A Lonely Runner Problem, Asymptotics of Toeplitz Determinants and Topological Recursion

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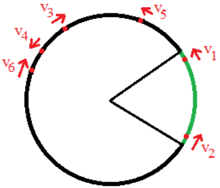
Unitary matrices

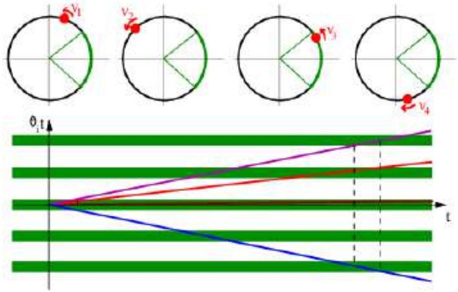
- **Random** (sampled uniformly according to Haar measure) **unitary matrix** U_N of size N.
- Eigenvalues: $(u_1, \ldots, u_N) = \left(e^{i\theta_1}, \ldots, e^{i\theta_N}\right)$, $(\theta_1, \ldots, \theta_N) \in [-\pi, \pi]^N$
- Define t^{th} **powers of eigenvalues**: $\left(e^{it\theta_1}, \ldots, e^{it\theta_N}\right)$ for $t \geq 0$.
- Questions: Let $\epsilon > 0$
  1. **Compute the probability $P_{N,\epsilon}(t)$ that at time $t > 0$, all eigenvalues $\left(e^{it\theta_1}, \ldots, e^{it\theta_N}\right)$ are located in $\{e^{i\theta}, \theta \in [-\pi\epsilon, \pi\epsilon]\}$.** t is called a **Strong Return Time** (SRT).
  2. Define $T_{N,\epsilon}$ the **first strong return time**:

$$T_{N,\epsilon} = \underset{t>0}{\text{Min}}\{t > 0 \text{ is SRT and } \not\exists t_0 < t \ / \ t_0 \text{ not STR}\}$$

Compute $\mathbb{E}(T_{N,\epsilon})$ and law of $T_{N,\epsilon}$



Lonely runner type problem



- **Particles running along the unit circle** $\Rightarrow$ Periodicity issues
- Velocities (θ_i= initial positions at $t = 1$) are NOT independent
- Times studied are "regroup times" around a specific point ($\theta = 0$) and not uniform spreading
- Real origin of the problem in quantum measurements theory (Poincaré reccurence time)

Measure on eigenvalues

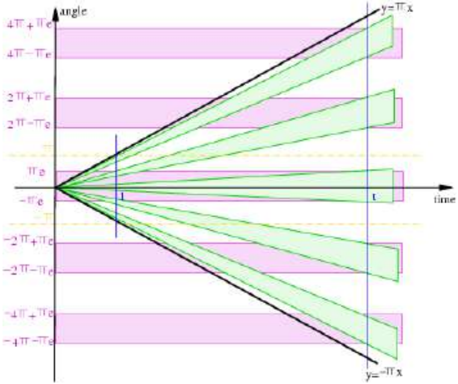
- **Induced Haar measure for eigenvalues**:

$$\begin{aligned} Z_N &= \int_{[-\pi,\pi]^N} d\theta_1 \dots d\theta_N \left(\prod_{i<j}^{N} \left| e^{i\theta_i} - e^{i\theta_j} \right|^2 \right) \\ &= (-1)^{\frac{N(N+1)}{2}} i^N \int_{\mathcal{C}^N} du_1 \dots du_N \left(\prod_{i<j}^{N} |u_i - u_j|^2 \right) e^{-N \sum_{k=1}^{N} \ln u_k} \\ &= (2\pi)^N N! \end{aligned}$$

- Z_N is a **Matrix Integral** with interactions $|\Delta(u_1, \dots, u_N)|^2$ with potential $V(x) = \ln x$. Compact closed contour $\mathcal{C}$.
- Integral over a union of intervals $I(t)$:

$$P_{N,\epsilon}(t) = \frac{1}{Z_N} \int_{I(t)^N} d\theta_1 \dots d\theta_N \left(\prod_{i<j}^{N} \left| e^{i\theta_i} - e^{i\theta_j} \right|^2 \right) \stackrel{\text{def}}{=} \frac{Z_{N,\epsilon}(t)}{Z_N}$$

Evolution of angles



$$\forall t \in [2k+\epsilon, 2(k+1)-\epsilon] : I(t) = \bigcup_{j=-k}^{k} \left[\frac{2\pi j - \pi\epsilon}{t}, \frac{2\pi j + \pi\epsilon}{t}\right]$$

$$\forall t \in [2k-\epsilon, 2k+\epsilon] : I(t) = \left(\bigcup_{j=-k+1}^{k-1} \left[\frac{2\pi j - \pi\epsilon}{t}, \frac{2\pi j + \pi\epsilon}{t}\right]\right) \cup \left[\frac{2\pi k - \pi\epsilon}{t}, \pi\right] \cup \left[-\pi, -\frac{2\pi k - \pi\epsilon}{t}\right]$$

Toeplitz determinants

- Toeplitz integrals:

$$\frac{1}{(2\pi)^N N!}\int_{[-\pi,\pi]^N}\left(\prod_{i<j}^{N}|e^{i\theta_i}-e^{i\theta_j}|^2\right)\left(\prod_{i=1}^{N} f(e^{i\theta_i})d\theta_i\right) = \det\left(T_{i,j}(f)=t_{i-j}\right)_{1\leq i,j\leq N}$$

with Fourier coefficients: $t_k = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})e^{-ik\theta}d\theta$

- Known results when **f is continuous** (Szegö) or **isolated jumps** (Fisher-Hartwig singularities): $\frac{1}{N}\ln\det T_N$ converges
- Known results (Widom) if f **supported on a single arc interval** $[\alpha, 2\pi-\alpha]$ ($\frac{1}{N^2}\ln\det T_N$ converges)
- Reformulation (efficient for **numeric finite N computations**):

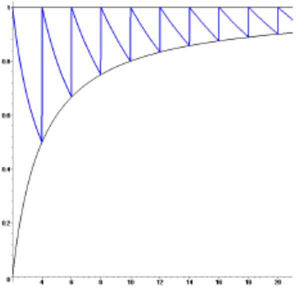
$$P_N\left(t\in[2R+\epsilon, 2(R+1)-\epsilon]\right) = \det\left[\frac{\sin\frac{(j-i)(2R+1)\pi}{t}\sin\frac{(j-i)\pi\epsilon}{t}}{\pi(j-i)\sin\frac{(j-i)\pi}{t}}\right]_{1\leq i,j\leq N}$$

$$P_N\left(t\in[2R-\epsilon, 2R+\epsilon]\right) = \det\left[\delta_{j-i=0}-\frac{\sin\frac{2(j-i)\pi R}{t}\sin\frac{(1-\epsilon)(j-i)\pi}{t}}{\pi(j-i)\sin\frac{(j-i)\pi}{t}}\right]_{1\leq i,j\leq N}$$

Trivial case when $t > N$

- Simplification for $t > N$ of the Toeplitz determinants:

$$\forall\, t > N \;:\; P_N(t) \underset{N\to\infty}{\sim} \epsilon^N \left(\frac{2}{t}\left\lfloor\frac{t}{2}\right\rfloor\right)^N = e^{N\epsilon \ln\left(\frac{2}{t}\lfloor\frac{t}{2}\rfloor\right)}$$



- New methods required for $N > t \Rightarrow$ **Matrix models techniques**

Expansion at large N

- - Eigenvalues condensate to a **absolutely continuous measure** $\rho_t(x)dx$ **on the unit circle when** $N \to \infty$
  - Support generically is $I(t) \Rightarrow$ Support is a union of g_t segments
  - **Stieljes transform of** $\rho_t(x)$ gives the "**spectral curve**"
- General theorem (Borot-Guionnet-Kozlowski):

$$Z_N = N^{N+\frac{1}{4}(g+1)} exp \left(\sum_{k=-1}^{\infty} N^{-2k} F_{\epsilon^\star}^{[2k]} \right)$$

$$\left\{ \sum_{m\geq 0} \sum_{\substack{l_1,\dots,l_m\geq 1 \\ k_1,\dots,k_m\geq -1 \\ \sum_{i=1}^{m} l_i+2k_i>0}} \frac{N^{-\sum_{i=1}^{m} l_i+2k_i}}{m!} \left(\bigotimes_{i=1}^{m} \frac{F_{\epsilon^\star}^{[2k_i],(l_i)}}{l_i!} \right) \cdot \nabla_\nu^{\otimes \left(\sum_{i=1}^{m} l_i\right)} \right\} \Theta_{-N\epsilon^\star} \left(\mathbf{0} \middle| F_{\epsilon^\star}^{[-2],(2)} \right)$$

- $g+1$ dimensional vector $\epsilon^\star$ is the **vector of optimal filling fractions** to spread over the various intervals of $\rho_t(x)$
- First orders:

$$\ln Z_N = N^2 F_{\epsilon^\star}^{[-2]} + N \ln N + \frac{1}{4}(g+1) \ln N + F_{\epsilon^\star}^{[0]} + \ln \left(\Theta_{-N\epsilon^\star} \left(\mathbf{0} \middle| F_{\epsilon^\star}^{[-2],(2)} \right) \right) + O\left(\frac{1}{N}\right)$$

Energy functional analysis

- Previous theorem is only valid under **certain restrictions**:
  1. **Decomposition of the interaction**:

$$\prod_{i<j}^{N} |e^{i\theta_i} - e^{i\theta_j}|^2 = \left(\prod_{i<j}^{N} |\theta_i - \theta_j|^2 \right) e^{\frac{1}{2}\sum_{i,j=1}^{N} T(\theta_i,\theta_j)}$$

     with $T(x_1, x_2)$ bounded on $[-\pi, \pi]^2$ and holomorphic on a neighborhood of $[-\pi, \pi]^2$: **OK**
  2. **Segments** of $I(t)$ are **not restricted to a single point** $\Rightarrow$ **Apart isolated times $\{t_k = 2k - \epsilon, k \in \mathbb{N}^*\}$: OK**
  3. **Minimum of the energy functional is unique**: **OK** (Fourier analysis)
  4. $\rho_t(x)$ **is non-critical** $\Leftrightarrow$ behaves like $\left(\sqrt{x - a_i}\right)^{\pm 1}$ at endpoints and strictly positive inside each intervals. **OK only for $t < 2 - \epsilon$ (1-cut case) and $t \in \mathbb{N}$ (additional discrete rotation symmetry)**
- Numeric simulations indicate non-criticality at all times
- Non-criticality often a difficult problem when no symmetry

One cut case: $t \leq 2 - \epsilon$

- $t < \epsilon$: all eigenvalues are inside $[-\pi\epsilon, \pi\epsilon]$: $P_{N,\epsilon}(t) = 1$
- $t < 2 - \epsilon$: $I(t) = [-t\pi, t\pi] \Rightarrow$ Special case of $I_{\theta_0,\theta_1} = [\theta_0\pi, \theta_1\pi]$
  1. **Loop equations** technique for matrix integrals $\Rightarrow$ **Spectral curve**:

$$y(x) = \frac{x-\alpha}{2x\sqrt{(x-a)(x-b)}}, a = e^{i\theta_0}, b = e^{i\theta_1}, \alpha = -e^{i\frac{\theta_0+\theta_1}{2}}$$

  2. Singular points $x \in \{0, \alpha\}$ outside I_{θ_0,θ_1} $\Rightarrow$ Non-criticality
  3. **Expansion of $Z_{N,\epsilon}(t)$ reduces to topological part**. **Symplectic invariants $F^{[g]}$** computed by **Topological Recursion**:

$$-F^{[-2]} = -\ln\left(\sin\frac{|\theta_1-\theta_0|}{4}\right)$$

$$-F^{[0]} = -\frac{1}{24}\ln 2 + \frac{1}{24}\ln\left(\tan\frac{|\theta_1-\theta_0|}{4}\right) - \frac{1}{8}\ln\left(\sin\frac{|\theta_1-\theta_0|}{2}\right)$$

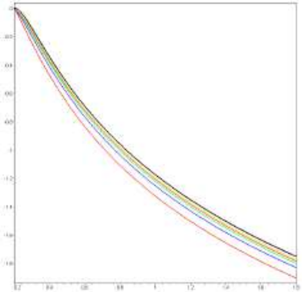
$$-F^{[2]} = \frac{3\cos\left(\frac{\theta_1-\theta_0}{2}\right)-1}{128\cos^2\left(\frac{\theta_1-\theta_0}{4}\right)}$$

Results for $t < 2 - \epsilon$

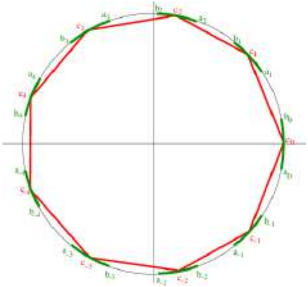
- Final result: $\forall \epsilon < t < 2 - \epsilon$:

$$\frac{1}{N^2}\ln P_{N,\epsilon}(t) + \frac{1}{N^2}\ln((2\pi)^N N!) \underset{N\to\infty}{=} \ln\left(\sin\frac{\pi\epsilon}{2t}\right) + \frac{\ln N}{N} + \frac{1}{4}\frac{\ln N}{N^2}$$
$$+ \frac{1}{24N^2}\ln\left(\frac{2\sin^3\frac{\pi\epsilon}{t}}{\tan\frac{\pi\epsilon}{2t}}\right) + \frac{1}{64N^4}\frac{1-3\cos\left(\frac{\pi\epsilon}{t}\right)}{1+\cos(\frac{\pi\epsilon}{t})} + O\left(\frac{1}{N^6}\right)$$

- **Improvement of Widom's result** (blue)
- Topological recursion can compute all next orders



Analysis at integer times



- Integer times $\Rightarrow$ **Additional symmetry** $\Rightarrow$ **Exact computation** of the **spectral curve** and **optimal filling fractions**
- Spectral curves at $t = 2k+1$ or $t = 2k$ ($k \in \mathbb{N}^*$):

$$y_{2k+1}(x) = \frac{\left(x^{2k+1}+1\right)}{2x\sqrt{\left(x^{2k+1}-e^{-i\pi\epsilon}\right)\left(x^{2k+1}-e^{i\pi\epsilon}\right)}}$$

$$y_{2k}(x) = \frac{\left(x^{2k}+1\right)}{2x\sqrt{\left(x^{2k}-e^{-i\pi\epsilon}\right)\left(x^{2k}-e^{i\pi\epsilon}\right)}}$$

Results at integer times

- Zeros of numerators outside $I(t) \Rightarrow$ Non-criticality
- **Symplectic transformation**
  $(X(z), Y(z)) = \left(x^{2k+1}(z), \frac{y(z)}{(2k+1)x^{2k}(z)}\right)$ gives for $t = 2k+1$:

$$\begin{cases} X(z) & = & \cos \pi\epsilon + \frac{1}{2} \sin \pi\epsilon \left(z - \frac{1}{z}\right) \\ Y(z) & = & \frac{1+X(z)}{(2k+1)X(z)\left(z+\frac{1}{z}\right) \sin \pi\epsilon} \end{cases}$$

- Preserves symplectic invariants $F^{[g]}$. $(X(z), Y(z))$ is a **genus** 0 **curve** $\Rightarrow$ Computation of Topological Recursion is possible
- **Symmetry** $\Rightarrow \epsilon^\star = \left(\frac{1}{2k+1}, \dots, \frac{1}{2k+1}\right)$
- Similar expressions for $t = 2k$
- Finally, $\underline{\forall\, t \in \mathbb{N}^*}$:

$$\frac{1}{N^2} \ln P_{N,\epsilon}(t) + \frac{1}{N^2} \ln((2\pi)^N N!) = \frac{1}{t} \ln\left(\sin \frac{\pi\epsilon}{2}\right) + \frac{\ln N}{N} + \frac{t}{4}\frac{\ln N}{N^2} - \frac{t}{24N^2}\left(2\ln t + \ln\left(4\tan\frac{\pi\epsilon}{2}\right)\right) + O\left(\frac{1}{N^2}\right)$$

Non Integer times

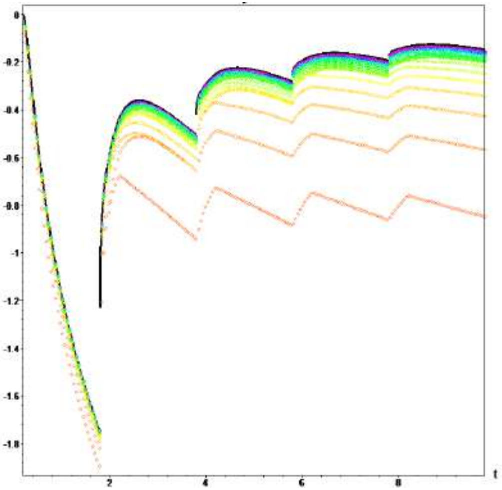
- Determining the spectral curve exactly remains open (Polynomial numerator of $y(x)$?)
- Non criticality condition remains open
- Determining algebraically the filling fractions $\epsilon^\star$ is known to be very challenging
- **Average Block Interaction Approximation** (ABIA): Approximate interactions for eigenvalues in different segments by mean interaction (i.e. concentration of eigenvalues in the center of the segment)
- Define $c_k(t)$ center of each segment $[a_k(t), b_k(t)]$ $(1 \leq k \leq g(t))$:

$$\frac{1}{N^2} \ln P_{N,\epsilon}(t) \approx 2\epsilon_k\epsilon_{k'} \sum_{k<j}^{g(t)} \ln |c_k(t) - c_j(t)| - \sum_{k=1}^{g(t)} F^{[-2]}(a_k(t), b_k(t), \epsilon_k) + O\left(\frac{1}{N}\right)$$

- Optimization relatively to ϵ $\Rightarrow$ quadratic form computations $\Rightarrow$ invert an explicit $g(t) \times g(t)$ matrix
- Integer times $\Rightarrow$ ϵ trivial $\Rightarrow$ Explicit computations:

$$P_{N,\epsilon}^{\text{ABIA}}(t) = \frac{1}{t} \ln \left(t \sin \frac{\pi\epsilon}{2t} \right) \quad \textit{instead of} \quad P_{N,\epsilon}(t) = \frac{1}{t} \ln \left(\sin \frac{\pi\epsilon}{2} \right)$$

Summary



Plot of $t \mapsto \frac{1}{N^2} \ln P_{N,\epsilon=\frac{1}{5}}(t)$. *Exact computations for* $2 \leq N \leq 35$ *in colored points. Black curve is ABIA*

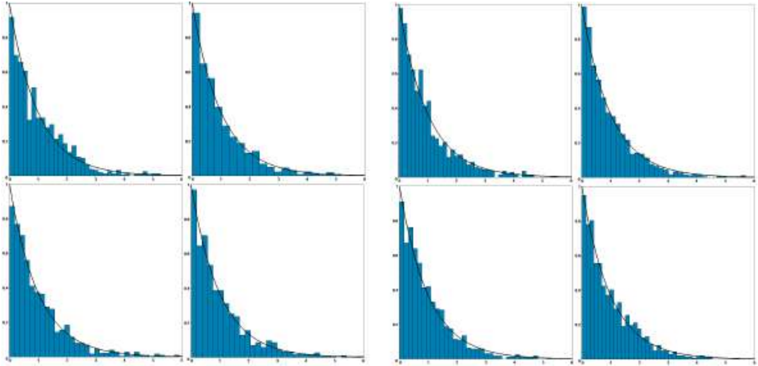
First Return Time

- For given θ_i's only $t_{i,k} = \frac{2\pi k}{|\theta_i|} - \frac{\epsilon}{2}\text{Sign}(\theta_i)$ with $1 \leq i \leq N$ and $k > 0$ are possible First Return Times $\Rightarrow$ Discrete problem
- $t_{i,k}$ are NOT independent $\Rightarrow$ very hard problem (Hitting time type problem)
- Assuming that $T_{N,\epsilon} = t_{i,k}$ does not provide a **tractable domain of integration** I (we need to rule out the lower $t_{j,l}$'s as first return times) $\Rightarrow$ **Spectral curve of very high genus**
- Topological Recursion should still apply as soon as the spectral curve is known
- The case of i.i.d. θ_i's corresponds to a number theory problem. Take $(X_i)_{1\leq i\leq N}$ i.i.d. uniform variables on $\left[-\frac{1}{2}, \frac{1}{2}\right]$
  Look at the first time $S_{N,\epsilon}$ where all tX_i's have a distance to their nearest integer less than $\frac{\epsilon}{2}$. Known as simultaneous Diophantine approximation type problem.

Conjecture

Conjecture

$$\frac{NT_{N,\epsilon}}{4\epsilon^{-(N-1)}} \xrightarrow[N\to\infty]{Law} \mathcal{E}xp(1) \text{ and } \frac{NS_{N,\epsilon}}{4\epsilon^{-(N-1)}} \xrightarrow[N\to\infty]{Law} \mathcal{E}xp(1)$$



Histograms of $\frac{NT_{N,\epsilon}}{4\epsilon^{-(N-1)}}$ *(left) and* $\frac{NS_{N,\epsilon}}{4\epsilon^{-(N-1)}}$ *(right) for* $N = 6$ *and* $\epsilon \in \{0.15, 0.2, 0.25, 0.3\}$ *(*10^3 *independent samples). Empirical estimation of* λ *decreases from* 1.021 *to* 1.002 *for* $T_{N,\epsilon}$ *and increases from* 0.96 *to* 0.91 *for* $S_{N,\epsilon}$

Conclusion

- Application of the **Topological Recursion** in probability for **unitary random matrices**
- **Toeplitz determinants** with symbols vanishing on several intervals rewritten as matrix integrals
- Computation of the spectral curve of the matrix integral
- Computation of the symplectic invariants by Topological Recursion $\Rightarrow$ Asymptotics of the Toeplitz determinant at large N *Rightarrow* Improvement of Widom's result.
- Method limited by the explicit computation of the spectral curve (limiting eigenvalues density and filling fractions)
- Explicit computations of the spectral curve when only **one cut** or when **additional symmetries**
- Good approximation (ABIA) when no symmetry to fall back into the one cut case
- Conjecture for the harder problem of first return time

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