Quantization of classical spectral curves via topological recursion

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Presentation of the problem

General position of the talk

General problem

How to quantize a "**classical spectral curve**" ($[y,x]=0$)

$$P(x,y)=0 \ , \ P \text{ rational in x, monic polynomial in y}$$

into a **linear differential equation** ($[\hbar\partial_x, x]=\hbar$):

$$\left(\hat{P}\left(x,\hbar\frac{d}{dx}\right)\right)\psi(x,\hbar)=0\,?$$

$\hat{P}$ rational in x with **same pole structure as** P.

Key ingredients

Key ingredient 1: Topological recursion [26].
Key ingredient 2: Integrable systems, Lax pairs:

$$\hbar\frac{\partial}{\partial x}\Psi(x,\hbar,t)=L(x,\hbar,t)\Psi(x,\hbar,t)\ ,\ \hbar\frac{\partial}{\partial t}\Psi(x,\hbar,t)=A(x,\hbar,t)\Psi(x,\hbar,t)$$

Strategy of the construction

1. Define proper initial data to apply topological recursion (TR) $\Leftrightarrow$ Minor technical restrictions on the classical spectral curve $P(x,y)=0$: "*Admissible initial data*"
2. Apply TR to initial data: $\Rightarrow$ Output: $(\omega_{\mathbf{h},\mathbf{n}})_{\mathbf{h},\mathbf{n}\geq \mathbf{0}}$: "*TR differentials*".
3. Stack the $\omega_{h,n}$ into some "*perturbative wave function*" $(\psi_i(z))_{i=1}^d$.

$$\psi_i(z)=\exp\left(\sum_{h,n\geq 0}\frac{\hbar^{2h-2+n}}{n!}\int_{D_i}\cdots\int_{D_i}\left(\omega_{h,n}(z_1,\dots,z_n)-\frac{\delta_{h,0}\delta_{n,2}dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2}\right)\right)$$

$\Rightarrow$ **formal WKB series in** $\hbar$.

4. Take kind of "formal Fourier transform" to get "*non-perturbative wave functions*" and regroup them into a wave matrix $\Psi^{\mathrm{NP}}(\lambda;\hbar)$ $\Rightarrow$ **Formal trans-series in** $\hbar$.
5. Prove that $\hbar\partial_\lambda\Psi^{\mathrm{NP}}(\lambda,\hbar)=L(\lambda,\hbar)\Psi^{\mathrm{NP}}(\lambda,\hbar)$ with L rational with controlled pole structure. $\Leftrightarrow$ "*Quantum curve*".
6. Obtain *auxiliary systems* $\hbar\partial_t\Psi^{\mathrm{NP}}(\lambda,\hbar,t)=A(\lambda,\hbar,t)\Psi^{\mathrm{NP}}(\lambda,\hbar,t)$ with A rational with controlled pole structure.

Known results and applications

- Review on TR and quantum curves by P. Norbury [35].
- Elements of the strategy already existing in the literature [7, 20, 22, 25, 26, 34].
- Non-perturbative construction is not necessary for **genus** 0 **classical spectral curves**.
- Several examples worked out in details [16, 17, 18, 19, 29, 31, 38].
- **Reverse approach also exists** [2, 5, 30, 33]:
  [Lax pair: $(L(\lambda,\hbar), A(\lambda,\hbar))$ + Topological type property] $\Rightarrow$
  Ψ reconstructed by TR applied on the associated classical spectral curve $\lim_{\hbar\to 0} \det(yI_d - L(\lambda,\hbar)) = 0$.
- Applications in enumerative geometry [1, 3, 4, 8, 13, 14, 21, 36, 37, 39, 27, 28].

Summary of our results

- Results presented following [32] for sl_2 case (hyper-elliptic case) and [24] **for the general gl_d case**. Similar works for sl_2 case in [23].
- Connection with isomonodromic deformations only in sl_2 case in [32].
- Technical assumptions include
  - **Pole of any degree including infinity.**
  - **Poles may be ramification points.**
  - Ramifications points are simple and smooth.
- Main results: Construction of the **matrix wave functions, quantum curve and some compatible auxiliary systems with same pole structure as the initial spectral curve.**
- Applications to two examples: gl_2 example (recovering Painlevé 2 equation) and a gl_3 example with only a single pole at infinity.

Classical spectral curve, TR

Classical spectral curve

Classical spectral curve

Let $(\Lambda_1, \dots, \Lambda_N)$ be $N \geq 0$ distinct points on $\mathbb{P}^1 \setminus \{\infty\}$. Let $\mathcal{H}_d(\Lambda_1, \dots, \Lambda_N, \infty)$ be the Hurwitz space of covers $x \colon \Sigma \to \mathbb{P}^1$ of degree d defined as the Riemann surface

$$\Sigma := \overline{\{(\lambda, y) \mid P(\lambda, y) = 0\}},$$

where $x(\lambda, y) := \lambda$ and

$$P(\lambda, y) = \sum_{l=0}^{d} (-1)^l y^{d-l} P_l(\lambda) = 0, \;\; P_0(\lambda) = 1$$

with each coefficient $(P_l)_{l \in [\![1,d]\!]}$ **being a rational function with possible poles at $\lambda \in \mathcal{P} := \{\Lambda_i\}_{i=1}^N \bigcup \{\infty\}$.**
A *classical spectral curve* (Σ, x) is the data of the Riemann surface Σ and its realization as a Hurwitz cover of $\mathbb{P}^1$.

Classical spectral curve with fixed pole structure

Classical spectral curve with fixed pole structure

For $l \in [\![1, d]\!]$, let $r_\infty^{(l)}$ and $\left(r_{\Lambda_i}^{(l)}\right)_{i=1}^{N}$ be some non-negative integers. We consider the subspace

$$\mathcal{H}_d\left((\Lambda_1, (r_{\Lambda_1}^{(l)})_{l=1}^d), \dots, (\Lambda_N, (r_{\Lambda_N}^{(l)})_{l=1}^d), (\infty, (r_\infty^{(l)})_{l=1}^d)\right) \subset \mathcal{H}_d(\Lambda_1, \dots, \Lambda_N, \infty)$$

of covers x such that the rational functions $(P_l)_{l=1}^d$ are of the form

$$P_l(\lambda) := \sum_{P \in \mathcal{P}} \sum_{k \in S_P^{(l)}} P_{P,k}^{(l)} \, \xi_P(\lambda)^{-k}, \text{ for } l \in [\![1, d]\!],$$

where we have defined

$$\forall\, i \in [\![1, N]\!] \,:\, \mathbf{S}_{\Lambda_i}^{(l)} := [\![\mathbf{1}, \mathbf{r}_{\Lambda_i}^{(l)}]\!] \qquad \text{and} \qquad \mathbf{S}_\infty^{(l)} := [\![\mathbf{0}, \mathbf{r}_\infty^{(l)}]\!],$$

and the local coordinates $\{\xi_P(\lambda)\}_{P \in \mathcal{P}}$ around $P \in \mathcal{P}$ are defined by

$$\forall\, i \in [\![1, N]\!] \,:\, \xi_{\Lambda_i}(\lambda) := (\lambda - \Lambda_i) \qquad \text{and} \qquad \xi_\infty(\lambda) := \lambda^{-1}.$$

Canonical local coordinates and spectral times

Canonical local coordinates

Let $P \in \mathbb{P}^1$ and $p \in x^{-1}(P)$. Canonical coordinates on $\mathbb{P}^1$ near P are

$$\xi_P(x) := x - P \quad \text{if } P \neq \infty\,, \quad \epsilon_P := 1,$$
$$\xi_P(x) := \frac{1}{x} \quad \text{if } P = \infty\,, \quad \epsilon_P := -1.$$

Canonical local coordinates near any $p \in x^{-1}(P)$ are

$$\zeta_p(z) = \xi_P(x(z))^{\frac{1}{d_p}}\,,\; d_p = \text{order}_p(\xi_P).$$

Spectral times (KP times)

The 1-form ydx has the following expansion:

$$ydx = \sum_{k=0}^{s_p-1} t_{p,k}\zeta_p^{-k-1}d\zeta_p + \text{analytic at } p.$$

$(t_{p,k})_{p\in x^{-1}(\mathcal{P}),k\in[\![0,s_p-1]\!]}$ are called "**spectral times**".

Ramification points and critical values

Ramification points and critical values

We denote by $\mathcal{R}_0$ the set of all ramification points of the cover x, and by $\mathcal{R}$ the set of all ramification points that are not poles (i.e. not in $x^{-1}(\mathcal{P})$),

$$\mathcal{R}_0 := \left\{ p \in \Sigma \,/\, 1 + \mathrm{order}_p \, dx \neq \pm 1 \right\},$$

$$\mathcal{R} := \left\{ p \in \Sigma \,/\, dx(p) = 0 \,, \;\; x(p) \notin \mathcal{P} \right\} = \mathcal{R}_0 \setminus x^{-1}(\mathcal{P}).$$

We shall refer to their images $x(\mathcal{R})$ as the *critical values* of x.

Admissible spectral curve

Admissible classical spectral curves

We say that a classical spectral curve (Σ, x) is *admissible* if it satisfies:

- The Riemann surface Σ defined by $P(\lambda, y) = 0$ is an irreducible algebraic curve, i.e. $P(\lambda, y)$ does not factorize.
- All ramification points are simple, i.e. dx has only a simple zero at $a \in \mathcal{R}$.
- Critical values are distinct: for any $(a_i, a_j) \in \mathcal{R} \times \mathcal{R}$ such that $a_i \neq a_j$ then $x(a_i) \neq x(a_j)$.
- Ramification points are smooth: for any $a \in \mathcal{R}$, $dy(a) \neq 0$ (i.e. the tangent vector $(dx(a), dy(a))$ to the immersed curve $\{(\lambda, y) \mid P(\lambda, y) = 0\}$ is not vanishing at a).
- Generic ramified poles: for any pole $p \in x^{-1}(\mathcal{P})$ ramified, the 1-form ydx has a pole of degree $r_p \geq 3$ at p, and the corresponding spectral times satisfy $t_{p, r_p - 2} \neq 0$.

Remarks on the technical assumptions

- Topology of admissible spectral curves relatively to spectral times is complicated. $\Rightarrow$ Spectral times are not independent. Tangent space and deformations hard to define for $d \geq 3$.
- Tangent space defined for $d = 2 \leftrightarrow$ Existence of deformations $\partial_{t_{p,k}}$.
- Ingredients to lift some technical assumptions already exist in the literature: simple ramification points, smooth ramification points, reducible algebraic curves.
- Defining properly the tangent space would allow to make the connection with isomonodromic deformations for $d \geq 3$.
- Last condition allows **not to include ramified poles in the residues of TR**.

Admissible initial data

Admissible initial data

Given an **admissible spectral curve** (Σ, x) of genus g, we add

- **Choice of Torelli marking** $(\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g$.
  $\Leftrightarrow$ Associated "Bergman" kernel (normalized fundamental second kind differential) $B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^g}$.
- A generic smooth point $o \in \Sigma \setminus x^{-1}(\mathcal{P})$ and some choice of non-intersecting homology chains $\mathcal{C}_{o \to p}$ for each $p \in x^{-1}(\mathcal{P})$ compatible with the Torelli marking:

$$\forall\, p \in x^{-1}(\mathcal{P}),\ \forall\, i \in [\![1, g]\!] \ , \qquad \mathcal{A}_i \cap \mathcal{C}_{o\to p} = 0 = \mathcal{B}_i \cap \mathcal{C}_{o\to p},$$

These three ingredients define some "**admissible initial data**" on which TR can be applied. Denoted $((\Sigma, x), (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^g)$.

General considerations

- Initial version [26] of TR dating back to 2007 is sufficient since ramification points are assumed simple.
- Some generalizations of TR exist to deal with non-simple ramification points, non-irreducible curves [6, 15].
- **TR takes admissible initial data as input and provides some TR differentials $(\omega_{h,n})_{h\geq 0, n\geq 0}$ as output.**

  https://en.wikipedia.org/wiki/Topological_recursion
- These differentials are computed by recursion on $s = n + 2h$ starting from

$$\omega_{0,1} := ydx, \qquad \omega_{0,2} := B^{(\mathcal{A}_i,\mathcal{B}_i)_{i=1}^{g}},$$

Definition of TR

Topological recursion

We have for $h \geq 0$, $n \geq 0$ with $(h,n) \notin \{(0,0),(0,1)\}$:

$$\omega_{h,n+1}(z_0,\mathbf{z}) := \sum_{a\in\mathcal{R}} \operatorname*{Res}_{z\to a} \frac{1}{2} \frac{\int_{\sigma_a(z)}^{z} \omega_{0,2}(z_0,\cdot)}{\omega_{0,1}(z) - \sigma_a^*\omega_{0,1}(z)} \widetilde{\mathcal{W}}^{(2)}_{h,n+1}(z,\sigma_a(z);\mathbf{z}),$$

with

$$\widetilde{\mathcal{W}}^{(2)}_{h,n+1}(z,z';\mathbf{z}) := \omega_{h-1,n+2}(z,z',\mathbf{z}) + \sum_{\substack{A\sqcup B=\mathbf{z},\, s\in[\![0,h]\!] \\ (s,|A|)\notin\{(0,0),(h,n)\}}} \omega_{s,|A|+1}(z,A)\,\omega_{h-s,|B|+1}(z',B)$$

and

$$\omega_{h,0} := \frac{1}{2-2h} \sum_{a\in\mathcal{R}} \operatorname*{Res}_{z\to a} \omega_{h,1}(z)\Phi(z), \ \forall\, h \geq 2$$

and $(\omega_{0,0},\omega_{1,0})$ defined by specific formulas (See [26])

Loop equations

- Some combinations of the TR differentials have interesting properties ⇒ "*Loop equations*"
- Following [7], for $(h,n,l) \in \mathbb{N}^3$:

$$
\begin{aligned}
Q^{(0)}_{h,n+1}(\lambda;\mathbf{z}) &= \hat{Q}^{(0)}_{h,n+1}(\lambda;\mathbf{z}) = \widetilde{Q}^{(0)}_{h,n+1}(\lambda;\mathbf{z}) \coloneqq \delta_{h,0}\delta_{n,0},\\
Q^{(l)}_{h,n+1}(\lambda;\mathbf{z}) &\coloneqq \sum_{\beta \underset{l}{\subseteq} x^{-1}(\lambda)} \sum_{\mu\in\mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i=\mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i = h+l(\mu)-l} \left[\prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i,J_i)\right]\\
\hat{Q}^{(l)}_{h,n+1}(z;\mathbf{z}) &\coloneqq \sum_{\beta \underset{l}{\subseteq} \left(x^{-1}(x(z))\setminus\{z\}\right)} \sum_{\mu\in\mathcal{S}(\beta)} \sum_{\bigsqcup_{i=1}^{l(\mu)} J_i=\mathbf{z}} \sum_{\sum_{i=1}^{l(\mu)} g_i = h+l(\mu)-l} \left[\prod_{i=1}^{l(\mu)} \omega_{g_i,|\mu_i|+|J_i|}(\mu_i,J_i)\right]\\
\widetilde{Q}^{(l)}_{h,n+1}(\lambda;\mathbf{z}) &\coloneqq \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l} - \sum_{j=1}^{n} d_{z_j}\left(\frac{1}{\lambda - x(z_j)}\,\frac{\hat{Q}^{(l-1)}_{h,n}(z_j;\mathbf{z}\setminus\{z_j\})}{(dx(z_j))^{l-1}}\right)
\end{aligned}
$$

Loop equations

For any $(h,n,l) \in \mathbb{N}^3$ and any $\mathbf{z} \in (\Sigma \setminus \mathcal{R})^n$, the function $\lambda \mapsto \frac{Q^{(l)}_{h,n+1}(\lambda;\mathbf{z})}{(d\lambda)^l}$ has no poles at critical values.

Perturbative wave functions

Generic perturbative wave functions

Perturbative wave functions

$((\Sigma, x), (\mathcal{A}_i, \mathcal{B}_i)_{i=1}^{g})$ admissible initial data, $D = \sum_{i=1}^{s} \alpha_i [p_i]$ generic divisor on Σ. *Perturbative wave functions* associated to D are

$$\psi(D, \hbar) := \exp\left(\sum_{h,n\geq 0} \frac{\hbar^{2h-2+n}}{n!} \int_D \cdots \int_D \omega_{h,n}(\mathbf{z}) - \delta_{h,0}\delta_{n,2} \frac{dx(z_1)dx(z_2)}{(x(z_1)-x(z_2))^2} \right)$$

$$\forall\, i \in [\![1, s]\!] \,:\, \psi_{0,i}(D, \hbar) := \psi(D, \hbar),$$

$$\forall\, i \in [\![1, s]\!]\,,\, l \geq 1 \,:\, \psi_{l,i}(D, \hbar) := \left[\sum_{h\geq 0} \sum_{n\geq 0} \frac{\hbar^{2h+n}}{n!} \overbrace{\int_D \cdots \int_D}^{n} \frac{\hat{Q}^{(l)}_{h,n+1}(p_i; \cdot)}{(dx(p_i))^l} \right] \psi(D, \hbar).$$

Remark

Definition as a **formal power series in $\hbar$ times exponential terms in finite negative powers of $\hbar$** (formal WKB series):

$$e^{-\hbar^{-2}\omega_{0,0}} e^{-\hbar^{-1}\int_D \omega_{0,1}} \psi(D, \hbar) \in \mathbb{C}[[\hbar]].$$

KZ equations

- Loop equations translates into Knizhnik–Zamolodchikov (KZ) equations [7]

Generic KZ equations

For $i \in [\![1,s]\!]$ and $l \in [\![0,d-1]\!]$, we have

$$\frac{\hbar}{\alpha_i}\frac{d\psi_{l,i}(D,\hbar)}{dx(p_i)} = -\psi_{l+1,i}(D,\hbar) - \hbar \sum_{j\in[\![1,s]\!]\setminus\{i\}} \alpha_j \frac{\psi_{l,i}(D,\hbar)-\psi_{l,j}(D,\hbar)}{x(p_i)-x(p_j)}$$
$$+\sum_{h\geq 0}\sum_{n\geq 0}\frac{\hbar^{2h+n}}{n!}\int_{z_1\in D}\cdots\int_{z_n\in D}\widetilde{Q}^{(l+1)}_{h,n+1}(x(p_i);\mathbf{z})\,\psi(D,\hbar)$$
$$+\left(\frac{1}{\alpha_i}-\alpha_i\right)\left[\sum_{(h,n)\in\mathbb{N}^2}\frac{\hbar^{2h+n+1}}{n!}\overbrace{\int_D\cdots\int_D}^{n}\frac{d}{dx(p_i)}\left(\frac{\hat{Q}^{(l)}_{h,n+1}(p_i;\cdot)}{(dx(p_i))^l}\right)\right]\psi(D,\hbar).$$

- Valid for generic divisors (p_i not a pole or a ramification point).
- Simplification for two points divisors with $(\alpha_1,\alpha_2)\in\{-1,+1\}^2$.

Remarks

- KZ equations allow to obtain PDEs for $\psi(D,\hbar)$.
- Generic divisors provide PDEs with derivatives $\frac{\partial}{\partial x(z)}$ up to order d^2 generically.
- Quantum curve is expected to be of order d and not d^2.
- At least two specific choices of divisors allow for order d: $D=[z]-[\infty^{(\alpha)}]$ or $D=[z]-[\sigma(z)]$.
- Other choices may also provide order d PDEs.

Regularization of perturbative wave functions for $D = [z] - [\infty^{(\alpha)}]$

Infinity is a pole of the classical spectral curve $\Rightarrow D = [z] - [\infty^{(\alpha)}]$ is **not** a generic divisor $\Rightarrow$ Some quantities requires regularization from $\lim_{p \to \infty^{(\alpha)}} ([z] - [p])$

Definition of regularized wave function

$$\psi^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) := \exp\left(\hbar^{-1}\left(V_{\infty^{(\alpha)}}(z) + \int_{\infty^{(\alpha)}}^{z} (ydx - dV_{\infty^{(\alpha)}})\right)\right)$$

$$\frac{1}{E(z, \infty^{(\alpha)})\sqrt{dx(z)d\zeta_{\infty^{(\alpha)}}(\infty^{(\alpha)})}} \exp\left(\sum_{n \geq 3\delta_{h,0}} \frac{\hbar^{2h-2+n}}{n!} \int_{\infty^{(\alpha)}}^{z} \cdots \int_{\infty^{(\alpha)}}^{z} \omega_{h,n}\right)$$

$$\psi_l^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) :=$$

$$\left(\sum_{n \geq 3\delta_{h,0}} \frac{\hbar^{2h+n}}{n!} \int_{\infty^{(\alpha)}}^{z} \cdots \int_{\infty^{(\alpha)}}^{z} \frac{\hat{Q}_{h,n+1}^{(l)}(z; z_1, \dots, z_n)}{dx(z)^l}\right) \psi^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

KZ equations for regularized wave functions

KZ equations for regularized wave functions

$$\hbar \frac{d}{dx(z)} \psi_l^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar) + \psi_{l+1}^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

$$= \Bigg[\sum_{h \geq 0} \sum_{n \geq 0} \frac{\hbar^{2h+n}}{n!} \sum_{P \in \mathcal{P}} \sum_{k \in S_P^{(l+1)}} \xi_P(x(z))^{-k} \operatorname*{Res}_{\lambda \to P} \xi_P(\lambda)^{k-1}$$

$$d\xi_P(\lambda) \int_{z_1 = \infty^{(\alpha)}}^{z_1 = z} \cdots \int_{z_n = \infty^{(\alpha)}}^{z_n = z} \frac{Q_{h,n+1}^{(l+1)}(\lambda; \mathbf{z})}{(d\lambda)^{l+1}} \Bigg] \psi^{\mathrm{reg}}(D = [z] - [\infty^{(\alpha)}], \hbar)$$

Comments and technical issue

- RHS of KZ equations uses residues, i.e. integrals.
- RHS may be rewritten using generalized integrals, i.e. **linear operators** $\mathcal{I}_{\mathcal{C}_{p,k}}$.
- $\mathcal{I}_{\mathcal{C}_{p,k}}$ is expected to correspond to $\partial_{t_{p,k}}$. Valid for $d = 2$ and examples.
- Action of these operators is defined only on a sub-algebra generated by $\int_{\mathcal{C}_1} \dots \int_{\mathcal{C}_n} \omega_{h,n}$. $\Leftrightarrow$ **Algebra of symbols**
- One need to check that these operators never act on something else.
- Avoid the problematic definition on all differential forms on Σ.

PDE form of KZ equations

PDE form of KZ equations

$$\hbar \frac{d}{dx(z)} \psi_l^{\mathrm{reg}}([z]-[\infty^{(\alpha)}]) + \psi_{l+1}^{\mathrm{reg}}([z]-[\infty^{(\alpha)}]) = \mathrm{ev.}\, \widetilde{\mathcal{L}}_l(x(z)) \left[\psi^{\mathrm{reg\ symb}}([z]-[\infty^{(\alpha)}])\right]$$

with

$$\widetilde{\mathcal{L}}_l(x(z)) = \sum_{P \in \mathcal{P}} \sum_{k \in S_P^{(l+1)}} \xi_P(x(z))^{-k} \widetilde{\mathcal{L}}_{P,k,l}$$

Definition of the operators

Definition of the operators $\widetilde{\mathcal{L}}_{P,k,l}$

$$\begin{aligned}
\widetilde{\mathcal{L}}_{P,k,l} \;:=\; & \epsilon_P^{l+1}\Bigg[\xi_P(x(z))^{-(l+1)\epsilon_P}\sum_{\ell'=0}^{l+1}\;\sum_{\nu'\subset_{\ell'}[\![1,d]\!]}\;\prod_{j\in\nu'}\left(\sum_{m=0}^{r_{P(j)}-1}\frac{t_{P(j),m}}{d_{P(j)}}\,\xi_P^{-\frac{m}{d_{P(j)}}}\right)\\
& \sum_{0\leq \ell''\leq \frac{l+1-\ell'}{2}}\;\sum_{\substack{\nu''\in\mathcal{S}^{(2)}([\![1,d]\!]\setminus\nu')\\ l(\nu'')=\ell''}}\;\prod_{i=1}^{\ell''}\frac{\hbar^2 R(P)_{\nu''_i}}{d_{P(\nu''_{i,+})}\,d_{P(\nu''_{i,-})}}\\
& \sum_{\nu\;\subseteq_{l+1-\ell'-2\ell''}\;[\![1,d]\!]\setminus(\nu'\cup\nu'')}\;\prod_{j\in\nu}\left(\hbar^2\sum_{m=1}^{\infty}\frac{\xi_P^{\frac{m}{d_{P(j)}}}}{d_{P(j)}}\,\mathcal{I}_{\mathcal{C}_{P(j),k}}\right)\Bigg]_{-k}\\
+\; & \hbar\,\delta_{P,\infty}\,\frac{\epsilon_\infty^{l+1}}{d_{\infty(\alpha)}}\Bigg[\xi_\infty(x(z))^{-(l+1)\epsilon_\infty}\sum_{\ell'=0}^{l+1}\;\sum_{\nu'\subset_{\ell'}[\![1,d]\!]\setminus\{\alpha\}}\;\prod_{j\in\nu'}\left(\sum_{m=0}^{r_{\infty(j)}-1}\frac{t_{\infty(j),k}}{d_{\infty(j)}}\,\xi_\infty^{-\frac{m}{d_{\infty(j)}}}\right)\\
& \sum_{0\leq \ell''\leq \frac{l+1-\ell'}{2}}\;\sum_{\substack{\nu''\in\mathcal{S}^{(2)}([\![1,d]\!]\setminus(\nu'\cup\{\alpha\}))\\ l(\nu'')=\ell''}}\;\prod_{i=1}^{\ell''}\frac{\hbar^2 R(\infty)_{\nu''_i}}{d_{\infty(\nu''_{i,+})}\,d_{\infty(\nu''_{i,-})}}\\
& \sum_{\nu\;\subseteq_{l-\ell'-2\ell''}\;[\![1,d]\!]\setminus(\nu'\cup\nu''\cup\{\alpha\})}\;\prod_{j\in\nu}\left(\hbar^2\sum_{m=1}^{\infty}\frac{\xi_\infty^{\frac{m}{d_{\infty(j)}}}}{d_{\infty(j)}}\,\mathcal{I}_{\mathcal{C}_{\infty(j),m}}\right)\Bigg]_{-k}
\end{aligned}$$

Monodromies

- Perturbative wave functions have bad monodromies on $\mathcal{B}$-cycles.
- Monodromies are directly connected to a shift of the filling fractions $\epsilon_i = \oint_{\mathcal{A}_i} \omega_{0,1}$ by $\hbar$.
- Monodromies issues only arises for genus $g > 0$ classical spectral curves.
- Solution is to "sum over filling fractions" $\Rightarrow$ Formal Fourier transform $\Rightarrow$ non-perturbative corrections.

Non-perturbative wave functions

Non perturbative wave functions

Non perturbative wave functions

$$\psi_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}) := e^{\hbar^{-2}\omega_{0,0}+\omega_{1,0}}\, e^{\hbar^{-1}\int_D \omega_{0,1}} \frac{1}{E(D)} \sum_{r=0}^{\infty} \hbar^r G^{(r)}(D;\boldsymbol{\rho})$$

where E prime form on Σ and

$$G^{(r)}(D;\boldsymbol{\rho}) := \sum_{k=0}^{3r} \sum_{(i_1,\dots,i_k)\in[\![1,g]\!]^k} \Theta^{(i_1,\dots,i_k)}(\mathbf{v},\tau)\, G^{(r)}_{(i_1,\dots,i_k)}(D)$$

with

$$v_j := \frac{\rho_j+\phi_j}{\hbar} + \mu_j^{(\alpha)}(z),\ \ \phi_j := \frac{1}{2\pi\mathrm{i}}\oint_{\mathcal{B}_j}\omega_{0,1},\ \ \mu_j^{(\alpha)}(z) := \frac{1}{2\pi\mathrm{i}}\int_D\oint_{\mathcal{B}_j}\omega_{0,2}.$$

Moreover

$$\psi^{\infty(\alpha)}_{l,\mathrm{NP}}(z,\hbar,\boldsymbol{\rho}) := \sum_{\beta \underset{l}{\subseteq} (x^{-1}(x(z))\setminus\{z\})} \frac{1}{l!}\,\mathrm{ev.}\left(\prod_{j=1}^{l}\mathcal{I}_{\mathcal{C}_{\beta_j,1}}\right)\psi^{\mathrm{symbol}}_{\mathrm{NP}}(D;\hbar,\boldsymbol{\rho}).$$

and $d\times d$ wave functions matrix

$$\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar,\boldsymbol{\rho}) := \left[\psi^{\infty(\alpha)}_{l-1,\mathrm{NP}}(z^{(\alpha)}(\lambda),\hbar,\boldsymbol{\rho})\right]_{1\le l,\alpha\le d},$$

Trans-series in $\hbar$

- Non-perturbative quantities are **formal trans-series in** $\hbar$ of the form

$$\sum_{r=0}^{\infty} \sum_{\mathbf{n}\in\mathbb{Z}^g} \hbar^r e^{\frac{1}{\hbar}\sum_{j=1}^{g} n_j\phi_j} F_{r,\mathbf{n}},$$

- Equalities should only be consider coefficients by coefficients in the trans-monomials.
- Non-perturbative wave functions satisfy same KZ equations as the perturbative wave functions.
- Non-perturbative wave functions have good monodromies. $\Rightarrow$ **rational functions** of λ.

Lax pairs

Lax systems

Lax systems

We have the Lax systems

$$\begin{aligned}\hbar\frac{d\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)}{d\lambda} &= \widehat{L}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)\\ \hbar^{-1}\mathrm{ev}.\mathcal{L}_{P,k,l}\widehat{\Psi}^{\mathrm{symbol}}_{\mathrm{NP}}(\lambda,\hbar) &= \widehat{A}_{P,k,l}(\lambda,\hbar)\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)\end{aligned}$$

with

$$\begin{aligned}\widehat{L}(\lambda,\hbar) &= \left[-\widehat{P}(\lambda)+\hbar\sum_{P\in\mathcal{P}}\sum_{k\in\mathbb{N}}\xi_P^{-k}(\lambda)\widehat{\Delta}_{P,k}(\lambda,\hbar)\right]\\ \left[\widehat{\Delta}_{P,k}(\lambda,\hbar)\right]_{2,j} &= \left[\widehat{A}_{P,k,l}(\lambda,\hbar)\right]_{1,j},\ \forall j\in[\![1,d]\!],\end{aligned}$$

and

$$\widehat{P}(\lambda) := \begin{bmatrix} -P_1(\lambda) & 1 & 0 & \dots & 0\\ -P_2(\lambda) & 0 & 1 & \dots & 0\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ -P_{d-1}(\lambda) & 0 & 0 & \dots & 1\\ -P_d(\lambda) & 0 & 0 & \dots & 0\end{bmatrix}$$

Gauge transformation to recover companion-like matrix when $\hbar \to 0$

Define

$$G(\lambda) := \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & 0 \\ P_1(\lambda) & -1 & 0 & \dots & 0 & 0 \\ P_2(\lambda) & -P_1(\lambda) & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{d-2}(\lambda) & -P_{d-3}(\lambda) & P_{d-4}(\lambda) & \dots & (-1)^{d-2} & 0 \\ P_{d-1}(\lambda) & -P_{d-2}(\lambda) & P_{d-3}(\lambda) & \dots & (-1)^{d-2}P_1(\lambda) & (-1)^{d-1} \end{bmatrix}$$

and

$$\begin{aligned} \widetilde{\Psi}(\lambda,\hbar) &:= (G(\lambda))^{-1}\,\widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar) \\ \hbar\frac{d\widetilde{\Psi}(\lambda,\hbar)}{d\lambda} &= \widetilde{L}(\lambda,\hbar)\widetilde{\Psi}(\lambda,\hbar) \\ \hbar^{-1}\mathrm{ev}.\mathcal{L}_{P,k,l}\widetilde{\Psi}(\lambda,\hbar) &= \widetilde{A}_{P,k,l}(\lambda,\hbar)\widetilde{\Psi}(\lambda) \end{aligned}$$

with

$$\widetilde{L}(\lambda,\hbar) = \left[\widetilde{P}(\lambda) + \hbar\sum_{P\in\mathcal{P}}\sum_{k\in\mathbb{N}}\xi_P^{-k}(\lambda)\widetilde{\Delta}_{P,k}(\lambda,\hbar)\right]$$

$\widetilde{\mathrm{P}}(\lambda)$ companion-like matrix associated to classical spectral curve.

Main result: pole structure of the Lax system

Pole structure of the Lax system

Matrices $\widetilde{A}_{P,k,l}(\lambda,\hbar)$ are **rational functions** of λ with **no pole at critical values** $u \in x(\mathcal{R})$.
Matrices $\widetilde{L}(\lambda,\hbar)$ and $\hat{L}(\lambda,\hbar)$ are **rational functions** of λ with possible **poles only at** $\lambda \in \mathcal{P}$ and at **zeroes of the Wronskian** $\det \widehat{\Psi}_{\mathrm{NP}}(\lambda,\hbar)$ (i.e. *apparent singularities*).

- Long and technical proof by induction on the order in the trans-series.
- Proof uses some of admissibility conditions (distinct critical values, smooth and simple ramification points).
- Proof should adapt without the admissibility conditions but involving more technical computations.

Quantum curve

Quantum curve

$\forall j \in [\![1,d]\!]$, $\psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(j)}(\lambda),\hbar)$ is solution to a degree d ODE of the form

$$\forall j \in [\![1,d]\!] \,:\, \sum_{k=0}^{d} b_{d-k}(\lambda,\hbar)\left(\hbar\frac{\partial}{\partial\lambda}\right)^k \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(j)}(\lambda),\hbar)=0,$$

Coefficients $(b_l(\lambda,\hbar))_{l\in[\![0,d]\!]}$ with $b_0(\lambda,\hbar)=1$ are **rational functions** of λ with **poles only at** $\lambda\in\mathcal{P}$ **and zeros of the Wronskian**.

$\Leftrightarrow$ Matrix form: $\Psi(\lambda,\hbar):=\left[\left(\hbar\frac{\partial}{\partial\lambda}\right)^{i-1}\psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(j)}(\lambda),\hbar)\right]_{1\leq i,j\leq d}$ satisfies:

$$\begin{aligned}\hbar\frac{\partial}{\partial\lambda}\Psi(\lambda,\hbar) &= \begin{bmatrix} 0 & 1 & \dots & 0\\ \vdots & \ddots & \ddots & \ddots\\ 0 & 0 & & 1\\ -b_d(\lambda,\hbar) & -b_{d-1}(\lambda,\hbar) & \dots & -b_1(\lambda,\hbar)\end{bmatrix}\Psi(\lambda,\hbar)\\ &:= L(\lambda,\hbar)\Psi(\lambda,\hbar)\end{aligned}$$

Gauge transformation to remove apparent singularities

- Apparent singularities $\Leftrightarrow$ zeros of Wronskian:

$$W(\lambda,\hbar) := \det \Psi(\lambda,\hbar) = \kappa \frac{\prod_{i=1}^{G}(\lambda - q_i(\hbar))}{\prod_{i=1}^{N}(\lambda-\Lambda_i)^{G_{\Lambda_i}}} \exp\left(\hbar^{-1}\int_0^{\lambda} P_1(\lambda)d\lambda\right),$$

- Explicit gauge transformation $J(\lambda,\hbar)$ to **remove apparent singularities**

$$\check{\Psi}(\lambda,\hbar) := \begin{bmatrix} 1 & \dots & 0 & 0 \\ \ddots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \\ \frac{Q_d(\lambda,\hbar)}{\prod_{i=1}^{G}(\lambda-q_i(\hbar))} & \dots & \frac{Q_2(\lambda,\hbar)}{\prod_{i=1}^{G}(\lambda-q_i(\hbar))} & \frac{Q_1(\lambda,\hbar)}{\prod_{i=1}^{G}(\lambda-q_i(\hbar))} \end{bmatrix} \Psi(\lambda,\hbar)$$

- Q_j: polynomial of degree $G-1$ at most defined by interpolation.
- Gauge transformation does not introduce new poles because

$$\det J(\lambda,\hbar) = \left(\prod_{k=1}^{N}(\lambda-\Lambda_k)^{G_{\Lambda_k}}\right)\left(\prod_{i=1}^{G}(\lambda - q_i(\hbar))\right)^{-1}$$

Remarks

4 equivalent gauges:

- Gauge $\hat{\Psi}(\lambda, \hbar)$: Natural gauge from KZ equations and provides compatible auxiliary systems. But leading order in $\hbar$ of $\hat{L}(\lambda, \hbar)$ is not companion-like $\Rightarrow$ Classical spectral curve is not easily recovered. Contains apparent singularities.
- Gauge $\widetilde{\Psi}(\lambda, \hbar)$: Same properties as the previous gauge ($\hbar^0$ gauge transformation) except leading order in $\hbar$ is companion-like and recovers the classical spectral curve.
- Gauge $\Psi(\lambda, \hbar)$: $L(\lambda, \hbar)$ is companion-like $\Rightarrow$ Quantum curve is directly read in the last line of $L(\lambda, \hbar)$. Classical spectral curve directly obtained as $\hbar \to 0$ limit of $L(\lambda, \hbar)$. But contains apparent singularities. Natural framework for Darboux coordinates and isomonodromic deformations.
- Gauge $\check{\Psi}$: $\check{L}(\lambda, \hbar)$ has no apparent singularity. But no longer companion like (last two lines are non-trivial) so less adapted to read the classical and quantum curves.

Example

Classical spectral curve

Classical spectral curve

We take $d = 2$, $N = 0$, $r_\infty^{(1)} = 2$ and $r_\infty^{(2)} = 4$. Two points above infinity denoted by $\infty^{(1)}$ and $\infty^{(2)}$ non-ramified.

$$y^2 - P_1(\lambda) y + P_2(\lambda) = 0,$$

with

$$\begin{aligned} P_1(\lambda) &= P^{(1)}_{\infty,2}\lambda^2 + P^{(1)}_{\infty,1}\lambda + P^{(1)}_{\infty,0} \\ P_2(\lambda) &= P^{(2)}_{\infty,4}\lambda^4 + P^{(2)}_{\infty,3}\lambda^3 + P^{(2)}_{\infty,2}\lambda^2 + P^{(2)}_{\infty,1}\lambda + P^{(2)}_{\infty,0} \end{aligned}$$

6 Spectral times $(t_{i,j})_{1\leq i\leq 2, 0\leq j\leq 3}$ are defined by $\forall\, i \in \{1,2\}$:

$$y(z) = -t_{i,3}x(z)^2 - t_{i,2}x(z) - t_{i,1} - t_{i,0}x(z)^{-1} + O\left(x(z)^{-2}\right), \text{ as } z \to \infty^{(i)}$$

Connection with spectral times

Relations between spectral times and Coefficients of the classical spectral curve:

$$\begin{aligned}
P^{(1)}_{\infty,2} &= -t_{1,3} - t_{2,3} \\
P^{(1)}_{\infty,1} &= -t_{1,2} - t_{2,2} \\
P^{(1)}_{\infty,0} &= -t_{1,1} - t_{2,1} \\
P^{(2)}_{\infty,4} &= t_{1,3}t_{2,3} \\
P^{(2)}_{\infty,3} &= t_{1,2}t_{2,3} + t_{1,3}t_{2,2} \\
P^{(2)}_{\infty,2} &= t_{1,2}t_{2,2} + t_{1,3}t_{2,1} + t_{1,1}t_{2,3} \\
P^{(2)}_{\infty,1} &= t_{1,3}t_{2,0} + t_{1,0}t_{2,3} + t_{1,2}t_{2,1} + t_{1,1}t_{2,2}
\end{aligned}$$

and $0 = -t_{1,0} - t_{2,0}$.

KZ equations

Using the general theory, we get:

KZ equations

$$\begin{cases} \hbar \frac{\partial \psi^{\infty^{(1)}}_{0,\mathrm{NP}}(z,\hbar)}{\partial x(z)} + \psi^{\infty^{(1)}}_{1,\mathrm{NP}}(z,\hbar) = P_1(x(z))\psi^{\infty^{(1)}}_{0,\mathrm{NP}}(z,\hbar), \\ \hbar \frac{\partial \psi^{\infty^{(1)}}_{1,\mathrm{NP}}(z,\hbar)}{\partial x(z)} = P_2(x(z))\psi^{\infty^{(1)}}_{0,\mathrm{NP}}(z,\hbar) + \hbar\, \mathrm{ev}.\, \mathcal{L}_{\mathrm{KZ}}(x(z)) \left[\psi^{\infty^{(1)},\,\mathrm{symbol}}_{0,\mathrm{NP}}(z,\hbar)\right] \end{cases}$$

where

$$\mathcal{L}_{\mathrm{KZ}}(\lambda) := \hbar \mathbf{t}_{1,3} \mathcal{I}_{\mathcal{C}_{\infty^{(2)},1}} + \hbar \mathbf{t}_{2,3} \mathcal{I}_{\mathcal{C}_{\infty^{(1)},1}} - \mathbf{t}_{2,3}\lambda - \mathbf{t}_{2,2}$$

Lax pair from KZ equations

Define $\Psi(\lambda,\hbar) = \begin{pmatrix} \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(1)}(\lambda),\hbar) & \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(2)}(\lambda),\hbar) \\ \hbar\partial_\lambda \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(1)}(\lambda),\hbar) & \hbar\partial_\lambda \psi^{\infty(\alpha)}_{0,\mathrm{NP}}(z^{(2)}(\lambda),\hbar) \end{pmatrix}$

KZ equations are equivalent to

$$\hbar\partial_\lambda \Psi(\lambda,\hbar) = \begin{pmatrix} 0 & 1 \\ -P_2(\lambda) + \hbar P_1'(\lambda) + H - \frac{p}{\lambda-q} + \hbar\alpha\lambda & P_1(\lambda) + \frac{\hbar}{\lambda-q} \end{pmatrix} \Psi(\lambda,\hbar)$$

$$\mathrm{ev}.\mathcal{L}_{KZ}(\lambda)[\Psi^{\mathrm{symbol}}(\lambda,\hbar)] = \begin{pmatrix} -\alpha\lambda - \frac{H}{\hbar} + \frac{p}{\hbar(\lambda-q)} & -\frac{1}{\lambda-q} \\ [A_{KZ}]_{2,1}(\lambda,\hbar) & [A_{KZ}]_{2,2}(\lambda,\hbar) \end{pmatrix} \Psi(\lambda,\hbar)$$

for $\alpha = t_{1,3} + 2t_{2,3}$ and some unknown H.
Equivalently defining

$$\mathcal{L} := \mathcal{L}_{\mathbf{KZ}}(\lambda) + \mathbf{t}_{2,3}\lambda + \mathbf{t}_{2,2} = \hbar \mathbf{t}_{1,3}\mathcal{I}_{\infty^{(2)},1} + \hbar \mathbf{t}_{2,3}\mathcal{I}_{\infty^{(1)},1}$$

we have

$$\begin{aligned} \mathrm{ev}.\mathcal{L}[\Psi^{\mathrm{symbol}}(\lambda,\hbar)] &= \begin{pmatrix} P^{(1)}_{\infty,2}\lambda + t_{2,2} - \frac{H}{\hbar} + \frac{p}{\hbar(\lambda-q)} & -\frac{1}{\lambda-q} \\ A_{2,1}(\lambda,\hbar) & A_{2,2}(\lambda,\hbar) \end{pmatrix} \Psi(\lambda,\hbar) \\ &:= A(\lambda,\hbar)\Psi(\lambda,\hbar) \end{aligned}$$

Evolution equations

- Compatibility equations $\mathcal{L}[L(\lambda,\hbar)] = \hbar\partial_\lambda A(\lambda,\hbar) + [A(\lambda,\hbar), L(\lambda,\hbar)]$:

$$\begin{aligned}
\mathcal{L}[P^{(2)}_{\infty,4}] &= \mathcal{L}[P^{(2)}_{\infty,3}] = 0\\
\mathcal{L}[P^{(2)}_{\infty,2}] &= -2\hbar P^{(2)}_{\infty,4} + \hbar\left[P^{(1)}_{\infty,2}\right]^2\\
\mathcal{L}[P^{(2)}_{\infty,1}] &= -\hbar P^{(2)}_{\infty,3} + \hbar P^{(1)}_{\infty,1} P^{(1)}_{\infty,2}\\
\mathcal{L}[P^{(2)}_{\infty,0}] - \mathcal{L}[H] &= 2\hbar P^{(2)}_{\infty,4} q^2 + \hbar P^{(2)}_{\infty,3} q - P^{(1)}_{\infty,2} p + \hbar P^{(1)}_{\infty,0} P^{(1)}_{\infty,2}\\
H &= \frac{p^2}{\hbar^2} - P_1(q)\frac{p}{\hbar} + P_2(q) - \hbar P_1'(q) + \hbar(P^{(1)}_{\infty,2} - t_{2,3})q\\
\mathcal{L}[q] &= P_1(q) - 2\frac{p}{\hbar}\\
\mathcal{L}[p] &= -P_1'(q)p + \hbar P_2'(q) + \hbar^2 t_{2,3}
\end{aligned}$$

- Equivalent to

$$\mathcal{L}[t_{1,3}] = \mathcal{L}[t_{2,3}] = \mathcal{L}[t_{1,2}] = \mathcal{L}[t_{1,0}] = \mathcal{L}[t_{2,0}] = 0\,,\ \mathcal{L}[t_{1,1}] = \hbar t_{2,3}\,,\ \mathcal{L}[t_{2,1}] = \hbar t_{1,3}$$

- Equivalent to $\mathcal{L} = \hbar t_{2,3}\partial_{t_{1,1}} + \hbar t_{1,3}\partial_{t_{2,1}}$

Hamiltonian evolution

Hamiltonian evolution

"Time" ($\mathcal{L}$)-evolution is Hamiltonian $\Leftrightarrow$ (p, q) are Darboux coordinates

$$\mathcal{L}[q] = -\hbar \frac{\partial H_0}{\partial p} \,, \quad \mathcal{L}[p] = \hbar \frac{\partial H_0}{\partial q}$$

for Hamiltonian $H_0(p, q, \hbar)$:

$$H_0(p, q, \hbar) = \frac{p^2}{\hbar^2} - P_1(q)\frac{p}{\hbar} + P_2(q) - \hbar P_1'(q) + \hbar q(2P^{(1)}_{\infty,2} - t_{2,3})$$

giving $H = H_0(p, q, \hbar) + \hbar(t_{1,3} + t_{2,3})q$.

Connection with the Painlevé 2 equation

- q satisfies the evolution equation:

$$\begin{aligned}\mathcal{L}^2[q] &= 2(t_{1,3}-t_{2,3})^2 q^3 + 3(t_{1,3}-t_{2,3})(t_{1,2}-t_{2,2})q^2 \\ &+ \left((t_{1,2}-t_{2,2})^2 + 2(t_{1,3}-t_{2,3})(t_{1,1}-t_{2,1})\right) q \\ &+ (t_{1,2}-t_{2,2})(t_{1,1}-t_{2,1}) + (2t_{1,0}-\hbar)(t_{1,3}-t_{2,3})\end{aligned}$$

- Change of variables $(t_{1,1}, t_{2,1}) \leftrightarrow (\tau, \tilde{\tau})$ and affine rescaling:

$$\begin{aligned}\tau &= \frac{1}{t_{1,3}-t_{2,3}}(t_{2,1}-t_{1,1}), \quad \tilde{\tau} = \frac{1}{t_{1,3}-t_{2,3}}(t_{1,3}t_{1,1}-t_{2,3}t_{2,1}) \\ t &= \left(-2(t_{1,3}-t_{2,3})^2\right)^{\frac{1}{3}}\left(\tau + \frac{(t_{1,2}-t_{2,2})^2}{4(t_{1,3}-t_{2,3})^2}\right) \\ \tilde{q} &= \left(\frac{-(t_{1,3}-t_{2,3})}{2}\right)^{\frac{1}{3}}\left(q + \frac{t_{1,2}-t_{2,2}}{2(t_{1,3}-t_{2,3})}\right)\end{aligned}$$

Then $\tilde{q}(t,\hbar)$ satisfies the Painlevé 2 equation

$$\hbar^2 \partial_{t^2}^2 \tilde{q} = 2\tilde{q}^3 + t\tilde{q} - \left(t_{1,0} - \frac{\hbar}{2}\right)$$

Gauge without apparent singularities

- Gauge transformation to remove apparent singularity:

$$\check{\Psi}(\lambda,\hbar)=\begin{pmatrix}1 & 0\\ -\frac{p}{\hbar(\lambda-q)} & \frac{1}{\lambda-q}\end{pmatrix}\Psi(\lambda,\hbar):=J(\lambda,\hbar)\Psi(\lambda,\hbar)$$

- Provides another Lax pair (**Jimbo-Miwa type**) without apparent singularity:

$$\begin{aligned}\check{L}(\lambda,\hbar) &= \begin{pmatrix}\frac{p}{\hbar} & \lambda-q\\ -((\lambda+q)(t_{1,3}+t_{2,3})+t_{2,2}+t_{1,2})\frac{p}{\hbar}+Q_3(\lambda,\hbar) & -\frac{p}{\hbar}+P_1(\lambda)\end{pmatrix}\\ \check{A}(\lambda,\hbar) &= \begin{pmatrix}-(t_{1,3}+t_{2,3})\lambda-\frac{H}{\hbar}+t_{2,2} & -1\\ (t_{1,3}+t_{2,3})\frac{p}{\hbar}+Q_2(\lambda,\hbar) & (t_{1,3}+t_{2,3})q+t_{1,2}+2t_{2,2}-\frac{H}{\hbar}\end{pmatrix}\end{aligned}$$

where

$$\begin{aligned}Q_3(\lambda,\hbar) &= -P^{(2)}_{\infty,4}\lambda^3-(P^{(2)}_{\infty,4}q+P^{(2)}_{\infty,3})\lambda^2-(P^{(2)}_{\infty,4}q^2+P^{(2)}_{\infty,3}q+P^{(2)}_{\infty,2})\lambda\\ &\quad +P^{(2)}_{\infty,4}q^3+P^{(2)}_{\infty,3}q^2+P^{(2)}_{\infty,2}q+P^{(2)}_{\infty,1}+\hbar t_{1,3})\\ Q_2(\lambda,\hbar) &= P^{(2)}_{\infty,4}\lambda^2+2P^{(2)}_{\infty,4}q\lambda+P^{(2)}_{\infty,3}\lambda+(3P^{(2)}_{\infty,4}q^2+2P^{(2)}_{\infty,3}q+P^{(2)}_{\infty,2})\end{aligned}$$

Open questions and outlooks

Open questions and outlooks

- Non-perturbative quantities (wave function, Lax pairs, etc.) are **formal $\hbar$ trans-series** $\Rightarrow$ Can we obtain convergent solutions? Possible solution: works of Costin [9, 10, 11, 12] $\Rightarrow$ Write down the RHP satisfied by $\Psi(\lambda, \hbar)$. Make connections with (bi)orthogonal polynomials RHP in the hermitian matrix models case.
- Remove some of the admissibility conditions: simple ramification points, smooth ramification points.
- General connections with isomonodromic deformations? Require to define in general the tangent space $\partial_{t_{i,j}}$ and "admissible" deformations of curves. Check that operators $\mathcal{L}$ may always be written using spectral times derivatives. Prove that time evolutions are Hamiltonian. Issue solved for $d = 2$ in [32, 33].
- Study the change of Torelli marking $\Rightarrow$ Hitchin's equations for choice of polarization in geometric quantization.
- Consider classical spectral curves over $\mathbb{C}^*$ (or more complicated base curve) to study of Gromov–Witten invariants of toric Calabi–Yau three-folds by mirror symmetry.

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