Quantization of spectral curves via integrable systems and topological recursion

Marchal Olivier

Université Jean Monnet St-Etienne, France
Institut Camille Jordan, Lyon, France

November 19th 2019

1. Presentation of the problem
2. Topological Recursion
   - Definition
3. Method 1
   - Determinantal formulas and classical spectral curves associated to a diff. system
   - Topological type property
4. Method 2
   - General setting
   - Perturbative quantities
   - Non-perturbative quantities
   - Results for $\phi \in \mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$
5. Outlook
   - Future works

Presentation of the problem

Position of the talk

General problem

How to quantize a "classical spectral curve"

$$P(x,y)=0 \ , \ P \text{ polynomial}$$

into a differential equation:

$$\hat{P}\left(x, \hbar \frac{d}{dx}\right)\Psi(x)=0\,?$$

Key ingredients

Key ingredient 1: Integrable systems and Lax pairs

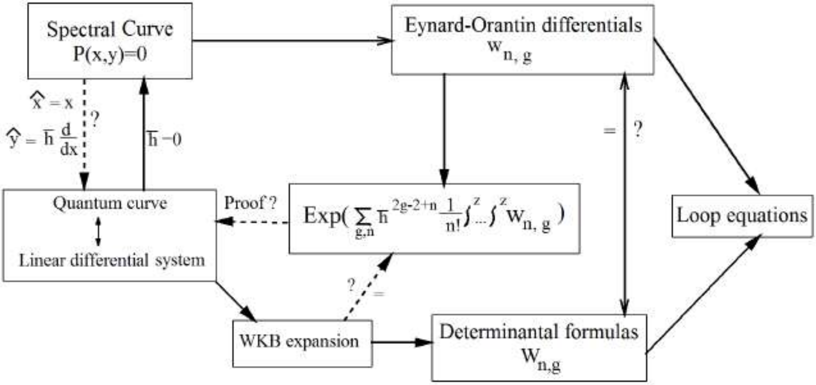
$$\frac{\partial}{\partial x}\Psi(x,t)=L(x,t)\Psi(x,t) \ , \ \frac{\partial}{\partial t}\Psi(x,t)=R(x,t)\Psi(x,t)$$

Key ingredient 2: Topological recursion introduced by Chekhov Eynard Orantin.

First method

- Start from a given $\hbar$-differential system $\hbar \frac{d}{dx}\Psi(x) = L(x)\Psi(x)$
- Define the classical spectral curve associated to it
- Show that interesting quantities (partition function, correlation functions, etc.) may be reconstructed from topological recursion applied on the classical spectral curve.
- Proof done by showing that the differential system $\partial_x \Psi = L(x)\Psi$ satisfy the Topological Type property (introduced by Bergère, Eynard, Borot).
- Showing the Topological Type property is hard without additional material... (Airy case).
- Method applied for Painlevé 2 (O.M. and K. Iwaki, 2014) generalized toall 6 Painlevé equations (O.M, K. Iwaki, A. Saenz, 2016) and recently (O.M, N. Orantin, 2019) for $\mathfrak{sl}_2(\mathbb{C})$ valued rational functions $L(x)$.
- Drawback: $\hbar$ introduced by rescaling of parameters corresponding to very specific time deformations.
- Classical spectral curves obtained this way are always of genus 0.

General view of method 1



Second method for hyper-elliptic curves

- Start with the space of quadratic differentials with prescribed poles (divisor) and partially prescribed coefficients.
- A given quadratic differential ϕ is represented by $y^2 = Q(x)$ with Q a rational function $\Rightarrow$ "Classical spectral curve"
- Define Eynard Orantin differentials and free energies associated to the classical spectral curve
- Assemble them into a formal series $\psi(x, \hbar)$ with formal parameter $\hbar$. $\Rightarrow$ "Perturbative wave function" (WKB expansion)
- Introduce an additional Fourier transform over filling fractions to obtain a non-perturbative wave function $\Psi(x, \hbar)$ (trans-series in $\hbar$)

Second method for hyper-elliptic curves 2

- Obtain a linear second order differential equation with rational coefficients satisfied by $\Psi(x,\hbar)$ using properties of Eynard-Orantin differentials: "Quantum curve"
- Use some time variations to rewrite the coefficients of the PDE in a "better" way using Darboux coordinates of the corresponding Hamiltonian system.

Aspects of the second method

- Allows to quantize a classical spectral curve with arbitrary genus.
- Isomonodromic deformations are essential to obtain a proper rewriting of the PDE.
- Method used by Iwaki for Painlevé 1 equation.
- Makes the connection with trans-series, Borel summability, exact WKB.

Connecting both settings

- Second method provides a $\hbar$ deformed Lax pair that may be seen as the starting point of method 1
- It should satisfy the Topological Type Property (with a classical spectral curve of positive genus) since reconstruction by the topological recursion is automatic by definition. (if you believe that Topological Type property is equivalent to reconstruction by TR).
- Open question: Is Topological Type Property equivalent to existence of some underlying isomonodromic deformations?
- All known cases of proof of TT property comes from isomonodromic deformations.

Topological Recursion

Initial data

- Original modern version of B. Eynard et N. Orantin, 2007. Many generalizations since (blubbed, refined, etc.).
- Initial data: **"classical spectral curve"**:
  1. Σ Riemann surface of genus g.
  2. Symplectic basis of non-contractibles cycles $(\mathcal{A}_i, \mathcal{B}_i)_{i \leq g}$ on Σ.
  3. Two meromorphic functions $x(z)$ et $y(z)$, $z \in \Sigma$ such that: $\Rightarrow$ $P(x, y) = 0$, with P polynomial.
  4. A symmetric bi-differential form $\omega_{0,2}$ on $\Sigma \times \Sigma$ such that $\omega_{0,2}(z_1, z_2) \underset{z_2 \to z_1}{\sim} \frac{dz_1\, dz_2}{(z_1 - z_2)^2} + \text{reg}$ with vanishing $\mathcal{A}$-cycles integrals.
- Regularity condition: Ramification points $(dx(a_i) = 0)$ are simple zeros of dx. $\Rightarrow$ existence of a local involution σ such that $x(z) = x(\sigma(z))$ around any ramification points.
- Topological Recursion gives by recursion n-**forms** $(\omega_{h,n})_{n \geq 1, h \geq 0}$ **(known as "Eynard Orantin differentials") and numbers** $(\omega_{h,0})_{h \geq 0}$ **(known as "free energies" or "symplectic invariants")**.

Topological recursion 2

- Recursion formula:

$$\omega_{h,n+1}(z,\mathbf{z_n}) = \sum_{i=1}^{r} \operatorname*{Res}_{q\to a_i} \frac{dE_q(z)}{(y(q)-y(\bar{q}))dx(q)} \Big[\omega_{h-1,n+2}(q,q,\mathbf{z_n}) + \sum_{m=0}^{h} \sum_{I\subset \mathbf{z_n}} \omega_{m,|I|+1}(q,I)\,\omega_{g-m,|\mathbf{z_n}\setminus I|+1}(q,\mathbf{z_n}\setminus I)\Big]$$

where $dE_q(z) = \frac{1}{2}\int_q^{\bar{q}} \omega_{0,2}(q,z)$.

- "Free energies" $\left(\omega_{h,0}\right)_{h\geq 2}$ given by:

$$\omega_{h,0} = \frac{1}{2-2h}\sum_{i=1}^{r} \operatorname*{Res}_{q\to a_i} \Phi(q)\,\omega_{h,1}(q) \text{ où } \Phi(q) = \int^{q} ydx$$

- Specific formula for $\omega_{0,0}$ and $\omega_{1,0}$

Method 1

Differential system and WKB expansion

- Let $\hbar \partial_x \Psi(x,\hbar) = L(x,\hbar)\Psi(x,\hbar)$ a differential system of dimension d. We assume:
  1. $L(x,\hbar) = \sum_{k=0}^{\infty} L^{(k)}(x)\hbar^k$ with $x \mapsto L^{(k)}(x)$ **rational functions**
  2. We look for **formal WKB solutions**:

$$\begin{aligned}\Psi(x,t,\hbar) &= \Psi_0(x,t)\left(\mathrm{Id} + \sum_{k=1}^{\infty} \Psi^{(k)}(x,t)\hbar^k\right) e^{\frac{1}{\hbar}\Psi^{(-1)}(x,t)} \\ &= \exp\left(\frac{1}{\hbar}\Psi^{(-1)}(x,t) + \sum_{k=0}^{\infty} \tilde{\Psi}^{(k)}(x,t)\hbar^k\right)\end{aligned}$$

with $\Psi^{(-1)}(x,t)$ diagonal (trivial gauge choice).

Determinantal formulas

Definition (Correlation functions associated to a diff. system)

Let $\hbar\partial_x\Psi(x,\hbar) = L(x,\hbar)\Psi(x,\hbar)$ a diff. system. Then we define correlation functions by "**determinantal formulas**":

$$W_n(x_1.E_{j_1},\dots,x_n.E_{j_n}) = \begin{cases} \hbar^{-1}\,\mathrm{Tr}\left(L(x_1)M(x_1.E_{j_1})\right)dx_1 & n=1 \\ \frac{1}{n}\sum_{\sigma\ \text{n-cycles}} \frac{\mathrm{Tr}\prod_{i=1}^{n} M(x_{\sigma(i)}.E_{j_{\sigma(i)}})}{\prod_{i=1}^{n}(x_{\sigma(i)}-x_{\sigma(i+1)})}\prod_{i=1}^{n}dx_i & n\geq 2 \end{cases}$$

where $M(x.E_i) = \Psi(x)E_i\Psi(x)^{-1}$ with $E_i = \mathrm{diag}(0,\dots,0,1,0,\dots,0)$

Properties

Correlation functions satisfied the set of equations known as "loop equations" also satisfied by Eynard Orantin differentials in the Topological Recursion.

Associated spectral curve

Definition (Classical spectral curve associated to diff. system)

We define the classical spectral curve by:

$$P(x,y) = \lim_{\hbar \to 0} \det \left(y \, \mathrm{Id} - L(x,\hbar) \right) = 0$$

giving a polynomial equation, i.e. a Riemann surface Σ. For non-zero genus curve, this must be completed with a choice of basis of symplectic cycles and a bi-differential form $\omega_2^{(0)}$.

Topological type property

Definition (Topological type property)

A diff. system $\hbar\partial_x\Psi(x,\hbar) = L(x,\hbar)\Psi(x,\hbar)$ is of "**topological type**" if:

1. Its correlation functions $W_n(x_1.E_{j_1},\dots,x_n.E_{j_n})$ admit a formal expansion in $\hbar$ of the form:

$$W_n(x_1.E_{j_1},\dots,x_n.E_{j_n}) = \sum_{k=0}^{\infty} W_n^{(k)}(x_1.E_{j_1},\dots,x_n.E_{j_n})\hbar^{n-2+2k}$$

2. Differentials $W_n^{(k)}(x_1.E_{j_1},\dots,x_n.E_{j_n})$ may only have pole singularities at branchpoints of the classical spectral curve.
3. Differentials $W_n^{(k)}(x_1.E_{j_1},\dots,x_n.E_{j_n})$ have null integrals over $\mathcal{A}$-cycles associated to the classical spectral curve.
4. The differential $W_2^{(0)}(x_1,x_2)$ identifies with $\omega_{0,2}$ of the classical spectral curve.

Topological type property 2

- Main interest: Sufficient condition for reconstruction by TR: Topological Type property $\Rightarrow$ Correlation functions $W_n^{(h)}$ identify with corresponding $\omega_{h,n}$ computed by TR applied to the classical spectral curve (Bergère, Borot, Eynard (2013)).
- General idea: Previous conditions $\Rightarrow$ uniqueness of the solutions of the loop equations.
- How to prove Topological Type property?

Simplification of genus 0 spectral curves

- Simplification of the Topological Type property in genus 0:
  1. Formal $\hbar$-expansion for $W_n \Rightarrow$ **Always true because we look for WKB solutions**.
  2. $\{$Singularities of $W_n^{(k)}\} \subset \{$Branchpoints$\}$.
  3. Parity of $\hbar$ powers in the expansion of W_n.
  4. Leading order of the expansion of W_n is $\hbar^{n-2}$.
- **General method showing 4 from 1 and 2 via loop equation.** (work with K. Iwaki)
- Conditions 2 and 3 are proved only in cases where the differential system comes from some Lax pair $\hbar \partial_t \Psi(x,t) = A(x,t)\Psi(x,t)$ with $A(x,t)$ rational in x with specific properties.

Some general results for $\mathfrak{sl}_2(\mathbb{C})$

Theory of isomonodromic deformations allows for a system $L(x) = \sum_{i=0}^{r_0} L_{0,i} x^i + \sum_{\nu=1}^{n} \sum_{i=1}^{r_\nu} \frac{L_{\nu,i}}{(x-a_\nu)^i}$ to introduce a primary time deformation:

Theorem

The integrable system defined on the coadjoint orbit through any $\mathfrak{sl}_2(\mathbb{C})$ valued rational function $L(x)$ can be deformed into an isomonodromic system

$$\begin{cases} \frac{\partial}{\partial x}\Psi(x,t) = L(x,t)\Psi(x,t) \\ \frac{\partial}{\partial t}\Psi(x,t) = A(x,t)\Psi(x,t) \end{cases}$$

where $A(x,t) = \frac{M(t)x + B(t)}{p(x)}$ with $p \in \mathbb{C}[X]$ and $(M,B) \in (\mathfrak{sl}_2(\mathbb{C}))^2$ and $L(x, t=0) = L(x)$.

Some general results for $\mathfrak{sl}_2(\mathbb{C})$

General results for $\mathfrak{sl}_2(\mathbb{C})$ (O.M., N. Orantin, (2019))

Introduction of $\hbar$ by rescaling of x, t, Hamiltonians, Ψ, etc in order to transform the system into:

$$\left\{ \begin{array}{l} \hbar\frac{\partial}{\partial x}\Psi(x,t,\hbar) = L(x,t,\hbar)\Psi(x,t,\hbar) \\ \hbar\frac{\partial}{\partial t}\Psi(x,t,\hbar) = A(x,t,\hbar)\Psi(x,t,\hbar) \end{array} \right.$$

with $L(x,t,\hbar)$ defining a classical spectral curve of genus 0 satisfying the Topological Type property.

Method 2

Quadratic differentials with prescribed pole structure

Definition

Let $n \geq 0$ and let $(X_\nu)_{\nu=1}^n$ be a set of distinct points on $\Sigma_0 = \mathbb{P}^1$ with $X_\nu \neq \infty$, for $\nu = 1, \dots, n$. We define the divisor

$$D = \sum_{\nu=1}^{n} r_\nu (X_\nu) + r_\infty(\infty)$$

Let $\mathcal{Q}(\mathbb{P}^1, D)$ **be the space of quadratic differentials on** $\mathbb{P}^1$ **such that any** $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$ **has a pole of order** $2r_\nu$ **at the finite pole** $X_\nu \in \mathcal{P}^{\text{finite}}$ **and a pole of order** $2r_\infty$ **or** $2r_\infty - 1$ **at infinity**.

Remark

Up to reparametrization, ∞ **is always part of the divisor**. *Infinity may be a pole of odd degree (i.e. a ramification point in what to follow) but all other finite poles are even degree.*

Quadratic differentials with prescribed pole structure 2

$\mathcal{Q}(\mathbb{P}^1, D)$

Let x be a coordinate on $\mathbb{C} \subset \mathbb{P}^1$. Any quadratic differential $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$ defines a compact Riemann surface Σ_ϕ by

$$\Sigma_\phi := \left\{ (x, y) \in \overline{\mathbb{C}} \times \overline{\mathbb{C}} \ / \ y^2 = \frac{\phi(x)}{(dx)^2} \right\}$$

$\frac{\phi(x)}{(dx)^2}$ is a meromorphic function on $\mathbb{P}^1$, i.e. a rational function of x.

Classical spectral curve associated to ϕ

For any $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$, we shall call "**classical spectral curve**" associated to ϕ the Riemann surface Σ_ϕ defined as a two-sheeted cover $x : \Sigma_\phi \to \mathbb{P}^1$. Generically, it has genus $g(\Sigma_\phi) = r - 3$ where

$$r = \sum_{\nu=1}^{n} r_\nu + r_\infty$$

Quadratic differentials with prescribed pole structure 3

Branchpoints

Σ_ϕ is branched over the odd zeros of ϕ and ∞ if ∞ is a pole of odd degree. We define:

$$\begin{aligned}\{b_\nu^+, b_\nu^-\} &:= x^{-1}(X_\nu) \text{ for } \nu = 1, \dots, n \\ \{b_\infty^+, b_\infty^-\} &:= x^{-1}(\infty) \text{ if } \infty \text{ pole of even degree} \\ \text{or } \{b_\infty\} &:= x^{-1}(\infty) \text{ if } \infty \text{ pole of odd degree}\end{aligned}$$

Filling fractions

Let $\eta = \phi^{\frac{1}{2}}$. We define the vector of filling fractions ϵ:

$$\forall\, i \in [\![1, g]\!] \,:\, \epsilon_i = \oint_{\mathcal{A}_i} \eta.$$

and its dual ϵ^* by:

$$\forall\, i \in [\![1, g]\!] \,:\, \epsilon_i^* = \oint_{\mathcal{B}_i} \eta.$$

Spectral Times

Definition (Spectral Times)

Given a divisor D, a *singular type* **T** is the data of

- a *formal residue* T_p at each finite pole and at $p = b_\nu^\pm$ satisfying $T_{b_\nu^+} = -T_{b_\nu^-}$;
- an *irregular type* given by a vector $(T_{p,k})_{k=1}^{r_p-1}$ at each pole $p \in \mathcal{P}$ satisfying $T_{b_\nu^+,k} = -T_{b_\nu^-,k}$.

For such a singular type **T**, let $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T}) \subset \mathcal{Q}(\mathbb{P}^1, D)$ be the space of quadratic differentials $\phi \in \mathcal{Q}(\mathbb{P}^1, D)$ such that $\eta = \phi^{\frac{1}{2}}$ satisfies

$$\forall\, b_\nu^\pm\,,\ \eta = \sum_{k=1}^{r_{b_\nu}} T_{b_\nu^\pm,k} \frac{dx}{(x - X_\nu)^k} + O\left(dx\right)$$

$$\eta = \sum_{k=1}^{r_\infty} T_{b_\infty^\pm,k} (x^{-1})^{-k} d(x^{-1}) + O(d(x^{-1})) = -\sum_{k=1}^{r_\infty} T_{b_\infty^\pm,k} x^{k-2} dx + O(x^{-2} dx)$$

if ∞ pole of even degree or

$$\eta = \sum_{k=1}^{r_\infty} T_{b_\infty,k} x^{k-1} d(x^{-\frac{1}{2}}) = -\sum_{k=1}^{r_\infty} \frac{T_{b_\infty,k}}{2} x^{k-\frac{5}{2}} dx$$

if ∞ pole of odd degree.

Symplectic structure

Theorem (Symplectic structure (T. Bridgeland (2018)))

$\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ is a symplectic vector space of dimension $2g$. *A basis of Darboux coordinates is given by the real part of periods of η along any symplectic basis $(\mathcal{A}_j, \mathcal{B}_j)_{j=1}^g$ of $H_1(\Sigma_\phi, \mathbb{Z})$. The associated coordinates are*

$$\forall\, i \in [\![1, g]\!] \;:\; \epsilon_i = \oint_{\mathcal{A}_i} \eta.$$

The dual coordinates are

$$\forall\, i \in [\![1, g]\!] \;:\; \epsilon_i^* = \oint_{\mathcal{B}_i} \eta.$$

Decomposition on $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$: Notation

- We denote $[f(x)]_{\infty,+}$ (resp. $[f(x)]_{X_\nu,-}$) the positive part of the expansion in x of a function $f(x)$ around ∞, including the constant term, (resp. the strictly negative part of the expansion in $x - X_\nu$ around X_ν) .
- We define $K_\infty = [\![2, r_\infty - 2]\!]$ and for all $k \in K_\infty$:

$$U_{\infty,k}(x) := (k-1) \sum_{l=k+2}^{r_\infty} T_{\infty,l}\, x^{l-k-2}$$

if ∞ pole of even degree and

$$U_{\infty,k}(x) := \left(k - \frac{3}{2}\right) \sum_{l=k+2}^{r_\infty} T_{\infty,l}\, x^{l-k-2}$$

if ∞ pole of odd degree.
- $K_\nu = [\![2, r_\nu + 1]\!]$ and for all $k \in K_\nu$:

$$U_{\nu,k}(x) := (k-1) \sum_{l=k-1}^{r_\nu} T_{\nu,l}\, (x - X_\nu)^{-l+k-2}$$

Decomposition on $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$

Lemma (Variational formulas)

A quadratic differential $\phi \in \mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ *reads* $\phi = f_\phi(x)(dx)^2$ *with*

$$f_\phi = \left[\left(\sum_{k=1}^{r_\infty} T_{\infty,k} x^{k-2}\right)^2\right]_{\infty,+} + \sum_{\nu=1}^{n}\left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k}\frac{dx}{(x-X_\nu)^k}\right)^2\right]_{X_\nu,-} + \sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^{n}\sum_{k\in K_\nu} U_{\nu,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\nu,k}}$$

if ∞ *pole of even degree and*

$$f_\phi = \left[\left(\sum_{k=2}^{r_\infty} \frac{T_{\infty,k}}{2} x^{k-\frac{5}{2}}\right)^2\right]_{\infty,+} + \sum_{\nu=1}^{n}\left[\left(\sum_{k=1}^{r_\nu} T_{\nu,k}\frac{dx}{(x-X_\nu)^k}\right)^2\right]_{X_\nu,-} + \sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\infty,k}} + \sum_{\nu=1}^{n}\sum_{k\in K_\nu} U_{\nu,k}(x)\frac{\partial\omega_{0,0}}{\partial T_{\nu,k}}$$

if ∞ *pole of odd degree*

Perturbative partition function

Definition (Perturbative partition function)

Given a classical spectral curve Σ, one defines the **perturbative partition function** as a function of a formal parameter $\hbar$ as

$$Z^{\text{pert}}(\hbar, \Sigma) := \exp\left(\sum_{h=0}^{\infty} \hbar^{2h-2} \omega_{h,0}(\Sigma)\right).$$

where $\omega_{h,0}$ are the Eynard-Orantin free energies associated to Σ.

Perturbative wave functions 1

Definition ($(F_{h,n})_{h\geq 0, n\geq 1}$ by integration of the correlators)

For $n \geq 1$ and $h \geq 0$ such that $2h - 2 + n \geq 1$, let us define

$$F_{h,n}(z_1, \dots, z_n) = \frac{1}{2^n} \int_{\sigma(z_1)}^{z_1} \dots \int_{\sigma(z_n)}^{z_n} \omega_{h,n}$$

where one integrates each of the n variables along paths linking two Gallois conjugate points inside a fundamental domain cut out by the chosen symplectic basis $(\mathcal{A}_j, \mathcal{B}_j)_{1\leq j\leq g}$.
For $(h, n) = (0, 1)$ we define:

$$F_{0,1}(z) := \frac{1}{2} \int_{\sigma(z)}^{z} \eta$$

For $(h, n) = (0, 2)$ regularization is required:

$$F_{0,2}(z_1, z_2) := \frac{1}{4} \int_{\sigma(z_1)}^{z_1} \int_{\sigma(z_2)}^{z_2} \omega_{0,2} - \frac{1}{2} \ln\left(x(z_1) - x(z_2)\right)$$

Perturbative wave functions 2

Definition (Definition of the perturbative wave functions)

We define first:

$$\begin{aligned} S_{-1}^{\pm}(x) &:= \pm F_{0,1}(z(x)) \\ S_0^{\pm}(x) &:= \frac{1}{2} F_{0,2}(z(x), z(x)) \\ \forall\, k \geq 1\,,\; S_k^{\pm}(\lambda) &:= \sum_{\substack{h\geq 0, n\geq 1 \\ 2h-2+n=k}} \frac{(\pm 1)^n}{n!} F_{h,n}(z(x),\dots,z(x)) \end{aligned}$$

where for $\lambda \in \mathbb{P}^1$, we define $z(\lambda) \in \Sigma_\phi$ as the unique point such that $x(z(\lambda)) = \lambda$ and $y(z(\lambda))dx(z(\lambda)) = \sqrt{\phi(\lambda)}$. The perturbative wave functions $\psi_\pm$ by:

$$\psi_\pm(\lambda, \hbar, \Sigma) := \exp\left(\sum_{k\geq -1} \hbar^k S_k^{\pm}(\lambda)\right)$$

Remarks

- Standard definitions used by K. Iwaki for Painlevé 1.
- Formulas do not require restriction to $\mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ but are well-defined for any classical spectral curve.
- $S^{\pm} = \ln(\psi_{\pm})$ are somehow more natural than $\psi_{\pm}$.
- $\psi_{\pm}$ **do not have nice monodromy properties**
  1. For $i \in [\![1, g]\!]$, the function $\psi_{\pm}(x, \hbar, \boldsymbol{\epsilon})$ has a formal monodromy along $\mathcal{A}_i$ given by

$$\psi_{\pm}(x, \hbar, \boldsymbol{\epsilon}) \mapsto e^{\pm 2\pi i \frac{\epsilon_i}{\hbar}} \psi_{\pm}(x, \hbar, \boldsymbol{\epsilon}).$$

  2. For $i \in [\![1, g]\!]$, the function $\psi_{\pm}(x, \hbar, \boldsymbol{\epsilon})$ has a formal monodromy along $\mathcal{B}_i$ given by

$$\psi_{\pm}(x, \hbar, \boldsymbol{\epsilon}) \mapsto \frac{Z^{\text{pert}}(\hbar, \boldsymbol{\epsilon} \pm \hbar\, \mathbf{e}_i)}{Z^{\text{pert}}(\hbar, \boldsymbol{\epsilon})} \psi_{\pm}(x, \hbar, \boldsymbol{\epsilon} \pm \hbar\, \mathbf{e}_i)$$

- Necessity of non-perturbative corrections (already known in the exact WKB literature).

Non-perturbative quantities

Definition

We define the non-perturbative partition function:

$$Z(\hbar,\Sigma,\boldsymbol{\rho}) := \sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2\pi i}{\hbar}\sum_{j=1}^{g} k_j\rho_j}\, Z^{pert}(\hbar,\boldsymbol{\epsilon}+\hbar\mathbf{k})$$

and the non-perturbative wave function:

$$\Psi_\pm(x,\hbar,\Sigma,\boldsymbol{\rho}) := \frac{\sum_{\mathbf{k}\in\mathbb{Z}^g} e^{\frac{2\pi i}{\hbar}\sum_{j=1}^{g} k_j\rho_j}\, Z^{pert}(\hbar,\boldsymbol{\epsilon}+\hbar\mathbf{k})\,\psi_\pm(x,\hbar,\boldsymbol{\epsilon}+\hbar\mathbf{k})}{Z(\hbar,\Sigma,\boldsymbol{\rho})}$$

Remarks

- Definitions similar to those of K. Iwaki for Painlevé 1 (genus 1)
- Discrete Fourier transforms of perturbative quantities
- **Provide good monodromy properties** (see next slide)
- **Dependence in $\hbar$ are no longer a WKB expansions**: trans-series:

$$\begin{aligned} Z(\hbar,\Sigma,\boldsymbol{\rho}) &= Z^{pert}(\hbar,\Sigma)\sum_{m=0}^{\infty}\hbar^m\Theta_m(\hbar,\Sigma,\boldsymbol{\rho}) \\ \Psi_{\pm}(x,\hbar,\Sigma,\boldsymbol{\rho}) &= \psi_{\pm}(x,\hbar,\Sigma)\frac{\sum_{m=0}^{\infty}\hbar^m\,\Xi_m(x,\hbar,\Sigma,\boldsymbol{\rho})}{\sum_{m=0}^{\infty}\hbar^m\,\Theta_m(\hbar,\Sigma,\boldsymbol{\rho})} \end{aligned}$$

Coefficients $\Theta_m(\hbar,\Sigma,\boldsymbol{\rho})$, $\Xi_m(x,\hbar,\Sigma,\boldsymbol{\rho})$ finite linear combinations of derivatives of theta functions.

Monodromy properties

- For $j=1,\dots,g$, $\Psi_\pm(x,\Sigma,\boldsymbol{\rho})$ has a formal monodromy along $\mathcal{A}_j$ given by
$$\Psi_\pm(x,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) \mapsto e^{\pm 2\pi i \frac{\epsilon_j}{\hbar}}\Psi_\pm(x,\Sigma,\boldsymbol{\rho}).$$
- For $j=1,\dots,g$, $\Psi_\pm(x,\Sigma,\boldsymbol{\rho})$ has a formal monodromy along $\mathcal{B}_j$ given by
$$\Psi_\pm(x,\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) \mapsto e^{\mp 2\pi i \frac{\rho_j}{\hbar}}\Psi_\pm(x,\Sigma,\boldsymbol{\rho}).$$

Wronskian

Wronskian

Let $\phi \in \mathcal{Q}(\mathbb{P}^1, D, \mathbf{T})$ defining a classical spectral curve Σ_ϕ. Then, the Wronskian $W(x;\hbar) = \hbar(\Psi_- \partial_x \Psi_+ - \Psi_+ \partial_x \Psi_-)$ is a rational function of the form:

$$W(x;\hbar) = w(\mathbf{T},\hbar) \frac{P_g(x)}{\prod_{\nu=1}^{n} (x - X_\nu)^{r_{b_\nu}}} = w(\mathbf{T},\hbar) \frac{\prod_{i=1}^{g} (x - q_i)}{\prod_{\nu=1}^{n} (x - X_\nu)^{r_{b_\nu}}}$$

with P_g a monic polynomial of degree g.

Remark

We denote $(q_i)_{i \leq g}$ the simple zeros of the Wronskian. Equivalent to

$$\forall i = 1, \dots, g \,, \quad \left. \frac{\partial \log \Psi_+}{\partial x} \right|_{x=q_i} = \left. \frac{\partial \log \Psi_-}{\partial x} \right|_{x=q_i} .$$

Quantum curve

Quantum Curve

The non-perturbative wave functions $\Psi_\pm$ satisfy a linear second order PDE with **rational coefficients**:

$$\left[\hbar^2 \frac{\partial^2}{\partial x^2} - \hbar^2 R(x)\frac{\partial}{\partial x} - \hbar Q(x) - \mathcal{H}(x)\right]\Psi_\pm = 0$$

with $R(x) = \frac{\partial \log W(x)}{\partial x}$ and

$$\begin{aligned}
\mathcal{H}(x) &= \left[\hbar^2 \sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial}{\partial T_{b_\infty,k}} + \hbar^2 \sum_{\nu=1}^{n}\sum_{k\in K_{b_\nu}} U_{b_\nu,k}(x)\frac{\partial}{\partial T_{b_\nu,k}}\right] \\
&\quad \left[\log Z(\mathbf{T},\boldsymbol{\epsilon},\boldsymbol{\rho}) - \hbar^{-2}\omega_{0,0}\right] + \frac{\phi(x)}{(dx)^2} \\
Q(x) &= \sum_{j=1}^{g}\frac{p_j}{x-q_j} + \frac{\hbar}{2}\left[\sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial(S_+(x)-S_-(x))}{\partial T_{\infty,k}}\right]_{\infty,+} \\
&\quad + \frac{\hbar}{2}\sum_{\nu=1}^{n}\left[\sum_{k\in K_\nu} U_{\nu,k}(x)\frac{\partial(S_+(x)-S_-(x))}{\partial T_{\nu,k}}\right]_{X_\nu,-}
\end{aligned}$$

Quantum curve 2

Additional relations

The pairs (q_i, p_i) satisfy $\forall i = 1, \dots, g$:

$$p_i^2 = \mathcal{H}(q_i) - \hbar p_i \left[\sum_{j \neq i} \frac{1}{q_i - q_j} - \sum_{\nu=1}^{n} \frac{r_\nu}{q_i - X_\nu} \right] \left. \frac{\partial \log \Psi_+(x)}{\partial x} \right|_{x=q_j} + \left[\frac{\partial \left(Q(x) - \frac{p_i}{x - q_i} \right)}{\partial x} \right]_{x=q_i}$$

Asymptotics $S_\pm(x)$ are given by:

$$S_\pm = \mp \hbar^{-1} \sum_{k=2}^{r_{b_\nu}} \frac{T_{b_\nu,k}}{k-1} \frac{1}{(x - X_\nu)^{k-1}} \pm \hbar^{-1} T_{b_\nu,1} \log(x - X_\nu) + \sum_{k=0}^{\infty} A^{\pm}_{\nu,k} (x - X_\nu)^k$$

$$S_\pm = \mp \hbar^{-1} \sum_{k=2}^{r_\infty} \frac{T_{b_\infty,k}}{k-1} x^{k-1} \mp \hbar^{-1} T_{b_\infty,1} \log(x) - \frac{\log x}{2} + \sum_{k=0}^{\infty} A^{\pm}_{\infty,k} x^{-k}$$

or

$$S_\pm = \mp \hbar^{-1} \sum_{k=2}^{r_\infty} \frac{T_{b_\infty,k}}{2k-3} x^{\frac{2k-3}{2}} \mp \hbar^{-1} T_{b_\infty,1} \log(x) - \frac{\log x}{4} + \sum_{k=0}^{\infty} A^{\pm}_{\infty,k} x^{-\frac{k}{2}}$$

Thus,

$$Q(x) = \sum_{j=1}^{g} \frac{p_j}{x - q_j} + \sum_{k=0}^{r_\infty - 4} Q_{\infty,k} x^k + \sum_{\nu=1}^{n} \sum_{k=1}^{r_\nu + 1} \frac{Q_{\nu,k}}{(x - X_\nu)^k}$$

Linearization and $\hbar$-deformed spectral curve

- Linearize the quantum curve, i.e. choose $\vec{\Psi}_\pm = \begin{pmatrix} \Psi_\pm \\ \alpha(x)\Psi_\pm + \beta(x)\partial_x\Psi_\pm \end{pmatrix}$ to have a 2×2 system

$$\hbar\partial_x\vec{\Psi}_\pm(x) = L(x)\vec{\Psi}_\pm(x) = \begin{pmatrix} P(x) & M(x) \\ W(x) & -P(x) \end{pmatrix}\vec{\Psi}_\pm(x)$$

- Define the $\hbar$-deformed spectral curve: $\det(ydx - L(x)dx) = 0 \Rightarrow y^2(dx)^2 = \phi_\hbar$:

$$\begin{aligned}\frac{\phi_\hbar}{(dx)^2} &= \mathcal{H}(x) + \hbar\sum_{j=1}^{g}\frac{p_j}{x-q_j} + \frac{\hbar^2}{2}\left[\sum_{k\in K_\infty} U_{\infty,k}(x)\frac{\partial(S_+(x)+S_-(x))}{\partial T_{\infty,k}}\right]_{\infty,+} \\ &+\frac{\hbar^2}{2}\sum_{\nu=1}^{n}\left[\sum_{k\in K_\nu} U_{\nu,k}(x)\frac{\partial(S_+(x)+S_-(x))}{\partial T_{\nu,k}}\right]_{X_\nu,-} + \hbar\frac{\partial P(x)}{\partial x} \\ &-\hbar\frac{\partial \log W(x)}{\partial x}P(x)\end{aligned}$$

Additional material 2

- Write the time differential systems

$$\partial_{T_{\nu,k}} \vec{\Psi}_{\pm} = R_{\nu,k}(x) \vec{\Psi}_{\pm}$$

- Define isomonodromic times $t_{\nu,k}$ and the map $(T_{\nu,k})_{\nu,k} \to (t_{\nu,k})_{\nu,k}$ and the differential systems $\partial_{t_{\nu,k}} \vec{\Psi}_{\pm} = L_{\nu,k}(x) \vec{\Psi}_{\pm}$
- Connected to the problem isospectral $\to$ isomonodromic: Existence of times t such that $\frac{\delta L(x)}{\delta t} = \frac{\partial L_t}{\partial x}$ where δ is the variation to explicit dependence on t only.
- Define g Hamiltonians $H_j(q_1, \dots, q_g, p_1, \dots, p_g, \hbar)$ so that $\hbar$-deformed Hamilton's equations are satisfied:

$$\hbar \partial_{t_j} q_j = \frac{\partial H_j}{\partial p_j} \text{ and } \hbar \partial_{t_j} p_j = -\frac{\partial H_j}{\partial q_j}$$

- Apply to all Painlevé equations and their hierarchies.

Outlook

Future works

- Check the topological type property (arbitrary genus case) of the previous Lax system.
- Extend results for non hyper-elliptic classical spectral curves: $\mathfrak{sl}_n(\mathbb{C})$
- Extend results to other manifolds than $\Sigma_0 = \mathbb{P}^1$
- Extend results for arbitrary Lie algebra $\mathfrak{g}$
- Extend results for Lie group (difference equations instead of differential equations)
- Extend results for β deformations?