Random matrices and applications

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March 17^{th} 2020

Random matrices

Definition

General problem

A random matrix is a $N \times N$ matrix whose entries are random variables:

$$M_N = \begin{pmatrix} X_{1,1} & \dots & X_{1,N} \\ \vdots & & \vdots \\ X_{N,1} & \dots & X_{N,N} \end{pmatrix}$$

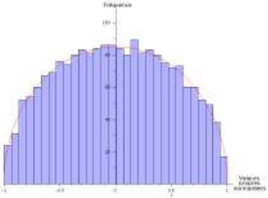
Standard questions on random matrices

- Does M_N admit N simple eigenvalues almost surely?
- Can we characterize the eigenvalues distribution, at least in the large N limit?
- Are eigenvalues independent? Can we characterize the correlations between them?
- Can we control the “largest” or “lowest” eigenvalue?
- Are there some applications of random matrix theory?

Answer

Answer

Answers depend on the **assumptions on the randomness of the entries**: symmetry of the matrix, independence, existence of moments, etc. But some **universal results** arise under weak conditions...

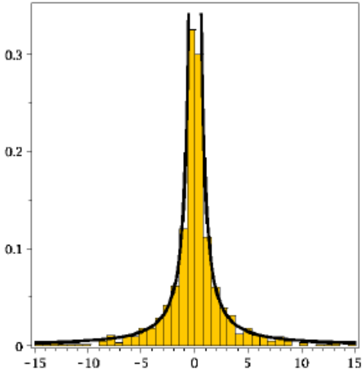


Histogram of the eigenvalues divided by $\sqrt{N}$ of a 100×100 Hermitian random matrix with i.i.d. $\mathcal{N}(0,1)$ entries.

Semi-circle law

Semi-circle law holds for other distributions. But not all of them...

Cauchy case



Histogram of the eigenvalues divided by N of a 100×100 symmetric random matrix with i.i.d. entries drawn from the Cauchy distribution. Black curve is $x \mapsto \frac{1}{2\pi x^{\frac{3}{2}}}$ (correct large x asymptotics)

A general result

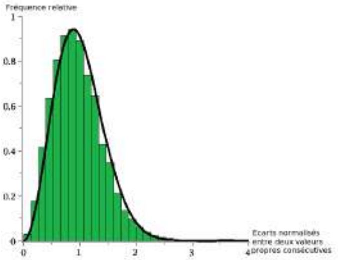
A sufficient condition

An Hermitian random matrix M with upper triangular i.i.d. entries drawn from a probability distribution with **zero mean** and $\mathbb{E}\left(|M_{i,j}|^{2+\epsilon}\right) < \infty$, for some $\epsilon > 0$, gives rise to Wigner semi-circle law (Tao and Vu [2008]).

Other results

Similar results are available for many other cases (relaxing the independence, not identically distributed entries, etc.). **Delocalization** results for **eigenvectors** are also available.

Universality results

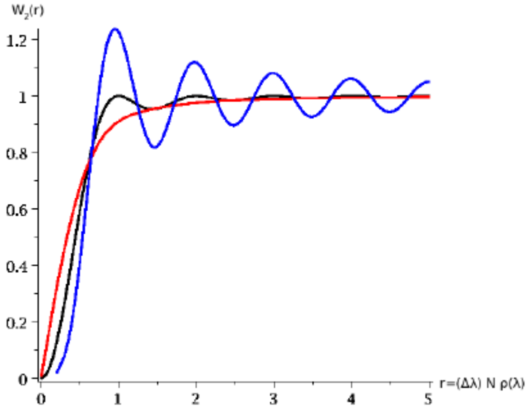


Histogram of gaps $\sqrt{N}(\lambda_{i+1} - \lambda_i)$ between consecutive eigenvalues of a 1000×1000 symmetric Gaussian random matrix with i.i.d. entries. Black curve is the Gaudin distribution.

Universal limiting correlations

Eigenvalues of a random matrix are **not independent even in the large N limit**. **Limiting local correlations are universal** and only depends on the **symmetry** of the matrix and the **local position in the limiting distribution** (bulk, edge,...) (Dyson [1970])

Universality in the bulk for the three classical ensembles



Two-points functions $W_2(r)$ with $r = N(\lambda_{i+1} - \lambda_i)\rho(\lambda_i)$ of the three classical ensembles around a bulk point. (Black: Hermitian, Red: Real-symmetric, Blue: Quaternionic self-dual)

Hermitian matrix integrals

Hermitian matrix integrals

Hermitian matrix integrals are drawn from partition functions:

$$Z_N(T) = \int_{\mathbb{E}_N} dM e^{-\frac{N}{T}\,\mathrm{Tr}(V(M))}$$

where $\mathbb{E}_N$ is a subset of Hermitian matrices. V is the "potential". T is a "temperature" parameter.

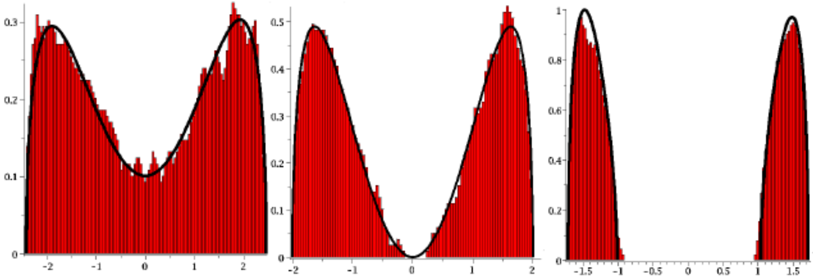
Diagonalization ⇒ Eigenvalues problem

$$Z_N = C_N \int_{E^N} d\lambda_1 \dots d\lambda_N \, \Delta(\boldsymbol{\lambda})^2 e^{-\frac{N}{T}\sum_{i=1}^{N} V(\lambda_i)} \ , \ C_N = \mathrm{Vol}(\mathcal{U}_N)$$

Balance between localization and repulsion

Competition between the potential (**accumulation around minima**) and the **Coulomb repulsion** $\Delta(\boldsymbol{\lambda})^2$.

Bigger diversity of the limiting eigenvalues distribution



Histogram of normalized eigenvalues of a 200 × 200 random Hermitian matrix with potential $V(x) = \left(\frac{x^4}{4} - x^2\right)$.
Center picture is for the **critical case** $T_c = 1$.
Results obtained from Metropolis-Hastings algorithm.

General results for the limiting eigenvalues distribution

Potential theory

For polynomial potentials of even degree, **empirical eigenvalues distribution almost surely converges to an absolutely continuous measure** (relatively to the Lebesgue measure) ρ_∞ which is **supported on a finite number of intervals** (Mehta [2004]).

Characterization

ρ_∞ can be computed in most cases $\Leftrightarrow$ Compute the limiting "**spectral curve**", i.e. its Stieljes transform that satisfies an algebraic equation $y^2 = Q(x)$.

New universality classes

Correlation around critical points

ρ_∞ may exhibit **critical points** where $\rho_\infty(x) \overset{x \to x_c}{\sim} \alpha(x - x_c)^{\frac{p}{q}}$ with $(p, q) \notin \{(0, 1), (1, 2)\}$. Local correlations between eigenvalues obeys **new universality laws** (Mehta [2004]).

Connection with integrable systems

Local correlations are characterized by **Fredholm determinants** whose kernel defines the universality class: Sine kernel for bulk point, Airy kernel for the regular soft edge case.

Open questions

Open questions

- Describe kernels for critical points and prove the Fredholm determinants representation
- Sine and Airy kernels: **integral representations** of the Fredholm determinants using **Painlevé transcendents** are available (Tracy and Widom [1994]):

$$F_{TW}(s) = \det(I - \chi_{[s,+\infty)} \mathbb{K}_{\mathrm{Ai}} \chi_{[s,+\infty)}) = \exp\left(-\int_s^{+\infty} (x-s) q(x)^2 dx\right)$$

where q is the unique solution (Hasting-McLeod) of the Painlevé 2 equation:

$$q''(s) = 2q(s)^3 + sq(s)\,,\ q(s) \overset{s\to+\infty}{\sim} e^{-\frac{2}{3}s^{\frac{3}{2}}}$$

Are there similar integral representations using other Painlevé solutions for other critical points?

Connection with integrable systems

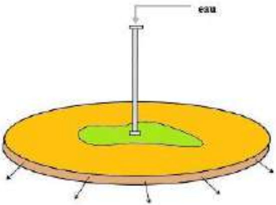
- Random matrix integrals have **deep connections with integrable systems** (Bertola et al. [2003, 2006]).
- Complete study of the case $(p, q) = (2m, 1)$ (Marchal and Cafasso [2011]).
- Partition functions of random matrix integrals are "**isomonodromic tau-function**" (Bertola and Marchal [2009])
- Personal recent results (Iwaki et al. [2018b], Marchal and Orantin [2019a,b]) are promising
  $\Rightarrow$ full understanding of the situation via the "topological recursion"
  $\Rightarrow$ **Reconstruction of the corresponding integrable systems** is on the way ("Quantum spectral curve").
- Very involved algebraic geometry: Riemann surfaces, moduli spaces of meromorphic connections...
- Interests for string theory and enumerative geometry.

Applications of random matrix theory

(Too?) Many fields of applications

- Energy gaps for heavy nuclei (Wigner and Weinberg [1958])
- Laplacian growth (oil interface) (Saffman and Taylor [1958], Zabrodin [2009])
- Random permutations (Baik et al. [2000], Okunkov [2000])
- Pavings (Aztek diamants,...) (Johansson [2002])
- Self-avoiding random walks (TASEP,...) (Eynard [2009])
- Dyson Brownian motions (Joyner and Smilansky [2015])
- Enumerative geometry (counting triangulations/quadrangulations of Riemann surfaces) (Eynard [2016])
- Telecommunications: multiple input-multiple output (MIMO): e.g. 5G network (Heath and Lozano [2018])
- String theory,2D gravity, Chern-Simons theory (Mariño [2005])
- Riemann hypothesis (Montgomery [1973])
- Fredholm determinants, integrable systems (Borodin and Okounkov [2000])
- Machine learning (Mai and Couillet [2018])
- And many more...

Hele-Shaw cell



Hele-Shaw cell: insertion of a viscous fluid in a liquid one.

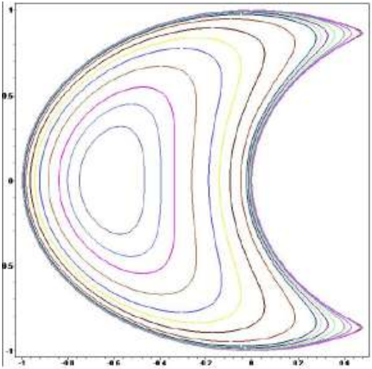
Theoretical model: **Normal matrix ensembles** (⇒ Complex eigenvalues)

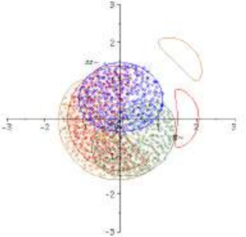
$$\mathbb{P}(z_1, \ldots, z_n) = \frac{1}{Z_n} |\Delta(\mathbf{z})|^2 e^{-N \sum_{i=1}^{N} |z_i|^2 + V(z_j) + \bar{V}(\bar{z}_j)}$$

First result: $\rho_{\text{emp.}}(z_1, \ldots, z_N) \underset{N\to\infty}{\overset{\text{a.s.}}{\to}} \frac{1}{\text{Area}(D)} \mathbb{1}_D$

$D \subset \mathbb{C}$. Its edge $\mathcal{C}$ represents the interface between the fluids.

Laplacian growth



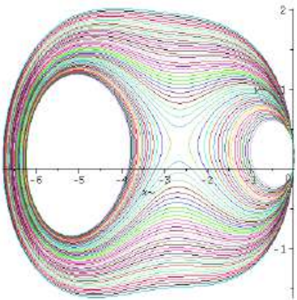


Growth of the interface for $V(z) = c\ln(z-a)$. (Personal talk at BIRS workshop 2011). Interface given by:

$$(-c - z(\bar{z}-a))\,(-c - \bar{z}(z-a)) - c^2 + \alpha = 0$$

α = Volume of fluid inserted.

Laplacian growth 2

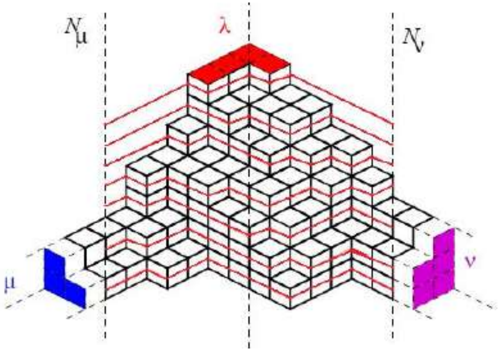


Another example of a Laplacian growth for $V(z) = 3t_3z^2$

Remark

These models exhibit singular domains in relation with universal classes...

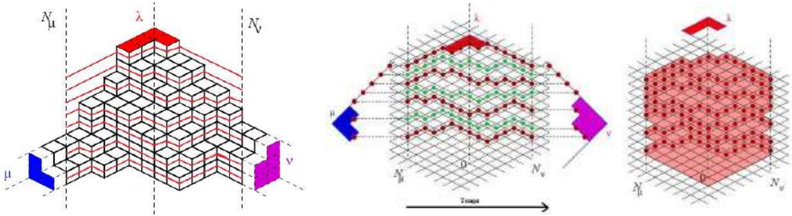
Self-avoiding walks and cristal growth



- Box stacking in the corner of a room (or cristal growth)
- Fixed edges described bythree given 2D partitions λ, μ, ν
- Partition function of the model:

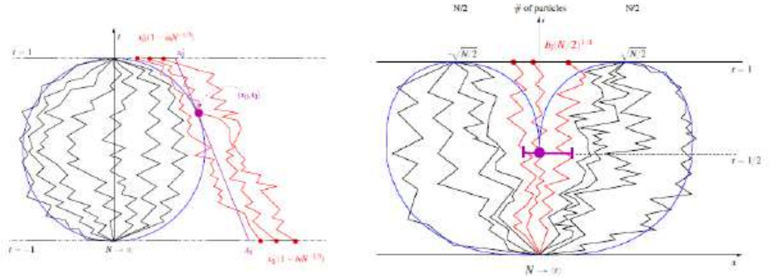
$$Z_{\lambda,\mu,\nu} = \sum_{\pi=\text{Config}} q^{|\pi|} \ , \ |\pi| = \text{volume of } \pi$$

Self-avoiding walks



- Existence of a underlying random matrix model (Eynard [2009])
- **Chain of matrices**: several random matrices coupled in a chain
- Tools and methods remain functional
- **Local behaviors remain universal**
- Global behavior (limiting distribution) is also computable ("artic zone")

Dyson Brownian paths



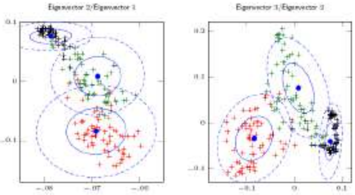
Non-colliding Brownian motions with specific initial and final conditions.

Left picture: Baik-Ben Arous-Péché kernel (Adler et al. [2010, 2013])

Right picture: Pearcey kernel (Adler et al. [2011])

Machine learning and RMT model

- Rand Matrix Theory (RMT) grasps and improves data processing (Couillet et al. [2018])
- RMT shows **good experimental results** for picture categorization problems: e.g. $\{0, 1, 2\}$ problem (Mai and Couillet [2018])
  1. Model the unknown image as a reference image + noise (i.e. random matrix)
  2. Use RMT results to obtain control over the eigenvectors.
  3. Compare your data to the reference domains and select the best one.



2D representation of eigenvectors for the MNIST dataset. Theoretical means and $1-2$ standard deviations in blue. 3 different classes in color (red, black, green). Picture extracted from a R. Couillet talk (2015).

Machine learning 2

- Very recent application of RMT (less than 10 years)
- Very in fashion in the "Big Data" context
- Most theoretical results used depend on Gaussian assumptions. But real data may not always have perfect Gaussian noise
- Data are so far modeled with very basic matrix models: more complex RM models should be considered to fit a bigger range of data
- Still many steps to applications like neural networks, real-life applications,...
- But **literature and simulations are developing very fast**...

A detailed example: Toeplitz determinants

Description of the problem

Statement of the problem

Let $\epsilon \in (0,1)$. Compute the following integral ($Z_n(\epsilon = 1) = 1$):

$$Z_n(\epsilon) = \frac{1}{(2\pi)^n n!} \int_{([-\pi\epsilon,\pi\epsilon])^n} d\theta_1 \dots d\theta_n \prod_{1 \le p < q \le n} \left| e^{i\theta_p} - e^{i\theta_q} \right|^2$$

Various reformulations

Standard results give for $a = \tan \frac{\pi\epsilon}{2}$:

$$\begin{aligned} Z_n(\epsilon) &= \frac{1}{(2\pi)^n n!} \int_{([-\pi\epsilon,\pi\epsilon])^n} d\theta_1 \dots d\theta_n \prod_{1 \le p < q \le n} \left| e^{i\theta_p} - e^{i\theta_q} \right|^2 \\ &= \det\left(T_{p,q} = t_{p-q} \right)_{1 \le p,q \le n} \text{ where } t_k = \epsilon \operatorname{sin}_c(k\pi\epsilon) \\ &= \frac{2^{n^2}}{(2\pi)^n n!} \int_{[-a,a]^n} dt_1 \dots dt_n \, \Delta(t_1,\dots,t_n)^2 e^{-n \sum_{k=1}^{n} \ln(1+t_k^2)} \end{aligned}$$

$(t_k)_{-(n-1) \le k \le n-1}$ are discrete Fourier coefficients of the symbol function $f = \mathbb{1}_{\{e^{it}, t \in [-\pi\epsilon,\pi\epsilon]\}}$.

Some remarks

- Connections between **Toeplitz determinants**, **Hermitian matrix integral**, **integral over the unit circle** is very general and apply for any symbol function (Duits and Johansson [2010])
- Toeplitz determinants have Fredholm determinant representations (Borodin and Okounkov [2000])
- Toeplitz determinant reformulation allows large n numerical simulations
- Hermitian matrix integrals allows the use of RMT results. Potential is $V(x) = \ln(1+x^2)\mathbb{1}_{[-a,a]}(x)$.

Widom's result

Widom's result

We have (Widom [1971]):

$$\ln Z_n(\epsilon) = n^2 \ln\left(\sin\frac{\pi\epsilon}{2}\right) - \frac{1}{4}\ln n - \frac{1}{4}\ln\left(\cos\frac{\pi\epsilon}{2}\right) + 3\,\zeta'(-1) + \frac{1}{12}\ln 2 + o(1)$$

where ζ denotes the Riemann zeta function.

Comment

- Widom's method does not say anything about the $o(1)$ **term**
- It does not generalize to other symbol functions

How to improve Widom's result

1. Compute the **limiting eigenvalues density** ρ_∞ and the corresponding spectral curve.
2. Check that conditions for application of (Borot and Guionnet [2011], Borot et al. [2014]) results apply (conditions on ρ_∞ and potential V)

$$\Rightarrow \ln Z_n(\epsilon) = -\frac{1}{4}\ln n + \sum_{k=-2}^{K} F^{\{k\}}(\epsilon) n^{-k} + o\left(n^{-K}\right) \ \forall K \geq -2$$

3. Coefficients $\left(F^{\{k\}}(\epsilon)\right)_{k\geq -2}$ can be computed by the **Eynard-Orantin topological recursion** (Eynard and Orantin [2007]) on the spectral curve up to some constants (independent of ϵ)

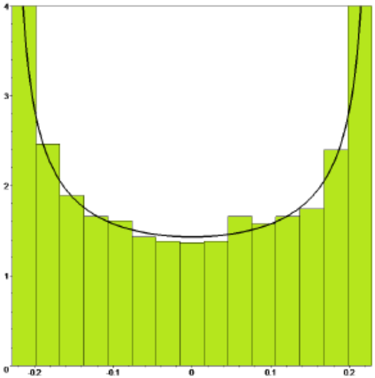
$$F^{\{k\}}(\epsilon) - F^{\{k\}}(\epsilon_0) = F^{\{k\}}_{\text{EO}}(\epsilon) - F^{\{k\}}_{\text{EO}}(\epsilon_0)$$

How to improve Widom's result 2

4. Carefully take the reference value $\epsilon_0 \to 0$:
   - $Z_n(\epsilon_0 = 0)$ is a known **Selberg integral** (Selberg [1944]) whose large n expansion is computable (Bernoulli numbers).
   - The spectral curve for $\epsilon_0 = 0$ is equivalent to **Legendre's spectral curve** $y = \frac{1}{\sqrt{1-x^2}}$.

     Results of (Iwaki et al. [2018a]) give all free energies $\left(F^{\{k\}}_{\text{Leg.}}\right)_{k \geq -2}$ in terms of Bernoulli numbers.
   - Turns out that both sets are identical except for $k = 0$.
5. **Compute the first terms** using Eynard-Orantin topological recursion and **compare with numerical simulations**.

Limiting eigenvalues density



Empirical eigenvalues density obtained from 100 independent Monte-Carlo simulations for $\epsilon = \frac{1}{7}$ and $n = 20$. Black curve is the theoretical density:

$$\rho_\infty(x) = \frac{1}{\pi \cos(\frac{\pi\epsilon}{2})(1+x^2)\sqrt{\tan^2(\frac{\pi\epsilon}{2}) - x^2}} \mathbb{1}_{[-\tan\frac{\pi\epsilon}{2}, \tan\frac{\pi\epsilon}{2}]}(x)$$

Remarks on ρ_∞

- Support of ρ_∞ is a **single interval** $\left[-\tan\frac{\pi\epsilon}{2}, \tan\frac{\pi\epsilon}{2}\right]$. $\Leftrightarrow$ **Genus 0 spectral curve**
- Single interval case is much simpler to deal with: **asymptotic expansion is purely "perturbative"**.
  No need to consider filling fractions (i.e. proportion of eigenvalues lying in each intervals of the support).
- Hard edges at $\pm\tan\frac{\pi\epsilon}{2} \Rightarrow \rho_\infty$ diverges at the edges.
- **Hard edges are regular** because $\rho_\infty(x) \sim \frac{c_\pm}{\sqrt{\tan(\frac{\pi\epsilon}{2}) \pm x}}$.
- Potential is **confining** (null) at $\pm\infty$.
- $\Rightarrow$ Sufficient conditions to apply (Borot and Guionnet [2011], Borot et al. [2014]) results.

Final result

Improvement of Widom's result (Marchal [2019])

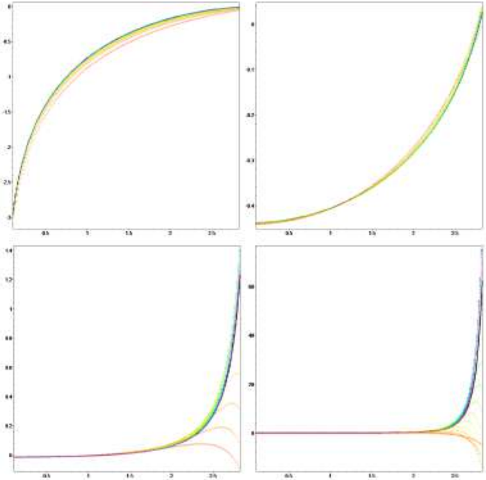
$$\ln Z_n(\epsilon) = n^2 \ln\left(\sin\left(\frac{\pi\epsilon}{2}\right)\right) - \frac{1}{4}\ln n - \frac{1}{4}\ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right)$$
$$+3\,\zeta'(-1) + \frac{1}{12}\ln 2 - \sum_{g=2}^{\infty} F_{\text{EO}}^{(g)}(\epsilon) n^{2-2g}$$

where $F_{\text{EO}}^{(g)}(\epsilon)$ are the Eynard-Orantin free energies associated to the spectral curve $y^2(x) = \frac{1}{\cos^2\left(\frac{\pi\epsilon}{2}\right)(1+x^2)^2\left(x^2 - \tan^2\left(\frac{\pi\epsilon}{2}\right)\right)}$

Computation of the first terms

$$\begin{aligned}\ln Z_n(\epsilon) &= n^2 \ln\left(\sin\left(\frac{\pi\epsilon}{2}\right)\right) - \frac{1}{4}\ln n - \frac{1}{4}\ln\left(\cos\left(\frac{\pi\epsilon}{2}\right)\right)\\ &+3\,\zeta'(-1) + \frac{1}{12}\ln 2 + \frac{1}{64n^2}\left(2\tan^2\left(\frac{\pi\epsilon}{2}\right) - 1\right)\\ &+\frac{1}{256n^4}\left(1 + 2\tan^2\left(\frac{\pi\epsilon}{2}\right) + 10\tan^4\left(\frac{\pi\epsilon}{2}\right)\right) + O\left(\frac{1}{n^6}\right)\end{aligned}$$

Numerical study



Computation of $\epsilon \mapsto \ln Z_n(\epsilon)$ with $0 < \epsilon < 1$ for $2 \leq n \leq 35$ with subtraction of the first coefficients of the large n expansion.

Possible generalizations

- Strategy applies to any symbol function supported on a subset of $\{e^{it}, t \in [a,b]\}$ with $-\pi < a < b < \pi$.
- ρ_∞ supported on several intervals $\Rightarrow$ Additional terms in the expansion involving Θ functions. Usually hard to compute analytically.
- Symbol function supported on the whole unit circle $\Rightarrow$ **Drastic changes expected**:
  1. **Szegö theorem** applies: $\ln Z_n \propto n$ and no longer $\ln Z_n \propto n^2$.
  2. Potential V is not sufficiently confining at $\pm\infty$ to apply Borot-Guionnet-Koslowski results.
  3. Does the Eynard-Orantin recursion still reconstruct the asymptotic expansion?
- What about Hankel determinants, Fredholm determinants, Toeplitz operators?

Conclusion

Conclusion

- **Random matrices** have **many fields of applications** from hard-core mathematics (number theory, enumerative geometry, integrable systems,...) to applied mathematics (statistics, telecommunications, nuclear physics, engineering,...)
- Many different kinds of mathematics behind random matrices: probability, statistics, linear algebra, geometry, algebraic geometry, PDE, SPDE, numerical simulations,...
- Still remains a "recent field" in mathematics (20^{th} century) with already some Fields medals (T. Tao, A. Okunkov, M. Kontsevich).
- **Various general methods to study RMT**: large deviations and CLT, potential theory, integrable systems, orthogonal polynomials, Riemann-Hilbert problems,...
- Coordination between different components of RMT is rising but still very low...

References I

M. Adler, P. Ferrari, and P. Van-Moerbeke. Airy processes with wanderers and new universality classes. *The Annals of Probability*, 38, 2010.

M. Adler, P. Ferrari, and P. Van-Moerbeke. A PDE for non-intersecting Brownian motions and applications. *Advances in Mathematics*, 226, 2011.

M. Adler, P. Ferrari, and P. Van-Moerbeke. Nonintersecting random walks in the neighborhood of a symmetric tacnode. *The Annals of Probability*, 40, 2013.

J. Baik, P. Deift, and K. Johansson. On the distribution of the length of the second row of a Young diagram under Plancherel measure. *Geometric and Functional Analysis*, 10, 2000.

M. Bertola and O. Marchal. The partition function of the two-matrix model as an isomonodromic tau-function. *Journal of Mathematical Physics*, 50, 2009.

M. Bertola, B. Eynard, and H. John. Partition functions for matrix models and isomonodromic tau functions. *Journal of Physics A*, 36:3067–3083, 2003.

M. Bertola, B. Eynard, and J. Harnad. Semiclassical orthogonal polynomials, matrix models and isomonodromic tau functions. *Communications in Mathematical Physics*, 263:401–437, 2006.

References II

A. Borodin and A. Okounkov. A Fredholm determinant formula for Toeplitz determinants. *Integral Equations and Operator Theory*, 37, 2000.

G. Borot and A. Guionnet. Asymptotic Expansion of β Matrix Models in the One-cut Regime. *Communications in Mathematical Physics*, 317, 2011.

G. Borot, A. Guionnet, and K. Kozlowski. Large-N asymptotic expansion for mean field models with Coulomb gas interaction. *International Mathematics Research Notices*, 20, 2014.

R. Couillet, M. Tiomoko, S. Zozor, and E. Moisan. Random matrix-improved estimation of covariance matrix distances. *arXiv preprint: arXiv:1810.04534*, 2018.

M. Duits and K. Johansson. Powers of large random unitary matrices and Toeplitz determinants. *Transactions of the American Mathematical Society*, 3, 2010.

F. Dyson. Correlations between the eigenvalues of a random matrix. *Communications in Mathematical Physics*, 19, 1970.

B. Eynard. A Matrix model for plane partitions. *Journal of Statistical Mechanics*, 2009.

B. Eynard. Counting Surfaces. *Progress in Mathematical Physics*, 70, 2016.

References III

B. Eynard and N. Orantin. Invariants of algebraic curves and topological recursion. *Communications in Number Theory and Physics*, 1:347–452, 2007.

R. Heath and A. Lozano. Foundations of MIMO Communication. *Cambridge University Press*, 2018.

K. Iwaki, T. Koike, and Y. Takei. Voros Coefficients for the Hypergeometric Differential Equations and Eynard-Orantin's Topological Recursion - Part II : For the Confluent Family of Hypergeometric Equations. *arXiv preprint: arXiv:1810.02946*, 2018a.

K. Iwaki, O. Marchal, and A. Saenz. Painlevé equations, topological type property and reconstruction by the topological recursion. *Journal of Geometry and Physics*, 124, 2018b.

K. Johansson. Non-intersecting paths, random tilings and random matrices. *Probability theory and related fields*, 2002.

C. Joyner and U. Smilansky. Dyson's Brownian-motion model for random matrix theory - revisited. *arXiv preprint: arXiv:1503.06417*, 2015.

X. Mai and R. Couillet. A Random Matrix Analysis and Improvement of Semi-Supervised Learning for Large Dimensional Data. *Journal of Machine Learning Research*, 19, 2018.

References IV

O. Marchal. Asymptotic expansions of some Toeplitz determinants via the topological recursion. *Letters in Mathematical Physics*, 2019.

O. Marchal and M. Cafasso. Double-scaling limits of random matrices and minimal $(2m, 1)$ models: the merging of two cuts in a degenerate case. *Journal of Statistical Mechanics*, 2011, 2011.

O. Marchal and N. Orantin. Isomonodromic deformations of a rational differential system and reconstruction with the topological recursion: the sl_2 case. *arXiv preprint: arXiv:1901.04344*, 2019a.

O. Marchal and N. Orantin. Quantization of hyper-elliptic curves from isomonodromic systems and topological recursion. *arXiv preprint: arXiv:1911.07739*, 2019b.

M. Mariño. Chern-Simons Theory, Matrix Models, and Topological Strings. *International Series of Monographs on Physics*, 2005.

M. Mehta. *Random matrices*, volume 142. Elsevier academic press, 2004.

H. Montgomery. The pair correlation of zeros of zeta function. *Proceedings of Symposia in Pure Mathematics*, 24, 1973.

A. Okunkov. Random Matrices and Random Permutations. *International Mathematics Research Notices*, 2000, 2000.

References V

P. Saffman and G. Taylor. The penetration of a fluid into a porous medium or Hele-Shaw cell containing a more viscous liquid. *Proceedings of the Royal Society A*, 245, 1958.

A. Selberg. Remarks on a multiple integral. *Norsk. Mat. Tidsskr.*, 26, 1944.

T. Tao and V. Vu. Random matrices: the circular law. *Communications in Contemporary Mathematics*, 10, 2008.

C. Tracy and H. Widom. Level-Spacing Distributions and the Airy Kernel. *Communications in Mathematical Physics*, 159, 1994.

H. Widom. Strong Szegö limit theorem on circular arcs. *Indiana University Mathematics Journal*, 21, 1971.

E. Wigner and A. Weinberg. Physical Theory of Neutron Chain Reactors. *University of Chicago Press*, 1958.

A. Zabrodin. Random matrices and Laplacian growth. *Oxford Handbook of Random Matrix Theory*, 2009.