Quantization of classical spectral curve and integrable systems

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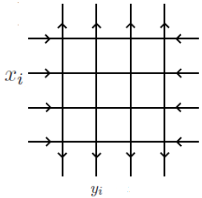
Six vertex model and reduction to Hermitian one matrix integrals

Six vertex models and random matrix integrals

- Take a DWBC 6V model [18] with $a = \sqrt{a_1 a_2} = q\rho - q^{-1}\rho^{-1}$, $b = \sqrt{b_1 b_2} = \rho - \rho^{-1}$, $c = \sqrt{c_1 c_2} = q - q^{-1}$:



- Korepin-Izergin determinant: $Z_N \propto \frac{\det(\phi(x_i, y_j))}{\Delta(X)\Delta(Y)}$ with $\phi(x, y)$ explicit. $\Delta(X)$ =Vandermonde determinant.
- Determinants of these types are **generalized** 2-**matrix models**:

$$\begin{aligned} Z_N &= \frac{1}{N!\Delta(X)\Delta(Y)} \int \cdots \int \prod_{i=1}^{N} d\mu(a_i, b_i) \det_{1\le i,j\le N}(e^{a_i x_j}) \det_{1\le i,j\le N}(e^{y_i b_j}) \ , \\ \phi(x, y) &= \int\int d\mu(a, b) e^{ax+by} \end{aligned}$$

Reduction to one matrix model

- If $\phi(x,y)$ only depends on $x-y$ the integral reduces to a generalized one-matrix integral:

$$Z_N \propto \int\cdots\int \prod_{i=1}^{N} d\mu(a_i) \det_{1\leq i,j\leq N}\left(e^{a_i x_j}\right) \det_{1\leq i,j\leq N}\left(e^{y_i a_j}\right),$$
$$\phi(z) = \int d\mu(a) e^{az}$$

- Homogeneous case $x_i - y_i = t$ for all $i \in [\![1,N]\!]$:

$$Z_N \propto \int\cdots\int \prod_{i=1}^{N} d\mu(a_i) \Delta(\mathbf{a})^2 e^{t\sum_{i=1}^{N} a_i},$$
$$\phi(z) = \int d\mu(a) e^{az}$$

- Hermitian one matrix integral with potential $e^{-tA-V(A)}$ and $e^{-V(z)}dz = d\mu(z)$.

Application to DWBC 6V model

- Partition function of the DWBC 6V model [1] with $a = \sin(\gamma - t)$, $b = \sin(\gamma + t)$, $c = \sin(2\gamma)$:

$$Z_N = \frac{(ab)^{N^2}}{\left(\prod_{k=1}^{N} k!\right)^2} \tau_N \ , \ \tau_N = \det \left(\frac{d^{i+k-2}\phi}{dt^{i+k-2}} \right)_{1 \leq i,k \leq N} \ , \ \phi = \frac{c}{ab} \ \text{Hankel det.}$$

- Hankel determinants $\Rightarrow$ **Hermitian matrix integrals** $\Rightarrow$ integrability:

$$\tau_N = \frac{\prod_{i=1}^{N} i!}{\pi^{\frac{N(N-1)}{2}}} \int_{\mathcal{H}_N} dM \, e^{\mathrm{Tr}(tM - V(M))} \ , \ m(\lambda) = e^{-V(\lambda)} = \frac{\sinh\left(\frac{\lambda(\pi - 2\gamma)}{2}\right)}{\sinh \frac{\lambda \pi}{2}}$$

- Eigenvalues distribution:

$$\tau_N \propto \int_{\mathbb{R}^N} d\lambda_1 \dots d\lambda_N \, \Delta(\boldsymbol{\lambda})^2 \, e^{\sum_{i=1}^{N} \mathrm{Tr}(t\lambda_i - V(\lambda_i))}$$

- Potential is not polynomial. Analysis of the limiting eigenvalue distributions (support, edges, etc.) is standard (saddle-point analysis).
- Many existing works [11, 17, 2]

Study of general Hermitian one matrix integrals

Hermitian random matrix models

- Hermitian random matrix integrals:

$$Z_N = \int_\Gamma \cdots \int_\Gamma d\lambda_1 \dots d\lambda_N \, \Delta(\boldsymbol{\lambda})^2 \, e^{-N\sum_{i=1}^{N} V(\lambda_i)}$$

- **Vandermonde-like interactions** (Coulomb gas):

$$\Delta(\boldsymbol{\lambda}) = \prod_{1\leq i<j\leq N} (\lambda_i - \lambda_j)$$

- **Potential** V (in general rational function) sufficiently confining.
- Contour $\Gamma \subset \mathbb{R}$ may have **hard edges**: $\Gamma = \bigcup_{i=1}^{n} [a_i, b_i]$.
- Several existing tools: Orthogonal polynomials and RHP, Eynard-Orantin topological recursion, integrable systems approach, Fredholm/Hankel determinants, etc.
- Specific local regimes lead to **universality** results.

Main questions arising in Hermitian random matrices

- Compute partition function Z_N and its large N expansion (under mild assumptions) [6]:

$$Z_N = \text{Prefactor}(N) \exp\left(\sum_{k=-2}^{\infty} F^{(k)} N^{-k}\right) \quad \text{(one-cut case)}$$

- Compute **correlation functions** (cumulants) and their large N expansions:

$$W_1(x) = \left\langle \frac{1}{N}\sum_{j=1}^{N} \frac{1}{x-\lambda_j} \right\rangle = \sum_{k=1}^{\infty} W_1^{(k)}(x) N^{1-2k} \quad \text{(one-cut case)}$$

$$W_n(x_1,\dots,x_n) = \left\langle \sum_{i_1,\dots,i_n=1}^{N} \frac{1}{x_1-\lambda_{i_1}} \cdots \frac{1}{x_n-\lambda_{i_n}} \right\rangle_c = \sum_{k=0}^{\infty} W_n^{(k)}(x_1,\dots,x_n) N^{-(2-n-2k)}$$

- Refined large N expansion exists (Theta functions) for several cuts case.

Orthogonal polynomials/RHP and Topological Recursion

- Orthogonal polynomials $(p_n)_{n\geq 1}$: p_n monic polynomial of degree n:

$$\int_\Gamma p_n(x)p_m(x)e^{-NV(x)} = \delta_{n,m}h_n \Rightarrow Z_N = N!\prod_{i=1}^{N} h_i$$

- Valid for any $N \in \mathbb{N}$. Efficient for computations at low N. Large N asymptotics is more difficult: solve RHP problem.
- Topological recursion (TR) approach: start with the large N limit: $W_1^{(0)}(x)$ ($\Leftrightarrow$ limiting density) and $W_2^{(0)}(x_1, x_2)$. Compute recursively all $\left(W_n^{(k)}\right)_{n\geq 0, k\geq 0}$. Recursion is on $2k+n$.
- Efficient for obtaining the large N asymptotics. Impossible to recover a finite N in practice. Looses the fact that N is an integer.
- Contains the full integrability structure (recover the full model starting only with its large N limit).

Getting the classical spectral curve

- Under mild assumptions [16], empirical eigenvalues distribution $d\nu_N(x;\boldsymbol{\lambda}) = \frac{1}{N}\sum_{j=1}^{N}\delta_N(x-\lambda_j)$ converges to an equilibrium density:

$$d\nu_N(x;\boldsymbol{\lambda}) \overset{N\to\infty}{\to} \rho_{\text{eq}}(x)dx$$

supported on a union of intervals.

- Stieltjes transform:

$$\omega(x) = \int \frac{\rho_{\text{eq}}(x')}{x-x'}dx'$$

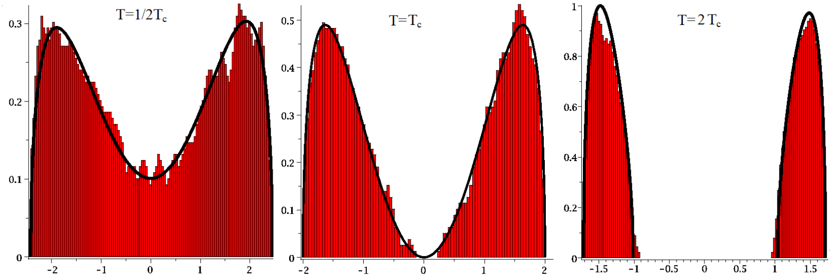
satisfies **an quadratic equation**

$$\omega(x)^2 - P_1(x)\omega(x) + P_2(x) = 0 \;\; (i)$$

with P_1, P_2 determined by potential V.

- Alternative derivation [8]: Define $W_1(x) = \left\langle \frac{1}{N}\sum_{j=1}^{N}\frac{1}{x-\lambda_j}\right\rangle$ and assume $W_1(x) = \sum_{h=1}^{\infty} W_1^{(h)}(x)N^{1-2h}$. $\omega(x) = W_1^{(0)}(x)$ satisfies (i).

Example



Limiting densities for

$$V(x, T, \epsilon) = \frac{1}{T}\left(\frac{x^4}{4} - \frac{4\cos(\pi\epsilon)x^3}{3} + \cos(2\pi\epsilon)x^2 + 8\cos(\pi\epsilon)x\right) \text{ for } \epsilon = \frac{1}{2}.$$

Phase transition at $T_c = 1 + 4\cos(\pi\epsilon)$.

Integrability at work using TR

Restriction to polynomial potentials

- We shall now restrict to $V'(\lambda) = \text{Pol}(\lambda)$.
- We authorize arbitrary number of hard edges (regular or irregular).
- Restrictions are made to have simpler formulas but general picture should hold for other cases.

Irregular times

- **Classical spectral curve** (**hyperelliptic Riemann surface** Σ of genus g): $\{(x(z), y(z)), z \in \Sigma\}$ $(y(z(x)) = W_1^{(0)}(x))$:

$$y^2 - P_1(x)y + P_2(x) = 0 \ , \ P_1 \ , \ P_2 \ \text{rational functions}$$

- y is singular at $\{\infty, X_1, \dots, X_n\}$. Assume (for simplicity) that poles are not ramified. Denote $x^{-1}(\{\infty\}) = \{\infty^{(1)}, \infty^{(2)}\}$ and $x^{-1}(\{X_i\}) = \{X_i^{(1)}, X_i^{(2)}\}$.

$$y(z) \overset{z\to\infty^{(i)}}{=} -\sum_{k=0}^{r_\infty -1} t_{\infty^{(i)},k} x(z)^{k-1} + O\left((x(z))^{-2}\right)$$

$$y(z) \overset{z\to X_s^{(i)}}{=} \sum_{k=0}^{r_s-1} t_{X_s^{(i)},k} (x(z) - X_s)^{-k-1} + O\left(1\right)$$

- $(t_{\infty^{(i)},k}, t_{X_s^{(i)},k})_{i,k,s}$ are "irregular times" [3, 4] in the study of **isomonodromic deformations of meromorphic connections**. "KP times" from isospectral perpective.

Irregular times 2

- Irregular times determine part of P_1 and P_2:

$$\begin{aligned} P_1(\lambda) &= \sum_{j=0}^{r_\infty-2} P^{(1)}_{\infty,j}\lambda^j + \sum_{s=1}^{n}\sum_{j=1}^{r_s} \frac{P^{(1)}_{X_s,j}}{(\lambda - X_s)^j} \\ P_2(\lambda) &= \sum_{j=0}^{2r_\infty-4} P^{(2)}_{\infty,j}\lambda^j + \sum_{s=1}^{n}\sum_{j=1}^{2r_s} \frac{P^{(2)}_{X_s,j}}{(\lambda - X_s)^j} \end{aligned}$$

- Only $g = r_\infty + \sum_{s=1}^{n} r_s - 3$ coefficients of P_2 remain unknown.

Interpretation of the g unknown coefficients

- **Potential V + hard edges + g filling fractions** $\Leftrightarrow$ Equilibrium measure $\Leftrightarrow$ Classical spectral curve
- Classical spectral curve $\Leftrightarrow$ Location of Poles + irregular times + g unknown coefficients
- g additional coefficients in one-to-one correspondence with **solutions** $(q_j)_{j=1}^{g}$ **of Hamiltonian systems** via isomonodromic deformations.
- In specific regimes:
    - Potential V and part of the hard edges do not play any role
    - Universal classical spectral curves
    - **Regimes are characterized by specific solutions of given Hamiltonian systems**
    - Universality.

Quantization and isomonodromic deformations

Quantization

1. Series of works [14, 13, 7, 15] in collaboration with N. Orantin, E. Garcia-Failde, M. Alameddine and B. Eynard from 2019 to 2022.
2. Apply topological recursion to the classical spectral curve $\Rightarrow$ $(\omega_{h,n}(z_1,\dots,z_n))_{h\geq 0, n\geq 0}$ multi-differentials on Σ.
3. Define 2 formal wave functions (mind regularizations) with $\hbar = N^{-1}$:
$$\begin{aligned}\psi_i(\lambda) &= \langle \det(\lambda I_2 - M)\rangle \\ &= \exp\left(\sum_{h,n\geq 0} \frac{\hbar^{2h+n-2}}{n!}\int_{\infty^{(1)}}^{z}\cdots\int_{\infty^{(1)}}^{z} \omega_{h,n}(z_1,\dots,z_n)dz_1\dots dz_n\right)\end{aligned}$$
4. Take "Fourier transform" (Theta functions: formal $\hbar$-transseries)
$$\Psi_i(z,\hbar;\boldsymbol{\epsilon},\boldsymbol{\rho}) := \sum_{\mathbf{n}\in\mathbb{Z}^g} e^{\frac{2\pi i}{\hbar}\sum_{j=1}^{g}\rho_j n_j}\,\psi_i(z,\hbar,\epsilon+\hbar\mathbf{n}).$$
5. (Ψ_1,Ψ_2) are formal $\hbar$-transseries solutions to an **ODE** ("**quantum curve**")
$$\left(\hbar^2\frac{\partial^2}{\partial\lambda^2} + b_1(\lambda,\hbar)\hbar\frac{\partial}{\partial\lambda} + b_2(\lambda,\hbar)\right)\Psi_i(\lambda,\hbar) = 0$$

Quantization 2

- Property of the quantum curve:

$$\left(\hbar^2 \frac{\partial^2}{\partial \lambda^2} + b_1(\lambda,\hbar)\hbar \frac{\partial}{\partial \lambda} + b_2(\lambda,\hbar)\right) \Psi_i(\lambda,\hbar) = 0$$

- Coefficients $b_1(\lambda,\hbar)$, $b_2(\lambda,\hbar)$ are **rational functions** of λ with same pole structure as initial classical spectral curve and g **apparent singularities**: $(q_1,\dots,q_g)$.
- Rewrite in companion matrix form $\Psi(\lambda,\hbar) = \begin{pmatrix} \Psi_1 & \Psi_2 \\ \hbar\partial_\lambda \Psi_1 & \hbar\partial_\lambda \Psi_2 \end{pmatrix}$

$$\hbar\partial_\lambda \Psi(\lambda,\hbar) = \begin{pmatrix} 0 & 1 \\ -b_2(\lambda,\hbar) & -b_1(\lambda,\hbar) \end{pmatrix} \Psi(\lambda,\hbar) \overset{\text{def}}{=} L(\lambda,\hbar)\Psi(\lambda,\hbar)$$

- Remove apparent singularities via gauge transformation.

Lax matrix

- Remove apparent singularities via gauge transformation:

$$\tilde{\Psi}(\lambda,\hbar) = J(\lambda,\hbar)\Psi(\lambda,\hbar) \text{ with } J(\lambda,\hbar) = \begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix}$$

- In this gauge:

$$\hbar\partial_\lambda \tilde{\Psi}(\lambda,\hbar) = \tilde{L}(\lambda,\hbar)\tilde{\Psi}(\lambda,\hbar)$$

  with $\tilde{L}(\lambda,\hbar)$ **rational in λ with poles only in $\{\infty, X_1, \ldots, X_n\}$**.
- No apparent singularities but matrices are no longer companion-like.
- Former gauge is more natural in geometry of integrable systems [9, 5].
- For $g = 0$ (i.e. one cut case), the Lax matrix is completely determined.

Isomonodromic deformations

- Study **general deformations** relatively to irregular times (except monodromies) and location of poles (tangent space):

$$\mathcal{L}_{\boldsymbol{\alpha}} = \hbar \sum_{i=1}^{2} \sum_{k=1}^{r_\infty -1} \alpha_{\infty^{(i)},k} \partial_{t_{\infty^{(i)},k}} + \hbar \sum_{i=1}^{2} \sum_{s=1}^{n} \sum_{k=1}^{r_s -1} \alpha_{X_s^{(i)},k} \partial_{t_{X_s^{(i)},k}} + \hbar \sum_{s=1}^{n} \alpha_{X_s} \partial_{X_s}$$

- Wave matrix $\Psi(\lambda, \hbar)$ satisfies

$$\mathcal{L}_{\boldsymbol{\alpha}}[\Psi(\lambda,\hbar)] = A_{\boldsymbol{\alpha}}(\lambda,\hbar)\Psi(\lambda,\hbar)$$

  with $A_{\boldsymbol{\alpha}}(\lambda,\hbar)$ **rational in λ with same pole structure as $L(\lambda,\hbar)$** $\Rightarrow$ **Lax pair**.

- Compatibility of the Lax system is

$$\mathcal{L}_{\boldsymbol{\alpha}}[L(\lambda)] = \hbar \partial_\lambda A_{\boldsymbol{\alpha}}(\lambda) + [A_{\boldsymbol{\alpha}}(\lambda), L(\lambda)]$$

Isomonodromic deformations 2

- Compatibility of the Lax system is

$$\mathcal{L}_{\boldsymbol{\alpha}}[L(\lambda)] = \hbar\partial_\lambda A_{\boldsymbol{\alpha}}(\lambda) + [A_{\boldsymbol{\alpha}}(\lambda), L(\lambda)]$$

- Provides complete expression of the matrices $L(\lambda, \hbar), A_{\boldsymbol{\alpha}}(\lambda, \hbar)$ in terms of irregular times and $(q_1, \dots, q_g)$ and their dual symplectic coordinates $(p_1, \dots, p_g)$.
- **Provides general and explicit evolution equations**:

$$(\mathcal{L}[q_j], \mathcal{L}[p_j])_{j=1}^{g}$$

- **Evolutions are Hamiltonians**. Expression of the general Hamiltonian $H_{\boldsymbol{\alpha}}(q_1, \dots, q_g, p_1, \dots, p_g)$ is explicit.

Reduction to isomonodromic deformations

- Space of deformations $\left(t_{\infty^{(i)},k}, t_{b_s^{(i)},k}, X_s\right)_{i,k,s}$ much bigger than $g =$ dimension of the expected symplectic space.
- Reduce the tangent space of deformations to only g isomonodromic times $(\tau_1, \dots, \tau_g)$ and some trivial times $(T_k)_k$.
- Trivial times must satisfy

$$\partial_{T_k} \check{q}_j = 0 = \partial_{T_k} \check{p}_j$$

  where $\check{q}_j = T_2 q_j + T_1 \ \ , \check{p}_j = T_2^{-1} \left(p_j - \frac{1}{2} P_1(q_j)\right)$ are shifted coordinates (T_1 and T_2 are explicit)
- **Reduction is explicit**.
- Hamiltonian evolutions of $(\check{q}_j, \check{p}_j)_{j=1}^{g}$ are independent of trivial times $\Rightarrow$ Canonical choice of trivial times (in particular $T_1 = 0$, $T_2 = 1$)
- For $g = 1$, one recovers all **Painlevé Lax pairs/equations**.

Summary and outlooks

Summary

- Construction from classical spectral curve (large N limit of Hermitian random matrix models) to formal wave functions via topological recursion.
- We obtain **rational Lax pairs** with **explicit isomonodromic Hamiltonian evolutions** and complete reduction to isomonodromic deformations.
- Construction is valid for **any arbitrary number of poles**. **Poles may be regular** (Fuchsian case) or **irregular poles of arbitrary degrees**.
- Similar construction is expected to hold for any classical spectral curve (not only hyperelliptic), i.e. two matrix models. Construction of the quantum curve is already done in [7].

Outlooks

- Wave functions are formal WKB series ($g = 0$) or formal transseries in $\hbar = N^{-1}$ ($g \geq 1$). **Borel resumation** is expected to provide analytic wave functions. Can we describe the analytic structure and Stokes phenomenon of these wave matrices?
- Can we use the Hamiltonian evolutions for Borel resumation?
- Can we relate the choice of solutions $(q_1, \ldots, q_g)$ to the choice of filling fractions $(\epsilon_1, \ldots, \epsilon_g)$?
- Make the connection with 2×2 matrices arising from orthogonal polynomials and RHP method.
- In the universal regimes, can we characterize (and prove existence and uniqueness) the specific solution of the Hamiltonian systems that arises in random matrices (For example: Hastings-McLeod solution of the Painlevé 2 equation [10], specific solution of Painlevé 5 for the sine kernel [12])?
- Can we deal with **non-polynomial potentials** (6V model)?

References I

[1] P. Bleher and L. Karl. Riemann-hilbert approach to the six-vertex model, 2012.

[2] P. Bleher and L. Karl. *Random Matrices and the Six-Vertex Model*. American Mathematical Society, 1st edition, 2013.

[3] P. Boalch. Symplectic Manifolds and Isomonodromic Deformations. *Adv. Math.*, 163(2):137–205, 2001.

[4] P. Boalch. Simply-laced isomonodromy systems. *Publ. Math. IHÉS*, 116:1–68, 2012.

[5] P. Boalch, J. Douçot, and G. Rembado. Twisted local wild mapping class groups: configuration spaces, fission trees and complex braids, 2022.

[6] G. Borot, A. Guionnet, and K. Kozlowski. Large-N asymptotic expansion for mean field models with Coulomb gas interaction. *Int. Math. Res. Not.*, 2015(20), 2015.

[7] B. Eynard, E. Garcia-Failde, O. Marchal, and N. Orantin. Quantization of classical spectral curves via topological recursion, 2021. math-ph/2106.04339.

References II

[8] B. Eynard and N. Orantin. Invariants of algebraic curves and topological expansion. *Commun. Number Theory and Physics*, 1(2), 2007.

[9] J. Harnad. Hamiltonian Dynamics, Classical R-matrices and Isomonodromic Deformations. *Lect. Notes Phys*, 502:137–205, 2007.

[10] S. Hastings and J. McLeod. A boundary value problem associated with the second Painlevé transcendent and the Korteweg-de Vries equation. *Arch. Ration. Mech. Anal.*, 73:31–51, 1980.

[11] A. Izerpin, D. Coker, and V. Korepin. Determinant formula for the six-vertex model. *J. Phys. A*, 25:4315–4334, 1992.

[12] M. Jimbo, T. Miwa, Y. Môri, and M. Sato. Density matrix of an impenetrable Bose gas and the fifth Painlevé transcendent. *Physica D*, 1(1):80–158, 1980.

[13] O. Marchal and N. Orantin. Isomonodromic deformations of a rational differential system and reconstruction with the topological recursion: the $\mathfrak{sl}_2$ case. *J. Math. Phys.*, 61(6), 2020.

[14] O. Marchal and N. Orantin. Quantization of hyper-elliptic curves from isomonodromic systems and topological recursion. *J. Geom. Phys.*, 177, 2022.

References III

[15] O. Marchal, N. Orantin, and M. Alameddine. Hamiltonian structure of isomonodromic deformations of gl_2 meromorphic connections. To appear.

[16] M. L. Mehta. *Random Matrices*. Elsevier Science, 3rd edition, 2004.

[17] P. Zinn-Justin. The six-vertex model on random lattices. *EPL*, 50(1):15–21, 2000.

[18] P. Zinn-Justin. Six-vertex, loop and tiling models: integrability and combinatorics, 2009.