronskianLambdabisZero)):
L22ZeroOrderlambdaMinus3:=factor(residue(L22Zero*lambda^2,
lambda=0));
```

```
L22ZeroOrderlambdaMinus2:=factor(residue(L22Zero*lambda^1,
lambda=0));
L22ZeroOrderlambdaMinus1:=factor(residue(L22Zero*lambda^0,
lambda=0));
L22ZeroOrderlambda0:=factor(residue(L22Zero*lambda^(-1),lambda=
0));
L22ZeroOrderlambda1:=factor(residue(L22Zero*lambda^(-2),lambda=
0)):
L22ZeroOrderlambda2:=factor(residue(L22Zero*lambda^(-3),lambda=
0)):
```

$$L22ZeroOrderlambdaMinus3 := 0 \tag{1.5}$$

$$L22ZeroOrderlambdaMinus2 := 0$$

$$L22ZeroOrderlambdaMinus1 := t010 - h + t020$$

$$L22ZeroOrderlambda0 := \frac{1}{t010 - t020} \left(h \left(B12\, h + B12\, t010 - B12\, t020 - B22\, h + B22\, t010 - B22\, t020\right)\right)$$

We get that L_{2,2} behaves at lambda=0 like (t010+t020-h)/lambda +O(1)

```
> L21Zero:=factor(simplify
(WronskianTildeLambdaZero/WronskianLambdabisZero)):
L21ZeroOrderlambdaMinus5:=factor(residue(L21Zero*lambda^4,
lambda=0));
L21ZeroOrderlambdaMinus4:=factor(residue(L21Zero*lambda^3,
lambda=0));
L21ZeroOrderlambdaMinus3:=factor(residue(L21Zero*lambda^2,
lambda=0));
L21ZeroOrderlambdaMinus2:=factor(residue(L21Zero*lambda^1,
lambda=0));
L21ZeroOrderlambdaMinus1:=factor(residue(L21Zero*lambda^0,
lambda=0));
L21ZeroOrderlambda0:=factor(residue(L21Zero*lambda^(-1),lambda=
0));
L21ZeroOrderlambda1:=factor(residue(L21Zero*lambda^(-2),lambda=
0)):
L21ZeroOrderlambda2:=factor(residue(L21Zero*lambda^(-3),lambda=
0)):
```

$$L21ZeroOrderlambdaMinus5 := 0 \tag{1.6}$$

$$L21ZeroOrderlambdaMinus4 := 0$$

$$L21ZeroOrderlambdaMinus3 := 0$$

$$L21ZeroOrderlambdaMinus2 := -t010\, t020$$

$$L21ZeroOrderlambdaMinus1 := -\frac{1}{t010 - t020} \left(h \left(B12\, h\, t020 + B12\, t010\, t020 - B12\, t020^2 - B22\, h\, t010 + B22\, t010^2 - B22\, t010\, t020\right)\right)$$

$$L21ZeroOrderlambda0 := \frac{1}{(t010-t020)^2}\left(h\left(B12^2 h^2 t020 - B12\, B22\, h^2\, t010 - B12\, B22\, h^2\, t020 - B12\, B22\, h\, t010^2 + 2\, B12\, B22\, h\, t010\, t020 - B12\, B22\, h\, t020^2 + B22^2 h^2\, t010 - 2\, B13\, h\, t010\, t020 + 2\, B13\, h\, t020^2 - B13\, t010^2\, t020 + 2\, B13\, t010\, t020^2 - B13\, t020^3 + 2\, B23\, h\, t010^2 - 2\, B23\, h\, t010\, t020 - B23\, t010^3 + 2\, B23\, t010^2\, t020 - B23\, t010\, t020^2\right)\right)$$

We get that L_{2,1} behaves at lambda=0 like -t010*t020/lambda^2+ O(1/lambda)

Study of the asymptotics at lambda=1

```
> logPsi1One:=t110/h*ln(lambda-1)+C10+C12/(2-1)*(lambda-1)^(2-1)+C13/(3-1)*(lambda-1)^(3-1)+C14/(4-1)*(lambda-1)^(4-1)+C15/(5-1)*(lambda-1)^(5-1)+C16/(6-1)*(lambda-1)^(6-1)+C17/(7-1)*(lambda-1)^(7-1)   ;
  logPsi2One:=t120/h*ln(lambda-1)+C20+C22/(2-1)*(lambda-1)^(2-1)+C23/(3-1)*(lambda-1)^(3-1)+C24/(4-1)*(lambda-1)^(4-1)+C25/(5-1)*(lambda-1)^(5-1)+C26/(6-1)*(lambda-1)^(6-1)+C27/(7-1)*(lambda-1)^(7-1)   ;
  Llogpsi1One:=Lt110/h*ln(lambda-1)+LC10+LC12/(2-1)*(lambda-1)^(2-1)+LC13/(3-1)*(lambda-1)^(3-1)+LC14/(4-1)*(lambda-1)^(4-1)+LC15/(5-1)*(lambda-1)^(5-1)+LC16/(6-1)*(lambda-1)^(6-1)+LC17/(7-1)*(lambda-1)^(7-1)   ;
  Llogpsi2One:=Lt120/h*ln(lambda-1)+LC20+LC22/(2-1)*(lambda-1)^(2-1)+LC23/(3-1)*(lambda-1)^(3-1)+LC24/(4-1)*(lambda-1)^(4-1)+LC25/(5-1)*(lambda-1)^(5-1)+LC26/(6-1)*(lambda-1)^(6-1)+LC27/(7-1)*(lambda-1)^(7-1)   ;
  Lpsi1One := exp((t110/h*ln(lambda-1)+C10+C12/(2-1)*(lambda-1)^(2-1)+C13/(3-1)*(lambda-1)^(3-1)+C14/(4-1)*(lambda-1)^(4-1)+C15/(5-1)*(lambda-1)^(5-1)+C16/(6-1)*(lambda-1)^(6-1)+C17/(7-1)*(lambda-1)^(7-1)))
  *(Lt110/h*ln(lambda-1)+LC10+LC12/(2-1)*(lambda-1)^(2-1)+LC13/(3-1)*(lambda-1)^(3-1)+LC14/(4-1)*(lambda-1)^(4-1)+LC15/(5-1)*(lambda-1)^(5-1)+LC16/(6-1)*(lambda-1)^(6-1)+LC17/(7-1)*(lambda-1)^(7-1));
  Lpsi2One := exp((t120/h*ln(lambda-1)+C20+C22/(2-1)*(lambda-1)^(2-1)+C23/(3-1)*(lambda-1)^(3-1)+C24/(4-1)*(lambda-1)^(4-1)+C25/(5-1)*(lambda-1)^(5-1)+C26/(6-1)*(lambda-1)^(6-1)+C27/(7-1)*(lambda-1)^(7-1)))
  *(Lt120/h*ln(lambda-1)+LC20+LC22/(2-1)*(lambda-1)^(2-1)+LC23/(3-1)*(lambda-1)^(3-1)+LC24/(4-1)*(lambda-1)^(4-1)+LC25/(5-1)*(lambda-1)^(5-1)+LC26/(6-1)*(lambda-1)^(6-1)+LC27/(7-1)*(lambda-1)^(7-1));
  psi1One:=exp(logPsi1One);
  psi2One:=exp(logPsi2One);
```

```
dpsi1dlambdaOne:=diff(psi1One,lambda):
dpsi2dlambdaOne:=diff(psi2One,lambda):
d2psi1dlambda2One:=diff(psi1One,lambda$2):
d2psi2dlambda2One:=diff(psi2One,lambda$2):
VOne1:=t110*ln(lambda-1);
VOne2:=t120*ln(lambda-1);


WronskianLambdaOne:=h*factor(psi1One*dpsi2dlambdaOne-psi2One*
dpsi1dlambdaOne):
WronskianLambdabisOne:=h*simplify(factor( (diff(logPsi2One,
lambda)-diff(logPsi1One,lambda))*exp(logPsi1One+logPsi2One))):

WronskianTildeLambdaOne:=h^3*factor(dpsi2dlambdaOne*
d2psi1dlambda2One-dpsi1dlambdaOne*d2psi2dlambda2One):
```

$$logPsi1One := \frac{t110\ln(\lambda-1)}{h} + C10 + C12\,(\lambda-1) + \frac{1}{2}\,C13\,(\lambda-1)^2 + \frac{1}{3}\,C14\,(\lambda-1)^3 + \frac{1}{4}\,C15\,(\lambda-1)^4 + \frac{1}{5}\,C16\,(\lambda-1)^5 + \frac{1}{6}\,C17\,(\lambda-1)^6 \quad (1.7)$$

$$logPsi2One := \frac{t120\ln(\lambda-1)}{h} + C20 + C22\,(\lambda-1) + \frac{1}{2}\,C23\,(\lambda-1)^2 + \frac{1}{3}\,C24\,(\lambda-1)^3 + \frac{1}{4}\,C25\,(\lambda-1)^4 + \frac{1}{5}\,C26\,(\lambda-1)^5 + \frac{1}{6}\,C27\,(\lambda-1)^6$$

$$Llogpsi1One := \frac{Lt110\ln(\lambda-1)}{h} + LC10 + LC12\,(\lambda-1) + \frac{1}{2}\,LC13\,(\lambda-1)^2 + \frac{1}{3}\,LC14\,(\lambda-1)^3 + \frac{1}{4}\,LC15\,(\lambda-1)^4 + \frac{1}{5}\,LC16\,(\lambda-1)^5 + \frac{1}{6}\,LC17\,(\lambda-1)^6$$

$$Llogpsi2One := \frac{Lt120\ln(\lambda-1)}{h} + LC20 + LC22\,(\lambda-1) + \frac{1}{2}\,LC23\,(\lambda-1)^2 + \frac{1}{3}\,LC24\,(\lambda-1)^3 + \frac{1}{4}\,LC25\,(\lambda-1)^4 + \frac{1}{5}\,LC26\,(\lambda-1)^5 + \frac{1}{6}\,LC27\,(\lambda-1)^6$$

$$Lpsi1One := e^{\frac{t110\ln(\lambda-1)}{h} + C10 + C12\,(\lambda-1) + \frac{1}{2}\,C13\,(\lambda-1)^2 + \frac{1}{3}\,C14\,(\lambda-1)^3 + \frac{1}{4}\,C15\,(\lambda-1)^4 + \frac{1}{5}\,C16\,(\lambda-1)^5 + \frac{1}{6}\,C17\,(\lambda-1)^6}\left(\frac{Lt110\ln(\lambda-1)}{h} + LC10 + LC12\,(\lambda-1) + \frac{1}{2}\,LC13\,(\lambda-1)^2 + \frac{1}{3}\,LC14\,(\lambda-1)^3 + \frac{1}{4}\,LC15\,(\lambda-1)^4 + \frac{1}{5}\,LC16\,(\lambda-1)^5 + \frac{1}{6}\,LC17\,(\lambda-1)^6\right)$$

$$Lpsi2One :=$$

$$e^{\frac{t120\ln(\lambda-1)}{h}+C20+C22(\lambda-1)+\frac{1}{2}C23(\lambda-1)^2+\frac{1}{3}C24(\lambda-1)^3+\frac{1}{4}C25(\lambda-1)^4+\frac{1}{5}C26(\lambda-1)^5+\frac{1}{6}C27(\lambda-1)^6}\left(\frac{Lt120\ln(\lambda-1)}{h}+LC20+LC22(\lambda-1)+\frac{1}{2}LC23(\lambda-1)^2+\frac{1}{3}LC24(\lambda-1)^3+\frac{1}{4}LC25(\lambda-1)^4+\frac{1}{5}LC26(\lambda-1)^5+\frac{1}{6}LC27(\lambda-1)^6\right)$$

$$psi1One := e^{\frac{t110\ln(\lambda-1)}{h}+C10+C12(\lambda-1)+\frac{1}{2}C13(\lambda-1)^2+\frac{1}{3}C14(\lambda-1)^3+\frac{1}{4}C15(\lambda-1)^4+\frac{1}{5}C16(\lambda-1)^5+\frac{1}{6}C17(\lambda-1)^6}$$

$$psi2One := e^{\frac{t120\ln(\lambda-1)}{h}+C20+C22(\lambda-1)+\frac{1}{2}C23(\lambda-1)^2+\frac{1}{3}C24(\lambda-1)^3+\frac{1}{4}C25(\lambda-1)^4+\frac{1}{5}C26(\lambda-1)^5+\frac{1}{6}C27(\lambda-1)^6}$$

$$VOne1 := t110\ln(\lambda-1)$$
$$VOne2 := t120\ln(\lambda-1)$$

```
> L22One:=factor(h*simplify(diff(WronskianLambdabisOne,lambda)
  /WronskianLambdabisOne)):
  L22OneOrderlambdaMinus3:=factor(residue(L22One*(lambda-1)^2,
  lambda=1));
  L22OneOrderlambdaMinus2:=factor(residue(L22One*(lambda-1)^1,
  lambda=1));
  L22OneOrderlambdaMinus1:=factor(residue(L22One*(lambda-1)^0,
  lambda=1));
  L22OneOrderlambda0:=factor(residue(L22One*(lambda-1)^(-1),
  lambda=1));
  L22OneOrderlambda1:=factor(residue(L22One*(lambda-1)^(-2),
  lambda=1)):
  L22OneOrderlambda2:=factor(residue(L22One*(lambda-1)^(-3),
  lambda=1)):
```

$$L22OneOrderlambdaMinus3 := 0 \qquad \textbf{(1.8)}$$
$$L22OneOrderlambdaMinus2 := 0$$
$$L22OneOrderlambdaMinus1 := t110-h+t120$$
$$L22OneOrderlambda0 := \frac{1}{t110-t120}\left(h\left(C12\,h+C12\,t110-C12\,t120-C22\,h+C22\,t110-C22\,t120\right)\right)$$

We get that L_{2,2} behaves at lambda=1 like (t110+t120-h)/(lambda-1) +O(1)

```
> L21One:=factor(simplify
  (WronskianTildeLambdaOne/WronskianLambdabisOne)):
```

```
L21OneOrderlambdaMinus5:=factor(residue(L21One*(lambda-1)^4,
lambda=1));
L21OneOrderlambdaMinus4:=factor(residue(L21One*(lambda-1)^3,
lambda=1));
L21OneOrderlambdaMinus3:=factor(residue(L21One*(lambda-1)^2,
lambda=1));
L21OneOrderlambdaMinus2:=factor(residue(L21One*(lambda-1)^1,
lambda=1));
L21OneOrderlambdaMinus1:=factor(residue(L21One*(lambda-1)^0,
lambda=1));
L21OneOrderlambda0:=factor(residue(L21One*(lambda-1)^(-1),
lambda=1));
L21OneOrderlambda1:=factor(residue(L21One*(lambda-1)^(-2),
lambda=1)):
L21OneOrderlambda2:=factor(residue(L21One*(lambda-1)^(-3),
lambda=1)):
```

$$L21OneOrderlambdaMinus5 := 0 \quad (1.9)$$

$$L21OneOrderlambdaMinus4 := 0$$

$$L21OneOrderlambdaMinus3 := 0$$

$$L21OneOrderlambdaMinus2 := -t120\, t110$$

$$L21OneOrderlambdaMinus1 := -\frac{1}{t110 - t120}\left(h\left(C12\, h\, t120 + C12\, t110\, t120 - C12\, t120^2 - C22\, h\, t110 + C22\, t110^2 - C22\, t110\, t120\right)\right)$$

$$L21OneOrderlambda0 := \frac{1}{(t110 - t120)^2}\left(h\left(C12^2\, h^2\, t120 - C12\, C22\, h^2\, t110 - C12\, C22\, h^2\, t120 - C12\, C22\, h\, t110^2 + 2\, C12\, C22\, h\, t110\, t120 - C12\, C22\, h\, t120^2 + C22^2\, h^2\, t110 - 2\, C13\, h\, t110\, t120 + 2\, C13\, h\, t120^2 - C13\, t110^2\, t120 + 2\, C13\, t110\, t120^2 - C13\, t120^3 + 2\, C23\, h\, t110^2 - 2\, C23\, h\, t110\, t120 - C23\, t110^3 + 2\, C23\, t110^2\, t120 - C23\, t110\, t120^2\right)\right)$$

We get that L_{2,1} behaves at lambda=1 like -t110*t120/(lambda-1)^2+ O(1/(lambda-1))

Study of the asymtotics at lambda=t with the deformation operator given by h* partial_t

```
> logPsi1T:=tt10/h*ln(lambda-t)+C10+C12/(2-1)*(lambda-t)^(2-1)+
C13/(3-1)*(lambda-t)^(3-1)+C14/(4-1)*(lambda-t)^(4-1)+C15/(5-1)
*(lambda-t)^(5-1)+C16/(6-1)*(lambda-t)^(6-1)+C17/(7-1)*(lambda-
t)^(7-1)   ;
logPsi2T:=tt20/h*ln(lambda-t)+C20+C22/(2-1)*(lambda-t)^(2-1)+
C23/(3-1)*(lambda-t)^(3-1)+C24/(4-1)*(lambda-t)^(4-1)+C25/(5-1)
*(lambda-t)^(5-1)+C26/(6-1)*(lambda-t)^(6-1)+C27/(7-1)*(lambda-
t)^(7-1)   ;
Llogpsi1T:=Ltt10/h*ln(lambda-t)+LC10+LC12/(2-1)*(lambda-t)^(2
-1)+LC13/(3-1)*(lambda-t)^(3-1)+LC14/(4-1)*(lambda-t)^(4-1)+
```

```
LC15/(5-1)*(lambda-t)^(5-1)+LC16/(6-1)*(lambda-t)^(6-1)+LC17/(7
-1)*(lambda-t)^(7-1)
+h*(-tt10/h/(lambda-t)-C12*(lambda-t)^(2-1-1)-C13*(lambda-t)^(3
-1-1)-C14*(lambda-t)^(4-1-1)-C15*(lambda-t)^(5-1-1)+C16*
(lambda-t)^(6-1-1)+C17*(lambda-t)^(7-1-1));
Llogpsi2T:=Ltt20/h*ln(lambda-t)+LC20+LC22/(2-1)*(lambda-t)^(2
-1)+LC23/(3-1)*(lambda-t)^(3-1)+LC24/(4-1)*(lambda-t)^(4-1)+
LC25/(5-1)*(lambda-t)^(5-1)+LC26/(6-1)*(lambda-t)^(6-1)+LC27/(7
-1)*(lambda-t)^(7-1)
+h*(-tt20/h/(lambda-t)-C22*(lambda-t)^(2-1-1)-C23*(lambda-t)^(3
-1-1)-C24*(lambda-t)^(4-1-1)-C25*(lambda-t)^(5-1-1)+C26*
(lambda-t)^(6-1-1)+C27*(lambda-t)^(7-1-1));
Lpsi1T := exp((tt10/h*ln(lambda-t)+C10+C12/(2-1)*(lambda-t)^(2
-1)+C13/(3-1)*(lambda-t)^(3-1)+C14/(4-1)*(lambda-t)^(4-1)+C15/
(5-1)*(lambda-t)^(5-1)+C16/(6-1)*(lambda-t)^(6-1)+C17/(7-1)*
(lambda-t)^(7-1)))
*(Ltt10/h*ln(lambda-t)+LC10+LC12/(2-1)*(lambda-t)^(2-1)+LC13/(3
-1)*(lambda-t)^(3-1)+LC14/(4-1)*(lambda-t)^(4-1)+LC15/(5-1)*
(lambda-t)^(5-1)+LC16/(6-1)*(lambda-t)^(6-1)+LC17/(7-1)*
(lambda-t)^(7-1)
+h*(-tt10/h/(lambda-t)-C12*(lambda-t)^(2-1-1)-C13*(lambda-t)^(3
-1-1)-C14*(lambda-t)^(4-1-1)-C15*(lambda-t)^(5-1-1)+C16*
(lambda-t)^(6-1-1)+C17*(lambda-t)^(7-1-1))
);
Lpsi2T := exp((tt20/h*ln(lambda-t)+C20+C22/(2-1)*(lambda-t)^(2
-1)+C23/(3-1)*(lambda-t)^(3-1)+C24/(4-1)*(lambda-t)^(4-1)+C25/
(5-1)*(lambda-t)^(5-1)+C26/(6-1)*(lambda-t)^(6-1)+C27/(7-1)*
(lambda-t)^(7-1)))
*(Ltt20/h*ln(lambda-t)+LC20+LC22/(2-1)*(lambda-t)^(2-1)+LC23/(3
-1)*(lambda-t)^(3-1)+LC24/(4-1)*(lambda-t)^(4-1)+LC25/(5-1)*
(lambda-t)^(5-1)+LC26/(6-1)*(lambda-t)^(6-1)+LC27/(7-1)*
(lambda-t)^(7-1)
+h*(-tt20/h/(lambda-t)-C22*(lambda-t)^(2-1-1)-C23*(lambda-t)^(3
-1-1)-C24*(lambda-t)^(4-1-1)-C25*(lambda-t)^(5-1-1)+C26*
(lambda-t)^(6-1-1)+C27*(lambda-t)^(7-1-1)));
psi1T:=exp(logPsi1T);
psi2T:=exp(logPsi2T);
dpsi1dlambdaT:=diff(psi1T,lambda):
dpsi2dlambdaT:=diff(psi2T,lambda):
d2psi1dlambda2T:=diff(psi1T,lambda$2):
d2psi2dlambda2T:=diff(psi2T,lambda$2):
VT1:=tt10*ln(lambda-t);
```

```
VT2:=tt20*ln(lambda-t);


WronskianLambdaT:=h*factor(psi1T*dpsi2dlambdaT-psi2T*
dpsi1dlambdaT):
WronskianLambdabisT:=h*simplify(factor( (diff(logPsi2T,lambda)-
diff(logPsi1T,lambda))*exp(logPsi1T+logPsi2T))):

WronskianTildeLambdaT:=h^3*factor(dpsi2dlambdaT*
d2psi1dlambda2T-dpsi1dlambdaT*d2psi2dlambda2T):
```

$$logPsi1T := \frac{tt10\ln(\lambda-t)}{h} + C10 + C12\,(\lambda-t) + \frac{1}{2}\,C13\,(\lambda-t)^2 + \frac{1}{3}\,C14\,(\lambda-t)^3 + \frac{1}{4}\,C15\,(\lambda-t)^4 + \frac{1}{5}\,C16\,(\lambda-t)^5 + \frac{1}{6}\,C17\,(\lambda-t)^6 \qquad (1.10)$$

$$logPsi2T := \frac{tt20\ln(\lambda-t)}{h} + C20 + C22\,(\lambda-t) + \frac{1}{2}\,C23\,(\lambda-t)^2 + \frac{1}{3}\,C24\,(\lambda-t)^3 + \frac{1}{4}\,C25\,(\lambda-t)^4 + \frac{1}{5}\,C26\,(\lambda-t)^5 + \frac{1}{6}\,C27\,(\lambda-t)^6$$

$$LlogpsiIT := \frac{Ltt10\ln(\lambda-t)}{h} + LC10 + LC12\,(\lambda-t) + \frac{1}{2}\,LC13\,(\lambda-t)^2 + \frac{1}{3}\,LC14\,(\lambda-t)^3 + \frac{1}{4}\,LC15\,(\lambda-t)^4 + \frac{1}{5}\,LC16\,(\lambda-t)^5 + \frac{1}{6}\,LC17\,(\lambda-t)^6 + h\left(-\frac{tt10}{h\,(\lambda-t)} - C12 - C13\,(\lambda-t) - C14\,(\lambda-t)^2 - C15\,(\lambda-t)^3 + C16\,(\lambda-t)^4 + C17\,(\lambda-t)^5\right)$$

$$Llogpsi2T := \frac{Ltt20\ln(\lambda-t)}{h} + LC20 + LC22\,(\lambda-t) + \frac{1}{2}\,LC23\,(\lambda-t)^2 + \frac{1}{3}\,LC24\,(\lambda-t)^3 + \frac{1}{4}\,LC25\,(\lambda-t)^4 + \frac{1}{5}\,LC26\,(\lambda-t)^5 + \frac{1}{6}\,LC27\,(\lambda-t)^6 + h\left(-\frac{tt20}{h\,(\lambda-t)} - C22 - C23\,(\lambda-t) - C24\,(\lambda-t)^2 - C25\,(\lambda-t)^3 + C26\,(\lambda-t)^4 + C27\,(\lambda-t)^5\right)$$

$$Lpsi1T := \mathrm{e}^{\frac{tt10\ln(\lambda-t)}{h} + C10 + C12\,(\lambda-t) + \frac{1}{2}\,C13\,(\lambda-t)^2 + \frac{1}{3}\,C14\,(\lambda-t)^3 + \frac{1}{4}\,C15\,(\lambda-t)^4 + \frac{1}{5}\,C16\,(\lambda-t)^5 + \frac{1}{6}\,C17\,(\lambda-t)^6}\left(\frac{Ltt10\ln(\lambda-t)}{h} + LC10 + LC12\,(\lambda-t) + \frac{1}{2}\,LC13\,(\lambda-t)^2 + \frac{1}{3}\,LC14\,(\lambda-t)^3 + \frac{1}{4}\,LC15\,(\lambda-t)^4 + \frac{1}{5}\,LC16\,(\lambda-t)^5\right.$$

$$+\frac{1}{6}\,LC17\,(\lambda-t)^6+h\left(-\frac{tt10}{h\,(\lambda-t)}-C12-C13\,(\lambda-t)-C14\,(\lambda-t)^2-C15\,(\lambda-t)^3+C16\,(\lambda-t)^4+C17\,(\lambda-t)^5\right)\Bigg)$$

$Lpsi2T :=$

$$e^{\frac{tt20\ln(\lambda-t)}{h}+C20+C22\,(\lambda-t)+\frac{1}{2}\,C23\,(\lambda-t)^2+\frac{1}{3}\,C24\,(\lambda-t)^3+\frac{1}{4}\,C25\,(\lambda-t)^4+\frac{1}{5}\,C26\,(\lambda-t)^5+\frac{1}{6}\,C27\,(\lambda-t)^6}\Bigg(\frac{Ltt20\ln(\lambda-t)}{h}+LC20+LC22\,(\lambda-t)+\frac{1}{2}\,LC23\,(\lambda-t)^2+\frac{1}{3}\,LC24\,(\lambda-t)^3+\frac{1}{4}\,LC25\,(\lambda-t)^4+\frac{1}{5}\,LC26\,(\lambda-t)^5+\frac{1}{6}\,LC27\,(\lambda-t)^6+h\left(-\frac{tt20}{h\,(\lambda-t)}-C22-C23\,(\lambda-t)-C24\,(\lambda-t)^2-C25\,(\lambda-t)^3+C26\,(\lambda-t)^4+C27\,(\lambda-t)^5\right)\Bigg)$$

$psi1T :=$

$$e^{\frac{tt10\ln(\lambda-t)}{h}+C10+C12\,(\lambda-t)+\frac{1}{2}\,C13\,(\lambda-t)^2+\frac{1}{3}\,C14\,(\lambda-t)^3+\frac{1}{4}\,C15\,(\lambda-t)^4+\frac{1}{5}\,C16\,(\lambda-t)^5+\frac{1}{6}\,C17\,(\lambda-t)^6}$$

$psi2T :=$

$$e^{\frac{tt20\ln(\lambda-t)}{h}+C20+C22\,(\lambda-t)+\frac{1}{2}\,C23\,(\lambda-t)^2+\frac{1}{3}\,C24\,(\lambda-t)^3+\frac{1}{4}\,C25\,(\lambda-t)^4+\frac{1}{5}\,C26\,(\lambda-t)^5+\frac{1}{6}\,C27\,(\lambda-t)^6}$$

$$VT1 := tt10\ln(\lambda-t)$$
$$VT2 := tt20\ln(\lambda-t)$$

```
> L22T:=factor(h*simplify(diff(WronskianLambdabisT,lambda)
/WronskianLambdabisT)):
L22TOrderlambdaMinus3:=factor(residue(L22T*(lambda-t)^2,lambda=
t));
L22TOrderlambdaMinus2:=factor(residue(L22T*(lambda-t)^1,lambda=
t));
L22TOrderlambdaMinus1:=factor(residue(L22T*(lambda-t)^0,lambda=
t));
L22TOrderlambda0:=factor(residue(L22T*(lambda-t)^(-1),lambda=t)
);
L22TOrderlambda1:=factor(residue(L22T*(lambda-t)^(-2),lambda=t)
):
L22TOrderlambda2:=factor(residue(L22T*(lambda-t)^(-3),lambda=t)
):
```

$$L22TOrderlambdaMinus3 := 0 \quad \textbf{(1.11)}$$

$$L22TOrderlambdaMinus2 := 0$$
$$L22TOrderlambdaMinus1 := tt10 - h + tt20$$
$$L22TOrderlambda0 := \frac{h\,(C12\,h + C12\,tt10 - C12\,tt20 - C22\,h + C22\,tt10 - C22\,tt20)}{tt10 - tt20}$$

We get that L_{2,2} behaves at lambda=t like (tt10+tt20-h)/(lambda-t) +O(1)

```
> L21T:=factor(simplify
  (WronskianTildeLambdaT/WronskianLambdabisT)):
  L21TOrderlambdaMinus5:=factor(residue(L21T*(lambda-t)^4,lambda=
  t));
  L21TOrderlambdaMinus4:=factor(residue(L21T*(lambda-t)^3,lambda=
  t));
  L21TOrderlambdaMinus3:=factor(residue(L21T*(lambda-t)^2,lambda=
  t));
  L21TOrderlambdaMinus2:=factor(residue(L21T*(lambda-t)^1,lambda=
  t));
  L21TOrderlambdaMinus1:=factor(residue(L21T*(lambda-t)^0,lambda=
  t));
  L21TOrderlambda0:=factor(residue(L21T*(lambda-t)^(-1),lambda=t)
  );
  L21TOrderlambda1:=factor(residue(L21T*(lambda-t)^(-2),lambda=t)
  ):
  L21TOrderlambda2:=factor(residue(L21T*(lambda-t)^(-3),lambda=t)
  ):
```

$$L21TOrderlambdaMinus5 := 0 \qquad \textbf{(1.12)}$$
$$L21TOrderlambdaMinus4 := 0$$
$$L21TOrderlambdaMinus3 := 0$$
$$L21TOrderlambdaMinus2 := -tt10\,tt20$$
$$L21TOrderlambdaMinus1 := -\frac{h\,(C12\,h\,tt20 + C12\,tt10\,tt20 - C12\,tt20^2 - C22\,h\,tt10 + C22\,tt10^2 - C22\,tt10\,tt20)}{tt10 - tt20}$$
$$L21TOrderlambda0 := \frac{1}{(tt10 - tt20)^2}\big(h\,(C12^2\,h^2\,tt20 - C12\,C22\,h^2\,tt10 - C12\,C22\,h^2\,tt20 - C12\,C22\,h\,tt10^2 + 2\,C12\,C22\,h\,tt10\,tt20 - C12\,C22\,h\,tt20^2 + C22^2\,h^2\,tt10 - 2\,C13\,h\,tt10\,tt20 + 2\,C13\,h\,tt20^2 - C13\,tt10^2\,tt20 + 2\,C13\,tt10\,tt20^2 - C13\,tt20^3 + 2\,C23\,h\,tt10^2 - 2\,C23\,h\,tt10\,tt20 - C23\,tt10^3 + 2\,C23\,tt10^2\,tt20 - C23\,tt10\,tt20^2)\big)$$

We get that L_{2,1} behaves at lambda=t like -tt10*tt20/(lambda-t)^2+ O(1/(lambda-t))

```
> L22Form:=(t110+t120-h)/(lambda-1)+ (t010+t020-h)/lambda+ (tt10+
  tt20-h)/(lambda-t)    +h/(lambda-q);
  TermInLambdaMinus1:=-residue(L22Form,lambda=infinity);
  simplify(TermInLambdaMinus1- (-(tinfty10+tinfty20+2*h))-
  CoherenceEquation1);
```

$$L22Form := \frac{t110 - h + t120}{\lambda - 1} + \frac{t010 - h + t020}{\lambda} + \frac{tt10 - h + tt20}{\lambda - t} + \frac{h}{\lambda - q} \qquad \textbf{(1.13)}$$

$TermInLambdaMinus1 := t110 - 2\,h + t120 + t010 + t020 + tt10 + tt20$

0

Formulas for L_{2,2} et L_{2,1}

We have L_{2,2} behaves at lambda=1 like (t110+t120-h)/(lambda-1) +O(1)
L_{2,2} behaves at lambda=0 like (t010+t020-h)/lambda +O(1)
L_{2,2} behaves at en lambda=t like (tt10+tt20-h)/(lambda-t) +O(1)
L_{2,2} behaves at lambda=infinity like -(tinfty10+tinfty20+2\hbar)/lambda+h*O(1/lambda^2)
Thus,

L_{2,2}= (t110+t120-h)/(lambda-1)+ (t010+t020-h)/lambda+ (tt10+tt20-h)/(lambda-t) +h/(lambda-q)

with the additional condition that (t110+t120-h)+(t010+t020-h)+(tt10+tt20-h)+h=t110+t120+t010+t020+tt10+tt20-2h= -(tinfty10+tinfty20 +2h) which is equivalent to the vanishing of the sum of monodromies: t110+t120+t010+t020+tt10+tt20+tinfty10+tinfty20=0

We have L_{2,1} behaves at lambda=1 like -t110*t120/(lambda-1)^2+ O(1/(lambda-1))
L_{2,1} behaves at lambda=0 like -t010*t020/lambda^2+ O(1/lambda)
L_{2,1} behaves at lambda=t like -tt10*tt20/(lambda-t)^2+ O(1/(lambda-t))
L_{2,1} behaves at lambda=infinity like (-tinfty10tinfty20 -h*tinfty10)/lambda^2 +O(lambda^{-1})
Thus,

L_{2,1}=-t110*t120/(lambda-1)^2 -a_1/(lambda-1) -t010*t020/lambda^2 -a_0/lambda -tt10*tt20/(lambda-t)^2 -a_t/(lambda-t) - p*h/(lambda-q)

with the conditions **a_0+a_1+a_t+ h*p=0** and **a_1+t*a_t+t010*t020+t110*t120+tt10*tt20-tinfty10*tinfty20+h*p*q -h*tinfty10=0**

```
> L21Form:=-t110*t120/(lambda-1)^2 -a_1/(lambda-1)   -t010*t020/lambda^2 -a_0/lambda -tt10*tt20/(lambda-t)^2 -a_t/(lambda-t) - p*h/(lambda-q);
L21FormOrderLambdaMinus1:=factor(-residue(L21Form,lambda=infinity));
L21FormOrderLambdaMinus2:=factor(-residue(L21Form*lambda,lambda=infinity));
CoherenceEquation4:= -(L21FormOrderLambdaMinus2-L21InftyOrderlambdaMinus2);
```

$$L21Form := -\frac{t110\,t120}{(\lambda-1)^2} - \frac{a\_1}{\lambda-1} - \frac{t010\,t020}{\lambda^2} - \frac{a\_0}{\lambda} - \frac{tt10\,tt20}{(\lambda-t)^2} - \frac{a\_t}{\lambda-t} - \frac{p\,h}{\lambda-q} \quad (1.14)$$

$$L21FormOrderLambdaMinus1 := -h\,p - a\_0 - a\_1 - a\_t$$

$$L21FormOrderLambdaMinus2 := -h\,p\,q - a\_t\,t - t010\,t020 - t110\,t120 - tt10\,tt20 - a\_1$$

$$CoherenceEquation4 := p\,h\,q + a\_t\,t + t010\,t020 + t120\,t110 + tt10\,tt20 + a\_1 - (h + tinfty20)\,tinfty10$$

Computation of the auxiliary Lax matrix A using the asymptotics of the wave functions

```
> WronskianLInfty:=factor(psi1Infty*Lpsi2Infty-psi2Infty*Lpsi1Infty):
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WronskianLZero:=factor(psi1Zero*Lpsi2Zero-psi2Zero*Lpsi1Zero):
WronskianLOne:=factor(psi1One*Lpsi2One-psi2One*Lpsi1One):
WronskianLT:=factor(psi1T*Lpsi2T-psi2T*Lpsi1T):
A12Infty:=factor(simplify(WronskianLInfty/WronskianLambdaInfty)
):
A12Zero:=factor(simplify(WronskianLZero/WronskianLambdaZero)):
A12One:=factor(simplify(WronskianLOne/WronskianLambdaOne)):
A12T:=factor(simplify(WronskianLT/WronskianLambdaT)):
Y1Infty:=h*factor(dpsi1dlambdaInfty/psi1Infty):
Y2Infty:=h*factor(dpsi2dlambdaInfty/psi2Infty):
Y1Zero:=h*factor(dpsi1dlambdaZero/psi1Zero):
Y2Zero:=h*factor(dpsi2dlambdaZero/psi2Zero):
Y1One:=h*factor(dpsi1dlambdaOne/psi1One):
Y2One:=h*factor(dpsi2dlambdaOne/psi2One):
Y1T:=h*factor(dpsi1dlambdaT/psi1T):
Y2T:=h*factor(dpsi2dlambdaT/psi2T):
Z1Infty:=factor(Lpsi1Infty/psi1Infty):
Z2Infty:=factor(Lpsi2Infty/psi2Infty):
Z1Zero:=factor(Lpsi1Zero/psi1Zero):
Z2Zero:=factor(Lpsi2Zero/psi2Zero):
Z1One:=factor(Lpsi1One/psi1One):
Z2One:=factor(Lpsi2One/psi2One):
Z1T:=factor(Lpsi1T/psi1T):
Z2T:=factor(Lpsi2T/psi2T):
A12bisInfty:=factor(simplify((Z2Infty-Z1Infty)/(Y2Infty-
Y1Infty))):
A12bisZero:=factor(simplify((Z2Zero-Z1Zero)/(Y2Zero-Y1Zero))):
A12bisOne:=factor(simplify((Z2One-Z1One)/(Y2One-Y1One))):
A12bisT:=factor(simplify((Z2T-Z1T)/(Y2T-Y1T))):
A11Infty:=factor(simplify( (Y2Infty*Z1Infty-Y1Infty*Z2Infty)/
(Y2Infty-Y1Infty) )):
A11Zero:=factor(simplify( (Y2Zero*Z1Zero-Y1Zero*Z2Zero)/
(Y2Zero-Y1Zero) )):
A11One:=factor(simplify( (Y2One*Z1One-Y1One*Z2One)/(Y2One-
Y1One) )):
A11T:=factor(simplify( (Y2T*Z1T-Y1T*Z2T)/(Y2T-Y1T) )):
factor(simplify(A12bisInfty-A12Infty));
factor(simplify(A12bisZero-A12Zero));
factor(simplify(A12bisZero-A12Zero));
```

(2.1)

$$0 \quad \textbf{(2.1)}$$
$$0$$
$$0$$

```
> Lt020:=Lt010:
  Lt120:=Lt110:
  Ltt20:=Ltt10:
  Lt010:=0:
  Lt110:=0:
  Ltt10:=0:
  Ltinfty20:=Ltinfty10:
  Ltinfty10:=0:
```

Study of A_{1,2}

```
> A12InftyLambda3:=factor(-residue(A12Infty/lambda^4,lambda=
  infinity));
  A12InftyLambda2:=factor(-residue(A12Infty/lambda^3,lambda=
  infinity));
  A12InftyLambda1:=factor(-residue(A12Infty/lambda^2,lambda=
  infinity));
  A12InftyLambda0:=factor(-residue(A12Infty/lambda^1,lambda=
  infinity));
  A12InftyLambdaMinus1:=factor(-residue(A12Infty/lambda^0,lambda=
  infinity)):
```

$$A12InftyLambda3 := 0 \quad \textbf{(2.2)}$$
$$A12InftyLambda2 := 0$$
$$A12InftyLambda1 := \frac{LA10 - LA20}{h - tinfty10 + tinfty20}$$
$$A12InftyLambda0 := -\frac{1}{(h - tinfty10 + tinfty20)^2} (A12\ LA10\ h - A12\ LA20\ h - A22\ LA10\ h + A22\ LA20\ h + LA12\ h - LA12\ tinfty10 + LA12\ tinfty20 - LA22\ h + LA22\ tinfty10 - LA22\ tinfty20)$$

```
> A12ZeroLambdaMinus3:=factor(residue(A12Zero*lambda^2,lambda=0))
  ;
  A12ZeroLambdaMinus2:=factor(residue(A12Zero*lambda^1,lambda=0))
  ;
  A12ZeroLambdaMinus1:=factor(residue(A12Zero*lambda^0,lambda=0))
  ;
  A12ZeroLambda0:=factor(residue(A12Zero*lambda^(-1),lambda=0));
  A12ZeroLambda1:=factor(residue(A12Zero*lambda^(-2),lambda=0));
```

$$A12ZeroLambdaMinus3 := 0 \quad \textbf{(2.3)}$$
$$A12ZeroLambdaMinus2 := 0$$

$$A12ZeroLambdaMinus1 := 0$$
$$A12ZeroLambda0 := 0$$
$$A12ZeroLambda1 := \frac{-LB20 + LB10}{t010 - t020}$$

```
> A12OneLambdaMinus3:=factor(residue(A12One*(lambda-1)^2,lambda=
1));
A12OneLambdaMinus2:=factor(residue(A12One*(lambda-1)^1,lambda=
1));
A12OneLambdaMinus1:=factor(residue(A12One*(lambda-1)^0,lambda=
1));
A12OneLambda0:=factor(residue(A12One*(lambda-1)^(-1),lambda=1))
;
A12OneLambda1:=factor(residue(A12One*(lambda-1)^(-2),lambda=1))
;
```

$$A12OneLambdaMinus3 := 0 \tag{2.4}$$
$$A12OneLambdaMinus2 := 0$$
$$A12OneLambdaMinus1 := 0$$
$$A12OneLambda0 := 0$$
$$A12OneLambda1 := \frac{LC10 - LC20}{t110 - t120}$$

```
> A12TLambdaMinus3:=factor(residue(A12T*(lambda-t)^2,lambda=t));
A12TLambdaMinus2:=factor(residue(A12T*(lambda-t)^1,lambda=t));
A12TLambdaMinus1:=factor(residue(A12T*(lambda-t)^0,lambda=t));
A12TLambda0:=factor(residue(A12T*(lambda-t)^(-1),lambda=t));
A12TLambda1:=factor(residue(A12T*(lambda-t)^(-2),lambda=t));
```

$$A12TLambdaMinus3 := 0 \tag{2.5}$$
$$A12TLambdaMinus2 := 0$$
$$A12TLambdaMinus1 := 0$$
$$A12TLambda0 := -1$$
$$A12TLambda1 := \frac{LC10 - LC20}{tt10 - tt20}$$

We thus obtain that **A_{1,2}= a*(lambda-t)+b-1+(q-t)*b/(lambda-q)**

```
> A12Form:=a*(lambda-t)+b-1+ (q-t)*b/(lambda-q);
factor( residue(A12Form/(lambda-t),lambda=t)-(-1));

A12Form:=(a*(lambda-t)^2+b*(lambda-t)+q-t)/(lambda-q);
factor( residue(A12Form/(lambda-t),lambda=t)-(-1));
```

$$A12Form := a\left(\lambda - t\right) + b - 1 + \frac{(q-t)\,b}{\lambda - q} \tag{2.6}$$
$$0$$
$$A12Form := \frac{a\left(\lambda - t\right)^2 + b\left(\lambda - t\right) + q - t}{\lambda - q}$$
$$0$$

Study of A_{1,1}

```
> A11InftyLambda3:=factor(-residue(A11Infty/lambda^4,lambda=
```

```
infinity));
A11InftyLambda2:=factor(-residue(A11Infty/lambda^3,lambda=
infinity));
A11InftyLambda1:=factor(-residue(A11Infty/lambda^2,lambda=
infinity));
A11InftyLambda0:=factor(-residue(A11Infty/lambda^1,lambda=
infinity));
A11InftyLambdaMinus1:=factor(-residue(A11Infty/lambda^0,lambda=
infinity)):

A11ZeroLambdaMinus3:=factor(residue(A11Zero*lambda^2,lambda=0))
;
A11ZeroLambdaMinus2:=factor(residue(A11Zero*lambda^1,lambda=0))
;
A11ZeroLambdaMinus1:=factor(residue(A11Zero*lambda^0,lambda=0))
;
A11ZeroLambda0:=factor(residue(A11Zero*lambda^(-1),lambda=0));
A11ZeroLambda1:=factor(residue(A11Zero*lambda^(-2),lambda=0)):

A11OneLambdaMinus3:=factor(residue(A11One*(lambda-1)^2,lambda=
1));
A11OneLambdaMinus2:=factor(residue(A11One*(lambda-1)^1,lambda=
1));
A11OneLambdaMinus1:=factor(residue(A11One*(lambda-1)^0,lambda=
1));
A11OneLambda0:=factor(residue(A11One*(lambda-1)^(-1),lambda=1))
;
A11OneLambda1:=factor(residue(A11One*(lambda-1)^(-2),lambda=1))
:

A11TLambdaMinus3:=factor(residue(A11T*(lambda-t)^2,lambda=t));
A11TLambdaMinus2:=factor(residue(A11T*(lambda-t)^1,lambda=t));
A11TLambdaMinus1:=factor(residue(A11T*(lambda-t)^0,lambda=t));
A11TLambda0:=factor(residue(A11T*(lambda-t)^(-1),lambda=t));
A11TLambda1:=factor(residue(A11T*(lambda-t)^(-2),lambda=t)):
```

$$A11InftyLambda3 := 0 \tag{2.7}$$

$$A11InftyLambda2 := 0$$

$$A11InftyLambda1 := 0$$

$$A11InftyLambda0 := \frac{LA10\,h + LA10\,tinfty20 - LA20\,tinfty10}{h - tinfty10 + tinfty20}$$

$$A11ZeroLambdaMinus3 := 0$$

$$A11ZeroLambdaMinus2 := 0$$
$$A11ZeroLambdaMinus1 := 0$$
$$A11ZeroLambda0 := -\frac{LB10\ t020 - LB20\ t010}{t010 - t020}$$
$$A11OneLambdaMinus3 := 0$$
$$A11OneLambdaMinus2 := 0$$
$$A11OneLambdaMinus1 := 0$$
$$A11OneLambda0 := -\frac{LC10\ t120 - LC20\ t110}{t110 - t120}$$
$$A11TLambdaMinus3 := 0$$
$$A11TLambdaMinus2 := 0$$
$$A11TLambdaMinus1 := 0$$
$$A11TLambda0 := -\frac{LC10\ tt20 - LC20\ tt10}{tt10 - tt20}$$

We thus obtain that **A_{1,1}=C+mu/(lambda-q)**

In the end the first line of the auxiliary matrix A is given by
A_{1,1}=C+ mu/(lambda-q)
A_{1,2}= a*(lambda-t)+b-1+(q-t)*b/(lambda-q)