Loop equations in differential systems

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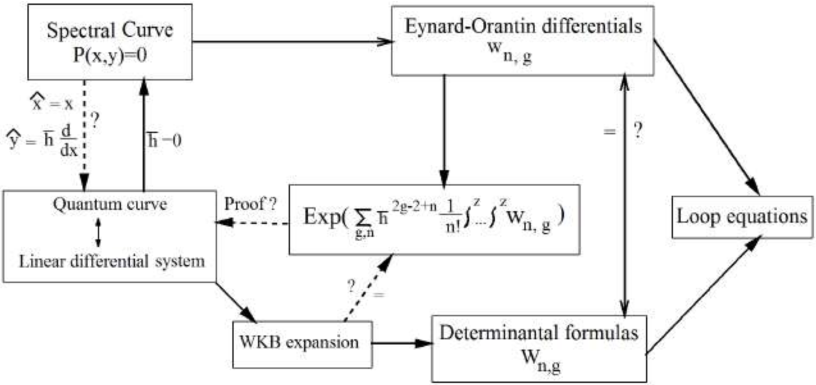
Position of the talk



Plan of the talk

- Differential system $d\Psi = \Phi\Psi$ on a Lie group: define a good set of correlators W_n
- Show that the correlators W_n satisfy a set of "loop equations" identical to the ones of matrix models and topological recursion
- Define the $\hbar$-deformation of the differential system and the "Topological Type property"
- Sufficient condition for "Topological Type property" and connection with reconstruction by the topological recursion
- Example for Painlevé 4 Lax pair and open questions

Remark: Joint work with **B. Eynard** and **R. Belliard**. Paper available at http://arxiv.org/abs/1602.01715

General setting

- Let $\mathfrak{g}$ be a **reductive Lie algebra** and $G = e^{\mathfrak{g}}$ its **connected Lie group**. (Think $G = Gl_n(\mathbb{C})$ and $\mathfrak{g} = \mathfrak{g}l_n(\mathbb{C})$)
- Take a linear differential equation: $\nabla\Psi = 0$ where
  - $\mathcal{P}$: a principal G-bundle over a complex curve Σ with connection ∇
  - $\Psi \in G$: flat section in $\mathcal{P}$
  - Locally equivalent to $d\Psi = \Phi\Psi$ with Φ a $\mathfrak{g}$-valued holomorphic 1-form
- **Faithful r-dimensional matrix representation ρ of $\mathfrak{g}$** with invariant form:

$$< a, b >= Tr(\rho(a)\rho(b)) \overset{def}{=} \mathrm{Tr}_\rho(ab)$$

General prime form

- Invariant form may depend on ρ but unique (up to a trivial global multiplication) if $\mathfrak{g}$ is semi-simple (Killing form)
- Σ is a **Riemann surface** possibly **non-compact**, with **punctures**, high genus, etc.
- Let $\mathcal{E}$ be **any "prime form"** on $\Sigma \times \Sigma$, i.e. a $\left(-\frac{1}{2}, -\frac{1}{2}\right)$ form behaving on the diagonal like:

$$\mathcal{E}(x,x') \underset{x \to x'}{\sim} \frac{x - x'}{\sqrt{dx dx'}}$$

  with no other zeros
- Connection ∇ is locally $d\Psi = \Phi\Psi$ with Ψ in the universal cover $\tilde{\Sigma}$ of Σ
- $\Phi(x)$ is called the "Higgs field"

General picture

- $\mathcal{P}_0$ is the trivialized $\mathfrak{g}$ bundle with constant fiber $\tilde{\Sigma} \times \mathfrak{g} \overset{\pi_0}{\mapsto} \tilde{\Sigma}$ and trivial flat sections (i.e. constant sections $\Leftrightarrow$ trivial connection d)

$$\begin{array}{ccc} \mathcal{P} & & \mathcal{P}_0 = pr^*\mathcal{P} = \tilde{\Sigma} \times \mathfrak{g} \\ p \downarrow & \overset{\pi}{\swarrow} & \downarrow \pi_0 \\ \Sigma & \underset{pr}{\longleftarrow} & \tilde{\Sigma} \end{array}$$

- Notation:
$X = \tilde{x}.E$ will define a point in $\mathcal{P}_0$ with $\pi_0(X) = \tilde{x} \in \tilde{\Sigma}$ and $E \in \mathfrak{g}$ and $\pi(X) = x = \text{pr}(\tilde{x}) \in \Sigma$

Bundle morphism M

Definition

Let M: $\mathcal{P}_0 \mapsto \text{Adj}\,\mathcal{P}$ (i.e. we need both $\tilde{x} \in \tilde{\Sigma}$ and an element $E \in \mathfrak{g}$ to define M) be defined by:

$$M(\tilde{x}.E) = \text{Ad}_{\Psi(\tilde{x})}(E) = "\Psi(\tilde{x})\, E\, \Psi(\tilde{x})^{-1}"$$

Transforms flat sections of $\mathcal{P}_0$ (i.e. constant E) into flat sections of $\text{d} - \text{adj}_\Phi$:

$$\text{d}M(X) = [\Phi(\pi(X)), M(X)]$$

Remark

Action of $\underline{\pi}_1(\Sigma)$: *Turning around a non-trivial loop on* Σ *implies:*

- *Monodromy for* Ψ: $\Psi(\tilde{x}+\gamma) = \Psi(\tilde{x})S_\gamma$
- *Action on* M: $M((\tilde{x}+\gamma).E) = M(\tilde{x}.(S_\gamma\, E\, S_\gamma^{-1}))$

Definition of $\hat{\Sigma}$

Definition

We can define the quotient:

$$\hat{\Sigma} = \mathcal{P}_0 / \underline{\pi}_1(\Sigma)$$

by identifying $\tilde{x}.E \equiv (\tilde{x} + \gamma).(S_\gamma^{-1}\, E\, S_\gamma))$

Notation: $X = [\tilde{x}.E]$ points of $\hat{\Sigma}$

Remark

Changing $\Psi \to \Psi C$*, the choice of the universal cover* $\tilde{\Sigma}$ *or the fundamental group* $\underline{\pi}_1(\Sigma)$ *is equivalent to conjugate the element* E *by a constant group element. Up to these isomorphisms,* **the upcoming** W_n **will only depend on** Φ **but not directly on local flat section** Ψ

Connected correlators $\hat{W}_n$

Definition (Connected correlators)

Let $X = [\tilde{x}.E]$, and $X_i = [\tilde{x}_i.E_i]$ be some points of $\hat{\Sigma}$, with distinct projections $x_i = \pi(X_i)$ on Σ, we define the **connected correlators**:

$$\hat{W}_1(X) =< M(X), \Phi(\pi(X)) >= \mathrm{Tr}_\rho \left(M(X)\Phi(\pi(X))\right),$$

$$\hat{W}_2(X_1, X_2) = -\frac{< M(X_1), M(X_2) >}{\mathcal{E}(x_1, x_2)\mathcal{E}(x_2, x_1)} = -\frac{\mathrm{Tr}_\rho M(X_1)M(X_2)}{\mathcal{E}(x_1, x_2)\mathcal{E}(x_2, x_1)},$$

and for $n \geq 3$,

$$\hat{W}_n(X_1, \dots, X_n) = \sum_{\sigma \in \Sigma_n^{1-\mathrm{cycle}}} (-1)^\sigma \frac{\mathrm{Tr}_\rho M(X_1)M(X_{\sigma(1)})M(X_{\sigma^2(1)}) \dots M(X_{\sigma^{n-1}(1)})}{\mathcal{E}(x_1, x_{\sigma(1)})\mathcal{E}(x_{\sigma(1)}, x_{\sigma^2(1)}) \dots \mathcal{E}(x_{\sigma^{n-1}(1)}, x_1)}$$

$\hat{W}_1$ is a 1-form on $\hat{\Sigma}$ while $\hat{W}_n$ is a **symmetric n-form on $\hat{\Sigma}^n$**

Correlators W_n

Definition (Correlators)

We define the (non-connected) **correlators** by:

$$W_n(X_1,\dots,X_n) = \sum_{\mu\vdash\{X_1,\dots,X_n\}} \prod_{i=1}^{\ell(\mu)} \hat{W}_{|\mu_i|}(\mu_i)$$

where we sum over all partitions of the set $\{X_1,\dots,X_n\}$ of n points.

$$W_1(X_1) = \hat{W}_1(X_1),$$

$$W_2(X_1,X_2) = \hat{W}_1(X_1)\hat{W}_1(X_2) + \hat{W}_2(X_1,X_2)$$

$$\begin{aligned} W_3(X_1,X_2,X_3) \quad = \quad & \hat{W}_1(X_1)\hat{W}_1(X_2)\hat{W}_1(X_3) + \hat{W}_1(X_1)\hat{W}_2(X_2,X_3) \\ & +\hat{W}_1(X_2)\hat{W}_2(X_1,X_3) + \hat{W}_1(X_3)\hat{W}_2(X_1,X_2) \\ & +\hat{W}_3(X_1,X_2,X_3) \end{aligned}$$

and so on. W_n is also a **symmetric n-form on $\hat{\Sigma}^n$**

Kernel $K(\tilde{x}_1, \tilde{x}_2)$

Definition (Fundamental kernel K)

Let $(\tilde{x}_1, \tilde{x}_2) \in \hat{\Sigma} \times \hat{\Sigma}$ and denote $(x_1, x_2) = (\mathrm{pr}(\tilde{x}_1), \mathrm{pr}(\tilde{x}_2)) \in \Sigma \times \Sigma$. We define the kernel $K(\tilde{x}_1, \tilde{x}_2)$ by:

$$K(\tilde{x}_1, \tilde{x}_2) = \begin{cases} \frac{\Psi(\tilde{x}_1)^{-1}\Psi(\tilde{x}_2)}{\mathcal{E}(x_1, x_2)} \in G_{x_1} \times G_{x_2} & \text{if} \quad x_1 \neq x_2 \\ \mathrm{Ad}_{\Psi(\tilde{x}_1)}(\Phi(x_1)) = \text{"}\Psi(\tilde{x}_1)^{-1}\Phi(x_1)\Psi(\tilde{x}_1)\text{"} \in \mathfrak{g} & \text{if} \quad x_1 = x_2 \end{cases}$$

It is a $\left(\frac{1}{2}, \frac{1}{2}\right)$ **form on** $\hat{\Sigma} \times \hat{\Sigma}$ **with a simple pole at** $x_1 = x_2$ (regularized by subtracting the pole at coinciding points)

Determinantal formulas

Theorem (Alternative expression for correlators)

Let $(\tilde{x}_1, \dots, \tilde{x}_n) \in \hat{\Sigma}^n$ *with distinct projections* $x_i = pr(\tilde{x}_i)$. *Let* $(E_1, \dots E_n) \in \mathfrak{g}^n$. *We have:*

$$W_n(\tilde{x}_1.E_1, \dots, \tilde{x}_n.E_n) = \mathrm{Tr} \sum_{\sigma \in S_n} (-1)^{|\sigma|} \prod_{i=1}^{n} \rho(E_i)\rho(K(\tilde{x}_i, \tilde{x}_{\sigma(i)}))$$

Equivalent to:

$$W_n(\tilde{x}_1.E_1, \dots, \tilde{x}_n.E_n) = \mathrm{Tr} \left(\det \left[\rho(E_i)\rho(K(\tilde{x}_i, \tilde{x}_j)) \right]_{1 \le i,j \le n} \right)$$

sometimes called "determinantal formulas"

Remark

"Determinant" must be understood as sum over permutations and not taking determinant of the matrix representation

Reminder on Casimirs

Definition

Let $(e_1, \dots, e_{\dim \mathfrak{g}})$ be a basis of the Lie algebra $\mathfrak{g}$. ρ **faithful** and $< a, b >= \mathrm{Tr}(\rho(a), \rho(b))$ implies **invariant form** $<,>$ **is non-degenerate on** $\mathfrak{g}$. Thus we can define the dual basis $(e^1, \dots, e^{\dim \mathfrak{g}})$ satisfying:

$$\forall\, i, j \in [\![1, \mathfrak{g}]\!] \;:\; \left\langle e_i, e^j \right\rangle = \delta_{i,j}$$

For any $v = \sum_{i=1}^{\dim \mathfrak{g}} v^i e_i$ we expand the characteristic polynomial:

$$\det(y\mathrm{Id}_r - \rho(v)) = \sum_{k=0}^{r} (-1)^k y^{r-k} \sum_{1 \leq i_1, \dots, i_k \leq \dim \mathfrak{g}} C_k(i_1, \dots, i_k) v^{i_1} \dots v^{i_k}$$

The Casimirs $(C_k)_{1 \leq k \leq r}$ of the Lie algebra are defined by:

$$C_k = \sum_{1 \leq i_1, \dots, i_k \leq \dim \mathfrak{g}} C_k(i_1, \dots, i_k) e^{i_1} \otimes \cdots \otimes e^{i_k}$$

Reminder on Casimirs 2

- Example: First non-trivial Casimir:

$$C_2 = -\frac{1}{2}\sum_{i=1}^{\dim \mathfrak{g}} e_i \otimes e^i$$

- The previous construction may not lead to independent Casimirs C_k
- The same construction can be performed with a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.
  Reduces all sums up to $\dim(\mathfrak{h})$ instead of $\dim(\mathfrak{g})$

W generators

Definition (W generators)

Given $X_1, \dots, X_n$ points of $\hat{\Sigma}$ with distinct projections on Σ, and $\tilde{x} \in \tilde{\Sigma}$, with $x = \mathrm{pr}(\tilde{x})$ distinct from the $\pi(X_i)$, we define:

$$W_{k;n}(C_k(x), X_1, \dots, X_n) \overset{\text{def}}{=} \sum_{1 \le i_1, \dots, i_k \le \dim \mathfrak{g}} C_k(i_1, \dots, i_k) W_{k+n}(\tilde{x}.e^{i_1}, \dots, \tilde{x}.e^{i_k}, X_1, \dots, X_n)$$

In case of identical projections, the previous regularization for K is used in the definition of W_{k+n}.

Remark

- *Definition depends* **only on** $x \in \Sigma$ *but not on* $\tilde{x} \in \hat{\Sigma}$
- *Definition is identical when using only a Cartan subalgebra* $\mathfrak{h}$ *instead of* $\mathfrak{g}$
- *Definition* **does not depend on the choice of the basis of** $\mathfrak{g}$ *(resp.* $\mathfrak{h}$*)*

Loop equations

Theorem (Loop equations)

For any $n \geq 0$, *and* $X_1, \dots, X_n$ *points of* $\hat{\Sigma}$ *with distinct projections* $x_i = \pi(X_i)$, *and* $\tilde{x} \in \tilde{\Sigma}$ *also with distinct projection* $x = pr(\tilde{x})$*:*

$$\sum_{k=0}^{r} (-1)^k y^{r-k} W_{k;n}(C_k(\mathbf{x}); X_1, \dots, X_n) =$$
$$[\epsilon_1 \dots \epsilon_n] \, det_\rho \left(y - (\Phi(x) + \mathcal{M}_\epsilon(x; X_1, \dots, X_n)) \right)$$

where:

$$\mathcal{M}_\epsilon(x; X_1, \dots, X_n) = \sum_{i=1}^{n} \epsilon_i \frac{M(X_i)}{\mathcal{E}(x, x_i)\mathcal{E}(x_i, x)}$$
$$+ \sum_{1 \leq i \neq j \leq n} \epsilon_i \epsilon_j \frac{M(X_i)M(X_j)}{\mathcal{E}(x, x_i)\mathcal{E}(x_i, x_j)\mathcal{E}(x_j, x)}$$
$$+ \sum_{k=3}^{n} \sum_{1 \leq i_1 \neq \dots \neq i_k \leq n} \epsilon_{i_1} \dots \epsilon_{i_k} \frac{M(X_{i_1}) \dots M(X_{i_k})}{\mathcal{E}(x, x_{i_1})\mathcal{E}(x_{i_1}, x_{i_2}) \dots \mathcal{E}(x_{i_k}, x)}$$

Loop equations

- $[\epsilon_1 \dots \epsilon_n]$ indicates the $\epsilon_1 \dots \epsilon_n$ coefficient of the Taylor expansion at $\vec{\epsilon} \to \vec{0}$.
- $\det_\rho \left(y - (\Phi(x) + \mathcal{M}_\epsilon(x; X_1, \dots, X_n))\right)$ only makes sense in the representation ρ
- R.h.s. is independent of the choice of basis in $\mathfrak{g}$ (or $\mathfrak{h}$)
- R.h.s is an analytic function of $x \in \Sigma$
- Loop equations proved $\Rightarrow$ previous properties apply to the l.h.s. $W_{k;n}(C_k(x); X_1, \dots, X_n)$
- If $G = \mathrm{Gl}_n(\mathbb{C})$ and $\Sigma = \bar{\mathbb{C}}$ and $\mathcal{E}(x, x') = \frac{x - x'}{\sqrt{dx dx'}}$ then we recover matrix models loop equations.

Sketch of the proof of the Loop equations for $n = 0$

- Start from l.h.s. $\sum_{k=0}^{r}(-1)^k y^{r-k} W_{k;0}(C_k(x))$ and use definitions at coinciding points for $C_k(x)$
- Obtain $W_{k;0}(C_k(x))$ replaced by a sum over permutations σ of $\mathrm{Tr}_\rho\, e^{\sigma(j)}\Psi(\tilde{x})^{-1}\Phi(x)\Psi(\tilde{x})$
- Use **cyclic property of trace** to get $\mathrm{Tr}_\rho\, \Psi(\tilde{x})e^{\sigma(j)}\Psi(\tilde{x})^{-1}\Phi(x)$
- Use **invariance of Casimirs under change of basis** to change $e_j \to \Psi(x)e_j\Psi(x)^{-1}$ to get $\mathrm{Tr}_\rho\, e^{\sigma(j)}\Phi(x)$
- Observe that the initial sum is:

$$\begin{aligned} &\sum_{k=0}^{r}(-1)^k y^{r-k} W_{k;0}(C_k(x)) \\ =\; &\sum_{k=0}^{r}(-1)^k y^{r-k} \sum_{1\leq i_1,\dots,i_k\leq \dim \mathfrak{g}} C_k(i_1,\dots,i_k) \sum_{\sigma\in\mathfrak{S}_k}(-1)^\sigma\, \mathrm{Tr}_\rho \prod_{j=1}^{k}\left(e^{\sigma(j)}\Phi(x)\right) \\ =\; &\det{}_\rho(y-\Phi(x))) \end{aligned}$$

- Same method used to get $W_{k;n}(C_k(x); X_1,\dots,X_n)$

$\hbar$ deformation

- Introduce a 1-**parameter family of deformations** of the connection:

$$\hbar\nabla = \hbar d - \Phi \ \Leftrightarrow \ \hbar d\Psi(\tilde{x},\hbar) = \Phi(x,\hbar)\Psi(\tilde{x},\hbar)$$

- Assume that $\Phi(x,\hbar)$ admits a formal expansion in $\hbar$:

$$\Phi(x,\hbar) = \sum_{k=0}^{\infty} \Phi^{(k)}(x)\hbar^k$$

- Questions:
  - $\hbar$-Expansion of the correlators W_n?
  - Definition of a spectral curve and reconstruction of correlators by topological recursion?

TT property

Definition

The $\hbar$-deformed system is said to be of Topological Type if the 4 following conditions are met

Condition 1: **Asymptotic expansion**

There exists some simply connected open domains of Σ and an Abelian subalgebra $\mathfrak{h}$ of $\mathfrak{g}$, in which the connected correlators $\hat{W}_n(X_1,\dots,X_n)$s with each $X_i \in \Sigma \times \mathfrak{h}$, have a Poincaré asymptotic $\hbar$ expansion

$$\hat{W}_n(X_1,\dots,X_n) = \frac{\delta_{n,1}}{\hbar}\hat{W}_1^{(0)}(X_1) + \sum_{k=0}^{\infty}\hbar^k\hat{W}_n^{(k)}(X_1,\dots,X_n), \qquad (4.1)$$

such that each $\hat{W}_n^{(k)}([x_1.E_1],\dots,[x_n.E_n])$ is, at fixed $E_i \in \mathfrak{h}$, an algebraic symmetric $n-$form of $x_1,\dots,x_n$. In other words, there must exist a (possibly nodal) Riemann surface $\mathcal{S}$ independent of k and n, which is a ramified cover of Σ, such that the pullbacks, at fixed $E_i \in \mathfrak{h}$, of $\hat{W}_n^{(k)}([x_1.E_1],\dots,[x_n.E_n])$ to $\mathcal{S}^n$ are meromorphic symmetric n-forms

TT property 2

Condition 2: **Pole structure**

For $(k,n) \notin \{(0,1),(0,2)\}$ and any $(E_1,\dots,E_n) \in \mathfrak{h}^n$, the connected correlators $\hat{W}_n^{(k)}([x_1.E_1],\dots,[x_n.E_n])$ pulled back to $\mathcal{S}$, **may only have poles at the ramification points of $\mathcal{S}$**

Remark: Correlators cannot have singularities at nodal points of $\mathcal{S}$ or at punctures (pullbacks of singularities of Φ)

Moreover $\hat{W}_2^{(0)}([x_1.E_1],[x_2.E_2])$ may only have a double pole along the diagonal of $\mathcal{S}\times\mathcal{S}$ of the form $\frac{dx_1 dx_2 < E_1, E_2 >}{(x_1-x_2)^2}$ but no other singularities.

TT property 3

Condition 3: **Parity**

Under the involution $\hbar \to -\hbar$:

$$\hat{W}_n|_{\hbar \mapsto -\hbar}([x_1.E_1], \dots, [x_n.E_n]) = (-1)^n \hat{W}_n([x_1.E_1], \dots, [x_n.E_n])$$

Condition 4: **Leading order**

For all $n \geq 1$, the leading order of the series expansion in $\hbar$ of the correlation function $\hat{W}_n$ is at least of order $\hbar^{n-2}$

Theorem (Reconstruction by topological recursion)

If the system is of Topological Type then **connected correlators** $\hat{W}_n^{(k)}$ **can be reconstructed by the topological recursion** *applied to the spectral curve* $\left(\mathcal{S}, \hat{W}_2^{(0)}\right)$

Sufficient conditions for TT

- Let $\mathfrak{h}$ be a Cartan subalgebra of $\mathfrak{g}$ (Ex: diagonal matrices in $\mathfrak{gl}_r(\mathbb{C})$). $\Phi^{(0)}(x)$ can be generically "diagonalized" into:

$$\Phi^{(0)}(x) = \mathrm{Adj}_{V(x)}(T'(x)) = \text{"} V(x) T'(x) V(x)^{-1} \text{"}$$

with $V(x) \in G_x$ and $T'(x)$ a $\mathfrak{h}$-valued 1-form.

- $V(x)$ and $T'(x)$ defined up to Weyl group action (permutation of eigenvalues) and torus action (right multiplication of $V(x)$ by constant)
- **Spectral curve** satisfied by $y = T'(x)$:

$$P(x,y) = \det_\rho (y - \Phi^{(0)}(x)) \;\Rightarrow\; \text{Riemann surface } \mathcal{S}$$

- $\mathcal{S}$ comes with the projection $x : \mathcal{S} \to \Sigma$ with some ramification points
- $T(x)$ can be taken as any anti-derivative of $T'(x)$ on the universal cover $\tilde{\Sigma}$ of Σ (base point will have no effect)

Sufficient condition for $\hbar$ expansion

Proposition (Formal WKB solution)

Under the previous conditions, one can construct recursively a formal solution

$$\Psi(x,\hbar) = V(x)\left(Id + \sum_{k=0}^{\infty} \Psi^{(k)}(x)\hbar^k\right) e^{\frac{1}{\hbar}T(x)}$$
$$\overset{def}{=} V(x)\hat{\Psi}(x,\hbar)e^{\frac{1}{\hbar}T(x)}$$

of the linear differential system. $\hat{\Psi}(x,\hbar)$ *satisfies:*

$$\hbar\, d\hat{\Psi} = (V^{-1}\Phi V - \hbar V^{-1} dV)\hat{\Psi} - \hat{\Psi} T'$$

Consequence: $M(x.E) = \mathbf{Ad}_{\Psi(x)}(E)$ **admits a $\hbar$ expansion** *and finally* **correlators $\hat{W}_n$ also admit a $\hbar$ expansion**

Sufficient condition for pole structure

- Spectral curve:

$$P(x,y) = \det_\rho(y - \Phi^{(0)}(x))$$

  defines an algebraic plane curve $\mathcal{S}$ immersed in the total space of the cotangent bundle $T^*\Sigma$
- Immersion may not be an embedding $\Rightarrow$ **nodal points**.
- Condition 2 requires that correlators $\hat{W}_n$ do not have singularities at the nodal points
- Non trivial condition $\Rightarrow$ **Specific choice of** $\Phi^{(0)}(x)$
- If Lax pair: $\hbar \partial_t \Psi(x,t) = \mathcal{R}(x,t,\hbar)\Psi(x,t)$ the Auxiliary curve $\det_\rho(z - \mathcal{R}^{(0)}(x,t))$ is usually an embedding $\Rightarrow$ Condition 2 satisfied.

Sufficient condition for parity

Proposition

If there exists $J \in G$ **(independent of** x**)** *such that:*

$$\rho(J)^{-1}\rho(\Phi(x;\hbar))^t\rho(J) = \rho(\Phi(x;-\hbar))$$

then the parity condition for the correlators is satisfied

Remark

1. *Necessary condition? No cases without existence of* J *but satisfying parity condition are known*
2. *Interpretation of the condition?*

Sufficient condition for leading order

- Property is **trivial for $\hat{W}_1$, $\hat{W}_2$ and $\hat{W}_3$** (under parity condition).
- Possible proof with an insertion operator of order $\hbar$: $\delta_{x_{n+1}} \hat{W}_n = \hat{W}_{n+1}$
- Alternative proof for rank 2 systems using only loop equations (simpler in dimension 2)
- **No general method for higher rank** (insertion operator not well-defined so far)
- Known examples: Six Painlevé cases, $(p, 2)$ minimal models and incomplete proof with insertion operator for (p, q) models
- Proof for any integrable system with genus 0 compact spectral curve in progress

Painlevé 4 Lax pair

- $G = Gl_2(\mathbb{C})$, $\mathfrak{g} = \mathfrak{gl}_2(\mathbb{C})$, ρ=Trivial rep., $< a, b >= \mathrm{Tr}(ab)$
- Natural abelian Cartan subalgebra generated by $E_1 = \mathrm{diag}(1, 0)$ and $E_2 = \mathrm{diag}(0, 1)$
- Painlevé 4 Lax pair: $\hbar \partial_x \Psi = \Phi \Psi$ and $\hbar \partial_t \Psi = \mathcal{R} \Psi$

$$\begin{aligned}\Phi(x,t) &= \begin{pmatrix} x+t+\frac{pq+\theta_0}{x} & 1-\frac{q}{x} \\ -2(pq+\theta_0+\theta_\infty)+\frac{p(pq+2\theta_0)}{x} & -\left(x+t+\frac{pq+\theta_0}{x}\right)\end{pmatrix}\\ \mathcal{R}(x,t) &= \begin{pmatrix} x+q+t & 1 \\ -2(pq+\theta_0+\theta_\infty) & -(x+q+t)\end{pmatrix}\end{aligned}$$

- $\hbar$-deformed Painlevé 4 equation:

$$\begin{aligned}\hbar^2 \ddot{q} &= \frac{\hbar^2}{2q}\dot{q}^2 + 2\left(3q^3 + 4tq^2 + \left(t^2 - 2\theta_\infty + \hbar\right) q - \frac{\theta_0^2}{q}\right)\\ H_4(p,q,t) &= qp^2 + 2(q^2 + tq + \theta_0)p + 2(\theta_0 + \theta_\infty) q\end{aligned}$$

Painlevé 4 spectral curve

- Spectral curve:

$$P(x,y) = y^2 - \frac{(x-q_0)^2\left(x^2 + 2(q_0+t)x + \frac{\theta_0^2}{q_0^2}\right)}{x^2}$$

- $\mathcal{S}$: genus 0 Riemann surface with 2 ramification points, a double point $x = q_0$ and poles at $x \in \{0, \infty\}$.
- Parity matrix: (found using deformed Hamiltonian structure)

$$J(t) = \begin{pmatrix} -2(pq+\theta_0+\theta_\infty) & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow J(t)\Phi(x,t;\hbar)^t J(t)^{-1} = \Phi(x,t;-\hbar)$$

- Auxiliary curve: $z^2 = -\det \mathcal{R}_4^{(0)} = -q_0^2\left(x^2 + 2(q_0+t)x + \frac{\theta_0^2}{q_0^2}\right)$ is regular at $x = q_0$ and $x = 0$.

Pole structure for Painlevé 4

- $M(x.E_1) = I_2 - M(x.E_2)$ in dimension 2.

$$M^{(0)}(x.E_1,t) = \begin{pmatrix} \frac{1}{2} + \frac{\mathcal{R}^{(0)}_{1,1}(x,t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x,t)}} & \frac{\mathcal{R}^{(0)}_{1,2}(x,t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x,t)}} \\ \frac{\mathcal{R}^{(0)}_{2,1}(x,t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x,t)}} & \frac{1}{2} - \frac{\mathcal{R}^{(0)}_{1,1}(x,t)}{2\sqrt{-\det \mathcal{R}^{(0)}(x,t)}} \end{pmatrix}$$

- Recursive system for $M^{(k)}(x.E_1,t)$ requires to invert a 3×3 matrix (same for all orders):

$$\det \begin{pmatrix} 0 & -\mathcal{R}^{(0)}_{2,1} & \mathcal{R}^{(0)}_{1,2} \\ -2\mathcal{R}^{(0)}_{1,2} & 2\mathcal{R}^{(0)}_{1,1} & 0 \\ \mathcal{R}^{(0)}_{1,1} & \frac{1}{2}\mathcal{R}^{(0)}_{2,1} & \frac{1}{2}\mathcal{R}^{(0)}_{1,2} \end{pmatrix} = -2\mathcal{R}^{(0)}_{1,2}(x,t)\det \mathcal{R}^{(0)}(x,t)$$

- No singularity is introduced at the double zero $x = q_0$
- Direct computation for $\hat{W}^{(0)}_2(x_1.E_i, x_2.E_j) = \frac{\delta_{i,j}\,dx_1\,dx_2}{(x_1-x_2)^2}$

Leading order condition for Painlevé 4

- Simpler form of loop equations ($X = x.E_1$ and $X_j = x_j.E_1$, $L_n = \{X_1, \dots, X_n\}$):

$$0 = \mathcal{P}_{1;n}(x; L_n) + \hat{W}_{n+2}(X, X, L_n) + 2\hat{W}_1(X)\hat{W}_{n+1}(X, L_n) + \sum_{J\subset L_n, J\notin\{\emptyset, L_n\}} \hat{W}_{1+|J|}(X, J)\hat{W}_{1+n-|J|}(X, L_n \setminus J) + \sum_{j=1}^{n} \frac{d}{dx_j} \frac{\hat{W}_n(X, L_n \setminus X_j) - \hat{W}_n(L_n)}{x - x_j}$$

- Analysis of the singularities of $\mathcal{P}_{1;n}(x; L_n)$ ($x \in \{0, \infty\}$)

$$x \mapsto \mathcal{P}_{1;n}(x, L_n) = \frac{C_{1;n}(L_n)}{x}$$

- If leading order $\hat{W}_n < \hbar^{n-2}$. Recursion leads to:

$$0 = \mathcal{P}^{(n-3)}_{1;i_0}(x; L_{i_0}) + 2y(x)\hat{W}^{(n-2)}_{i_0+1}(X, L_{i_0})$$

$$\Rightarrow \hat{W}^{(n-2)}_{i_0+1}(X, L_{i_0}) = \frac{C^{(n-3)}_{1;i_0}(L_{i_0})}{2(x - q_0)\sqrt{x^2 + 2(q_0 + t)x + \frac{\theta_0^2}{q_0^2}}}$$

- Contradiction with the pole structure of $\hat{W}^{(n-2)}_{i_0+1}(X, L_{i_0})$

Conclusion

- General derivation of loop equations in a Lie algebra setting
- Generalization of Topological Type property and corresponding sufficient conditions
- Valid for **any reductive Lie algebra**, **any Riemann surface** Σ and any choice of **prime form** $\mathcal{E}$
- Recover known results in simple cases (Painlevé, minimal models)
- May be useful for the inverse problem: (Spectral curve $\mathcal{S}$ + Top. Rec.) $\Rightarrow$ Correlators $W_n^{(g)} \overset{?}{\Rightarrow}$ Differential system $\hbar d\Psi = \Phi\Psi$ (i.e. a quantum curve)
- Application to usual Lie groups?