Minimal k-partition for the p-norm of the eigenvalues

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Calculus of variations, optimal transportation, and geometric measure theory: from theory to applications

Lyon

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Notation

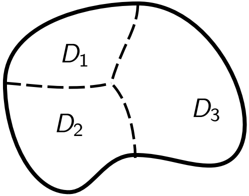
- $\Omega \subset \mathbf{R}^2$: bounded and connected domain

- $\lambda_1(D) < \lambda_2(D) \leqslant \cdots$ eigenvalues of the Dirichlet-Laplacian on D

- $\mathcal{D} = (D_i)_{i=1,\dots,k}$: k-partition of Ω

(i.e. D_i open, $D_i \cap D_j = \emptyset$, and $D_i \subset \Omega$)

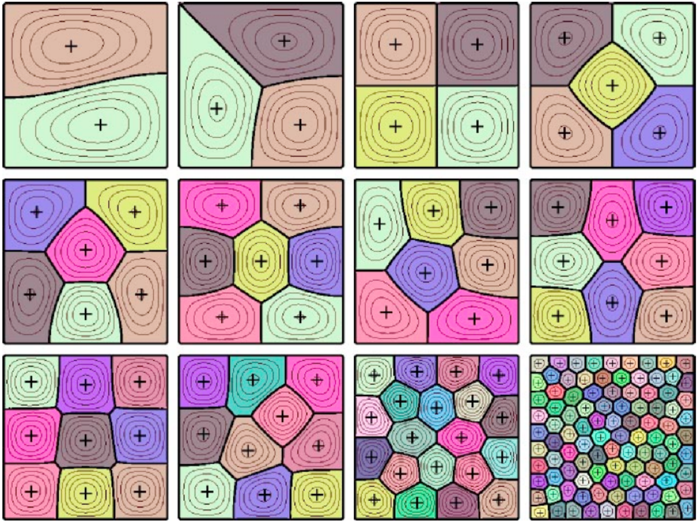
strong if $\mathrm{Int}\overline{D_i} \setminus \partial\Omega = D_i$ and $(\overline{\cup D_i}) \setminus \partial\Omega = \Omega$



- $\mathfrak{O}_k(\Omega) = \{$strong k-partitions of $\Omega\}$

k-partitions

Examples



p-minimal k-partition

Definitions

- p-energy

$\mathcal{D} = (D_i)_{i=1,\dots,k}$: k-partition of Ω

$1 \le p < +\infty$	$p = +\infty$
$\Lambda_{k,p}(\mathcal{D}) = \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_1(D_i)^p \right)^{1/p}$	$\Lambda_{k,\infty}(\mathcal{D}) = \max_{1 \leqslant i \leqslant k} \lambda_1(D_i)$

- Optimization problem: let $1 \le p \le \infty$,

$$\mathfrak{L}_{k,p}(\Omega) = \inf_{\mathcal{D} \in \mathfrak{O}_k(\Omega)} \Lambda_{k,p}(\mathcal{D})$$

- Comparison $\forall k \ge 2, \quad \forall 1 \le p \le q < \infty$

$$\frac{1}{k^{1/p}} \Lambda_{k,\infty}(\mathcal{D}) \le \Lambda_{k,p}(\mathcal{D}) \le \Lambda_{k,q}(\mathcal{D}) \le \Lambda_{k,\infty}(\mathcal{D})$$

$$\frac{1}{k^{1/p}} \mathfrak{L}_{k,\infty}(\Omega) \le \mathfrak{L}_{k,p}(\Omega) \le \mathfrak{L}_{k,q}(\Omega) \le \mathfrak{L}_{k,\infty}(\Omega)$$

- $\mathcal{D}^*$ is called a p-minimal k-partition if $\Lambda_{k,p}(\mathcal{D}^*) = \mathfrak{L}_{k,p}(\Omega)$

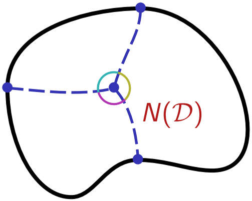
p-minimal k-partition

Existence of minimal partition

Theorem

For any $k \geq 2$ and $p \in [1, +\infty]$,
there exists a regular strong p-minimal k-partition

[*Bucur–Buttazzo-Henrot, Caffarelli–Lin, Conti–Terracini–Verzini, Helffer–Hoffmann-Ostenhof–Terracini*]



$$N(\mathcal{D}) = \overline{\cup(\partial D_i \cap \Omega)}$$

Regular : $N(\mathcal{D})$ is smooth curve except at finitely many points and

- $N(\mathcal{D}) \cap \partial\Omega$ is finite (boundary singular points)
- $N(\mathcal{D})$ satisfies the Equal Angle Property

Nodal partition

Let u be an eigenfunction of $-\Delta$ on Ω

- The nodal domains of u are the connected components of

$$\Omega \setminus N(u) \qquad \text{with} \qquad N(u) = \overline{\{x \in \Omega |\, u(x) = 0\}}$$

- nodal partition $=$ {nodal domains}

Regularity

$N(u)$ is a $\mathcal{C}^\infty$ curve except on some critical points $\{x\}$
If $x \in \Omega$, $N(u)$ is locally the union of an **even** number of half-curves ending at x with equal angle
If $x \in \partial\Omega$, $N(u)$ is locally the union of half-curves ending at x with equal angle

Theorem

Any eigenfunction u associated with λ_k has at most k nodal domains

[*Courant*]

u is said Courant-sharp if it has exactly k nodal domains
For $k \geq 1$, $L_k(\Omega)$ denotes the smallest eigenvalue (if any) for which there exists an eigenfunction with k nodal domains
We set $L_k(\Omega) = +\infty$ if there is no eigenfunction with k nodal domains

$$\lambda_k(\Omega) \leq L_k(\Omega)$$

Properties

Equipartition

Proposition

- *If* $\mathcal{D}^* = (D_i)_{1\le i\le k}$ *is a* ∞*-minimal* k*-partition,*
  then $\mathcal{D}^*$ *is an equipartition*

$$\lambda_1(D_i) = \lambda_1(D_j)\,, \qquad \text{for any} \quad 1 \le i,j \le k$$

- *Let* $p \ge 1$ *and* $\mathcal{D}^*$ *a* p*-minimal* k*-partition*
  If $\mathcal{D}^*$ *is an equipartition, then*

$$\mathfrak{L}_{k,q}(\Omega) = \mathfrak{L}_{k,p}(\Omega), \qquad \text{for any} \quad q \ge p$$

We set

$$p_\infty(\Omega,k) = \inf\{p \ge 1, \mathfrak{L}_{k,p}(\Omega) = \mathfrak{L}_{k,\infty}(\Omega)\}$$

2-partition

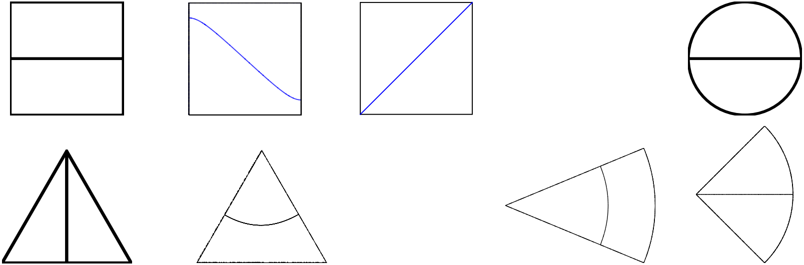
$p = +\infty$

Proposition

$$\mathfrak{L}_{2,\infty}(\Omega) = \lambda_2(\Omega) = L_2(\Omega)$$

The nodal partition of any eigenfunction associated with $\lambda_2(\Omega)$ gives a ∞-minimal 2-partition

Examples



2-partition

$p = 1$ - $p = \infty$

Proposition

Let $\mathcal{D} = (D_1, D_2)$ be a ∞-minimal 2-partition
Suppose that there exists a second eigenfunction φ_2 of $-\Delta$ on Ω having D_1 and D_2 as nodal domains and such that

$$\int_{D_1} |\varphi_2|^2 \neq \int_{D_2} |\varphi_2|^2$$

Then

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

[*Helffer–Hoffman-Ostenhof*]

2-partition

$p = 1$ - $p = \infty$

Applications

Let $\mathcal{D} = (D_i)_{1 \le i \le k}$ be a ∞-minimal k-partition

Let $D_i \sim D_j$ be a pair of neighbors. We denote

$$D_{ij} = \mathrm{Int}\,\overline{D_i \cup D_i}$$

- $\lambda_2(D_{ij}) = \mathfrak{L}_{2,\infty}(\Omega)$
- Suppose that there exists a second eigenfunction φ_{ij} of $-\Delta$ on D_{ij} having D_i and D_j as nodal domains and such that

$$\int_{D_i} |\varphi_{ij}|^2 \neq \int_{D_j} |\varphi_{ij}|^2$$

Then

$$\mathfrak{L}_{k,1}(\Omega) < \Lambda_{k,\infty}(\mathcal{D})$$

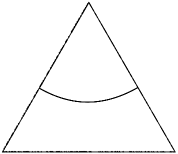
[*Helffer–Hoffman-Ostenhof*]

2-partition

$p = 1$

- $\Omega = \square, \ \bigcirc$?

- $\Omega = \triangle$



φ_2: symmetric eigenfunction associated with $\lambda_2(\Omega)$

$$0.495 \simeq \int_{D_1} |\varphi_2|^2 < \int_{D_2} |\varphi_2|^2 \simeq 0.505$$

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

2-partition

$p = 1$

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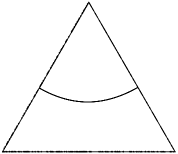
- $\Omega = \triangle$



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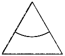
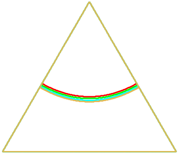
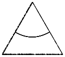
 is a ∞-minimal 2-partition but not a 1-minimal 2-partition

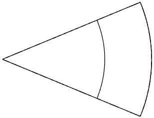


2-partition

$p = 1$

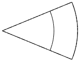
- $\Omega = \square, \ \bigcirc$?
-  is a ∞-minimal 2-partition but not a 1-minimal 2-partition
- Angular sector with opening $\pi/4$

φ_2: symmetric eigenfunction associated with $\lambda_2(\Omega)$



$$0.37 \simeq \int_{D_1} |\varphi_2|^2 < \int_{D_2} |\varphi_2|^2 \simeq 0.63$$

$$\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$$

 is a ∞-minimal 2-partition but not a 1-minimal 2-partition

- The inequality $\mathfrak{L}_{2,1}(\Omega) < \mathfrak{L}_{2,\infty}(\Omega)$ is "generically" satisfied

[*Helffer–Hoffmann-Ostenhof*]

Lower bounds

Square, equilateral triangle, disk

$$\left(\frac{1}{k}\sum_{i=1}^{k}\lambda_i(\Omega)^p\right)^{1/p} \leq \mathfrak{L}_{k,p}(\Omega)$$

Explicit eigenvalues for $\square$, $\triangle$, $\bigcirc$

Ω	$\lambda_{m,n}(\Omega)$	m, n
$\square$	$\pi^2(m^2+n^2)$	$m, n \geq 1$
$\triangle$	$\frac{16}{9}\pi^2(m^2+mn+n^2)$	$m, n \geq 1$
$\bigcirc$	$j_{m,n}^2$	$m \geq 0,\ n \geq 1$ (multiplicity)

Bounds for $p = \infty$

Theorem

$$\lambda_k(\Omega) \le \mathfrak{L}_{k,\infty}(\Omega) \le L_k(\Omega)$$

If $\mathfrak{L}_{k,\infty} = L_k$ *or* $\mathfrak{L}_{k,\infty} = \lambda_k$, *then* $\lambda_k(\Omega) = \mathfrak{L}_{k,\infty}(\Omega) = L_k(\Omega)$ *with a Courant sharp eigenfunction associated with* $\lambda_k(\Omega)$

[*Helffer–Hoffmann-Ostenhof–Terracini*]

Theorem

- *There exists* k_0 *such that* $\lambda_k < L_k$ *for* $k \ge k_0$ [*Pleijel*]
- *Explicit upper-bound for* k_0 [*Bérard-Helffer 16, van den Berg-Gittins 16*]

Upper bounds

Square, disk

$$\mathfrak{L}_{k,p}(\Omega) \leq \Lambda_{k,\infty}(\mathcal{D}_\star)$$

Explicit upper bound for $\bigcirc$

$$\mathfrak{L}_{k,p}(\bigcirc) \leq \lambda_1(\Sigma_{2\pi/k})$$

with $\Sigma_{2\pi/k}$: angular sector of opening $2\pi/k$

Explicit upper bound for $\square$

$$\mathfrak{L}_{k,p}(\square) \leq \inf_{m,n\geq 1}\{\lambda_{m,n}(\square) | mn = k\} \leq \lambda_{k,1}(\square)$$

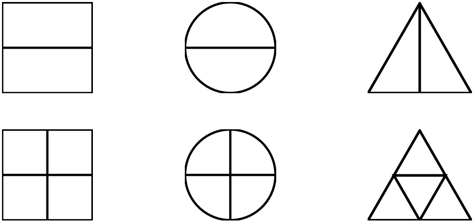
Examples for $p = \infty$

Minimal nodal partitions

- Let $\Omega = \square$, $\bigcirc$ or $\triangle$,

$$\lambda_k(\Omega) = \mathfrak{L}_{k,\infty}(\Omega) = L_k(\Omega) \qquad \text{iff} \qquad k = 1, 2, 4$$

∞-minimal nodal partitions



Properties

Dichotomy for the case $p = \infty$

Let $k > 2$
To determine a ∞-minimal k-partition,
we consider the eigenspace E_k associated with λ_k

Two cases:

- If there exists $u \in E_k$ with k nodal domains, then u produces a minimal k-partition and any minimal k-partition is nodal

$$\mathfrak{L}_{k,\infty}(\Omega) = \lambda_k(\Omega) = L_k(\Omega)$$

[Bipartite case]

- If $\mu(u) < k$ for any $u \in E_k$...

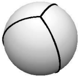
...we have to find another strategy
[Non bipartite case]

Known results in the non bipartite case, $p = \infty$

Sphere and fine flat torus

Theorem

The minimal 3-partition for the sphere is



[*Helffer–Hoffmann-Ostenhof–Terracini*]

Theorem

Let $0 < b \leq a$ *and* $T(a,b) = (\mathbf{R}/a\mathbf{Z}) \times (\mathbf{R}/b\mathbf{Z})$ *the flat torus*

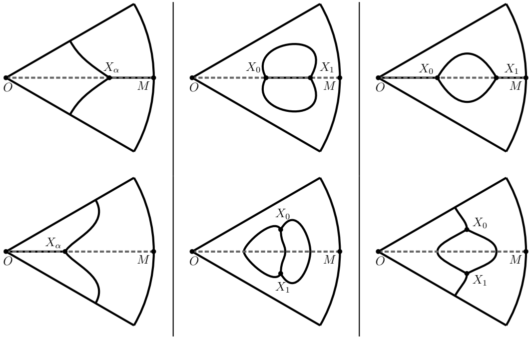
$$\mathcal{D}_k(a,b) = \{]\tfrac{i-1}{k}a, \tfrac{i}{k}a[\times]0,b[, 1 \leq i \leq k \}$$

- k *even and* $\frac{b}{a} \leq \frac{2}{k} \Rightarrow \mathcal{D}_k(a,b)$ *is minimal*
- k *odd and* $\frac{b}{a} < \frac{1}{k} \Rightarrow \mathcal{D}_k(a,b)$ *is minimal* [*Helffer–Hoffmann-Ostenhof*]
- k *odd and* $\frac{1}{k} \leq \frac{b}{a} \leq \ell_* \Rightarrow \mathcal{D}_k(a,b)$ *is minimal* [*BN-Léna 16*]

The question is open for any other domain (in the non bipartite case)

Topological configurations

Euler formula $\Rightarrow$ 3 types of configurations

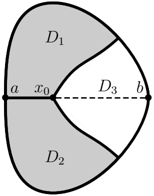


Question

If Ω is symmetric, does it exist a symmetric minimal 3-partition ?

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis



$\mathcal{D} = (D_1, D_2, D_3)$ minimal 3-partition

$\Rightarrow (D_1, D_3)$ minimal 2-partition for $\mathrm{Int}(\overline{D_1} \cup \overline{D_3})$

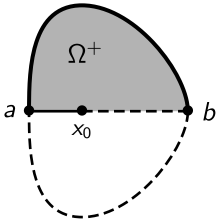
$\Rightarrow$ nodal partition on $\mathrm{Int}(\overline{D_1} \cup \overline{D_3})$

[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

First configuration: One critical point on the symmetry axis

Introduce a mixed Dirichlet-Neumann problem



$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

- $(\lambda_2(x_0), \varphi_{x_0})$ second eigenmode
- $x_0 \mapsto \lambda_2(x_0)$ is increasing
- the nodal line starts from (a, b) and reaches the boundary

[*BN–Helffer–Vial 10*]

Non bipartite symmetric ∞-minimal 3-partition

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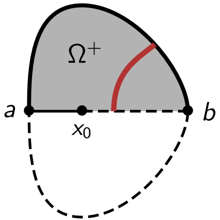
Introduce a mixed Dirichlet-Neumann problem



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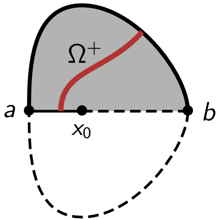

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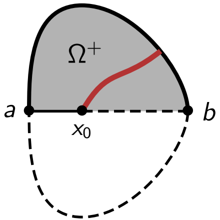

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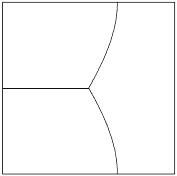
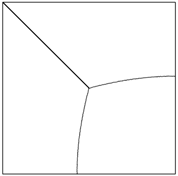
[BN–Helffer–Vial 10]

Non bipartite symmetric ∞-minimal 3-partition

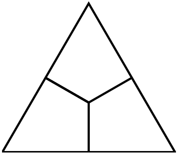
First configuration: examples



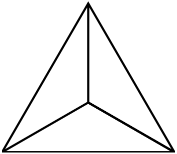
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.581$



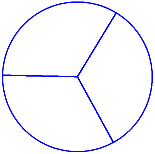
$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 66.581$

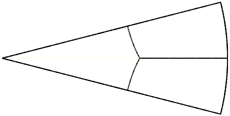


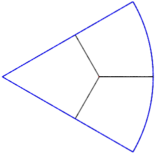
$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 61.872$

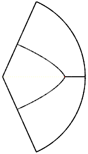


$\Lambda_{3,\infty}(\mathcal{D}_1) \simeq 93.156$



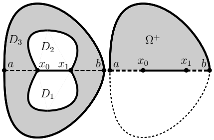






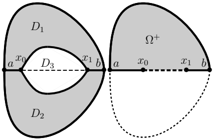
Non bipartite symmetric ∞-minimal 3-partition

Second and third configurations: Two critical points on the symmetry axis



Mixed Neumann-Dirichlet-Neumann problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [a, x_0] \cup [x_1, b] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$



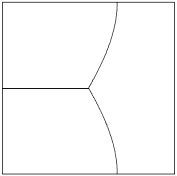
Mixed Dirichlet-Neumann-Dirichlet problem

$$\begin{cases} -\Delta\varphi = \lambda\varphi & \text{in } \Omega^+ \\ \partial_{\mathbf{n}}\varphi = 0 & \text{on } [x_0, x_1] \\ \varphi = 0 & \text{elsewhere} \end{cases}$$

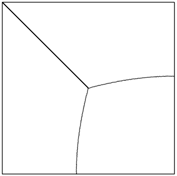
No candidate for the square, disk, angular sectors with two critical points!

∞-minimal 3-partition

Candidates



$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 66.58$



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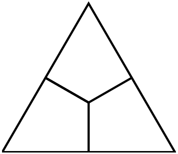
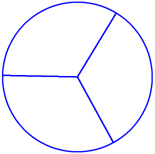
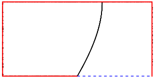


$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 61.872$

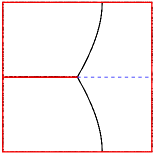


$\Lambda_{3,\infty}(\mathcal{D}_0) \simeq 20.20$

Applications



$0.75 \simeq \int_{D_1} |\varphi_2|^2 > 2 \int_{D_2} |\varphi_2|^2 \simeq 0.51$



$\mathfrak{L}_{3,1}(\square) < \Lambda_{3,\infty}(\mathcal{D})$

Numerical simulations

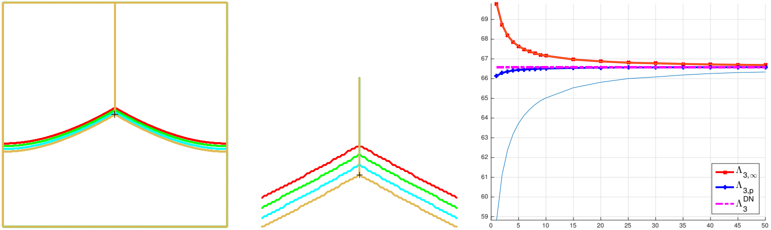
p-minimal 3-partition for the square

Since $\Lambda_3^{DN} \simeq 66.581$ and $L_3 = 10\pi^2 \simeq 98.696$

$$49.35 \simeq 5\pi^2 = \lambda_3 < \mathfrak{L}_{3,\infty} \leq \Lambda_3^{DN} \simeq 66.581$$

$$\pi^2 \left(\frac{2^p + 5^p + 5^p}{3} \right)^{1/p} \leq \mathfrak{L}_{3,p} \leq \Lambda_3^{DN} \quad \Rightarrow \quad 39.48 \simeq 4\pi^2 \leq \mathfrak{L}_{3,1} \leq 66.58$$

$p = 1, 2, 5, 50$



[Bogosel-BN16]

Numerical simulations

p-minimal 3-partition

Conjecture

For the square :

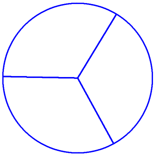
- $p \mapsto \mathfrak{L}_{3,p}(\square)$ is increasing
- $p_\infty(\square, 3) = +\infty$

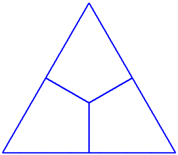
For the disk:

- $p_\infty(\bigcirc, 3) = 1$

For the equilateral triangle:

- $p_\infty(\triangle, 3) = 1$

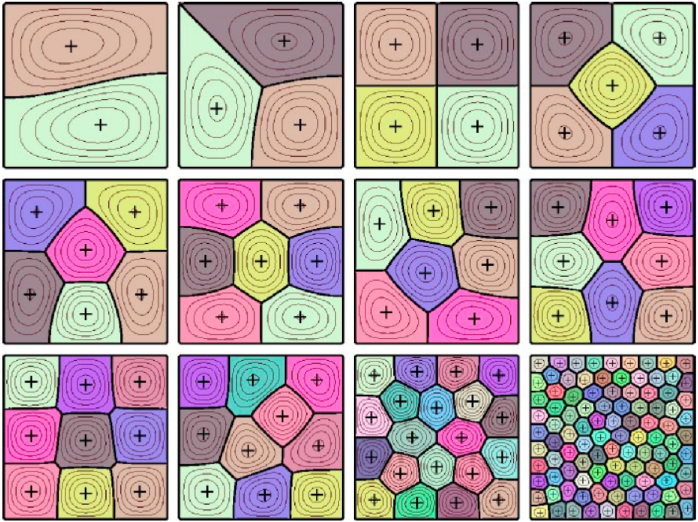




is a p-minimal 3-partition for any $p \geq 1$

k-partitions

Examples



[*Cybulski-Babin-Holyst 05*]

Iterative methods

Penalization

1. Instead of looking for k domains $(D_1, \dots, D_K)$, we look for a k-upple of functions $(\varphi_1, \dots, \varphi_k) \in M$ with

$$M = \left\{ (\varphi_1, \dots, \varphi_k), \varphi_i : \Omega \to [0,1] \text{ measurable }, \sum_{i=1}^{k} \varphi_i = 1 \text{ a.e. } \Omega \right\}.$$

2. Penalized eigenvalue problem on Ω

$$-\Delta v_i + \frac{1}{\varepsilon}(1 - \varphi_i) v_i = \lambda(\varepsilon, \varphi_i) v_i \qquad \text{in } \Omega$$

   Note that $\lim_{\varepsilon \to 0} \lambda(\varepsilon, \varphi_i) = \lambda_1(D_i)$

3. Penalized optimization problem

$$\mathcal{M}(\varepsilon, k) = \inf \left\{ \left(\frac{1}{k} \sum_{i=1}^{k} \lambda_1^p(\varepsilon, \varphi_i) \right)^{1/p}, (\varphi_1, \dots, \varphi_k) \in M \right\}$$

   In some sense, $\lim_{\varepsilon \to 0} \mathcal{M}(\varepsilon, k) = \mathfrak{L}_{k,p}(\Omega)$

4. Projected-gradient descent with adaptive step

[Bourdin-Bucur-Oudet 09]

Algorithm

Let $\rho > 0$, $\varepsilon > 0$

Initialisation k vectors Φ_ℓ^0 given randomly

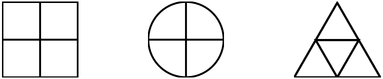
Iteration Step p: for any $\ell = 1, \ldots, k$:

1. Compute the first eigenmode $(\lambda(\Phi_\ell), U(\Phi_\ell))$ of $\mathbb{A}(\varepsilon, \Phi_{\ell)}$
2. Gradient descent : $\tilde{\Phi}_\ell^{p+1} = \Phi_\ell^{p+1} - \rho \nabla_{\Phi_\ell^p} \lambda(\Phi_\ell)$
3. Projection on $\mathcal{S}$: $\tilde{\Phi}_\ell^{p+1} = \Pi_{\mathcal{S}} \tilde{\Phi}_\ell^{p+1}$

p-minimal 4-partition

$$\lambda_4(\Omega) = \mathfrak{L}_{4,\infty}(\Omega) = L_4(\Omega) \qquad \text{if} \qquad \Omega = \square, \bigcirc, \triangle$$

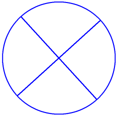
∞-minimal 4-partitions :



Conjecture

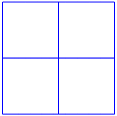
For the disk:

- $p_\infty(\bigcirc, 4) = 1$



For the square :

- $p_\infty(\square, 4) = 1$



is a p-minimal 4-partition for any $p \geq 1$

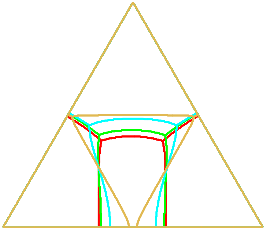
p-minimal 4-partition

Equilateral triangle

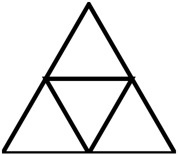
Conjecture

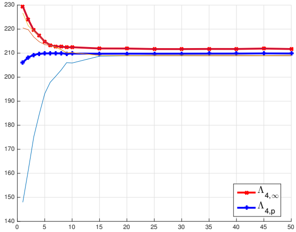
- $p \mapsto \mathfrak{L}_{4,p}(\triangle)$ is increasing

$p = 1, 2, 5, 50$



$p = \infty$

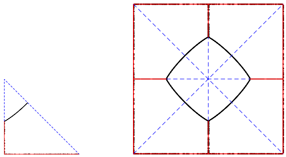




p-minimal 5-partition

Symmetric candidates for the square

Use a Dirichlet-Neumann approach to find some symmetric equipartition



$$0.72 \simeq \int_{D_1} |\varphi_2|^2 < 4 \int_{D_2} |\varphi_2|^2 \simeq 1.12$$

$$\mathfrak{L}_{5,1}(\square) < \Lambda_{5,\infty}(\mathcal{D})$$

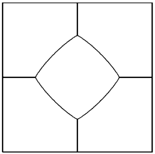
p-minimal 5-partition

Square

$$98.7 \simeq 10\pi^2 = \lambda_5 < \mathfrak{L}_{5,\infty} < L_5 \leq 26\pi^2 \simeq 256.6$$

$$\pi^2 \left(\frac{2^p+5^p+5^p+8^p+10^p}{4}\right)^{1/p} \leq \mathfrak{L}_{5,p} \qquad \Rightarrow \qquad 59.22 \simeq 6\pi^2 \leq \mathfrak{L}_{5,1}$$

Mixed Dirichlet-Neumann approach



$$\mathfrak{L}_{5,p} \leq \Lambda_5^{DN} \simeq 104.294$$

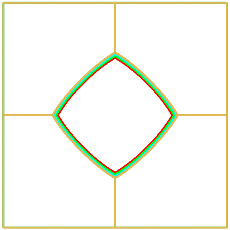
p-minimal 5-partition

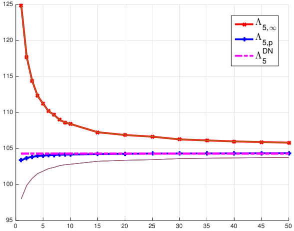
Square

$$98.7 \simeq 10\pi^2 = \lambda_5 < \mathfrak{L}_{5,\infty} \leq \Lambda_5^{DN} \simeq 104.29$$

$$\pi^2 \left(\frac{2^p+5^p+5^p+8^p+10^p}{5} \right)^{1/p} \leq \mathfrak{L}_{5,p} \quad \Rightarrow \quad 59.22 \simeq 6\pi^2 \leq \mathfrak{L}_{5,1} \leq \Lambda_5^{DN} \simeq 104.29$$

$p = 1, 2, 5, 50$

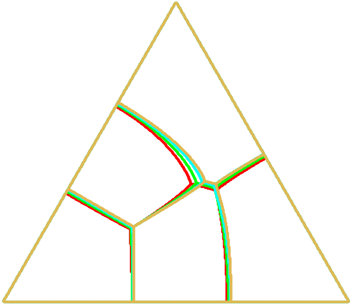


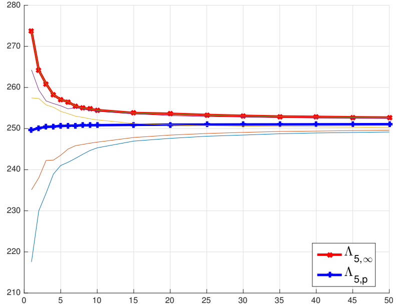


[Bogosel-BN16]

p-minimal 5-partition

Equilateral triangle





[*Bogosel-BN16*]

p-minimal 5-partition

Conjecture

For the square :

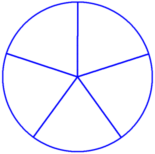
- $p \mapsto \mathfrak{L}_{5,p}(\square)$ is increasing
- $p_\infty(\square, 5) = +\infty$

For the equilateral triangle :

- $p \mapsto \mathfrak{L}_{5,p}(\triangle)$ is increasing
- $p_\infty(\triangle, 5) = +\infty$

For the disk:

- $p_\infty(\bigcirc, 5) = 1$
- For any $p \geq 1$, a p-minimal 5-partition is

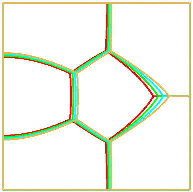


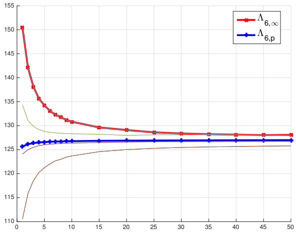
p-minimal 6-partition

Square

$$98.7 \simeq 10\pi^2 = \lambda_6 < \mathfrak{L}_{6,\infty} < L_6 = 13\pi^2 \simeq 128.3$$

$p = 1, 2, 5, 10$



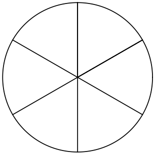


Conjecture

- $p \mapsto \mathfrak{L}_{6,p}$ is increasing
- $p_\infty(\square, 6) = +\infty$

p-minimal 6-partition

Disk



is not Courant-sharp

then not minimal

$\lambda_1(\Sigma_{\pi/3}) \simeq 40.73$

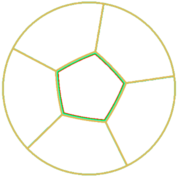
$$\mathfrak{L}_{6,p} < \lambda_1(\Sigma_{\pi/3}) \qquad \forall p \geq 1$$

p-minimal 6-partition

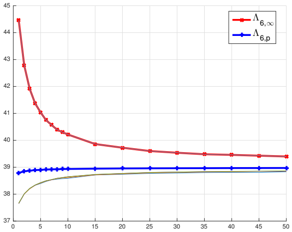
Disk

$$\mathfrak{L}_{6,p} < \lambda_1(\Sigma_{\pi/3}) \qquad \forall p \geq 1$$

$p = 1, 2, 5, 50$



[*Bogosel-BN16*]

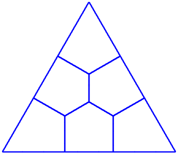


Conjecture

- $p \mapsto \mathfrak{L}_{6,p}(\bigcirc)$ is increasing
- $p_\infty(\bigcirc, 6) = +\infty$

p-minimal 6-partition

Equilateral triangle



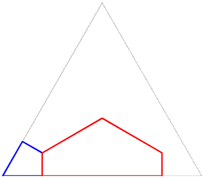
Conjecture

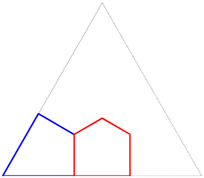
- $p_\infty(\triangle, 6) = 1$

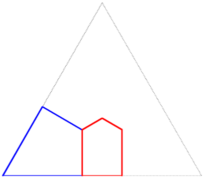
p-minimal 6-partition

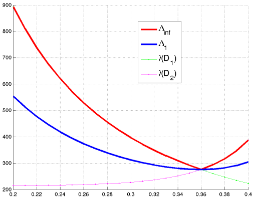
Equilateral triangle

Candidates of 6-partitions for $p = \infty$









Best candidate

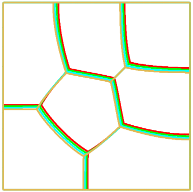
$x_{opt} \simeq 0.3598$

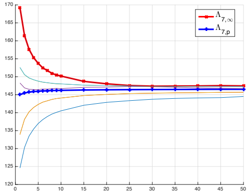
Eigenvalues : 275.94, 275.97

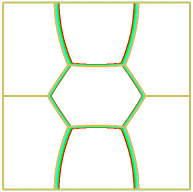
p-minimal 7-partition

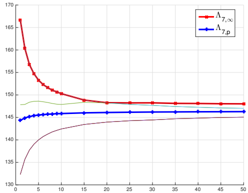
Square

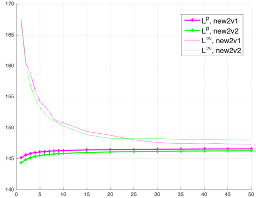
$$128.3 \simeq 13\pi^2 = \lambda_7 < \mathfrak{L}_{7,\infty} < L_7 \leq 50\pi^2$$







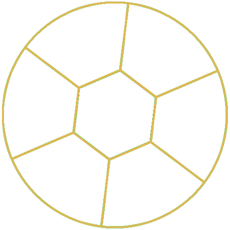


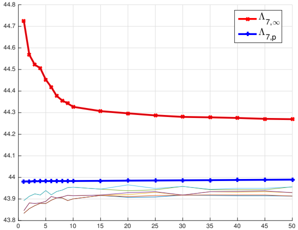


p-minimal 7-partition

Disk

$\lambda_1(\Sigma_{2\pi/7}) \simeq 48.86, \qquad \mathfrak{L}_{7,\infty} < \lambda_1(\Sigma_{2\pi/7})$

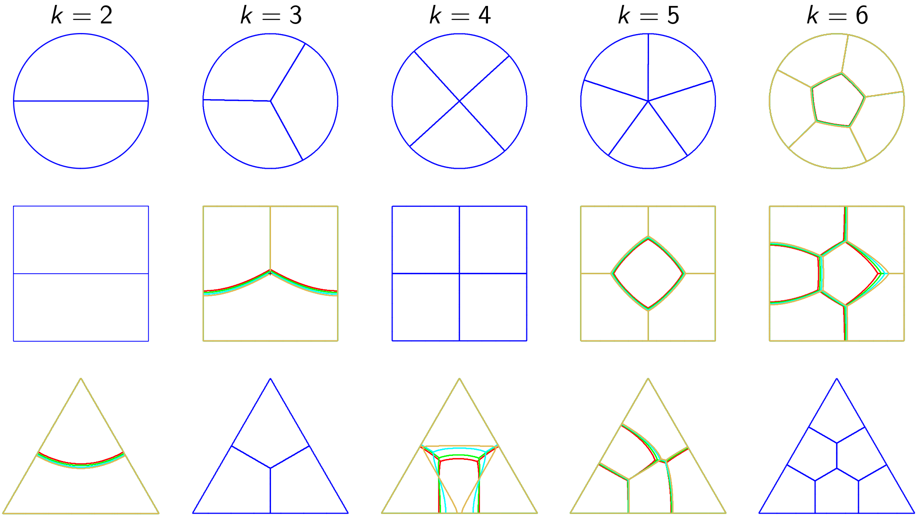




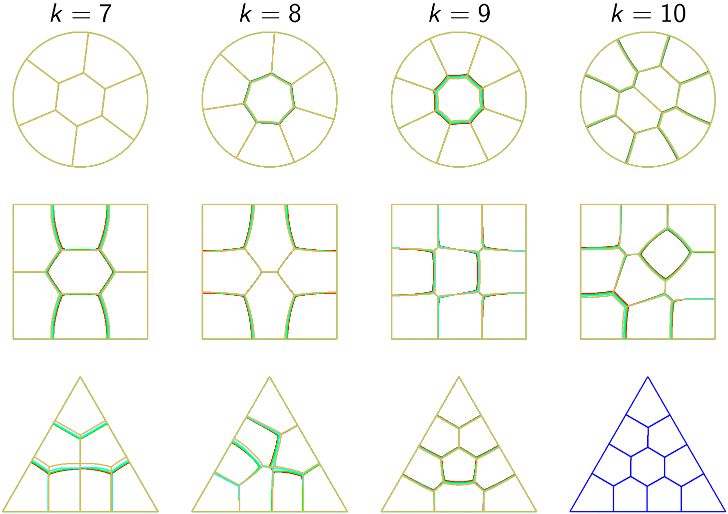
- $p_\infty(\bigcirc, 7) = 1$?

[*Bogosel-BN16*]

Conclusion



Conclusion



Asymptotics $k \to \infty$

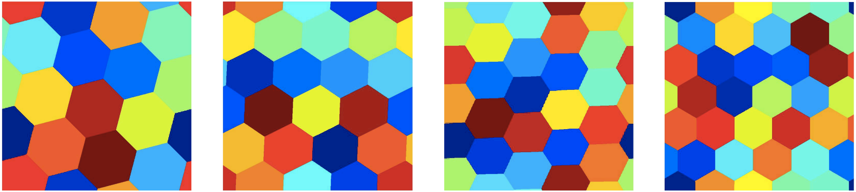
Hexagonal conjecture

- The limit of $\mathfrak{L}_{k,\infty}(\Omega)/k$ as $k \to +\infty$ exists and

$$\lim_{k \to +\infty} \frac{\mathfrak{L}_k(\Omega)}{k} = \frac{\lambda_1(\hexagon)}{|\Omega|}$$

- The limit of $\mathfrak{L}_{k,1}(\Omega)/k$ as $k \to +\infty$ exists and

$$\lim_{k \to +\infty} \frac{\mathfrak{L}_{k,1}(\Omega)}{k} = \frac{\lambda_1(\hexagon)}{|\Omega|}$$



$k = 15$ $k = 20$ $k = 25$ $k = 30$

[Bourdin–Bucur–Oudet, BN–Léna]