Isomonodromic deformations, exact WKB analysis and Painlevé 1

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Main objectives of the talk

1. Introduce the theory of isomonodromic deformations of ($\hbar$-deformed) rational connections in $\mathfrak{gl}_2(\mathbb{C})$ that includes the Painlevé equations.
2. Show how to obtain the symplectic structure (Hamiltonians) for a specific set of Darboux coordinates.
3. Reverse way: Formal reconstruction using Topological Recursion via quantization of classical spectral curves.
4. Application to Exact WKB reconstruction for 0-parameter solutions of the Painlevé 1 equation.

Isomonodromic deformations in $\mathfrak{gl}_2(\mathbb{C})$

History

- Isomonodromic deformations dates back to the **beginning of** 20^{th} **century**. Big names in the theory are Picard, Fuchs, Painlevé, Garnier, Okamoto, Malmquist, Schlesinger, then Sato, Jimbo, Miwa, Ueno, Lax and more recently Harnad, Hurtibise, Bertola and Boalch, Yamakawa, Woodhouse, Komyo and many others.
- Geometric point of view presented in this talk is mostly based on P. Boalch and D. Yamakawa's approach.
- Topic belongs to "*integrable systems*" at the border of geometry (differential and symplectic), PDEs and mathematical physics.
- **Literature is vast and diverse** (from very abstract geometry to big formulas or applications) and with remaining open questions.

Rational connections in $\mathfrak{gl}_2(\mathbb{C})$

Rational connections in $\mathfrak{gl}_2(\mathbb{C})$

Let $\{X_i\}_{i=1}^n$ be n distinct points in the complex plane. Take $\mathbf{r} := (r_\infty, r_1, \dots, r_n) \in (\mathbb{N} \setminus \{0\})^{n+1}$, and define

$$F_{\mathcal{R},\mathbf{r}} := \left\{ \hat{L}(\lambda) = \sum_{k=1}^{r_\infty - 1} \hat{L}^{[\infty,k]} \lambda^{k-1} + \sum_{s=1}^{n} \sum_{k=0}^{r_s - 1} \frac{\hat{L}^{[X_s,k]}}{(\lambda - X_s)^{k+1}} \text{ with } \left\{ \hat{L}^{[p,k]} \right\} \in \left(\mathfrak{gl}_2 \right)^{r-1} \right\} / \mathrm{GL}_2(\mathbb{C})$$

where $r = r_\infty + \sum_{s=1}^{n} r_s$ and $\mathrm{GL}_2(\mathbb{C})$ acts simultaneously by conjugation on all coefficients $\{\hat{L}^{[p,k]}\}_{p,k}$.

Short version

$\hat{L}(\lambda)$ is a rational function with fixed poles (including ∞) of given order with values in $\mathfrak{gl}_2(\mathbb{C})$. Global conjugation action shall be used to select a representative normalized at infinity.

Connections and gauge transformation

Connections and horizontal sections

The linear differential system

$$\partial_\lambda \hat{\Psi}(\lambda) = \hat{L}(\lambda)\hat{\Psi}(\lambda)$$

defines a rational connection on $\mathfrak{gl}_2(\mathbb{C})$. $\hat{\Psi}(\lambda)$ is called the horizontal section or wave matrix. $\hat{L}(\lambda)$ is called the Lax matrix.

Gauge transformations

Performing a gauge transformation $\hat{\Psi} \to G(\lambda)\hat{\Psi}$ implies that

$$\hat{L}(\lambda) \to G(\lambda)\hat{L}(\lambda)G^{-1}(\lambda) + (\partial_\lambda G)G(\lambda)^{-1}$$

Local diagonalization of the singular parts

Generic case: Local diagonalization of the singular part at each pole

Let $\hat{L}(\lambda) \in \hat{F}_{\mathcal{R},\mathbf{r}}$ the subset of $F_{\mathcal{R},\mathbf{r}}$ such that all coefficients have distinct eigenvalues (generic case). At any pole $p \in \{X_1, \dots, X_n, \infty\}$there exists a local gauge transformation $G_p(\lambda)$ locally holomorphic in λ such that $\Psi_p = G_p(\lambda)\hat{\Psi}(\lambda)$ is

$$\Psi_p(\lambda) = \Psi_p^{(\text{reg})}(\lambda)\,\text{diag}\left(\exp\left(-\sum_{k=1}^{r_p-1} \frac{t_{p^{(1)},k}}{k z_p(\lambda)^k} + t_{p^{(1)},0}\ln z_p(\lambda)\right), \exp\left(-\sum_{k=1}^{r_p-1} \frac{t_{p^{(2)},k}}{k z_p(\lambda)^k} + t_{p^{(2)},0}\ln z_p(\lambda)\right)\right)$$

with $z_{X_s}(\lambda) = (\lambda - X_s)$ (or $z_\infty(\lambda) = \lambda^{-1}$ at infinity) and $\Psi_p^{(\text{reg})}(\lambda)$ is regular at $\lambda \to p$. The Lax matrix has a locally diagonal singular part:

$$L_p(\lambda) = \text{diag}\left(\sum_{k=1}^{r_p-1} \frac{t_{p^{(1)},k}}{z_p(\lambda)^{k+1}} + \frac{t_{p^{(1)},0}}{z_p(\lambda)}, \sum_{k=1}^{r_p-1} \frac{t_{p^{(2)},k}}{z_p(\lambda)^{k+1}} + \frac{t_{p^{(2)},0}}{z_p(\lambda)}\right) + O(1)$$

Comments on local diagonalizations

- Local diagonalization is known as "**Birkhoff factorization**" or "formal normal solution" or "Turritin-Levelt fundamental form".
- Definition needs adaptation if the matrices are not diagonalizable (e.g. Painlevé 1) using $z_p(\lambda) = (\lambda - X_s)^{\frac{1}{2}}$ and holomorphic in z_p and $z_p G_p$ is locally holomorphic in z_p. Case known as "**twisted case**".
- Local diagonalizations provide a **canonical set of irregular times** $\mathbf{t} := (t_{p^{(i)},k})_{p,i,k\geq 1}$ **and monodromies** $\mathbf{t}_0 := (t_{p^{(i)},0})_{p,i}$ **to parametrize the connections** in addition to the **location of poles** $(X_i)_i$.
- Singularities with $r_p = 1$ are called *Fuchsian singularities* (no irregular times, only location of pole).
- Construction is similar for connections in $\mathfrak{gl}_d(\mathbb{C})$ with $d \geq 2$, but many more ways to twist depending on the Jordan blocks of the singular parts.

General picture

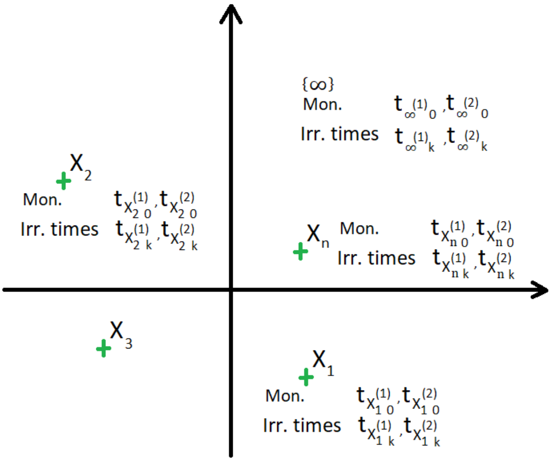


Figure: *Summary of the notation for poles, monodromies and irregular times parametrizing the family of connections*

Moduli space and symplectic manifold

Symplectic manifold

$$\hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t_0}} := \left\{ \hat{L}(\lambda) \in \hat{F}_{\mathcal{R},\mathbf{r}} \ / \ \hat{L}(\lambda) \text{ has irregular times } \mathbf{t} \text{ and monodromies } \mathbf{t_0} \right\}$$

is a symplectic manifold of dimension

$$\dim \hat{\mathcal{M}}_{\mathcal{R},\mathbf{r},\mathbf{t},\mathbf{t}_0} = 4r - 7 - (2r-1) = 2g \quad \text{where} \quad g := r - 3$$

g is the genus of the **spectral curve defined by the algebraic equation** $\det(yI_2 - \hat{L}(\lambda)) = 0$.

Darboux coordinates

The Lax matrix $\hat{L}(\lambda)$ is completely determined by the **poles**, **irregular times**, **monodromies** and $2g$ Darboux coordinates $(q_j, p_j)_{1\leq j\leq g}$ whose evolutions relatively to the irregular times and position of poles (i.e. **isomonodromic deformations**) are Hamiltonians.

Introduction of $\hbar$

Introduction of a formal $\hbar$ parameter

One can perform a rescaling of the quantities:

$$\begin{array}{rcl} t_{\infty^{(i)},k} & \to & \hbar^{k-1} t_{\infty^{(i)},k} \;,\; \forall (i,k) \in [\![1,2]\!] \times [\![0, r_\infty - 1]\!], \\ t_{X_s^{(i)},k} & \to & \hbar^{-1-k} t_{X_s^{(i)},k} \;,\; \forall (i,s,k) \in [\![1,2]\!] \times [\![1,n]\!] \times [\![0, r_s - 1]\!], \\ X_s & \to & \hbar^{-1} X_s \;,\; \forall s \in [\![1,n]\!], \\ \lambda & \to & \hbar^{-1}\lambda \\ \hat{\Psi} & \to & \mathrm{diag}\left(\hbar^{-\frac{r_\infty - 3}{2}}, \hbar^{\frac{r_\infty - 3}{2}}\right) \hat{\Psi} \end{array}$$

so that the differential system reads ("$\hbar$-deformed connections")

$$\hbar \partial_\lambda \hat{\Psi}(\lambda,\hbar) = \hat{L}(\lambda,\hbar)\hat{\Psi}(\lambda,\hbar)$$

Gauge transformations $\hat{\Psi} \to G\hat{\Psi}$ become

$$\hat{L} \to G\hat{L}G^{-1} + \hbar(\partial_\lambda G)G^{-1}$$

$\hbar$ interpolates between usual isomonodromic world $(\hbar = 1)$ and isospectral world $(\hbar \to 0)$.

Lax pairs and isomonodromic deformations

- Construction of a ($\hbar$-deformed) rational connection (Lax matrix $\hat{L}$) in $\mathfrak{gl}_2(\mathbb{C})$ with given pole structure.
- It is parametrized by
  1. Location of poles: $(X_s)_{1\leq s\leq n}$.
  2. Irregular times $\left(t_{p^{(i)},k}\right)_{p,i,k}$ (from local diagonalization at each pole).
  3. Monodromies $\left(t_{p^{(i)},0}\right)_{p,i}$ (from local diagonalization at each pole).
- **Isomonodromic def. $\Leftrightarrow$ deformations relatively to irregular times and location of poles**. **Compatible auxiliary systems**:

$$\hbar\partial_t\hat{\Psi}(\lambda,\mathbf{t};\hbar)=\hat{A}_t(\lambda,\mathbf{t};\hbar)\hat{\Psi}(\lambda,\mathbf{t};\hbar)$$

  with $\hat{A}_t(\lambda,\mathbf{t};\hbar)$ **rational in λ with same pole structure as** $\hat{L}$.
- Compatibility of the systems implies compatibility equations ("zero-curvature equation")

$$\begin{aligned} 0 &= \hbar\partial_t\hat{L}-\hbar\partial_\lambda\hat{A}_t+\left[\hat{L},\hat{A}_t\right] \\ 0 &= \hbar\partial_{t'}\hat{A}_t-\hbar\partial_{t'}\hat{A}_t+\left[\hat{A}_{t'},\hat{A}_t\right] \end{aligned}$$

- $\left(\hat{L},\hat{A}_t\right)$ are called **Lax pairs**.

Next steps

1. Define **suitable Darboux coordinates** $(\mathbf{q},\mathbf{p}) := (q_i,p_i)_{i=1}^{g}$ and **express the Lax pair $\left(\hat{L},\hat{A}_t\right)$ in terms of Darboux coordinates, location of poles, irregular times and monodromies.**
2. **Solve the compatibility equations** to obtain the **Hamiltonian evolutions of the Darboux coordinates**.

$$\hbar\partial_t q = \frac{\partial \,\mathrm{Ham}_t(\mathbf{q},\mathbf{p},\mathbf{t};\hbar)}{\partial p} \;,\; \hbar\partial_t p = -\frac{\partial \,\mathrm{Ham}_t(\mathbf{q},\mathbf{p},\mathbf{t};\hbar)}{\partial q}$$

3. Reduce the (big) deformation space to only **g non-trivial directions** to get Arnold-Liouville form of the Hamiltonian system (*symplectic reduction*).

Oper gauge and choice of Darboux coordinates

Oper gauge or companion-like gauge

Let $G(\lambda) := \begin{pmatrix} 1 & 0 \\ \hat{L}_{1,1}(\lambda) & \hat{L}_{1,2}(\lambda) \end{pmatrix}$, $\Psi(\lambda) := G(\lambda)\hat{\Psi}(\lambda) = \begin{pmatrix} \hat{\Psi}_{1,1} & \hat{\Psi}_{1,2} \\ \hbar\partial_\lambda \hat{\Psi}_{1,1} & \hbar\partial_\lambda \hat{\Psi}_{1,2} \end{pmatrix}$

$$\text{Then : } \hbar\partial_\lambda \Psi = \begin{pmatrix} 0 & 1 \\ L_{2,1} & L_{2,2} \end{pmatrix} \Psi := L(\lambda)\Psi \quad \text{and } \hbar\partial_t \Psi := A_t(\lambda)\Psi$$

i.e. $\Psi_{1,1} = \hat{\Psi}_{1,1}$ and $\Psi_{1,2} = \hat{\Psi}_{1,2}$ satisfies the **quantum curve**

$$\left[\hbar^2 \frac{\partial^2}{\partial \lambda^2} - L_{2,2}(\lambda)\hbar \frac{\partial}{\partial \lambda} - L_{2,1}(\lambda)\right] \Psi_{1,j}(\lambda) = 0$$

Apparent singularities

$$L_{2,1} = -\det \hat{L} + \hbar\partial_\lambda \hat{L}_{1,1} - \hbar \hat{L}_{1,1} \frac{\partial_\lambda \hat{L}_{1,2}}{\hat{L}_{1,2}} \;,\; L_{2,2} = \operatorname{Tr} \hat{L} + \hbar \frac{\partial_\lambda \hat{L}_{1,2}}{\hat{L}_{1,2}}$$

$\Rightarrow L(\lambda)$ has apparent singularities at the zeros of $\hat{L}_{1,2}(\lambda)$ that we shall denote $(q_j)_{1\leq j\leq g}$ **and take as half of the Darboux coordinates.**

Choice of Darboux coordinates

- Idea to use the oper gauge and the apparent singularities as natural Darboux coordinates dates back at least to Jimbo, Miwa, Ueno.
- We complement the Darboux coordinates by

$$p_i := -\frac{1}{\hbar} \operatorname*{Res}_{\lambda \to q_i} L_{2,1}(\lambda) = \hat{L}_{1,1}(q_i) \ , \ \forall\, i \in [\![1, g]\!]$$

$\det(p_i I_2 - \hat{L}(q_i)) = 0 \Rightarrow (\mathbf{q_i}, \mathbf{p_i})_{\mathbf{i=1}}^{\mathbf{g}}$ **are points on the spectral curve**.

- Oper gauge has computational advantages: only $L_{2,1}$ and $L_{2,2}$.
- Define the **general auxiliary matrix** $A_{\boldsymbol{\alpha}}(\lambda)$ such that $\mathcal{L}_{\boldsymbol{\alpha}}\Psi := A_{\boldsymbol{\alpha}}(\lambda)\Psi$ where $\boldsymbol{\alpha} := (\alpha_{p^{(i)},k})_{p,i,k}$ describes the **full tangent space of isomonodromic deformations**:

$$\mathcal{L}_{\boldsymbol{\alpha}} := \hbar \sum_{i=1}^{2} \sum_{k=1}^{r_\infty - 1} \alpha_{\infty^{(i)},k} \partial_{t_{\infty^{(i)},k}} + \hbar \sum_{i=1}^{2} \sum_{s=1}^{n} \sum_{k=1}^{r_s - 1} \alpha_{X_s^{(i)},k} \partial_{t_{X_s^{(i)},k}} + \hbar \sum_{s=1}^{n} \alpha_{X_s} \partial_{X_s}$$

Compatibility equation provides the second line of $A_{\boldsymbol{\alpha}}(\lambda)$:

$$\begin{aligned}
[A_{\boldsymbol{\alpha}}(\lambda)]_{2,1} &= \hbar\partial_\lambda [A_{\boldsymbol{\alpha}}(\lambda)]_{1,1} + [A_{\boldsymbol{\alpha}}(\lambda)]_{1,2}\, L_{2,1}(\lambda), \\
[A_{\boldsymbol{\alpha}}(\lambda)]_{2,2} &= \hbar\partial_\lambda [A_{\boldsymbol{\alpha}}(\lambda)]_{1,2} + [A_{\boldsymbol{\alpha}}(\lambda)]_{1,1} + [A_{\boldsymbol{\alpha}}(\lambda)]_{1,2}\, L_{2,2}(\lambda),
\end{aligned}$$

Explicit expressions for the Hamiltonians and Lax matrices in the oper gauge

General expressions

There exist explicit expressions of the Hamiltonians, the Lax matrix L and auxiliary matrix A_α in the oper gauge in terms of poles, irregular times, monodromies and our choice of Darboux coordinates [8, 9].

- One can also obtain explicit expressions for the Lax matrix $\hat{L}$ and auxiliary matrix $\hat{A}_\alpha$in the initial geometric gauge by expressing the gauge transformation $G(\lambda)$ in terms of those quantities (Prop. 2.1 of [9]).
- Due to normalization at infinity of $\hat{L}$, expressions require special attention for $r_\infty \leq 2$.
- General strategy to obtain formulas is to **solve the compatibility equations in the oper gauge** giving evolutions of $(\mathbf{q}, \mathbf{p})$. Requires substantial computations.

Expression for the Lax matrix

Analysis of the singular part at each pole provides information on the most singular coefficients of the entries $L_{1,1}$ and $L_{1,2}$.

Functions P_1 and P_2

We define the **rational functions** $P_1(\lambda)$ and $P_2(\lambda)$ in terms of **irregular times and monodromies** by

$$P_1(\lambda) = -\sum_{j=0}^{r_\infty-2}\left(t_{\infty^{(1)},k+1}+t_{\infty^{(2)},k+1}\right)\lambda^j+\sum_{s=1}^{n}\sum_{j=1}^{r_s}\frac{t_{X_s^{(1)},k-1}+t_{X_s^{(2)},k-1}}{(\lambda-X_s)^j}$$

$$P_2(\lambda) = \sum_{j=\max(0,r_\infty-3)}^{2r_\infty-4} P^{(2)}_{\infty,j}\lambda^j+\sum_{s=1}^{n}\sum_{j=r_s+1}^{2r_s}\frac{P^{(2)}_{X_s,j}}{(\lambda-X_s)^j}$$

$$P^{(2)}_{\infty,2r_\infty-4-k} = \sum_{j=0}^{k} t_{\infty^{(1)},r_\infty-1-j}\, t_{\infty^{(2)},r_\infty-1-(k-j)}\ ,\quad \forall\, k\in [\![0,r_\infty-1]\!]$$

$$P^{(2)}_{X_s,2r_s-k} = \sum_{j=0}^{k} t_{X_s^{(1)},r_s-1-j}\, t_{X_s^{(2)},r_s-1-(k-j)}\ ,\quad \forall\, s\in[\![1,n]\!]\ ,\quad \forall\, k\in[\![0,r_s-1]\!]$$

Expression for the Lax matrix

Expression of the Lax matrix in the oper gauge

The Lax matrix $L(\lambda)$ in the oper gauge is of the form:

$$\begin{aligned}
L_{1,1}(\lambda) &= 0\\
L_{1,2}(\lambda) &= 1\\
L_{2,2}(\lambda) &= P_1(\lambda) + \sum_{j=1}^{g} \frac{\hbar}{\lambda - q_j} - \sum_{s=1}^{n} \frac{\hbar r_s}{\lambda - X_s}\\
L_{2,1}(\lambda) &= -P_2(\lambda) - \sum_{j=1}^{g} \frac{\hbar p_j}{\lambda - q_j} - \hbar t_{\infty^{(1)}, r_\infty - 1} \lambda^{r_\infty - 3} \delta_{r_\infty \geq 3}\\
&\quad + \sum_{j=0}^{r_\infty - 4} H_{\infty,j} \lambda^j + \sum_{s=1}^{n} \sum_{j=1}^{r_s} \frac{H_{X_s,j}}{(\lambda - X_s)^j}
\end{aligned}$$

Only g coefficients $(H_{p,j})_{p,j}$ (often called "***spectral invariants***") **remain to determine**. Regroup them into vectors $\mathbf{H}_\infty := (H_{\infty,0}, \dots H_{\infty, r_\infty - 4})^t$ and $\mathbf{H}_{X_s} := (H_{X_s,1}, \dots, H_{X_s, r_s})^t$.

Expression for the Lax matrix 2

Expression for coefficients $(H_{p,j})_{p,j}$

Coefficients $(H_{p,j})_{p,j}$ are determined by:

$$\begin{pmatrix} (V_\infty)^t & (V_1)^t & \dots & (V_n)^t \end{pmatrix} \begin{pmatrix} \mathbf{H}_\infty \\ \mathbf{H}_{X_1} \\ \vdots \\ \mathbf{H}_{X_n} \end{pmatrix} =$$

$$\begin{pmatrix} p_1^2 - P_1(q_1)p_1 + p_1 \sum_{s=1}^{n} \frac{\hbar r_s}{q_1 - X_s} + P_2(q_1) + \hbar \sum_{i \neq 1} \frac{p_i - p_1}{q_1 - q_i} + \hbar t_{\infty^{(1)}, r_\infty - 1} q_1^{r_\infty - 3} \delta_{r_\infty \geq 3} \\ \vdots \\ \vdots \\ p_g^2 - P_1(q_g)p_g + p_g \sum_{s=1}^{n} \frac{\hbar r_s}{q_g - X_s} + P_2(q_g) + \hbar \sum_{i \neq g} \frac{p_i - p_g}{q_g - q_i} + \hbar t_{\infty^{(1)}, r_\infty - 1} q_g^{r_\infty - 3} \delta_{r_\infty \geq 3} \end{pmatrix}$$

where matrices $(V_\infty, V_1, \dots, V_n)$ are rectangular **Vandermonde matrices** with entries given by the apparent singularities. Coefficients $(H_{p,j})_{p,j}$ depend on the whole pole structure not only pole by pole.

Expression for the Lax matrix 3

Expression for the Vandermonde matrices

The Vandermonde matrices $(V_\infty, V_1, \dots, V_n)$ are given by

$$V_\infty := \begin{pmatrix} 1 & 1 & \dots & \dots & 1 \\ q_1 & q_2 & \dots & \dots & q_g \\ \vdots & & & & \vdots \\ \vdots & & & & \vdots \\ q_1^{r_\infty - 4} & q_2^{r_\infty - 4} & \dots & \dots & q_g^{r_\infty - 4} \end{pmatrix}$$

$$V_s := \begin{pmatrix} \frac{1}{q_1 - X_s} & \dots & \dots & \frac{1}{q_g - X_s} \\ \frac{1}{(q_1 - X_s)^2} & \dots & \dots & \frac{1}{(q_g - X_s)^2} \\ \vdots & & & \vdots \\ \vdots & & & \vdots \\ \frac{1}{(q_1 - X_s)^{r_s}} & \dots & \dots & \frac{1}{(q_g - X_s)^{r_s}} \end{pmatrix}, \ \forall\, s \in [\![1, n]\!]$$

Expression for the general Hamiltonian

General isomonodromic deformation

For a vector $\boldsymbol{\alpha} \in \mathbb{C}^{2g+4-n}$, define

$$\mathcal{L}_{\boldsymbol{\alpha}} := \hbar \sum_{i=1}^{2} \sum_{k=1}^{r_\infty - 1} \alpha_{\infty^{(i)},k} \partial_{t_{\infty^{(i)},k}} + \hbar \sum_{i=1}^{2} \sum_{s=1}^{n} \sum_{k=1}^{r_s - 1} \alpha_{X_s^{(i)},k} \partial_{t_{X_s^{(i)},k}} + \hbar \sum_{s=1}^{n} \alpha_{X_s} \partial_{X_s}$$

the general isomonodromic deformation (i.e. a general vector in the tangent space)

Hamiltonian evolutions

The Darboux coordinates $(q_j, p_j)_{1 \leq j \leq g}$ have Hamiltonian evolutions:

$$\forall j \in [\![1, g]\!] \,:\, \mathcal{L}_{\boldsymbol{\alpha}}[q_j] = \frac{\partial \mathrm{Ham}^{(\boldsymbol{\alpha})}(\mathbf{q}, \mathbf{p})}{\partial p_j} \text{ and } \mathcal{L}_{\boldsymbol{\alpha}}[p_j] = -\frac{\partial \mathrm{Ham}^{(\boldsymbol{\alpha})}(\mathbf{q}, \mathbf{p})}{\partial q_j}$$

and the expression of the general Hamiltonian $\mathrm{Ham}^{(\boldsymbol{\alpha})}(\mathbf{q}, \mathbf{p})$ is explicit.

Expression for the general Hamiltonian 2

Expression of the general Hamiltonian

For any $\boldsymbol{\alpha} \in \mathbb{C}^{2g+4-n}$ we have

$$\begin{aligned}
\mathrm{Ham}^{(\boldsymbol{\alpha})}(\mathbf{q},\mathbf{p}) \quad = \quad & \sum_{k=0}^{r_\infty-4} \nu_{\infty,k+1}^{(\boldsymbol{\alpha})} H_{\infty,k} - \sum_{s=1}^{n}\sum_{k=2}^{r_s} \nu_{X_s,k-1}^{(\boldsymbol{\alpha})} H_{X_s,k} + \sum_{s=1}^{n} \alpha_{X_s}^{(\boldsymbol{\alpha})} H_{X_s,1} \\
& -\hbar \sum_{j=1}^{g} \left[\sum_{k=0}^{r_\infty-1} c_{\infty,k}^{(\boldsymbol{\alpha})} q_j^k + \sum_{s=1}^{n}\sum_{k=1}^{r_s-1} c_{X_s,k}^{(\boldsymbol{\alpha})} (q_j - X_s)^{-k} \right] \\
& +\nu_{\infty,-1}^{(\boldsymbol{\alpha})} \sum_{s=1}^{n} \left(X_s H_{X_s,1} + H_{X_s,2}\delta_{r_s\geq 2} \right) + \nu_{\infty,0}^{(\boldsymbol{\alpha})} \sum_{s=1}^{n} H_{X_s,1} \\
& -\delta_{r_\infty \in \{1,2\}} \left(\sum_{s=1}^{n} H_{X_s,1} - \hbar \sum_{j=1}^{g} p_j \right) \nu_{\infty,0}^{(\boldsymbol{\alpha})} \\
& -\delta_{r_\infty=1} \left(\sum_{s=1}^{n} X_s H_{X_s,1} + \sum_{s=1}^{n} H_{X_s,2}\delta_{r_s\geq 2} - \hbar \sum_{j=1}^{g} q_j p_j \right) \nu_{\infty,-1}^{(\boldsymbol{\alpha})} \\
& -\hbar \nu_{\infty,0}^{(\boldsymbol{\alpha})} \sum_{j=1}^{g} p_j - \hbar \nu_{\infty,-1}^{(\boldsymbol{\alpha})} \sum_{j=1}^{g} q_j p_j ,
\end{aligned}$$

Expression for the general Hamiltonian 3

Expression of the coefficients $(\nu^{(\boldsymbol{\alpha})}_{p,k})_{p,k}$

Coefficients $(\nu^{(\boldsymbol{\alpha})}_{p,k})_{p,k}$ are independent of Darboux coordinates. They are time-dependent linear combinations of the vector of deformation $\boldsymbol{\alpha}$:

$$\forall\, s \in [\![1,n]\!] \,:\, \nu^{(\boldsymbol{\alpha})}_{X_s,0} = -\alpha_{X_s} \text{ and } M_s \begin{pmatrix} \nu^{(\boldsymbol{\alpha})}_{X_s,1} \\ \vdots \\ \vdots \\ \nu^{(\boldsymbol{\alpha})}_{X_s,r_s-1} \end{pmatrix} = \begin{pmatrix} -\frac{\alpha_{X_s^{(1)},r_s-1} - \alpha_{X_s^{(2)},r_s-1}}{r_s-1} \\ \vdots \\ \vdots \\ -\frac{\alpha_{X_s^{(1)},1} - \alpha_{X_s^{(2)},1}}{1} \end{pmatrix}$$

$$M_\infty \begin{pmatrix} \nu^{(\boldsymbol{\alpha})}_{\infty,-1} \\ \nu^{(\boldsymbol{\alpha})}_{\infty,0} \\ \vdots \\ \nu^{(\boldsymbol{\alpha})}_{\infty,r_\infty-3} \end{pmatrix} = \begin{pmatrix} \frac{\alpha_{\infty^{(1)},r_\infty-1} - \alpha_{\infty^{(2)},r_\infty-1}}{r_\infty-1} \\ \frac{\alpha_{\infty^{(1)},r_\infty-2} - \alpha_{\infty^{(2)},r_\infty-2}}{r_\infty-2} \\ \vdots \\ \frac{\alpha_{\infty^{(1)},1} - \alpha_{\infty^{(2)},1}}{1} \end{pmatrix}$$

where $(M_\infty, M_1, \dots, M_n)$ are **lower triangular Toeplitz matrices** with coefficients given by irregular times at each pole.

Expression for the general Hamiltonian 4

Expression of the lower triangular Toeplitz matrices $(M_\infty, M_1, \dots, M_n)$

$$M_s := \begin{pmatrix} (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) & 0 & \dots & \dots & 0 \\ (t_{X_s^{(1)},r_s-2} - t_{X_s^{(2)},r_s-2}) & (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) & 0 & & \vdots \\ \vdots & \ddots & \ddots \quad \ddots & & \vdots \\ \vdots & \ddots & \ddots \quad \ddots & 0 & \vdots \\ (t_{X_s^{(1)},2} - t_{X_s^{(2)},2}) & \ddots & \ddots \quad \ddots & (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) & 0 \\ (t_{X_s^{(1)},1} - t_{X_s^{(2)},1}) & (t_{X_s^{(1)},2} - t_{X_s^{(2)},2}) & \dots & (t_{X_s^{(1)},r_s-2} - t_{X_s^{(2)},r_s-2}) & (t_{X_s^{(1)},r_s-1} - t_{X_s^{(2)},r_s-1}) \end{pmatrix}$$

Properties induced by the explicit expressions

- Expressions are rational functions of Darboux coordinates, irregular times and location of poles $\Rightarrow$ "There exists a birational map between the symplectic Ehresmann connection and the Jimbo-Miwa-Ueno/Boalch symplectic isomonodromy connection"
- Roughly: **Hamiltonians are time-dependent linear combinations** (coefficients $\nu_{p,j}^{(\boldsymbol{\alpha})}$) **of the spectral invariants** $H_{p,j}$ (independent of the deformation).
- Increase the order at a pole $\Rightarrow$ increase the size of Toeplitz matrix.
- Fuchsian singularities provide only $-\alpha_{X_s} H_{X_s,1}$ in the Hamiltonian $\Rightarrow$ simpler formulas as known from Schlesinger.
- Many directions in the tangent space (specific choice of $\boldsymbol{\alpha}$) gives trivial Hamiltonian evolutions $\Rightarrow$ Existence of a **symplectic reduction** to obtain Arnold-Liouville form (i.e. same number of Darboux coordinates as non-trivial deformation parameters).

Shifted Darboux coordinates

Shifted Darboux coordinates and trivial/non-trivial times for $r_\infty \geq 3$

Define $\check{\mathbf{q}}_{\mathbf{j}} := \mathbf{T_2 q_j + T_1}$, $\check{\mathbf{p}}_{\mathbf{j}} := \mathbf{T}_2^{-1}\left(\mathbf{p_j} - \frac{1}{2}\mathbf{P_1(q_j)}\right)$ with

$$T_1 := \frac{t_{\infty^{(1)}, r_\infty - 2} - t_{\infty^{(2)}, r_\infty - 2}}{2^{\frac{1}{r_\infty - 1}}(r_\infty - 2)(t_{\infty^{(1)}, r_\infty - 1} - t_{\infty^{(2)}, r_\infty - 1})^{\frac{r_\infty - 2}{r_\infty - 1}}} \; , \; T_2 := \left(\frac{t_{\infty^{(1)}, r_\infty - 1} - t_{\infty^{(2)}, r_\infty - 1}}{2}\right)^{\frac{1}{r_\infty - 1}}$$

Define also:

$$\begin{aligned}
T_{\infty,k} &= t_{\infty^{(1)},k} + t_{\infty^{(2)},k} \; , \; T_{X_s,k} = t_{X_s^{(1)},k} + t_{X_s^{(2)},k} \\
\tau_{\infty,j} &= 2^{\frac{j}{r_\infty - 1}}\Bigg[\sum_{i=0}^{r_\infty - j - 3} \frac{(-1)^i (j+i-1)!}{i!(j-1)!(r_\infty - 2)^i} \frac{(t_{\infty^{(1)}, r_\infty - 2} - t_{\infty^{(2)}, r_\infty - 2})^i (t_{\infty^{(1)}, j+i} - t_{\infty^{(2)}, j+i})}{(t_{\infty^{(1)}, r_\infty - 1} - t_{\infty^{(2)}, r_\infty - 1})^{\frac{i(r_\infty - 1)+j}{r_\infty - 1}}} \\
&\quad + \frac{(-1)^{r_\infty - j - 2}(r_\infty - 3)!}{(r_\infty - 1 - j)(r_\infty - j - 3)!(j-1)!(r_\infty - 2)^{r_\infty - j - 2}} \frac{(t_{\infty^{(1)}, r_\infty - 2} - t_{\infty^{(2)}, r_\infty - 2})^{r_\infty - 1 - j}}{(t_{\infty^{(1)}, r_\infty - 1} - t_{\infty^{(2)}, r_\infty - 1})^{\frac{(r_\infty - 2)(r_\infty - 1 - j)}{r_\infty - 1}}}\Bigg] \\
\tau_{X_s,k} &= (t_{X_s^{(1)},k} - t_{X_s^{(2)},k}) T_2^k \; , \; \forall\, k \in [\![1, r_s - 1]\!] \\
\tilde{X}_s &= T_2 X_s + T_1
\end{aligned}$$

Properties of the symplectic decomposition

- One-to-one map between $(t_{p,k}, X_s) \leftrightarrow (T_1, T_2, T_{p,k}, \tau_{p,k}, \tilde{X}_s)$.
- $(T_1, T_2, T_{p,k})$ are trivial times, i.e. $\partial_T \check{q}_j = \partial_T \check{p}_j = 0$.
- Shifted Darboux coordinates $(\check{q}_j, \check{p}_j)$ are independent of the trivial times $\Rightarrow$ only depend on non-trivial times $(\tilde{X}_s, \tau_{p,j})$
- Hamiltonian evolutions of $(\check{q}_j, \check{p}_j)$ only depend on non-trivial times. Non-trivial directions give $c_{p,k}^{(\boldsymbol{\alpha})} = 0$ and other simplifications:

$$\mathrm{Ham}^{(\boldsymbol{\alpha}_\tau)}(\check{\mathbf{q}}, \check{\mathbf{p}}) = \sum_{k=0}^{r_\infty - 4} \nu_{\infty,k+1}^{(\boldsymbol{\alpha}_\tau)} H_{\infty,k} - \sum_{s=1}^{n} \sum_{k=2}^{r_s} \nu_{X_s,k-1}^{(\boldsymbol{\alpha}_\tau)} H_{X_s,k} + \sum_{s=1}^{n} \alpha_{X_s}^{(\boldsymbol{\alpha}_\tau)} H_{X_s,1}$$

- Canonical choice is to take $T_2 = 1,\ T_1 = 0,\ T_{p,k} = 0$ so that $(q_j, p_j) = (\check{q}_j, \check{p}_j)$
- Canonical choice **kills the trace** $(T_{p,k} = 0 \Leftrightarrow P_1 = 0)$ of $\hat{L}(\lambda)$ and the action of **Möbius transformations** $\lambda \to \frac{a\lambda+b}{c\lambda+d}$ $(T_2 = 1, T_1 = 0)$

Properties of the symplectic decomposition 2

Reduction of the symplectic two-form

The symplectic two-form Ω characterizing the symplectic structure reduces:

$$\begin{aligned}\Omega \;:=\;& \hbar \sum_{j=1}^{g} dq_j \wedge dp_j - \sum_{s=1}^{n}\sum_{i=1}^{2}\sum_{k=1}^{r_s-1} dt_{X_s^{(i)},k} \wedge d\mathrm{Ham}^{(\mathbf{e}_{X_s^{(i)},k})} \\ & - \sum_{i=1}^{2}\sum_{k=1}^{r_\infty-1} dt_{\infty^{(i)},k} \wedge d\mathrm{Ham}^{(\mathbf{e}_{\infty^{(i)},k})} - \sum_{s=1}^{n} dX_s \wedge d\,\mathrm{Ham}^{(\mathbf{e}_{X_s})} \\ =\;& \hbar \sum_{j=1}^{g} d\check{q}_j \wedge d\check{p}_j - \sum_{\tau \in \mathcal{T}_{\text{non triv.}}} d\tau \wedge d\mathrm{Ham}^{(\alpha_\tau)}\end{aligned}$$

where $\mathcal{T}_{\text{non triv.}}$ is the set of non-trivial times.

- Provides **Arnold-Liouville form**
- $\mathfrak{gl}_2 \to \mathfrak{sl}_2$ reduction was already known geometrically
- Möbius reduction also known: fixes either location of 3 poles (P6) or one pole (∞) and the two most singular coefficients (P2)

Examples from direct application of the general formulas

Examples:

- $n = 3$ with $r_\infty = r_1 = r_2 = r_3 = 1$ gives Painlevé 6 in Jimbo-Miwa form after canonical reduction
- $n = 2$ with $r_\infty = 1$, $r_1 = 1$ and $r_2 = 2$: Painlevé 5 in Jimbo-Miwa form after canonical reduction
- $n = 1$ with $r_\infty = 1$ and $r_1 = 3$. Painlevé 4 case. To get Jimbo-Miwa case, another choice of canonical trivial times is necessary
- $n = 1$ with $r_\infty = 2$ and $r_1 = 1$: Painlevé 3 case in Jimbo-Miwa form after canonical reduction
- $n = 0$ with $r_\infty = 4$: Painlevé 2 case in Jimbo-Miwa form after canonical reduction
- $n = 0$ arbitrary r_∞: Full Painlevé 2 hierarchy ($r_\infty = 5$ already known in the literature by H. Chiba)

Twisted cases and Painlevé 1 hierarchy

- Similar results available for the **twisted case** (pole=ramification point) in [8]
- Also gives rise to a symplectic reduction and Arnold-Liouville form
- Includes Painlevé 1 case and the full Painlevé 1 hierarchy
- Birkhoff factorization is different but in the end Hamiltonian formulas and Lax matrices have very similar form to the non-twisted cases (lower triangular Toeplitz matrices, Vandermonde matrices, symplectic reduction, etc.)
- Cover all possible cases arising in $\mathfrak{gl}_2(\mathbb{C})$
- Explicit formulas enables direct link with **isospectral coordinates** developed by the Montréal school

Quantization and reverse way

General idea

- Use the formal $\hbar$ parameter to define formal power series/transseries in $\hbar \to 0$ for Darboux coordinates and formal WKB series/transseries for wave matrix $\hat{\Psi}$.
- Solve **recursively** the formal power series/transseries
- Find a way to resum formal power series/transseries to get analytic quantities for Darboux coordinates and wave matrix (Laplace Borel resummation or other ways)
- Take $\hbar = 1$ if the analytic continuation can reach this point.
- Many interesting features in enumerative geometry and mathematical physics
- Situation is much simpler when the spectral curve is of genus 0 (no need for transseries)
- Formal recursive part can be dealt with Topological Recursion (TR) of Chekhov-Eynard-Orantin
- Going from formal to analytic rigorously is still hard

Topological recursion as a black box

Classical spectral and TR

A classical spectral curve is defined by an **algebraic curve**

$$0 = P(x,y) = \sum_{k=0}^{d} P_k(x) y^k$$

with $(P_k)_{0\leq k\leq d}$ rational functions with given pole structure. It defines a Riemann surface Σ of genus g and we choose a Torelli marking $(\mathcal{A}_j, \mathcal{B}_j)_{j=1}^{g}$. Add *admissible* conditions like: irreducibility, simple and smooth ramification points, distinct critical values, etc.

Initial quantities for TR

We define

$$\omega_{0,1} = ydx \ , \ \omega_{0,2} = \text{Bergman kernel}$$

where Bergman kernel is the unique symmetric $(1 \boxtimes 1)$-form B on Σ^2 with a unique double pole on the diagonal Δ, without residue, bi-residue equal to 1 and normalized on the $\mathcal{A}$-cycles by $\oint_{z_1 \in \mathcal{A}_i} B(z_1, z_2) = 0$.

Parametrization of classical spectral curve

Parametrization of the classical spectral curve

The classical spectral curve is parametrized by **spectral (irregular) times** $(t_{p,k})_{p,k}$ given by the singular part of ydx at each pole p and g filling fractions $(\epsilon_i)_{i=1}^{g}$:

$$\epsilon_i := \oint_{\mathcal{A}_i} ydx$$

- Connection with isomonodromic deformations is that classical spectral curve: $P(x,y) := \lim_{\hbar\to 0} \det(yI_d - \hat{L}(x)) = 0$
- Limit $\hbar \to 0$ independent of gauge choice: $\hbar(\partial_\lambda G(x))G(x)^{-1} \to 0$
- Problem: Requires to define the "limit $\hbar \to 0$ of Darboux coordinates $(\mathbf{q},\mathbf{p})$"

Topological recursion

Black box TR

- Topological recursion is a **recursive procedure** (sum of residue at ramification points) starting from $\omega_{0,1}$ and $\omega_{0,2}$ that produces $(\omega_{h,n})_{h\geq 0, n\geq 0}$ by induction on $2h+n-2$.
- Formulas can be found in [3] with special cases for $(\omega_{h,0})_{h\geq 0}$.
- $(\omega_{h,n})_{h\geq 0, n\geq 1}$ are called **Chekhov-Eynard-Orantin differentials** (or TR differentials) and are symmetric n-forms on Σ^n with only poles at ramification points when $(h,n) \notin \{(0,1),(0,2)\}$.
- $(\omega_{h,0})_{h\geq 0}$ are just numbers sometimes called "free energies" or "symplectic invariants" and sometimes denoted $(F_h)_{h\geq 0}$.
- Many generalizations of TR exist to deal with non-admissible curves.

Step 1: Formal WKB wave functions

Formal WKB wave functions

For any λ not a pole, define $\left(z^{(j)}(\lambda)\right)_{j=1}^{d}$ the d points on Σ such that $x(z^{(j)}(\lambda)) = \lambda$. Then, define the **formal pertubative WKB wave functions**:

$$\psi_j(\lambda,\hbar) := \exp\Bigg(\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h-2+n}}{n!}\overbrace{\int_{\infty^{(1)}}^{z^{(j)}(\lambda)}\cdots\int_{\infty^{(1)}}^{z^{(j)}(\lambda)}}^{n}\Big(\omega_{h,n}(z_1,\dots,z_n) - \delta_{h,0}\delta_{n,2}\frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\Big)\Bigg)$$

and the **formal perturbative partition function**:

$$Z(\hbar) := \exp\left(\sum_{h\geq 0}\hbar^{2h-2}\omega_{h,0}\right)$$

Definitions are chosen to satisfy the KZ equations.

Monodromies around $\mathcal{A}$ and $\mathcal{B}$ cycles

Monodromies

The formal perturbative wave functions have good monodromies on $\mathcal{A}$-cycles:

$$\psi_j(\lambda + \mathcal{A}_i, \hbar) = e^{\frac{2\pi i}{\hbar}\epsilon_i}\psi_j(\lambda, \hbar)$$

They have bad monodromies on the $\mathcal{B}$-cycles:

$$\begin{aligned}\psi_j(\lambda + \mathcal{B}_i, \hbar) &= \exp\left(\sum_{(h,n)\in\mathbb{N}^2} \frac{\hbar^{2h-2+n}}{n!} \overbrace{\int_{\infty^{(1)}}^{z} \cdots \int_{\infty^{(1)}}^{z}}^{n} \sum_{m\geq 0} \frac{1}{m!}\left(\hbar \frac{\partial}{\partial \epsilon_i}\right)^m \omega_{h,n}\right) \\ &= \psi_j(\lambda, \epsilon_i \to \epsilon_i + \hbar, \hbar),\end{aligned}$$

Requires to formally "sum on filling fractions" to obtain good monodromies $\Rightarrow$ creates Theta functions evaluated at $\frac{\rho}{\hbar}$ $\Rightarrow$ formal transseries.

Quantum curve and formal solutions

Quantum curve

After "sum on filling fractions", i.e. going from $\psi_j(\lambda,\hbar) \to \psi_{j,\text{NP}}(\lambda,\hbar)$ by adding formal theta series terms (Cf. [2]), we get that $(\psi_{j,\text{NP}}(\lambda,\hbar))$ are formal solutions to the ODE

$$\sum_{k=0}^{d} b_{d-k}(\lambda,\hbar)\left(\hbar\frac{\partial}{\partial\lambda}\right)^k \psi_{j,\text{NP}}(\lambda,\hbar)=0,$$

with coefficients $b_j(\lambda,\hbar)$ **rational in λ with same pole structure as classical spectral curve and simple poles at some apparent singularities** $(q_j)_{1\leq j\leq g}$ defined by $\det \Psi_{\text{NP}}=0$ with $\Psi_{\text{NP}}\in\mathcal{M}_d(\mathbb{C})$:

$$[\Psi_{\text{NP}}]_{i,j} := (\hbar\partial_\lambda)^i\psi_{j,\text{NP}}(\lambda,\hbar) \;\Leftrightarrow \hbar\partial_\lambda\Psi_{\text{NP}} = L_{\text{NP}}\Psi_{\text{NP}}\;,\, L_{\text{NP}} \text{ companion}$$

Remark

$b_0(\lambda,\hbar)=1$ and $b_l(\lambda,\hbar) \overset{\hbar\to 0}{\to} (-1)^l P_l(\lambda) \Rightarrow$ Formal quantization of the classical spectral curve $\Rightarrow$ Terminology: **quantum curve**

Connection with isomonodromic deformations

- One can derive formal auxiliary matrices $A_{t,\text{NP}}(\lambda, \hbar)$ such that $\hbar\partial_t \Psi_{NP}(\lambda, \hbar) := A_{t,\text{NP}}(\lambda, \hbar)\Psi_{\text{NP}}(\lambda, \hbar)$ for any spectral time t or any position of poles with good pole structure.
- One can perform an explicit gauge transformation to remove the apparent singularities. For $d = 2$, one obtains $\hat{L}_{\text{NP}}$ and $\hat{A}_{\text{NP}}$.
- **Starting from a classical spectral curve ($\hbar \to 0$ limit), we have reconstructed formal Lax systems and formal wave matrices that arises in $\hbar$-deformed isomonodromic deformations.**
- For genus 0 spectral curves, there is no need for NP quantities: simple power series for Darboux coordinates and WKB formal series for wave functions
- Construction is made for arbitrary rank $d \geq 2$
- Only a **formal** reconstruction since all series/transseries are **divergent**. What sense to give to $\hbar = 1$?

0-parameter solutions of the Painlevé 1 equation

Lax system and Painlevé 1 equation

Painlevé 1 Lax system

The Painlevé 1 system correspond to $n=0$ and a **twisted singularity at infinity** $r_\infty = 4$ (genus $g=1$ case). The $\hbar$-deformed Lax matrices are

$$\hat{L}(\lambda) := \begin{pmatrix} p & 4(\lambda - q) \\ \lambda^2 + q\lambda + q^2 + \frac{1}{2}t & -p \end{pmatrix}$$
$$\hat{A}(\lambda) := \frac{1}{2}\begin{pmatrix} 0 & 4 \\ \lambda + 2q & 0 \end{pmatrix}$$

Compatibility implies the **Painlevé 1 Hamiltonian system**

$$\begin{cases} \hbar\frac{\partial}{\partial t} q = p = \hbar\frac{\partial}{\partial p}\mathrm{Ham}(q,p;t), \\ \hbar\frac{\partial}{\partial t} p = 6q^2 + t = -\hbar\frac{\partial}{\partial q}\mathrm{Ham}(q,p;t), \end{cases}$$

with Hamiltonian $\mathrm{Ham}(q,p;t) = \frac{1}{2}p^2 - 2q^3 - tq$. $q(t)$ satisfies P1:

$$\hbar^2 \frac{\partial^2}{\partial t^2} q = 6q^2 + t$$

0-parameter solutions of the Painlevé 1 equation

0-parameter solutions

We look for **formal** 0-**parameter solutions** (also known as tritronquées solutions) of the Painlevé 1 equation:

$$\hat{q}(t;\hbar) = \sum_{k=0}^{\infty} q_k(t)\hbar^k \;\Rightarrow\; \hat{p}(t;\hbar) = \sum_{k=0}^{\infty} p_k(t)\hbar^k = \sum_{k=1}^{\infty} \dot{q}_{k-1}(t)\hbar^k$$

It implies **formal** $\hbar$ **power series for the Lax matrices**

$$\hat{L}(\lambda, t;\hbar) = \sum_{k=0}^{\infty} \hat{L}_k(\lambda, t)\hbar^k \;,\; \hat{A}(\lambda, t;\hbar) = \sum_{k=0}^{\infty} \hat{A}_k(\lambda, t)\hbar^k$$

and **formal WKB expansion for** $\hat{\Psi}(\lambda, t;\hbar)$:

$$\hat{\Psi}(\lambda, t;\hbar) = \exp\left(\sum_{k=-1}^{\infty} \Psi_k(\lambda, t)\hbar^k\right)$$

Degenerate genus 0 family of classical spectral curve

- Leading order: $q_0(t) = \left(-\frac{t}{6}\right)^{\frac{1}{2}}$ and $p_0 = 0$
- The **classical spectral curve** is defined as

$$\mathcal{S}_0 := \left\{(\lambda, y)\ \det(yI_2 - \hat{L}_0(\lambda, t)) = 0 = \lim_{\hbar \to 0} \det(yI_2 - \hat{L}(\lambda, t))\right\}$$

- It gives a (time-dependent) family of **singular hyperelliptic genus 0 curves**:

$$\mathbf{y^2 = 4(x - q_0(t))^2(x + 2q_0(t))}$$

- Apply TR $\Rightarrow$ Coefficients of formal WKB expansion are given by integrals of Eynard-Orantin differentials (Cf. Topological Type Property of [6])
- Explicit induction for formal coefficients $(q_k(t))_{k\geq 1}$ and $(\Psi_k(\lambda, t))_{k\geq -1}$ $\Rightarrow$ divergent but Gevrey 1-series

Borel-resummation for $\hat{q}(t)$

- Works of N. Nikolaev providing mathematically rigorous Laplace-Borel resummation both in for $\hat{q}(t;\hbar)$ and $\Psi(\lambda,t;\hbar)$. (See [10, 11, 12]) in λ for fixed t.
- Full geometric description for Painlevé I is done in [1] using groupoids.
- Results already conjectured and used by mathematical physicists.
- Natural coordinate is $q_0 \in \mathbb{C}$ rather than t ($q_0(t) = \left(-\frac{t}{6}\right)^{\frac{1}{2}}$) to avoid square root branch.
- Existence of 5 sectors in the q_0-plane.

Sketch of the strategy

- Existence/uniqueness of formal power series $\hat{q} = \sum_{n=0}^{\infty} q_n(t)\hbar^n$. It is of **factorial type** (i.e. Gevrey 1).
- Borel transform: $\hat{\mathcal{B}}[\hat{q}](\xi) = \sum_{n=0}^{\infty} \frac{1}{n!} q_{n+1}(t)\xi^n$ is **locally analytic around** $\xi = 0$ (Germ). Local solution the Borel transform of P1 equation.
- New vision of the Laplace transform and Borel "plane":

$$\mathcal{L}[f](z,\hbar) = \int_{e^{i\alpha}\mathbb{R}_+} e^{-\xi/\hbar} f(z,\xi)d\xi \to \int_\gamma e^{-Z(\gamma_s)/\hbar} f(\gamma_s)\, Ds$$

where $\gamma \in \Pi_1(\mathbb{C}^*)$, γ_s is the arc-length parametrization, $Z : \Pi_1(\mathbb{C}^*) \mapsto \mathbb{C}$ is the central charge:

$$\xi = Z(\gamma) = \int_\gamma y dx = \int_z^{e^u z} \left(-\tfrac{1}{6}s^4\right) ds = \tfrac{1}{30}(1 - e^{5u})z^5$$

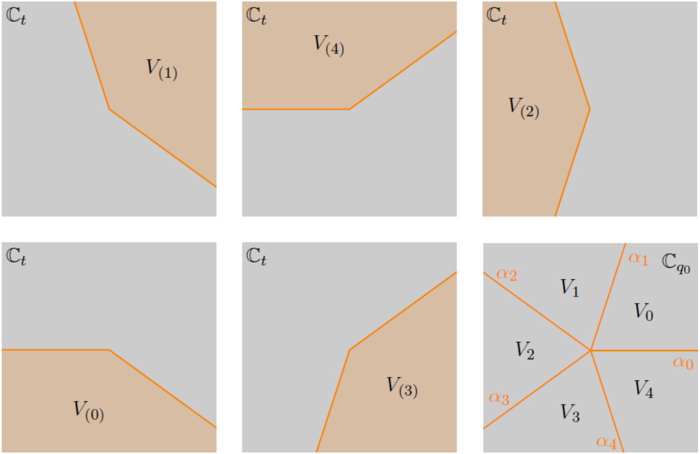
Sketch of the strategy 2

- Rewrite the Borel transform of P1 and all quantities in terms of paths, central charge, arc-length parametrization, anchor map, source, target of paths, etc. $\Leftrightarrow$ Borel "plane" $\rightarrow$ Borel covering space.
- Local Borel transform $\hat{\mathcal{B}}[\hat{q}](\xi)$ provides an **Initial Value Problem on groupoids of paths**.
- Prove that the IVP admits a unique solution on the Borel covering space using Contraction Mapping Principle on a Banach algebra with norm

$$||f||_K := \sup_{\gamma \in \tilde{B}^\circ} \inf \int_0^{|\gamma|} e^{-Ks} \left| f(\gamma_s)\, Ds \right|$$

- Allows to **analytically continue $\hat{\mathcal{B}}[\hat{q}]$ on the Borel covering space** $\Rightarrow$ Define Laplace transform in each Stokes sector in the q_0 plane.

Stokes sectors in the t or q_0-plane



Borel resummation theorems for $q(t;\hbar)$

Existence of uniqueness of 0-parameter solutions from Borel resummation

Choose a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and a Stokes sector $V_{(k)}$ in the t-plane. Define

$$\mathbb{H}_\theta := \left\{ re^{i\vartheta} \,\middle|\, r > 0 \quad \text{and} \quad \vartheta \in \left(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right) \right\}$$

Then, there is a domain $\mathbb{V}_{(k)} \subset V_{(k)} \times \mathbb{H}_\theta$ such that the Painlevé 1 equation has a unique holomorphic solution $q_{(k)}$ on $\mathbb{V}_{(k)}$ which admits an asymptotic expansion of factorial type:

$$q_{(k)}(t,\hbar) \sim \hat{q}_{(k)}(t,\hbar) \qquad \text{as } \hbar \to 0 \text{ unif. along } \mathbb{H}_\theta,$$

locally uniformly for all $t \in V_{(k)}$.

Remark

The domain $\mathbb{V}_{(k)}$ satisfies that every point $t_0 \in V_{(k)}$ has a neighborhood $V \subset V_{(k)}$ such that there is a sector $U \subset \mathbb{H}_\theta$ with opening $\left(\theta - \frac{\pi}{2}, \theta + \frac{\pi}{2}\right)$ with the property that $V \times U \subset \mathbb{V}_{(k)}$.

Towards Exact WKB wave matrices: Step 1 Stokes lines

- Insert $q_{(k)}$ in the Lax matrices to have a well defined analytic Lax matrices admitting formal power series expansions in $\hbar$
- Solve recursively the formal system (careful on compatibility). No singularity for coefficients of $\Psi_k(\lambda, t)$ at the double point $\lambda = q_0$
- Exact WKB resummation implies to avoid **Stokes curves** defining the **Stokes graph** (or "spectral network") defined by

$$\mathrm{Im}\Big(\Phi(\lambda) - \Phi(-2q_0)\Big) = 0 \qquad \text{and} \qquad \mathrm{Im}\Big(\Phi(\lambda) - \Phi(q_0)\Big) = 0$$

where $y^2 = 4(\lambda - q_0)^2(\lambda + 2q_0)$ is the classical spectral curve and

$$\Phi(\lambda) := e^{-i\theta} \int^{\lambda} y dx = e^{-i\theta} \left(\tfrac{4}{5}(\lambda - 3q_0)(\lambda + 2q_0)^{\frac{3}{2}} \right)$$

- Corresponds to Stokes trajectories ending at the ramification point $-2q_0$ or the double point q_0
- **Stokes graph defines several Stokes sectors**
- Critical Stokes graphs (i.e. with a Stokes curve connecting $-2q_0$ to q_0) only when t belongs to a Stokes line in the t-plane

Example of Stokes graphs

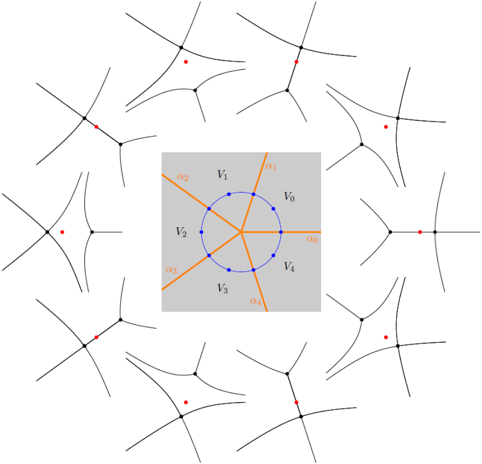


Figure: Stokes graphs in the λ-plane for $\theta = 0$ and various values of q_0

Stokes regions in the λ-plane for non-critical configurations

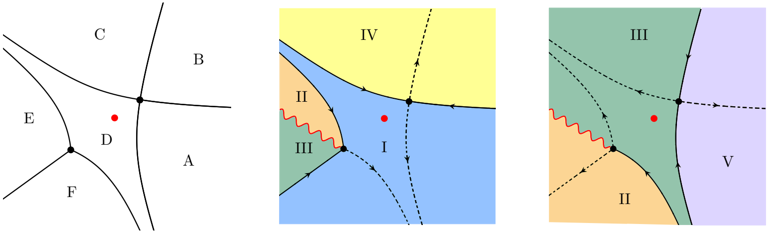


Figure: Labelling convention for the Stokes regions in the λ-plane (left) and the Stokes cells in the classical spectral curve (middle and right).

Step 2: Exact WKB wave matrices

Existence and uniqueness in each Stokes sector (work in progress)

- Fix a phase $\theta \in \mathbb{R}/2\pi\mathbb{Z}$. Select $t \in \mathbb{C}$ in a Stokes sector $V := V_{(k)}$ and associated holomorphic $q_{(k)}$.
- Let $\hat{\Psi}$ be a formal WKB wave matrix of the formal $\hbar$-deformed P1.
- Select a t-dependent Stokes region $U \subset \mathbb{C}_\lambda$.

Then, there is a canonical WKB wave matrix Ψ over U. Namely, there is a domain $\mathbb{U} \subset U \times \mathbb{H}_\theta$ such that the $\hbar$-deformed Painlevé 1 system has a **unique holomorphic fundamental solution Ψ on** $\mathbb{U}$ with the property that

$$\Psi(\lambda, t, \hbar) \sim \hat{\Psi}(\lambda, t, \hbar) \qquad \text{as } \hbar \to 0 \text{ unif. along } \mathbb{H}_\theta$$

of factorial/WKB type, locally uniformly for all $\lambda \in U$.
Specifically, Ψ is the Borel resummation of $\hat{\Psi}$ with phase θ, locally uniformly for all $\lambda \in U$.

Stokes phenomenon and jump matrices

- **Previous theorem defines** $(\Psi_A, \dots, \Psi_F)$ **solutions to the** $\hbar$**-deformed P1 system**.
- Each solution can be analytically continued and is holomorphic in the full $\mathbb{C}_\lambda$ plane (ODE has no finite singularity)
- Lax system is linear $\Rightarrow$ Existence of **Stokes matrices**: $\Psi_U = \Psi_{U'} S_{UU'}$. Stokes matrices are time-independent.
- Asymptotics only valid in the Stokes sector indexing the wave matrix
- On classical spectral curve one scalar solution does not jump $\Rightarrow S_{UU'}$ are lower or upper triangular matrices for contiguous Stokes sectors.
- Upon proper normalization of the columns (i.e. normalization of wave functions) we get Stokes matrices of the form

$$S_{U,U'} = \begin{pmatrix} 1 & \star \\ 0 & 1 \end{pmatrix} \quad \text{or} \quad S_{U,U'} = \begin{pmatrix} 1 & 0 \\ \star & 1 \end{pmatrix}$$

  for contiguous Stokes sectors U and U'.
- Branchcut (exchange sheets) $\Rightarrow$ "Stokes matrix" $\begin{pmatrix} 0 & -\alpha \\ \alpha^{-1} & 0 \end{pmatrix}$

Stokes matrices version 1

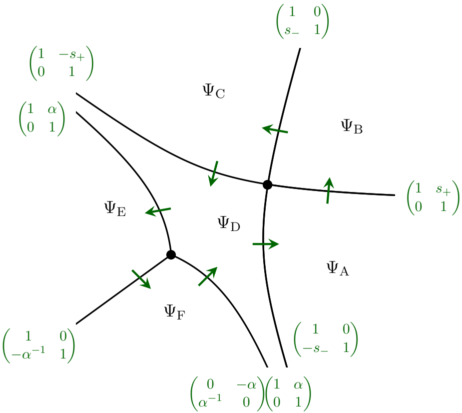


Figure: Reduction of the Stokes matrices after turning around the two vertices: 3 parameters: (α, s_-, s_+)

Stokes matrices around the double turning point

- The **value of α is irrelevant**. Corresponds to a choice of normalization between the two sheets. Usually set to $\alpha = i$ by physicists (normalization at the ramification point) or to $\alpha = 1$ (normalization at infinity)
- Last step is to prove that $s_- = s_+ = 0$, i.e. no active Stokes matrices at the double turning point
- Consequence of the fact that formal WKB solutions have regular coefficients at $\lambda = q_0$
- Difficult technical part is to integrate the flows in the Borel planes (both in (λ, ξ) and (t, ξ)) and keep them compatible (work in progress) using the adapted terminology of groupoids.
- Recover conjectured Kapaev's Stokes matrices [7] for connections associated to tritronquée (0-parameter) solutions of $P1$ in a different context.

Stokes matrices version 2

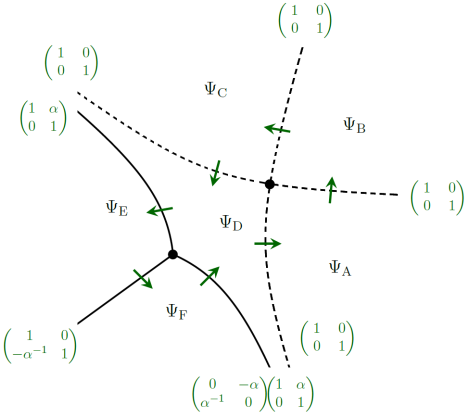


Figure: Final Stokes matrices for wave matrices associated to 0-parameter (tritronquées) solutions of the P1 equation.

Riemann-Hilbert problem for 0-parameter solutions of P1

RHP for 0-parameter solutions of P1 (work in progress)

Let $t \in V_{(k)}$. We look for $\Psi(\lambda, t; \hbar)$ such that

1. Ψ is holomorphic for $\lambda \in \mathbb{C}$ except on the previous Stokes lines where it has jumps given by the previous Stokes matrices
2. Ψ admits the following expansion at $\lambda \to \infty$ (consequence of the local Birkhoff factorization):

   $$\lim_{\lambda\to\infty} \lambda^{\frac{1}{2}} \left[\frac{1}{2}(\sigma_1+\sigma_3)\lambda^{\frac{1}{4}\sigma_3}\begin{pmatrix}\frac{1}{2} & 0\\ 0 & 1\end{pmatrix}\Psi(\lambda,t;\hbar)e^{-\frac{\theta(\lambda,t)}{\hbar}} - I_2\right] = D(\hbar) \text{ where}$$

   $\theta(\lambda, t) = \frac{1}{\hbar}\left(\frac{4}{5}\lambda^{\frac{5}{2}} + t\lambda^{\frac{1}{2}}\right)\sigma_3$ and $D(\hbar)$ is a $\hbar$-dependent diagonal matrix.

Work in progress

The previous RHP admits a unique solution which is obtained as the Borel-resummation of $\hat{\Psi}$

Possible generalizations

- Main interest is to obtain a **Riemann-Hilbert problem formulation**. Make the link with Hermitian matrix models and orthogonal polynomials approach.
- Generalization to 0-parameter solutions of Hamiltonian systems arising from isomonodromic deformations (at least rank 2) is very likely. Difficulty is to describe general properties of Stokes graphs both in the t-plane and the λ-plane.
- Physicists [4, 5, 7, 13] conjectured similar results for general 2-parameters solutions of the Painlevé 1 equation described by formal transseries. Stokes graphs and Stokes matrices are very similar (but all are active). Mathematically difficult because no clear $\hbar \to 0$ limit in transseries. What is the meaning of Gevrey 1-series? Requires a proper geometric description of the Borel space?
- Could give a mathematically rigorous understanding of *resurgence* in physics

Thank You

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