Perturbative expansion of the Painlevé Lax systems and topological recursion

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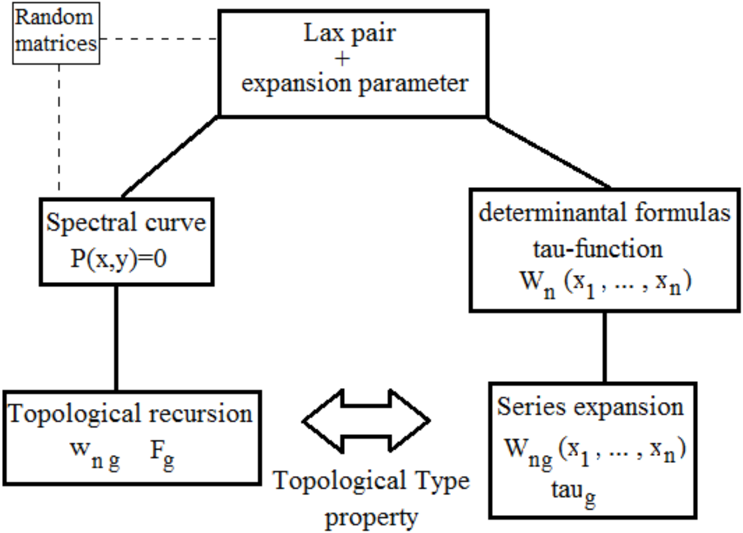
General picture



Eigenvalues correlation functions

- Let $Z_N = \int_{\mathcal{H}_N} dM_N e^{-N \operatorname{Tr} V(M_N)}$ with $V(z)$ monic polynomial potential of even degree.
- **Eigenvalues correlation functions** (Stieltjes transforms):

$$
\begin{aligned}
W_1(x) &= \left\langle \sum_{i=1}^{N} \frac{1}{x-\lambda_i} \right\rangle_N \\
W_2(x_1,x_2) &= \left\langle \sum_{i,j=1}^{N} \frac{1}{(x_1-\lambda_i)(x_2-\lambda_j)} \right\rangle_N - W_1(x_1)W_1(x_2) \\
W_p(x_1,\dots,x_p) &= \left\langle \sum_{i_1,\dots,i_p}^{N} \frac{1}{x_1-\lambda_{i_1}} \cdots \frac{1}{x_p-\lambda_{i_p}} \right\rangle_{N,\text{cumulant}}
\end{aligned}
$$

- Generating series of joint **moments** $\left\langle \sum_{i=1}^{N} \lambda_i^k \right\rangle_N$, $\left\langle \sum_{i,j=1}^{N} \lambda_i^r \lambda_j^s \right\rangle_N$
- Hermitian case: Correlation functions satisfy algebraic relations known as loop equations, Schwinger-Dyson equations, Virasoro constraints, etc.

Loop equations

- Let:

$$P_p(x_1;x_2,\dots,x_p) = \left\langle \sum_{i_1,\dots i_p} \frac{V'(x_1)-V'(\lambda_{i_1})}{x_1-\lambda_{i_1}} \frac{1}{x_2-\lambda_{i_2}} \cdots \frac{1}{x_p-\lambda_{i_p}} \right\rangle_{N,\textbf{cumulant}}$$

- Loop equations (notation $L_p=\{x_2,\dots,x_p\}$):

$$-P_1(x) = W_1^2(x) - V'(x)W_1(x) + \frac{1}{N^2}W_2(x,x)$$

$$P_p(x_1;L_p) = (2W_1(x_1)-V'(x_1))W_p(L_p) + \frac{1}{N^2}W_{p+1}(x_1,x_1,L_p)$$

$$+\sum_{I\subset L_p} W_{|I|+1}(x_1,L_I)W_{p-|I|}(x_1,L_{J\setminus I})$$

$$-\sum_{j=2}^{p} \frac{\partial}{\partial x_j} \frac{W_{p-1}(L_p)-W_{p-1}(x_1,L_p\setminus x_j)}{x_1-x_j}$$

- Property: $x\mapsto P_p(x;L_p)$ is a polynomial. Is it enough to solve the equations and find W_p?

Perturbative solutions

- $Z_N = \int_{\mathcal{H}_N} dM_N e^{-N \operatorname{Tr} V(M_N)}$. **Series expansion** at large N: We **assume**:

$$F_N \overset{\text{def}}{=} \ln Z_N = \sum_{g=0}^{\infty} F^{(g)} \left(\frac{1}{N}\right)^{2g-2}$$

$$W_p(x_1,\dots,x_p) = \sum_{g=0}^{\infty} \omega_p^{(g)}(x_1,\dots,x_p) \left(\frac{1}{N}\right)^{N+2g-2}$$

- May also work for other parameters:

$$Z_N[t_4] = \int_{\mathcal{H}_N} dM_N e^{-\frac{N}{2} \operatorname{Tr}(M_N^2) - \frac{t_4}{4} N \operatorname{Tr}(M_N^4)}$$

we may assume:

$$\ln Z_N[t_4] = \sum_{g=0}^{\infty}\sum_{v=0}^{\infty} F^{(g,v)} (t_4)^v \left(\frac{1}{N}\right)^{2g-2} + \text{similar dev. for } W_p$$

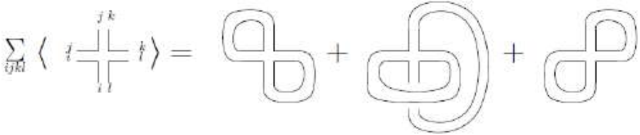
- Allow to solve recursively the loop equations.

Applications in combinatorics

- Interesting in combinatorics:

$$Z_N[t_4] = \int_{\mathcal{H}_N} dM_N e^{-\frac{N}{2}\mathrm{Tr}\,(M_N^2) - \frac{t_4}{4} N\,\mathrm{Tr}\,(M_N^4)}$$

Perturbative series expansion in $t_4 \Rightarrow$ enumeration of **fat ribbon graph (similar to Feynman expansion)**:



$F^{(g,v)}$ count the number of such connex graphs with v vertices (4 legs) and of genus g:

$$F[t_4] = \ln Z_N[t_4] = \sum_{\mathcal{G}=4-\text{ribbon graph}} \frac{1}{|\mathrm{Aut}\ \mathcal{G}|} t_4^{\#v(\mathcal{G})} \left(\frac{1}{N}\right)^{-\chi(\mathcal{G})}$$

Applications in geometry

- **Kontsevich integral**: Intersection theory of Riemann surfaces moduli spaces:

$$\langle \tau_{d_1} \dots \tau_{d_n} \rangle = \int_{\bar{\mathcal{M}}_{g,n}} \psi_1^{d_1} \dots \psi_n^{d_n}$$

  may be computed through the **formal expansion** of the Kontsevich integral of $F = \ln Z$ with:

$$Z[t_0, t_1, \dots] = (\det \Lambda)^Q \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}\,(M\Lambda M) + \frac{1}{3!}\mathrm{Tr}\,(M^3)\right)$$

  where $t_i = -(2i-1)!!\,\mathrm{Tr}\,(\Lambda^{-(2i-1)})$

- Remark: $F[t_0, t_1, \dots]$ in connection with the KdV equation: $u \overset{\text{def}}{=} \frac{\partial^2 F}{\partial t_1^2}$ satisfies: $\frac{\partial u}{\partial t_3} = u\frac{\partial u}{\partial t_1} + \frac{1}{12}\frac{\partial^3 u}{\partial t_1^3}$ Generalization: Kontsevitch-Penner model (Safnuk, Alekandrov: open intersection numbers):

$$Z[Q, t_i] = (\det \Lambda)^Q \int dM \exp\left(-\frac{1}{2}\mathrm{Tr}\,(M\Lambda M) + \frac{1}{3}\mathrm{Tr}\,(M^3) - Q\ln M\right)$$

Spectral curve

- **Formal solution of the loop equations** with the **assumption** that:

$$F_N = \ln Z_N = \sum_{g=0}^{\infty} F^{(g)} \left(\frac{1}{N}\right)^{2g-2}$$

$$W_p(x_1,\dots,x_p) = \sum_{g=0}^{\infty} \omega_p^{(g)}(x_1,\dots,x_p) \left(\frac{1}{N}\right)^{2g+N-2}$$

- Central element = Spectral curve:

$$Y(x) = \omega_1^{(0)}(x) - \frac{V'(x)}{2} = \int \frac{\rho_{\lim}(\lambda)d\lambda}{x-\lambda} - \frac{V'(x)}{2}$$

$$\text{satisfies } Y^2(x) = \frac{V'(x)^2}{4} - P_1^{(0)}(x)$$

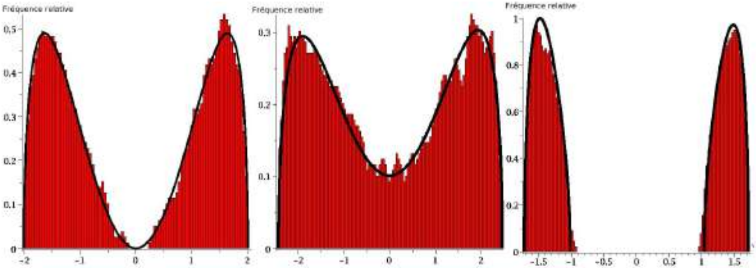
$\Leftrightarrow$ **Riemann surface** (hyperelliptic) of genus g.

- Undetermined coefficients of $P_1^{(0)}(x)$ are equivalent to fixing filling fractions:

$$\epsilon_i = \frac{1}{2\pi i} \oint_{\mathcal{A}_i} Y(x)dx$$

Filling fractions and $\omega_2^{(0)}$

- Random matrix theory:



Summation of filling fractions ↔ Oscillating terms to get convergence?

- Combinatorics: Usually fixed at “natural” values.
- Never a problem when the spectral curve is of genus 0.

Topological recursion

Theorem (Eynard-Orantin-Chekhov)

The spectral curve allow to recursively compute all orders $\omega_n^{(g)}(x_1, \dots, x_n)$ and $F^{(g)}$ through a recursive procedure known as the topological recursion.

Ingredients: Normalized bi-differential $\omega_2^{(0)}(z_1, z_2)$, recursion kernel and integration kernel:

$$K(z_0, z) = \frac{\frac{1}{2}\int_z^{\bar{z}} \omega_2^{(0)}(s, z_0) ds}{(Y(z) - Y(\bar{z}))dx(z)} \text{ and } \Phi(z) = \int^z Ydx$$

then (notation $I_n = \{z_1, \dots, z_n\}$):

$$\omega_{n+1}^{(g)}(z_1, \dots, z_n) = \sum_i \mathop{\mathrm{Res}}_{z \to a_i} K(z_1, z)\Big(\omega_{n+2}^{(g-1)}(z, \bar{z}, p_{I_n}) + \sum_{m=0}^{g} \sum_{I_1 \sqcup I_2 = I}^{\prime} \omega_{|I_1|+1}^{(m)}(z, z_{I_1})\omega_{|I_2|+1}^{(g-m)}(\bar{z}, z_{I_2})\Big)$$

Conversely (with $F^{(g)} = \omega_0^{(g)}$):

$$\omega_n^{(g)}(I_n) = \frac{1}{2 - 2g - n} \sum_i \mathop{\mathrm{Res}}_{z \to a_i} \Phi(z)\omega_{n+1}^{(g)}(z, I_n)$$

Main features of the topological recursion

- **Formal** of the loop equation under the assumption of existence of series expansions $\Rightarrow$ Natural question of series convergence is open (Borel sumability, non zero radius of convergence, etc.).
- Fixed filling fractions: hard to determine in practice (static or dynamical determination).
- Genus 0 spectral curves are easier to handle: global parametrization + explicit expression of the normalized bi-differential $\omega_2^{(0)}(z_1, z_2)$.
- **Topological recursion generalized outside any underlying random hermitian matrix model**. Only a spectral curve is required.

Formal approach

- General idea: **Find corresponding definitions of quantities arising in the topological recursion directly into integrable systems formalism**.
- Interesting quantities: formal expansion parameter $\hbar$ equivalent to $\frac{1}{N}$, spectral curve, quantities similar to correlation functions, etc.
- Recent solution proposed by Bergère, Borot and Eynard starting from a given **Lax pair**.
- Prove that the topological recursion is satisfied: **Topological Type property** (sufficient (and necessary?) condition)

Lax pair

- Definition for 2×2 system by Bergère and Eynard, generalized for $n \times n$ systems by Bergère, Borot and Eynard.
- Lax pair:

$$\partial_x \Psi(x,t) = \mathcal{D}(x,t)\Psi(x,t)\ ,\ \partial_t \Psi(x,t) = \mathcal{R}(x,t)\Psi(x,t)$$

- Example for Painlevé 4:

$$\begin{aligned}
\mathcal{D}(x,t) &= \begin{pmatrix} x+t+\frac{pq+\theta_0}{x} & 1-\frac{q}{x} \\ -2(pq+\theta_0+\theta_\infty)+\frac{p(pq+2\theta_0)}{x} & -\left(x+t+\frac{pq+\theta_0}{x}\right) \end{pmatrix} \\
\mathcal{R}(x,t) &= \begin{pmatrix} x+q+t & 1 \\ -2(pq+\theta_0+\theta_\infty) & -(x+q+t) \end{pmatrix}
\end{aligned}$$

- Compatibility equation (zero-curvature equation):

$$\partial_t \mathcal{D}(x,t) - \partial_x \mathcal{R}(x,t) + [\mathcal{D}(x,t), \mathcal{R}(x,t)] = 0$$

- Equivalent **Hamiltonian formalism**:

$$H_4(p,q,t) = qp^2 + 2(q^2+tq+\theta_0)p + 2(\theta_0+\theta_\infty)q$$

- Jimbo-Miwa τ-function at infinity equals the Hamiltonian

Determinantal formulas: version 1

Let:

$$\Psi(x,t) = \begin{pmatrix} \psi(x,t) & \phi(x,t) \\ \tilde{\psi}(x,t) & \tilde{\phi}(x,t) \end{pmatrix}$$

We define the **Christoffel-Darboux kernel**:

$$K(x_1,x_2) = \frac{\psi(x_1)\tilde{\phi}(x_2) - \tilde{\psi}(x_1)\phi(x_2)}{x_1 - x_2}$$

and then the **correlation functions**:

$$\begin{aligned} W_1(x) &= \frac{\partial \psi}{\partial x}(x)\tilde{\phi}(x) - \frac{\partial \tilde{\psi}}{\partial x}(x)\phi(x) \\ W_n(x_1,\dots,x_n) &= -\frac{\delta_{n,2}}{(x_1-x_2)^2} + (-1)^{n+1} \sum_{\sigma: n\text{-cycles}} \prod_{i=1}^{n} K(x_i, x_{\sigma(i)}) \end{aligned}$$

Determinantal formulas: version 2

"Alternative" definition in terms of the resolvent matrix $M(x,t)$

$$M(x) = \Psi(x)\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\Psi^{-1}(x) = \begin{pmatrix}\psi\tilde{\phi} & -\psi\phi\\ \tilde{\psi}\tilde{\phi} & -\phi\tilde{\psi}\end{pmatrix}$$

then we can rewrite the **correlation functions**:

$$W_1(x) = -\frac{1}{\hbar}\,\mathrm{Tr}\,(\mathcal{D}(x)M(x))$$

$$W_2(x_1,x_2) = \frac{\mathrm{Tr}\,(M(x_1)M(x_2)) - 1}{(x_1-x_2)^2}$$

$$W_n(x_1,\dots,x_n) = (-1)^{n+1}\,\mathrm{Tr}\sum_{\sigma:n\text{-cycles}}\prod_{i=1}^{n}\frac{M(x_{\sigma(i)})}{x_{\sigma(i)}-x_{\sigma(i+1)}}$$

$$= \frac{(-1)^{n+1}}{n}\sum_{\sigma\in S_n}\frac{\mathrm{Tr}\,M(x_{\sigma(1)})\dots M(x_{\sigma(n)})}{(x_{\sigma(1)}-x_{\sigma(2)})\dots(x_{\sigma(n-1)}-x_{\sigma(n)})(x_{\sigma(n)}-x_{\sigma(1)})}$$

Properties of the determinantal formulas

- Valid for any **linear differential system**: $\partial_x \Psi(x) = L(x)\Psi(x)$ and not only for Lax pairs
- $W_n(x_1, \dots, x_n)$ are invariant under **"admissible" gauge transformations**: $\tilde{\Psi}(x,t) = U(x,t)\Psi(x,t)$ with:
  - $U(x,t) = U(t)$ independent of x
  - $U(x,t)$ proportional to I_2 (special case for $W_1(x)$). $U(x,t) = \frac{f'(x,t)}{f(x,t)} I_2$ gives:

$$\tilde{W}_1(x) = W_1(x) + \frac{f'(x,t)}{f(x,t)}$$

- These gauge transformations allow to get "good" Lax pairs from Jimbo-Miwa's (See Lax pair for Painlevé 4).

Connection with the topological recursion

Theorem (Bergère-Borot-Eynard)

If the determinantal formulas $W_n(x_1,\dots,x_n)$ have a series expansion in a parameter $\hbar$ of the form:

$$W_n(x_1,\dots,x_n)=\sum_{g=0}^{\infty}\hbar^{n-2+2g}W_n^{(g)}(x_1,\dots,x_n)\quad \text{for } n\geq 1$$

then we can obtain the $W_n^{(g)}$ through the topological recursion applied to the spectral curve attached to the Lax pair:

$$E(x,Y)=\det(Y-\mathcal{D}(x,t))_{|\hbar\to 0}=0$$

Moreover, the τ-function admits a series expansion of the form:

$$\frac{1}{\hbar^2}\ln\tau=\sum_{g=0}^{\infty}\tau^{(2g)}\hbar^{2g-2}$$

with $\tau^{(g)}(t)=F^{(g)}(t)+C^{(g)}$ computed from the topological recursion.

Topological Type property

Theorem (Bergère-Borot-Eynard)

Is the spectral curve is of genus 0*, the following conditions (known as Topological Type property) are* **sufficient conditions** *to prove that the determinantal formulas satisfy the previous theorem:*

(1) *Existence of a formal $\hbar$ series expansion: The determinantal formulas admit a series expansion in $\hbar$:*

$$W_n(x_1,\dots,x_n) = \sum_{g=0}^{\infty} W_n^{(g)}(x_1,\dots,x_n)\hbar^g$$

(2) *Parity:* $W_n|_{\hbar\mapsto -\hbar} = (-1)^n W_n$ *for* $n \geq 1$

(3) *Pole structure: The functions $W_n^{(g)}(x_1,\dots,x_n)$ are regular at the even zeros of the spectral curve.*

(4) *Leading order: The $\hbar$ series expansion of W_n is at least of order $\hbar^{n-2}$.*

Plan of the proof for Painlevé equations

1. Presentation of the Lax pair and introduction of a formal parameter $\hbar$
2. Computation of the spectral curve (genus 0)
3. Proof of the topological type property
   - Existence of formal series expansion in $\hbar$ for $W_n \Leftrightarrow$ Gauge choice
   - Study of the $\hbar \leftrightarrow -\hbar$ operator
   - Control of the pole structure of W_n
   - Leading order of series expansion of W_n using pole structure and loop equations.

Introduction of $\hbar$

- Introduction through a **rescaling of the parameters**:

$$P4: \quad (t,x,q,p,\theta_0,\theta_\infty) \to \left(\hbar^{\frac{1}{2}} t, \hbar^{\frac{1}{2}} x, \hbar^{\frac{1}{2}} q, \hbar^{\frac{1}{2}} p, \hbar\theta_0, \hbar\theta_\infty\right)$$

$$\Psi(x,t) \to \begin{pmatrix} \hbar^{-\frac{1}{4}} & 0 \\ 0 & \hbar^{\frac{1}{4}} \end{pmatrix} \Psi(x,t)$$

- Equivalent to the new differential system:

$$\hbar\partial_x \Psi(x,t) = \mathcal{D}(x,t)\Psi(x,t) \text{ with } \hbar\partial_t\Psi(x,t) = \mathcal{R}(x,t)\Psi(x,t)$$

- Similar transformations are available for the other Painlevé equations.
- Specific regime. $\hbar = 1 \Leftrightarrow$ usual formulation
- Deformation of the Painlevé equation:

$$\hbar^2 \ddot{q} = \frac{\hbar^2}{2q}\dot{q}^2 + 2\left(3q^3 + 4tq^2 + \left(t^2 - 2\theta_\infty + \hbar\right) q - \frac{\theta_0^2}{q}\right)$$

- Deformation of the Hamiltonian formalism:

$$\begin{aligned} H_4(p,q,t) &= qp^2 + 2(q^2 + tq + \theta_0)p + 2(\theta_0 + \theta_\infty)q \\ \hbar\dot{q} &= \frac{\partial H_4}{\partial p}(p,q) \text{ with } \hbar\dot{p} = -\frac{\partial H_4}{\partial q}(p,q) \end{aligned}$$

Modified Painlevé equations

- (P_{I}): $\hbar^2 \ddot{q} = 6q^2 + t$
- (P_{II}): $\hbar^2 \ddot{q} = 2q^3 + tq + \frac{\hbar}{2} - \theta$
- (P_{III}): $\hbar^2 \ddot{q} = \frac{\hbar^2}{q}\dot{q}^2 - \frac{\hbar^2}{t}\dot{q} + \frac{4}{t}\left(\theta_0 q^2 - \theta_\infty + \hbar\right) + 4q^3 - \frac{4}{q}$
- (P_{IV}): $\hbar^2 \ddot{q} = \frac{\hbar^2}{2q}\dot{q}^2 + 2\left(3q^3 + 4tq^2 + \left(t^2 - 2\theta_\infty + \hbar\right) q - \frac{\theta_0^2}{q}\right)$
- (P_{V}):
  $\hbar^2 \ddot{q} = \left(\frac{1}{2q} + \frac{1}{q-1}\right)(\hbar\dot{q})^2 - \hbar^2 \frac{\dot{q}}{t} + \frac{(q-1)^2}{t^2}\left(\alpha q + \frac{\beta}{q}\right) + \frac{\gamma q}{t} + \frac{\delta q(q+1)}{q-1}$
  where
  $\alpha = \frac{(\theta_0 - \theta_1 - \theta_\infty)^2}{8}$, $\beta = -\frac{(\theta_0 - \theta_1 + \theta_\infty)^2}{8}$, $\gamma = \theta_0 + \theta_1 - \hbar$, $\delta = -\frac{1}{2}$
- (P_{VI}): $\hbar^2 \ddot{q} = \frac{\hbar^2}{2}\left(\frac{1}{q} + \frac{1}{q-1} + \frac{1}{q-t}\right)\dot{q}^2 - \hbar^2\left(\frac{1}{t} + \frac{1}{t-1} + \frac{1}{q-t}\right)\dot{q} +$
  $\frac{q(q-1)(q-t)}{t^2(t-1)^2}\left[\alpha + \beta\frac{t}{q^2} + \gamma\frac{t-1}{(q-1)^2} + \delta\frac{t(t-1)}{(q-t)^2}\right]$ where
  $\alpha = \frac{1}{2}(\theta_\infty - \hbar)^2$, $\beta = -\frac{\theta_0^2}{2}$, $\gamma = \frac{\theta_1^2}{2}$, $\delta = \frac{\hbar^2 - \theta_t^2}{2}$

Modified Hamiltonians

- $H_1(p,q,t) = \frac{1}{2}p^2 - 2q^3 - tq$
- $H_2(p,q,t) = \frac{1}{2}p^2 + (q^2 + \frac{t}{2})p + \theta q$
- $H_3(p,q,t,\hbar) = \frac{1}{t}\left[2q^2p^2 + 2(-tq^2 + \theta_\infty q + t)p - (\theta_0 + \theta_\infty)tq - t^2 - \frac{1}{4}(\theta_0^2 - \theta_\infty^2) - \hbar pq\right]$
- $H_4(p,q,t) = qp^2 + 2(q^2 + tq + \theta_0)p + 2(\theta_0 + \theta_\infty)q$
- $H_5(p,q,t) = \frac{1}{t}\left[q(q-1)^2p^2 + \left(\frac{\theta_0 - \theta_1 + \theta_\infty}{2}(q-1)^2 + (\theta_0 + \theta_1)q(q-1) - tq\right)p + \frac{1}{2}\theta_0(\theta_0 + \theta_1 + \theta_\infty)q\right]$
- $H_6(p,q,t,\hbar) = \frac{1}{t(t-1)}\left[q(q-1)(q-t)p^2 - p\left(\theta_0(q-1)(q-t) + \theta_1 q(q-t) + (\theta_t - \hbar)q(q-1)\right) + \frac{1}{4}(\theta_0 + \theta_1 + \theta_t - \theta_\infty)(\theta_0 + \theta_1 + \theta_t + \theta_\infty - \hbar)(q-t) + \frac{1}{2}((t-1)\theta_0 + t\theta_1)(\theta_t - \hbar)\right]$

Remark: In all cases we observe the property:

$$\ln \tau_J = H_J(p(t), q(t), t, \hbar = 0) \quad \text{for } 1 \leq J \leq 6$$

Formal series expansion

Assumption (Assumption of a formal $\hbar$ series expansion)

We assume that the solution $q(t)$ of the deformed Painlevé equation admits a series expansion in $\hbar$:

$$q(t) = \sum_{k=0}^{\infty} q^{(k)}(t)\hbar^k$$

- Formal series expansion? Equivalent to specific initial conditions?
- If radius of convergence $R \geq 1$, we can reconstruct the initial Painlevé solution.
- Leading order may only be $\hbar^0$ because of the Painlevé equation.
- Inserting back into $P4$we can express $q^{(k)}$ for $k \geq 1$ as a rational function of $q^{(0)}$. Same holds for $\frac{d^k}{dt^k} q^{(0)}$

Gauge choice

Proposition (Good gauge choice)

There exists an admissible gauge choice for which the previous assumption implies that $\mathcal{D}(x,t,\hbar)$ and $\mathcal{R}(x,t,\hbar)$ admit a $\hbar$ series expansion of the form:

$$\mathcal{D}(x,t,\hbar)=\sum_{k=0}^{\infty}\mathcal{D}^{(k)}(x,t)\hbar^k \quad \text{with} \quad \mathcal{R}(x,t,\hbar)=\sum_{k=0}^{\infty}\mathcal{R}^{(k)}(x,t)\hbar^k$$

- Our Lax pairs are chosen in this gauge.
- Gauge is a little different from Jimbo-Miwa's but explicit connections are available.
- Main results are independent of the admissible gauge choice.
- Consequence: $M(x,t,\hbar)$ and $W_n(x_1,\dots,x_n)$ have a $\hbar$ series expansion: 1^{st} **condition of the Topological Type property is satisfied**.

Parity property $\hbar \leftrightarrow -\hbar$

Proposition (Sufficient condition for parity (Bergère-Borot-Eynard))

Let $\dagger$ be the operator changing $\hbar$ into $-\hbar$. If there exists an invertible matrix $\Gamma(t)$ (independent of x) such that:

$$\Gamma^{-1}(t)\mathcal{D}^t(x,t)\Gamma(t) = \mathcal{D}^\dagger(x,t)$$

then the determinantal formulas W_n satisfy $W_n^\dagger = (-1)^n W_n$ (Parity condition of the Topological Type property)

Theorem (Existence of $\Gamma(t)$ matrices)

We can find explicit $\Gamma(t)$ matrices in our six Painlevé cases and $\ln \tau$ (as well as Okamoto's σ functions) are always even functions of $\hbar$.

Operator $\dagger$

- $P1$: $q^\dagger = q\,,\; p^\dagger = -p$
- $P2$: $q^\dagger = -q - \frac{\theta}{p}\,,\; p^\dagger = p$
- $P3$: $q^\dagger = \frac{-2qp^2+2(tq-\theta_\infty)p+t(\theta_0+\theta_\infty)}{2(p-t)p}\,,\; p^\dagger = p$
- $P4$:
$$q^\dagger = \frac{p(pq+2\theta_0)}{2(pq+\theta_0+\theta_\infty)}\,,\; p^\dagger = \frac{2q(pq+\theta_0+\theta_\infty)}{pq+2\theta_0}$$
- $P5$:
$$q^\dagger = \frac{p(2pq+\theta_0-\theta_1+\theta_\infty)}{(pq+\theta_0)(2pq+\theta_0+\theta_1+\theta_\infty)}\,,\; p^\dagger = \frac{q(pq+\theta_0)(2pq+\theta_0+\theta_1+\theta_\infty)}{2pq+\theta_0-\theta_1+\theta_\infty}$$
- $P6$:
$$q^\dagger = \frac{t^2 z_0(z_0+\theta_0)(q-1)}{t^2 z_0(z_0+\theta_0)(q-1)-(t-1)^2 z_1(z_1+\theta_1)q}\,,\; p^\dagger = \frac{z_0+\theta_0}{q^\dagger}+\frac{z_1+\theta_1}{q^\dagger-1}+\frac{z_t+\theta_t}{q^\dagger-t}$$

$\Gamma(t)$ matrices

- Painlevé 1: $\Gamma_1(t) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$
- Painlevé 2: $\Gamma_2(t) = \begin{pmatrix} -2p & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé 3: $\Gamma_3(t) = \begin{pmatrix} -\frac{p-t}{t} & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé 4: $\Gamma_4(t) = \begin{pmatrix} -2(pq + \theta_0 + \theta_\infty) & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé 5: $\Gamma_5(t) = \begin{pmatrix} -\frac{pq}{pq+\theta_0} & 0 \\ 0 & 1 \end{pmatrix}$
- Painlevé 6: $\Gamma_6(t) = \begin{pmatrix} -\frac{t^2 z_0(z_0+\theta_0)}{q} + \frac{(t-1)^2 z_1(z_1+\theta_1)}{q-1} & 0 \\ 0 & 1 \end{pmatrix}$

Spectral curves

Theorem (Spectral curves)

The six deformed Painlevé Lax pairs have genus 0 *spectral curves:*

$$
\begin{aligned}
(P_{\mathrm{I}}) &: \quad Y^2 = 4(x+2q_0)(x-q_0)^2 \\
(P_{\mathrm{II}}) &: \quad Y^2 = (x-q_0)^2\left(x^2+2q_0x+q_0^2+\frac{\theta}{q_0}\right) \\
(P_{\mathrm{III}}) &: \quad Y^2 = \frac{t(q_0x+1)^2\left((\theta_\infty-\theta_0q_0^2)x^2-2xq_0(\theta_\infty q_0^2-\theta_0)+q_0^2(\theta_\infty-\theta_0q_0^2)\right)}{4x^4(q_0^4-1)q_0} \\
(P_{\mathrm{IV}}) &: \quad Y^2 = \frac{(x-q_0)^2\left(x^2+2(q_0+t)x+\frac{\theta_0^2}{q_0^2}\right)}{x^2} \\
(P_{\mathrm{V}}) &: \quad Y^2 = \frac{t^2(x-Q_0)^2(x-Q_1)(x-Q_2)}{4x^2(x-1)^2} \\
(P_{\mathrm{VI}}) &: \quad Y^2 = \frac{\theta_\infty^2(x-q_0)^2P_2(x)}{4x^2(x-1)^2(x-t)^2}
\end{aligned}
$$

where $P_2(x) = x^2+\left(-1-\frac{\theta_0^2t^2}{\theta_\infty^2q_0^2}+\frac{\theta_1^2(t-1)^2}{\theta_\infty^2(q_0-1)^2}\right)x+\frac{\theta_0^2t^2}{\theta_\infty^2q_0^2}$

Pole structure

- The six spectral curves have a double zero ⇒ We need to prove that the W_n do not have singularities at these points (3rd condition of the Topological Type property).
- Crucial use of the time differential equation.
- Two steps proof dependent of the gauge choice:
  1. Explicit computation of $M^{(0)}(x,t)$ and direct verification that it is regular at the double zero.
  2. Recursive system giving $M^{(k+1)}(x,t)$ in terms of lower orders. Verification that the recursion does not introduce singularity at the double zero.

Step 1: Example for Painlevé 4

- In the good gauge ($\mathrm{Tr}\,\mathcal{D}(x,t)=0$ and $\mathrm{Tr}\,\mathcal{R}(x,t)=0$):

$$M^{(0)}(x,t)=\begin{pmatrix} \frac{1}{2}+\frac{\mathcal{R}^{(0)}_{1,1}(x,t)}{2\sqrt{-\det\mathcal{R}^{(0)}(x,t)}} & \frac{\mathcal{R}^{(0)}_{1,2}(x,t)}{2\sqrt{-\det\mathcal{R}^{(0)}(x,t)}} \\ \frac{\mathcal{R}^{(0)}_{2,1}(x,t)}{2\sqrt{-\det\mathcal{R}^{(0)}(x,t)}} & \frac{1}{2}-\frac{\mathcal{R}^{(0)}_{1,1}(x,t)}{2\sqrt{-\det\mathcal{R}^{(0)}(x,t)}} \end{pmatrix}$$

- For Painlevé 4: $x\mapsto\mathcal{R}^{(0)}(x,t)$ is singular at $x=0$ and $x=\infty$ only and:

$$\det\mathcal{R}^{(0)}_4=q_0^2\left(x^2+2(q_0+t)x+\frac{\theta_0^2}{q_0^2}\right)$$

- Reminder of the spectral curve: $Y^2=\frac{(x-q_0)^2\left(x^2+2(q_0+t)x+\frac{\theta_0^2}{q_0^2}\right)}{x^2}$
- Previous formula is valid if we change $\mathcal{R}^{(0)}(x,t)\leftrightarrow\mathcal{D}^{(0)}(x,t)$ but conclusion at the double zero is no longer possible.

Step 2: Example for Painlevé 4

- In the good gauge $M^{(k)}(x,t)$ is characterized by $\mathrm{Tr}\, M^{(k)} = 0$, $(\det M)^{(k)} = 0$ and $[\mathcal{R}, M]^{(k)} = 0$:

$$\begin{pmatrix} 0 & -\mathcal{R}^{(0)}_{2,1} & \mathcal{R}^{(0)}_{1,2} \\ -2\mathcal{R}^{(0)}_{1,2} & 2\mathcal{R}^{(0)}_{1,1} & 0 \\ \mathcal{R}^{(0)}_{1,1} & \frac{1}{2}\mathcal{R}^{(0)}_{2,1} & \frac{1}{2}\mathcal{R}^{(0)}_{1,2} \end{pmatrix} \begin{pmatrix} M^{(k)}(x,t)_{1,1} \\ M^{(k)}(x,t)_{1,2} \\ M^{(k)}(x,t)_{2,1} \end{pmatrix}$$
$$= \begin{pmatrix} \partial_t M^{(k-1)}(x,t)_{1,1} - \sum_{i=0}^{k-1} \left[\mathcal{R}^{(k-i)}(x,t), M^{(i)}(x,t)\right]_{1,1} \\ \partial_t M^{(k-1)}(x,t)_{1,2} - \sum_{i=0}^{k-1} \left[\mathcal{R}^{(k-i)}(x,t), M^{(i)}(x,t)\right]_{1,2} \\ \sqrt{-\det \mathcal{R}^{(0)}} \sum_{i=1}^{k-1} \left(M^{(i)}(x,t)_{1,1} M^{(k-i)}(x,t)_{1,1} + M^{(i)}(x,t)_{1,2} M^{(k-i)}(x,t)_{2,1}\right) \end{pmatrix}$$

- **Recursive** system requires to invert a 3×3 matrix (same for all orders):

$$\det \begin{pmatrix} 0 & -\mathcal{R}^{(0)}_{2,1} & \mathcal{R}^{(0)}_{1,2} \\ -2\mathcal{R}^{(0)}_{1,2} & 2\mathcal{R}^{(0)}_{1,1} & 0 \\ \mathcal{R}^{(0)}_{1,1} & \frac{1}{2}\mathcal{R}^{(0)}_{2,1} & \frac{1}{2}\mathcal{R}^{(0)}_{1,2} \end{pmatrix} = -2\mathcal{R}^{(0)}_{1,2}(x,t) \det \mathcal{R}^{(0)}(x,t)$$

- No singularity is introduced at the double zero $x = q_0$.

Recursion for the leading order (4^{th} condition of the Topological Type property)

- New proof only based on **loop equations**:

$$\begin{aligned} 0 \quad = \quad & P_{n+1}(x;L_n) + W_{n+2}(x,x,L_n) + 2W_1(x)W_{n+1}(x,L_n) + \\ & \sum_{J\subset L_n, J\notin\{\emptyset,L_n\}} W_{1+|J|}(x,J)W_{1+n-|J|}(x,L_n\setminus J) \\ & + \sum_{j=1}^{n} \frac{d}{dx_j}\frac{W_n(x,L_n\setminus x_j) - W_n(L_n)}{x-x_j} \end{aligned}$$

- Analysis of the singularities of $P_{n+1}(x;L_n)$ $(x\in\{0,1,t,\infty\})$

$$P4: x \mapsto P_{n+1}(x,L_n) = \frac{\tilde{P}_{n+1}(L_n)}{x}$$

- If leading order: $W_n \leq \hbar^{n-2}$. Recursion leads to:

$$0 = P^{(n-3)}_{i_0+1}(x;L_{i_0}) + 2Y(x)W^{(n-2)}_{i_0+1}(x,L_{i_0})$$

$$\text{For } P4: \; W^{(n-2)}_{i_0+1}(x,L_{i_0}) = \frac{\tilde{P}^{(n-3)}_{i_0+1}(L_{i_0})}{2(x-q_0)\sqrt{x^2+2(q_0+t)x+\frac{\theta_0^2}{q_0^2}}}$$

- Contradiction with the pole structure of $W^{(n-2)}_{i_0+1}(x,L_{i_0})$

Recursion for the leading order 2

- Proof **can be directly adapted for all six Painlevé cases**.
- It is always by counting the orders of all poles that we get the contradiction.
- Contradiction is always the presence of a pole at the double zero of the spectral curve $\Rightarrow$ Importance of the presence of a double zero in the spectral curve.
- Proof depends on the gauge choice (existence of $M^{(k)}(x,t)$) but the final result is independent of the gauge choice (W_n are invariant under admissible gauge transformations)
- Possibility to rewrite the proof with an "insertion operator"?

Main result

Theorem (O.M., K. Iwaki, A. Saenz)

The six ($1 \leq J \leq 6$) deformed **Painlevé Lax pairs** *(with $\hbar$ and arbitrary monodromies) satisfy the* **Topological Type property** *under the existence of a formal series expansion in $\hbar$ of the solution $q(t)$ of the Painlevé equations. Consequently* **the determinantal formulas can be reconstructed from the topological recursion** *applied to the spectral curve of the Lax pair:*

$$\frac{1}{\hbar^2}\ln \tau_{\mathrm{J}}(t) = \sum_{g=0}^{\infty} F_{\mathrm{J}}^{(g)}(t)\hbar^{2g-2}$$

$$W_n(x(z_1),\dots,x(z_n))\,dx(z_1)\cdots dx(z_n) = \sum_{g=0}^{\infty} \omega_n^{(g)}(z_1,\dots,z_n)\hbar^{2g-2+n}$$

Open questions

- Existence of a general proof for 2×2 systems?
- If we fix $\mathcal{D}(x,t)$ with poles at $x \in \{0, 1, t, \infty\}$ and satisfying the Topological Type property, do we always recover a Painlevé system?
- Systematic property satisfied by all 2×2 integrable systems?
- Generalization to $n \times n$ systems (Schlesinger, (p,q) models, cluster algebra (M. Shapiro talk), Lie Algebra (B. Dubrovin talk)?
- Assumptions are equivalent to a WKB series expansion for $\Psi(x,t,\hbar)$. Existence of convergent solutions? (Borel summability at $\hbar = 0$ but $\hbar = 0$ at border of the convergence domain?)
- Is $\Psi(x,t)$ an interesting quantity? $M(x,t)$ has much better property under gauge transformations.
- Is the symplectic invariance property for $F^{(g)}$ obvious on the integrable system side?