Partial functional correspondence

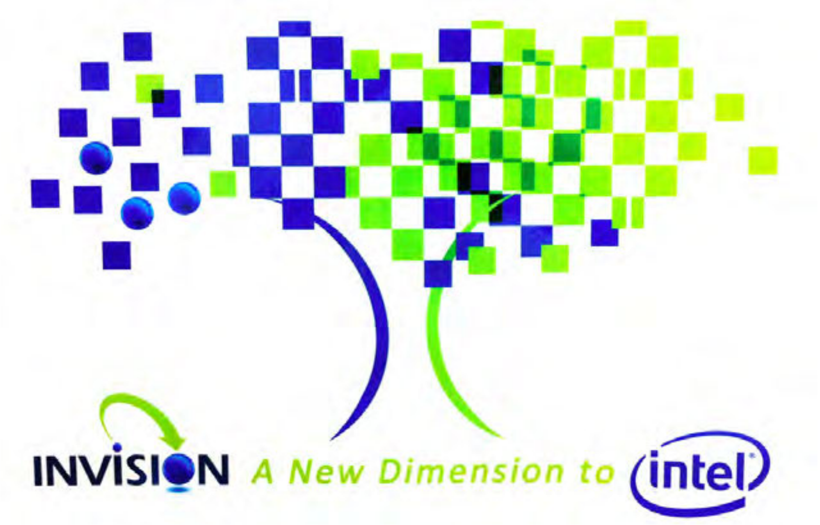
Michael Bronstein

University of Lugano

Intel Corporation

Lyon, 7 July 2016

Microsoft Kinect 2010



(Acquired by Intel in 2012)


intel
REALSENSE
TECHNOLOGY

Different form factor computers featuring Intel RealSense 3D camera

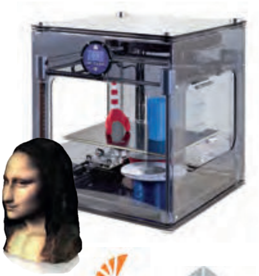
Deluge of geometric data

KINECT for XBOX 360
SoftKinetic
intel REALSENSE

3D sensors

Google 3D warehouse
shapeways

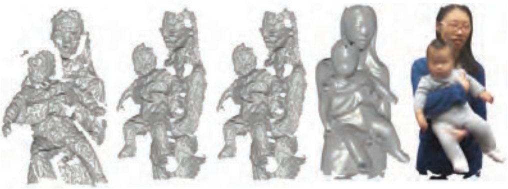
Repositories



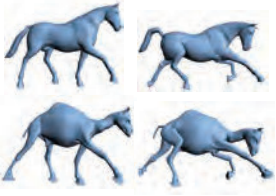
Stratasys
3D SYSTEMS

3D printers

Applications



Deformable fusion



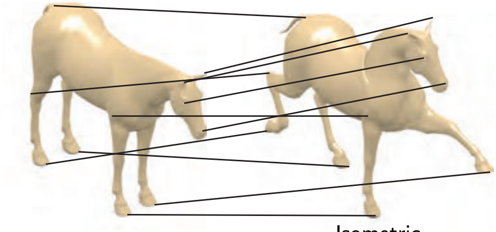
Motion transfer

Motion capture

Texture mapping

Dou et al. 2015; Sumner, Popović 2004; Faceshift; Cow image: Moore 2014

Shape correspondence problem



Isometric

Shape correspondence problem

Isometric

Partial

Shape correspondence problem

Isometric

Partial

Different representation

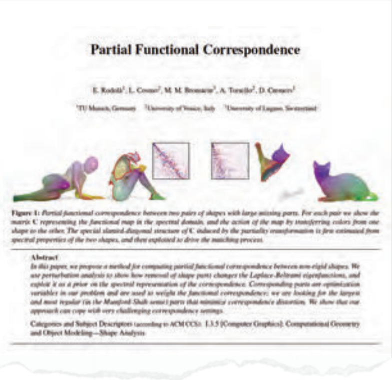
Shape correspondence problem

Isometric

Partial

Different representation

Non-isometric



Partial Functional Correspondence

Computer Graphics Forum
SGP 2016

Non-Rigid Puzzles

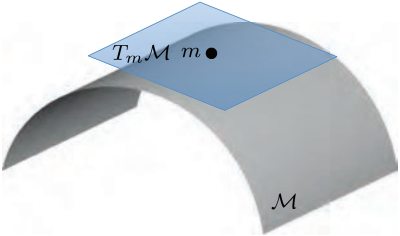
Computer Graphics Forum
SGP 2016 **Best paper award**

Outline

- **Background: Spectral analysis on manifolds**
- Functional correspondence
- Partial functional correspondence
- Non-rigid puzzles

Riemannian geometry in one minute

- Tangent plane $T_m\mathcal{M}$ = local Euclidean representation of manifold (surface) $\mathcal{M}$ around m

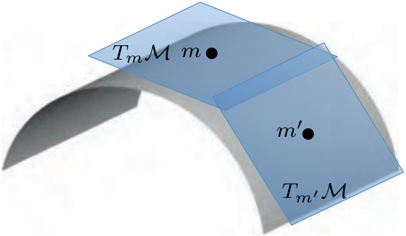


Riemannian geometry in one minute

- Tangent plane $T_m\mathcal{M}$ = local Euclidean representation of manifold (surface) $\mathcal{M}$ around m
- Riemannian metric

  $$\langle \cdot, \cdot \rangle_{T_m\mathcal{M}} : T_m\mathcal{M} \times T_m\mathcal{M} \to \mathbb{R}$$

  depending smoothly on m



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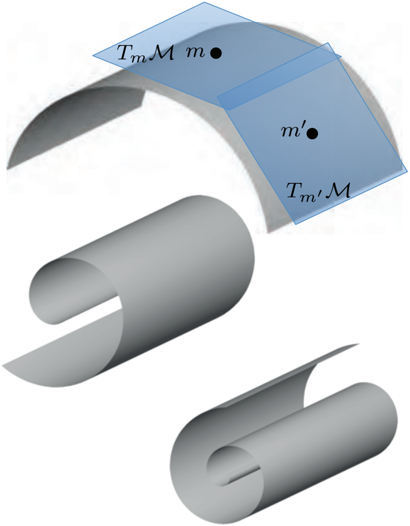
  Isometry = metric-preserving shape deformation



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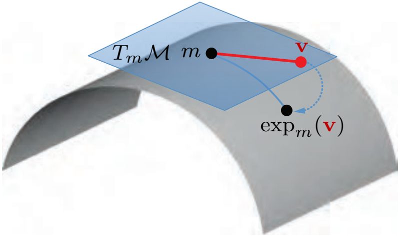
  depending smoothly on m

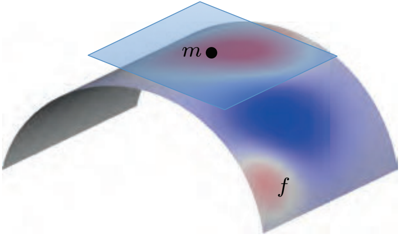
  Isometry = metric-preserving shape deformation
- Exponential map

  $$\exp_m : T_m\mathcal{M} \to \mathcal{M}$$

  'unit step along geodesic'

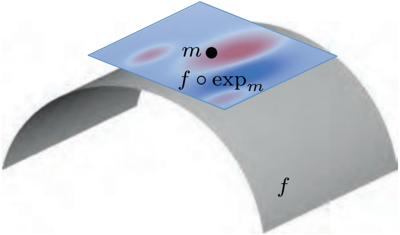


Laplace-Beltrami operator



Smooth field $f : \mathcal{M} \to \mathbb{R}$

Laplace-Beltrami operator



Smooth field $f \circ \exp_m : T_m\mathcal{M} \to \mathbb{R}$

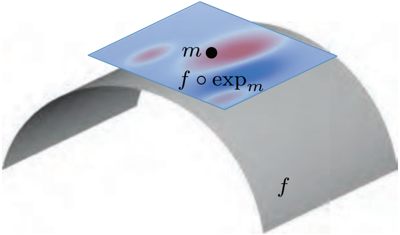
Laplace-Beltrami operator

- Intrinsic gradient

  $$\nabla_{\mathcal{M}} f(m) = \nabla(f \circ \exp_m)(\mathbf{0})$$

  Taylor expansion

  $$(f \circ \exp_m)(\mathbf{v}) \quad \approx \quad f(m) + \langle \nabla_{\mathcal{M}} f(m), \mathbf{v} \rangle_{T_m\mathcal{M}}$$



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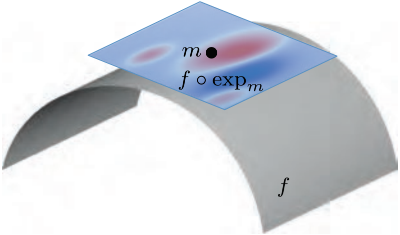

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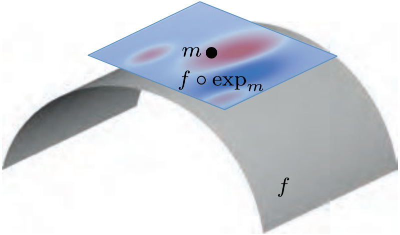
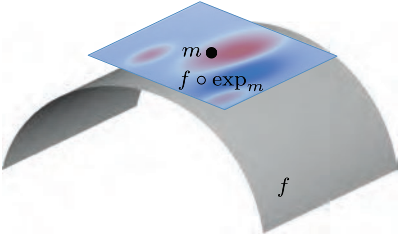


- Laplace-Beltrami operator

  $$\Delta_{\mathcal{M}} f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$

  - Intrinsic (expressed solely in terms of the Riemannian metric)

Laplace-Beltrami operator



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$$\Delta_{\mathcal{M}} f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$

  - Intrinsic (expressed solely in terms of the Riemannian metric)
  - Isometry-invariant

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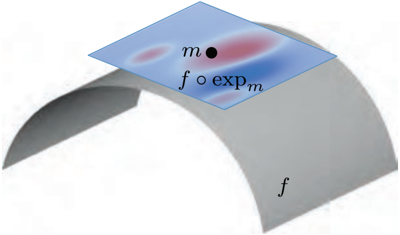

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  $$\Delta_{\mathcal{M}} f(m) = \Delta(f \circ \exp_m)(\mathbf{0})$$

  - Intrinsic (expressed solely in terms of the Riemannian metric)
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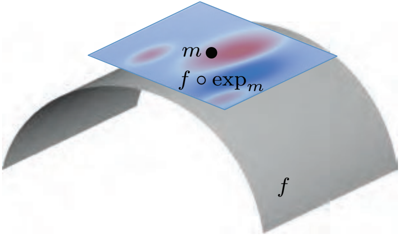
  - Self-adjoint $\langle \Delta_{\mathcal{M}} f, g \rangle_{L^2(\mathcal{M})} = \langle f, \Delta_{\mathcal{M}} g \rangle_{L^2(\mathcal{M})}$

Laplace-Beltrami operator



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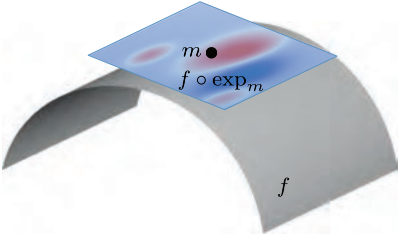

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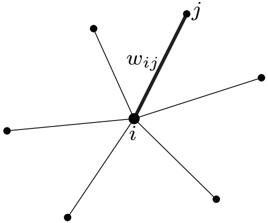
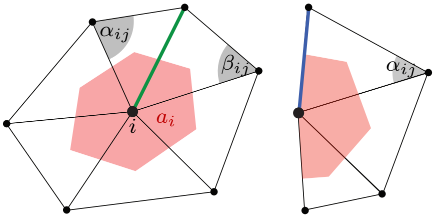
  - Positive semidefinite $\Rightarrow$ non-negative eigenvalues

Discrete Laplacian



Undirected graph (V, E)

$$(\Delta f)_i \approx \sum_{(i,j)\in E} w_{ij}(f_i - f_j)$$



Triangular mesh (V, E, F)

$$(\Delta f)_i \approx \frac{1}{a_i} \sum_{(i,j)\in E} w_{ij}(f_i - f_j)$$

$$w_{ij} = \begin{cases} \frac{\cot\alpha_{ij} + \cot\beta_{ij}}{2} & (i,j) \in E_\mathrm{i} \\ \frac{1}{2}\cot\alpha_{ij} & (i,j) \in E_\mathrm{b} \\ -\sum_{k\neq i} w_{ik} & i = j \\ 0 & \text{else} \end{cases}$$

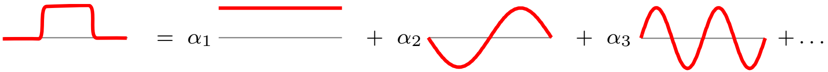
a_i = local area element

Tutte 1963; MacNeal 1949; Duffin 1959; Pinkall, Polthier 1993

Fourier analysis (Euclidean spaces)

A function $f : [-\pi, \pi] \to \mathbb{R}$ can be written as Fourier series

$$f(x) = \sum_{\omega} \quad \frac{1}{2\pi} \int_{-\pi}^{\pi} f(\xi) e^{i\omega\xi} d\xi \quad e^{-i\omega x}$$



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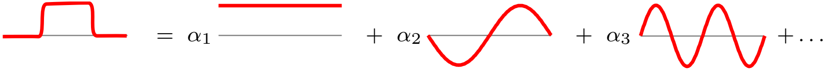

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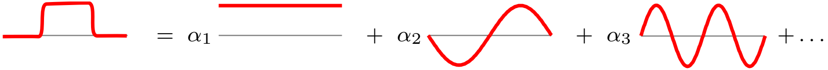
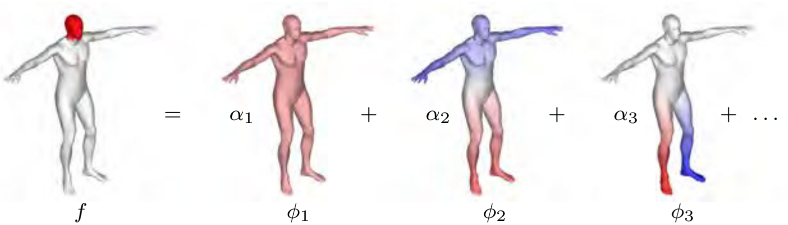


Fourier basis = Laplacian eigenfunctions: $\Delta e^{-i\omega x} = \omega^2 e^{-i\omega x}$

Fourier analysis (non-Euclidean spaces)

A function $f : \mathcal{M} \to \mathbb{R}$ can be written as Fourier series

$$f(m) = \sum_{k \geq 1} \underbrace{\int_{\mathcal{M}} f(\xi)\phi_k(\xi)d\xi}_{\hat{f}_k = \langle f, \phi_k \rangle_{L^2(\mathcal{M})}} \; \phi_k(m)$$

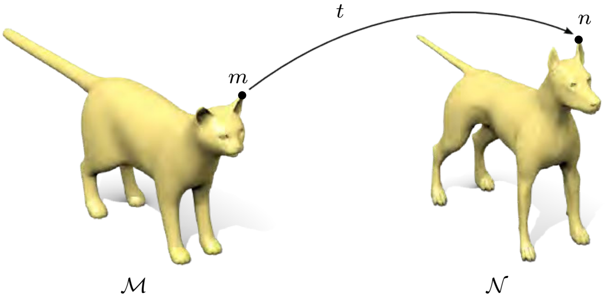


Fourier basis = Laplacian eigenfunctions: $\Delta_{\mathcal{M}}\phi_k = \lambda_k \phi_k$

Outline

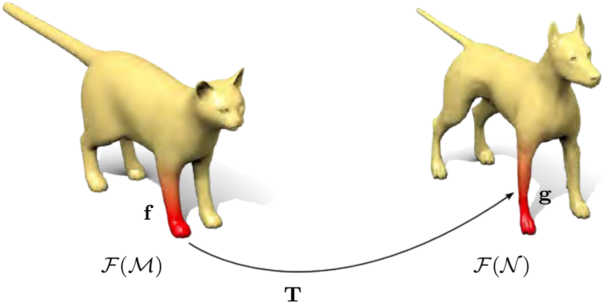
- Background: Spectral analysis on manifolds
- **Functional correspondence**
- Partial functional correspondence
- Non-rigid puzzles

Point-wise correspondence



Point-wise maps $t\colon \mathcal{M} \to \mathcal{N}$

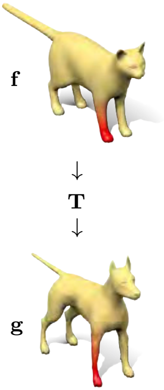
Functional correspondence



Functional maps $\mathbf{T}\colon \mathcal{F}(\mathcal{M}) \to \mathcal{F}(\mathcal{N})$

Ovsjanikov et al. 2012

Functional correspondence



Ovsjanikov et al. 2012

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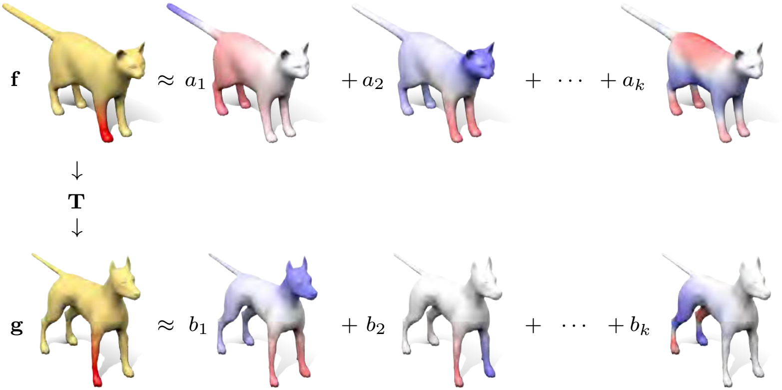

Ovsjanikov et al. 2012

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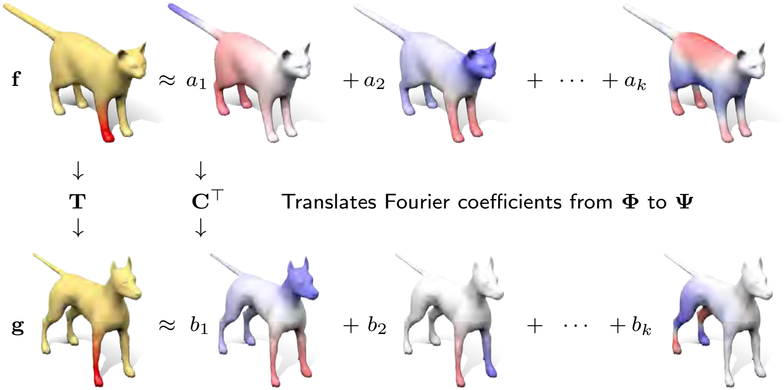
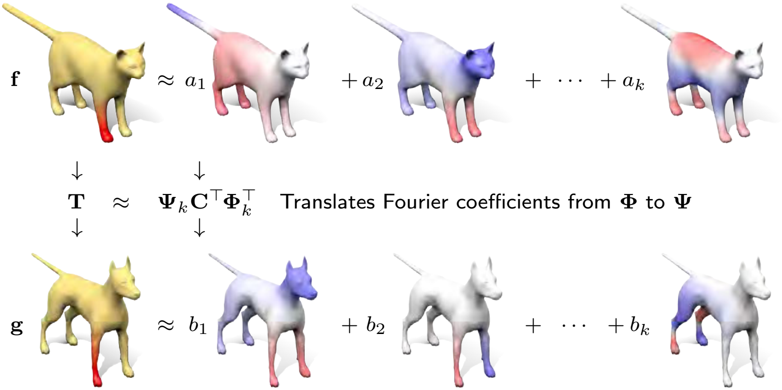


Ovsjanikov et al. 2012

Functional correspondence

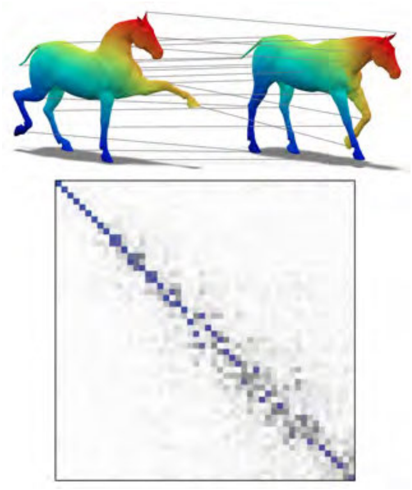


$$\mathbf{g}^\top \boldsymbol{\Psi}_k = \mathbf{f}^\top \boldsymbol{\Phi}_k \mathbf{C}$$

where $\boldsymbol{\Phi}_k = (\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_k)$, $\boldsymbol{\Psi}_k = (\boldsymbol{\psi}_1, \ldots, \boldsymbol{\psi}_k)$ are truncated Laplace-Beltrami eigenbases

Ovsjanikov et al. 2012

Functional correspondence in Laplacian eigenbases

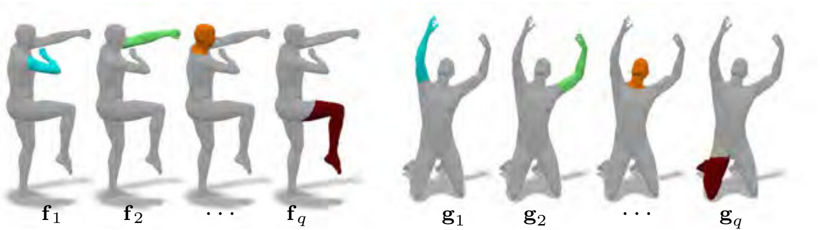


For isometric simple spectrum shapes $\mathbf{C}$ is diagonal since $\boldsymbol{\psi}_i = \pm \mathbf{T} \boldsymbol{\phi}_i$

Computing functional correspondence

Ovsjanikov et al. 2012

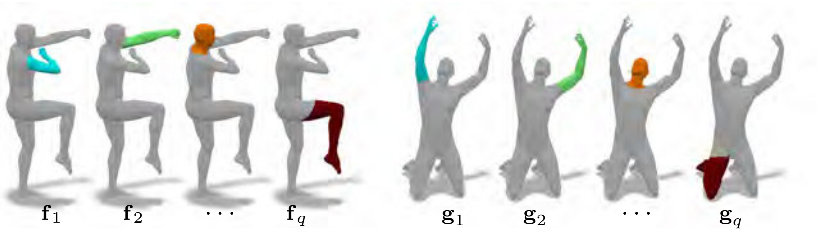
Computing functional correspondence



- Given ordered set of functions $\mathbf{f}_1, \ldots, \mathbf{f}_q$ on $\mathcal{M}$ and corresponding functions $\mathbf{g}_1, \ldots, \mathbf{g}_q$ on $\mathcal{N}$ ($\mathbf{g}_i \approx \mathbf{T}\mathbf{f}_i$)

Ovsjanikov et al. 2012

Computing functional correspondence



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- $\mathbf{C}$ found by solving a system of qk equations with k^2 variables

$$\mathbf{G}^\top \mathbf{\Psi}_k = \mathbf{F}^\top \mathbf{\Phi}_k \mathbf{C}$$

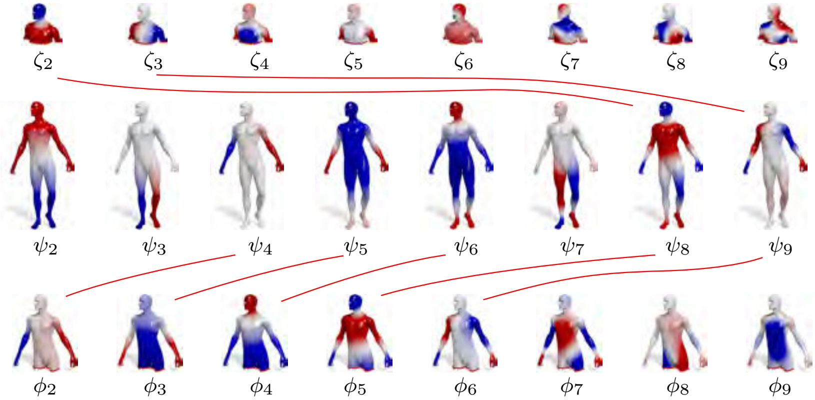
where $\mathbf{F} = (\mathbf{f}_1, \dots, \mathbf{f}_q)$ and $\mathbf{G} = (\mathbf{g}_1, \dots, \mathbf{g}_q)$ are $n \times q$ and $m \times q$ matrices

Ovsjanikov et al. 2012

Key issues

- How to recover point-wise correspondence with some guarantees (e.g. bijectivity)?
- How to automatically find corresponding functions $\mathbf{F}$, $\mathbf{G}$?
- Near isometric shapes: easy (a lot of structure!)
- Non-isometric shapes: hard
- Does not work well in case of **missing parts** and topological noise

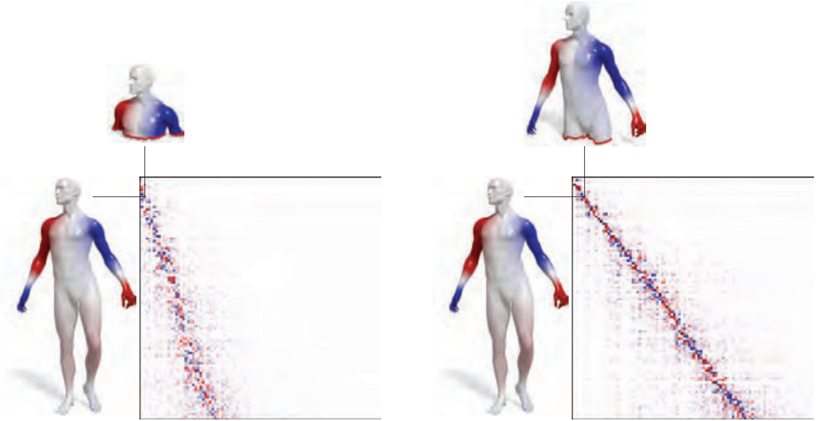
Partial Laplacian eigenvectors



Laplacian eigenvectors of a shape with missing parts
(Neumann boundary conditions)

Rodolà, Cosmo, B, Torsello, Cremers 2016

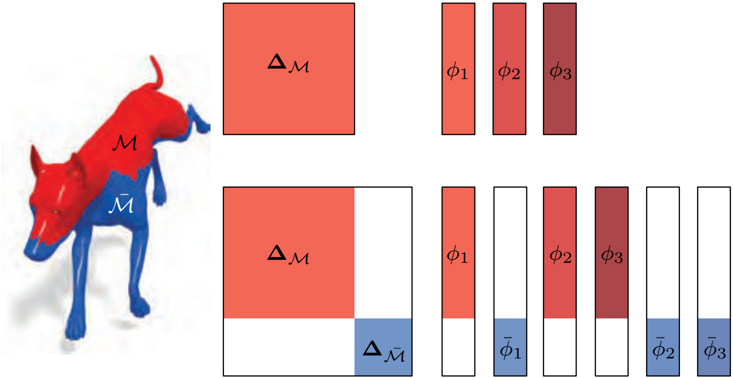
Partial Laplacian eigenvectors



Functional correspondence matrix $\mathbf{C}$

Rodolà, Cosmo, B, Torsello, Cremers 2016

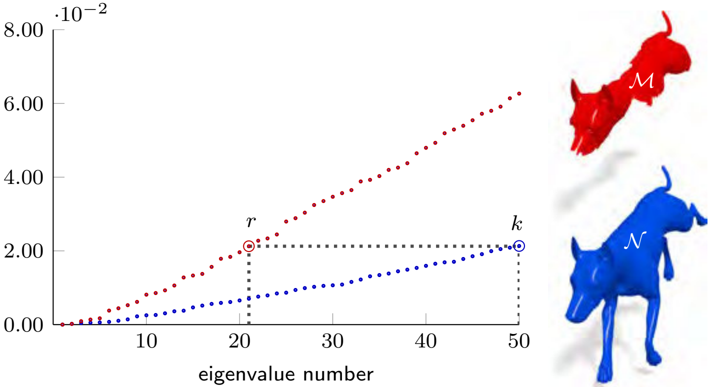
Perturbation analysis: intuition



- Ignoring boundary interaction: disjoint parts (block-diagonal matrix)
- Eigenvectors = Mixture of eigenvectors of the parts

Rodolà, Cosmo, B, Torsello, Cremers 2016

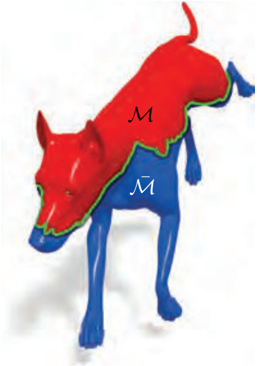
Perturbation analysis: eigenvalues

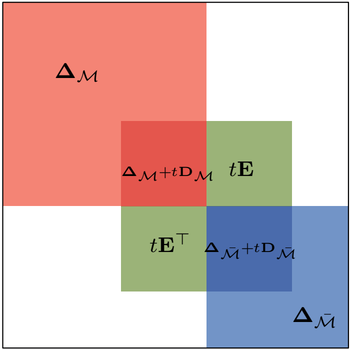


- Slope $\frac{r}{k} \approx \frac{|\mathcal{M}|}{|\mathcal{N}|}$ (depends on the area of the cut)
- Consistent with Weil's law

Rodolà, Cosmo, B, Torsello, Cremers 2016

Perturbation analysis: details





Rodolà, Cosmo, B, Torsello, Cremers 2016

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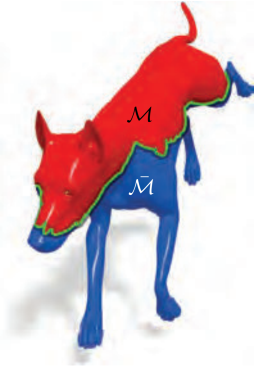
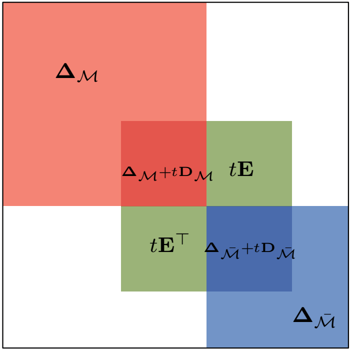
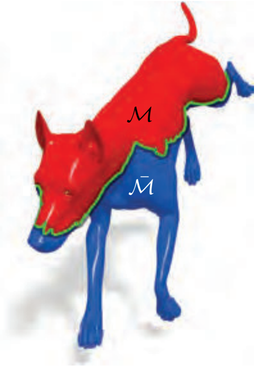


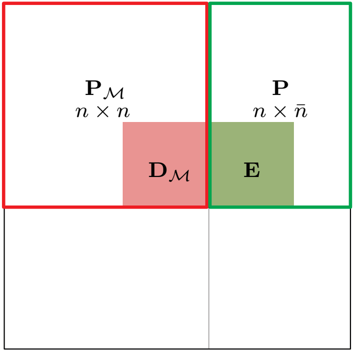


"How would the Laplacian eigenvalues and eigenvectors of the red part change if we attached a blue part to it?"

Rodolà, Cosmo, B, Torsello, Cremers 2016

Perturbation analysis: details





"How would the Laplacian eigenvalues and eigenvectors of the red part change if we attached a blue part to it?"

Rodolà, Cosmo, B, Torsello, Cremers 2016

Perturbation analysis: details

Denote $\mathbf{\Delta}_{\mathcal{M}} + t\mathbf{P}_{\mathcal{M}} = \mathbf{\Phi}(t)\mathbf{\Lambda}(t)\mathbf{\Phi}(t)^\top$, $\mathbf{\Delta}_{\bar{\mathcal{M}}} = \bar{\mathbf{\Phi}}\bar{\mathbf{\Lambda}}\bar{\mathbf{\Phi}}^\top$, $\mathbf{\Phi} = \mathbf{\Phi}(0)$, and $\mathbf{\Lambda} = \mathbf{\Lambda}(0)$.

Theorem 1 (eigenvalues) The derivative of the non-trivial eigenvalues is given by

$$\frac{d}{dt}\lambda_i = \boldsymbol{\phi}_i^\top \mathbf{P}_{\mathcal{M}} \boldsymbol{\phi}_i \qquad \mathbf{P}_{\mathcal{M}} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{\mathcal{M}} \end{pmatrix}$$

Rodolà, Cosmo, B, Torsello, Cremers 2016

Perturbation analysis: details

Denote $\mathbf{\Delta}_{\mathcal{M}} + t\mathbf{P}_{\mathcal{M}} = \mathbf{\Phi}(t)\mathbf{\Lambda}(t)\mathbf{\Phi}(t)^\top$, $\mathbf{\Delta}_{\bar{\mathcal{M}}} = \bar{\mathbf{\Phi}}\bar{\mathbf{\Lambda}}\bar{\mathbf{\Phi}}^\top$, $\mathbf{\Phi} = \mathbf{\Phi}(0)$, and $\mathbf{\Lambda} = \mathbf{\Lambda}(0)$.

Theorem 1 (eigenvalues) The derivative of the non-trivial eigenvalues is given by

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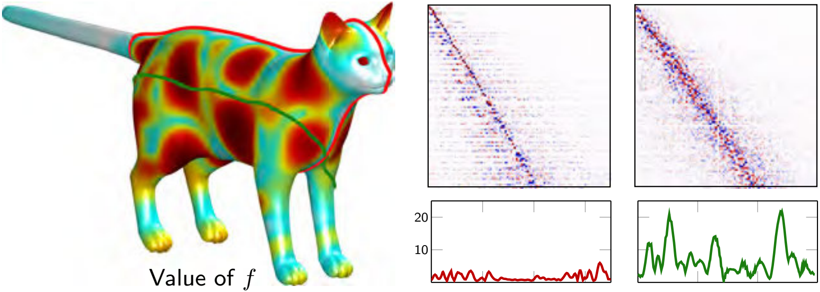
Theorem 2 (eigenvectors) Assuming $\lambda_i \neq \lambda_j$ for $i \neq j$ and $\lambda_i \neq \bar{\lambda}_j$ for all i, j, the derivative of the non-trivial eigenvectors is given by

$$\frac{d}{dt}\boldsymbol{\phi}_i = \sum_{\substack{j=1 \\ j\neq i}}^{n} \frac{\boldsymbol{\phi}_i^\top \mathbf{P}_{\mathcal{M}} \boldsymbol{\phi}_j}{\lambda_i - \lambda_j}\boldsymbol{\phi}_j + \sum_{j=1}^{\bar{n}} \frac{\boldsymbol{\phi}_i^\top \mathbf{P}\, \bar{\boldsymbol{\phi}}_j}{\lambda_i - \bar{\lambda}_j}\bar{\boldsymbol{\phi}}_j \qquad \mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{E} & \mathbf{0} \end{pmatrix}$$

Rodolà, Cosmo, B, Torsello, Cremers 2016

Perturbation analysis: boundary interaction strength

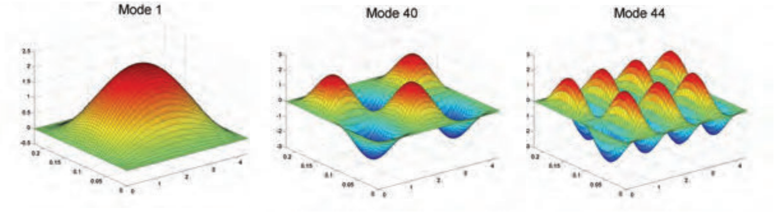


- Eigenvector perturbation depends on length and position of the boundary
- Perturbation strength $\|\frac{d}{dt}\mathbf{\Phi}\|_{\mathrm{F}} \leq c \int_{\partial\mathcal{M}} f(m)dm$, where

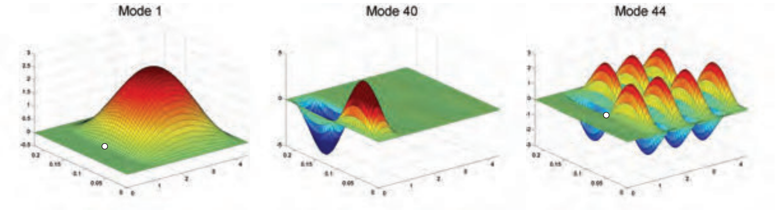
$$f(m) = \sum_{\substack{i,j=1 \\ j \neq i}}^{n} \left(\frac{\phi_i(m)\phi_j(m)}{\lambda_i - \lambda_j} \right)^2$$

Rodolà, Cosmo, B, Torsello, Cremers 2016

Laplacian perturbation: typical picture



Plate



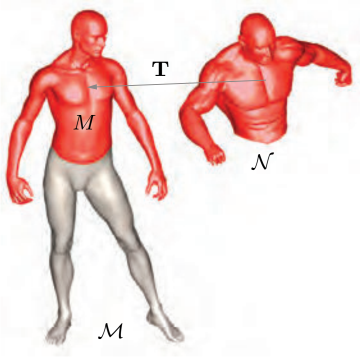
Punctured plate

Figure: Filoche, Mayboroda 2009

Partial functional maps

- Model shape $\mathcal{M}$
- Query shape $\mathcal{N}$
- Part $M \subseteq \mathcal{M} \approx$ isometric to $\mathcal{N}$
- Data $\mathbf{F}$, $\mathbf{G}$
- Partial functional map

$$\mathbf{TG} \approx \mathbf{F}(M)$$



Rodolà, Cosmo, B, Torsello, Cremers 2016

Partial functional maps

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- Query shape $\mathcal{N}$
- Part $M \subseteq \mathcal{M} \approx$ isometric to $\mathcal{N}$
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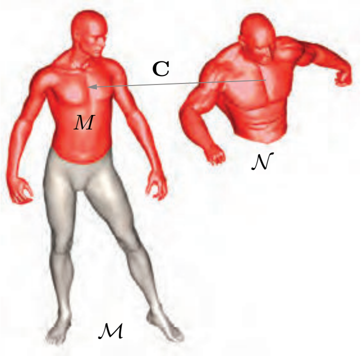
$$\mathbf{G}^\top \mathbf{\Psi} \mathbf{C} \approx \mathbf{F}(M)^\top \mathbf{\Phi}$$



Rodolà, Cosmo, B, Torsello, Cremers 2016

Partial functional maps

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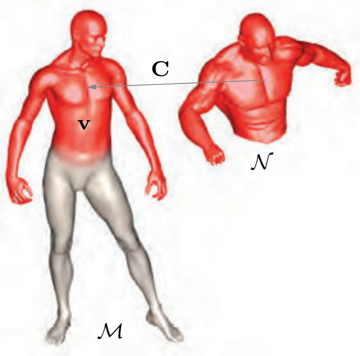
$$\mathbf{G}^\top \mathbf{\Psi} \mathbf{C} \approx \mathbf{F}^\top \mathrm{diag}(\mathbf{v}) \mathbf{\Phi}$$

$\mathbf{v} \in \mathcal{F}(\mathcal{M})$ indicator function of M



Rodolà, Cosmo, B, Torsello, Cremers 2016

Partial functional maps

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- Data $\mathbf{F}$, $\mathbf{G}$
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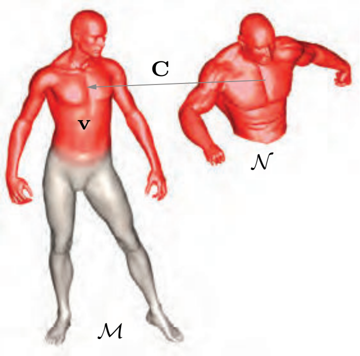
$$\mathbf{G}^\top \mathbf{\Psi} \mathbf{C} \approx \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v}))\mathbf{\Phi}$$

$\mathbf{v} \in \mathcal{F}(\mathcal{M})$ indicator function of M

$\eta(t) = \frac{1}{2}(\tanh(2t-1)+1)$



Optimization problem w.r.t. correspondence $\mathbf{C}$ and part $\mathbf{v}$

$$\min_{\mathbf{C},\mathbf{v}} \|\mathbf{G}^\top \mathbf{\Psi} \mathbf{C} - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v}))\mathbf{\Phi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C}) + \rho_{\mathrm{part}}(\mathbf{v})$$

Rodolà, Cosmo, B, Torsello, Cremers 2016

Partial functional maps

$$\min_{\mathbf{C},\mathbf{v}} \|\mathbf{G}^\top \boldsymbol{\Psi} \mathbf{C} - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v})) \boldsymbol{\Phi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C}) + \rho_{\mathrm{part}}(\mathbf{v})$$

Rodolà, Cosmo, B, Torsello, Cremers 2016

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- **Part regularization**

$$\rho_{\mathrm{part}}(\mathbf{v}) = \mu_1 \left(|\mathcal{N}| - \int_{\mathcal{M}} \eta(\mathbf{v}) dm \right)^2 + \mu_2 \int_{\mathcal{M}} \xi(\mathbf{v}) \|\nabla_{\mathcal{M}} \mathbf{v}\| dm$$

where $\xi(t) \approx \delta\left(\eta(t) - \frac{1}{2}\right)$

Rodolà, Cosmo, B, Torsello, Cremers 2016; BB 2008

Partial functional maps

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Partial functional maps

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- **Correspondence regularization**

$$\rho_{\mathrm{corr}}(\mathbf{C}) = \mu_3 \|\mathbf{C} \circ \mathbf{W}\|_{\mathrm{F}}^2 + \mu_4 \sum_{i \neq j} (\mathbf{C}^\top \mathbf{C})_{ij}^2 + \mu_5 \sum_i ((\mathbf{C}^\top \mathbf{C})_{ii} - d_i)^2$$

Rodolà, Cosmo, B, Torsello, Cremers 2016; BB 2008

Partial functional maps

$$\min_{\mathbf{C},\mathbf{v}} \|\mathbf{G}^\top \mathbf{\Psi}\mathbf{C} - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v}))\mathbf{\Phi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C}) + \rho_{\mathrm{part}}(\mathbf{v})$$

- **Part regularization**

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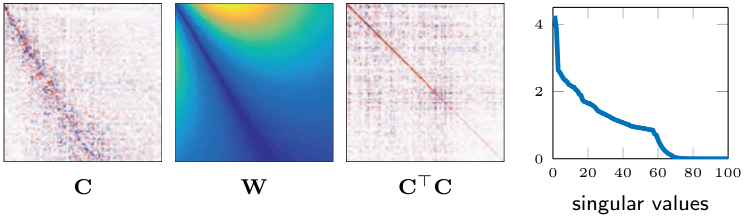
$$\rho_{\mathrm{corr}}(\mathbf{C}) = \underbrace{\mu_3 \|\mathbf{C} \circ \mathbf{W}\|_{\mathrm{F}}^2}_{\text{slant}} + \underbrace{\mu_4 \sum_{i \neq j} (\mathbf{C}^\top \mathbf{C})_{ij}^2}_{\approx \text{orthogonality}} + \underbrace{\mu_5 \sum_{i} ((\mathbf{C}^\top \mathbf{C})_{ii} - d_i)^2}_{\text{rank} \approx r}$$

Rodolà, Cosmo, B, Torsello, Cremers 2016; BB 2008

Structure of partial functional correspondence



Rodolà, Cosmo, B, Torsello, Cremers 2016

Alternating minimization

- **$\mathbf{C}$-step:** fix $\mathbf{v}^*$, solve for correspondence $\mathbf{C}$

$$\min_{\mathbf{C}} \|\mathbf{G}^\top \mathbf{\Psi} \mathbf{C} - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v}^*)) \mathbf{\Phi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C})$$

- **$\mathbf{v}$-step:** fix $\mathbf{C}^*$, solve for part $\mathbf{v}$

$$\min_{\mathbf{v}} \|\mathbf{G}^\top \mathbf{\Psi} \mathbf{C}^* - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v})) \mathbf{\Phi}\|_{2,1} + \rho_{\mathrm{part}}(\mathbf{v})$$

Rodolà, Cosmo, B, Torsello, Cremers 2016

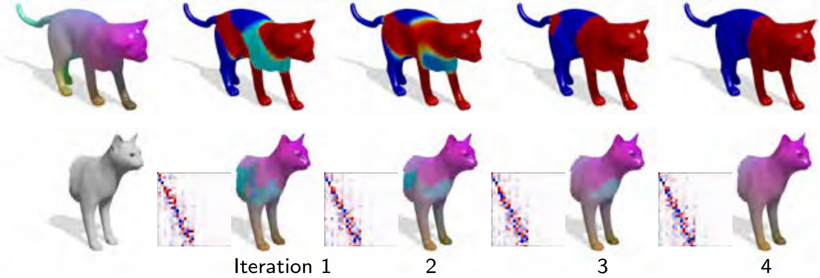
Alternating minimization

- **C-step:** fix $\mathbf{v}^*$, solve for correspondence $\mathbf{C}$

$$\min_{\mathbf{C}} \|\mathbf{G}^\top \mathbf{\Psi} \mathbf{C} - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v}^*))\mathbf{\Phi}\|_{2,1} + \rho_{\mathrm{corr}}(\mathbf{C})$$

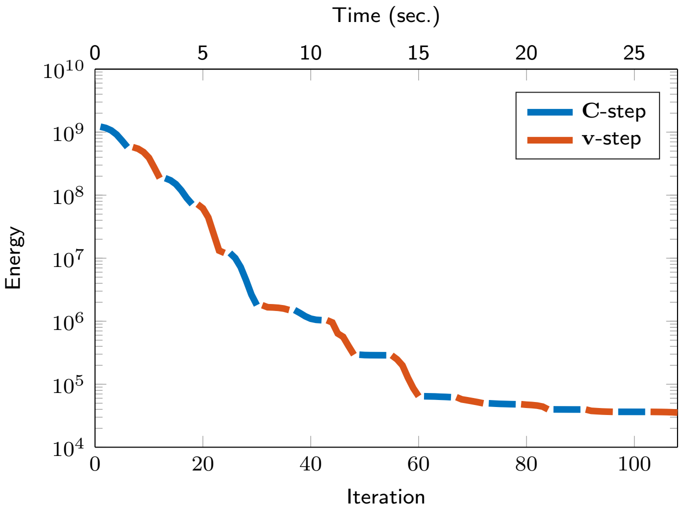
- **v-step:** fix $\mathbf{C}^*$, solve for part $\mathbf{v}$

$$\min_{\mathbf{v}} \|\mathbf{G}^\top \mathbf{\Psi} \mathbf{C}^* - \mathbf{F}^\top \mathrm{diag}(\eta(\mathbf{v}))\mathbf{\Phi}\|_{2,1} + \rho_{\mathrm{part}}(\mathbf{v})$$



Rodolà, Cosmo, B, Torsello, Cremers 2016

Example of convergence

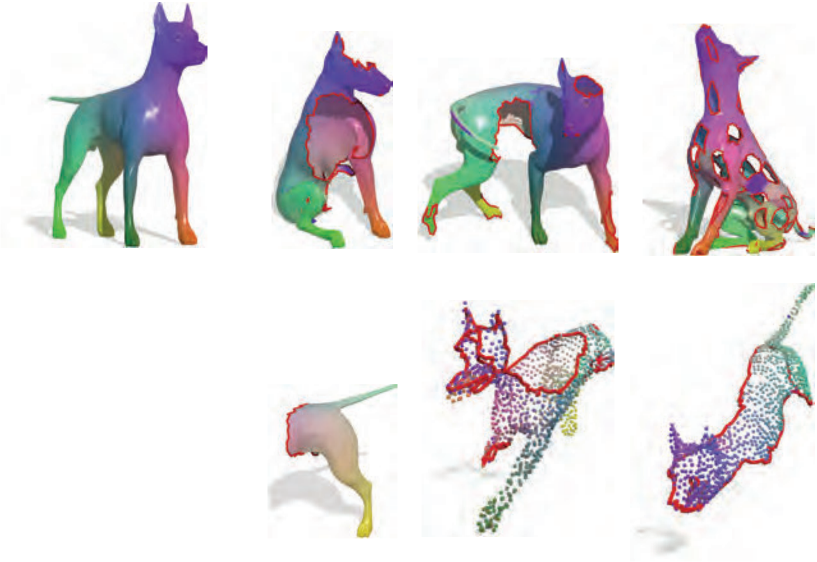


Rodolà, Cosmo, B, Torsello, Cremers 2016

Examples of partial functional maps

Rodolà, Cosmo, B, Torsello, Cremers 2016

Examples of partial functional maps



Rodolà, Cosmo, B, Torsello, Cremers 2016

Examples of partial functional maps

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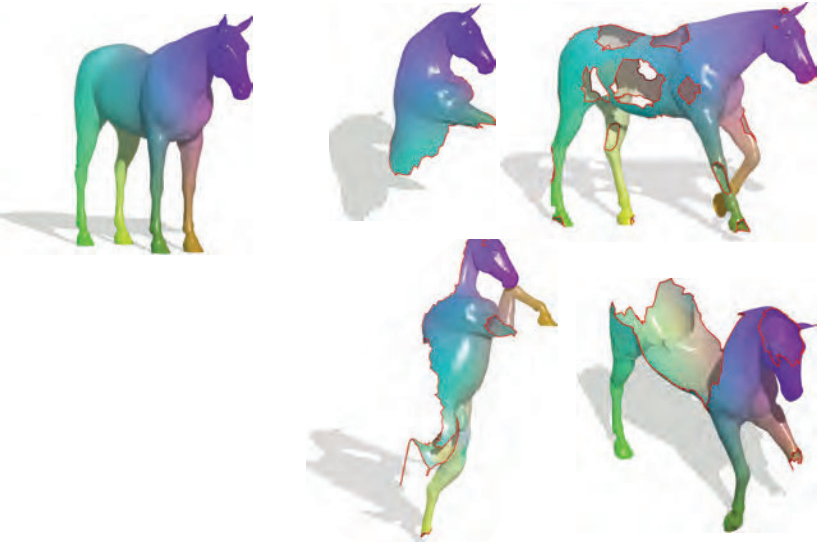
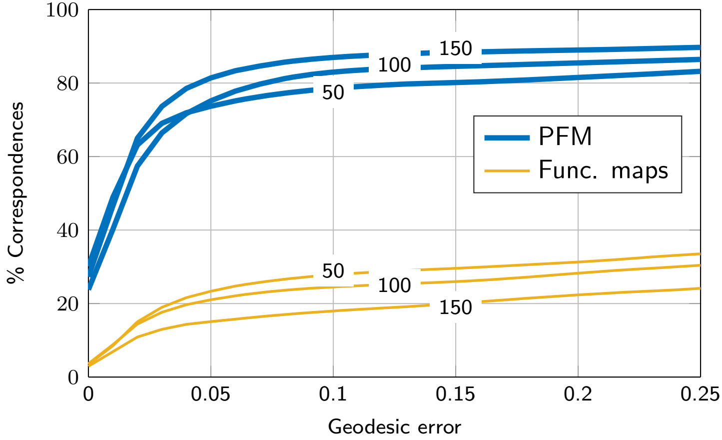
Examples of partial functional maps



Rodolà, Cosmo, B, Torsello, Cremers 2016

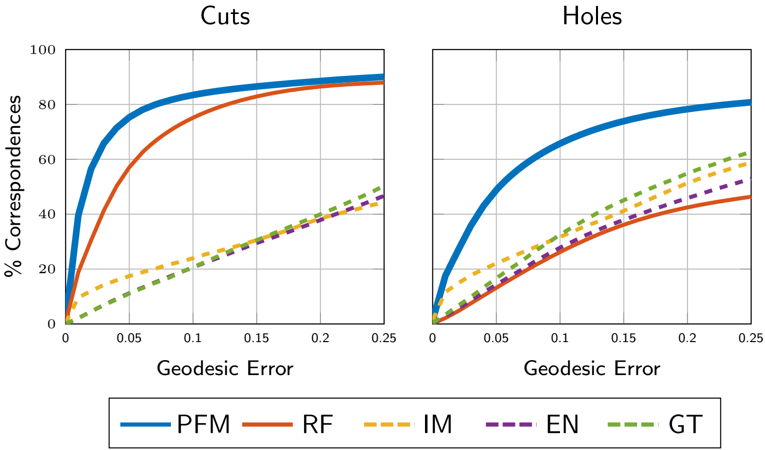
Partial functional maps vs Functional maps



Correspondence performance for different rank values k

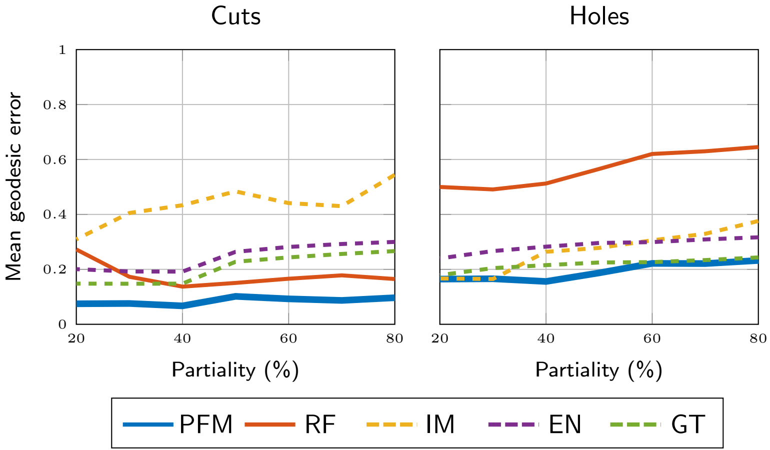
Rodolà, Cosmo, B, Torsello, Cremers 2016

Partial correspondence performance



SHREC'16 Partial Matching benchmark Rodolà et al. 2016; Methods: Rodolà, Cosmo, B, Torsello, Cremers 2016 (**PFM**); Sahillioğlu, Yemez 2012 (IM); Rodolà, Bronstein, Albarelli, Bergamasco, Torsello 2012 (GT); Rodolà et al. 2013 (EN); Rodolà et al. 2014 (RF)

Partial correspondence performance



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Deep learning + Partial functional maps

Correspondence

Correspondence error

Boscaini, Masci, Rodolà, B 2016

Deep learning + Partial functional maps


Correspondence

Correspondence error

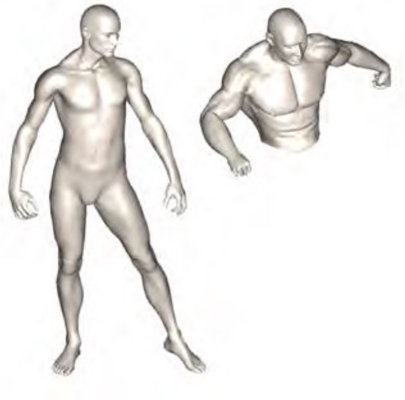
Boscaini, Masci, Rodolà, B 2016

Outline

- Background: Spectral analysis on manifolds
- Functional correspondence
- Partial functional correspondence
- **Non-rigid puzzles**

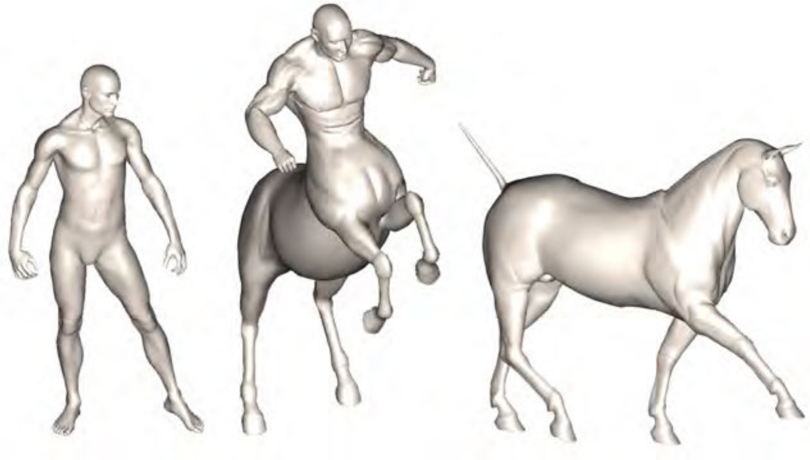
Litani, BB 2012

Partial correspondence



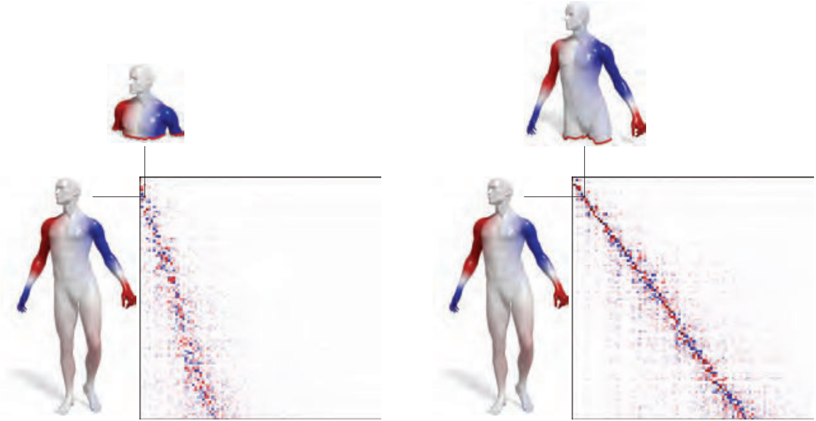
Rodolà, Cosmo, B, Torsello, Cremers 2016

Non-rigid puzzle



Litani, Rodolà, BB, Cremers 2016

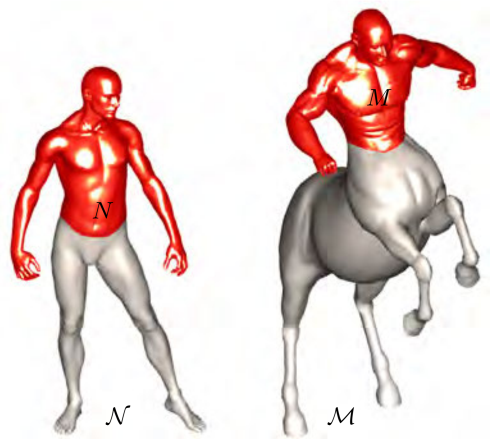
Partial Laplacian eigenvectors



Functional correspondence matrix $\mathbf{C}$

Rodolà, Cosmo, B, Torsello, Cremers 2016

Key observation

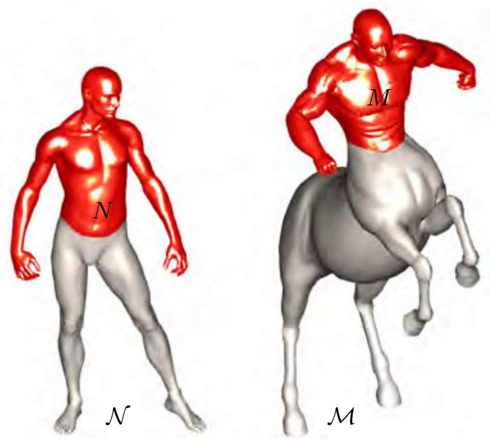


$\mathbf{C}_{N\mathcal{N}}$ $\mathbf{C}_{M\mathcal{M}}$

slant $\propto \frac{|N|}{|\mathcal{N}|}$ slant $\propto \frac{|M|}{|\mathcal{M}|}$

Litani, Rodolà, BB, Cremers 2016

Key observation

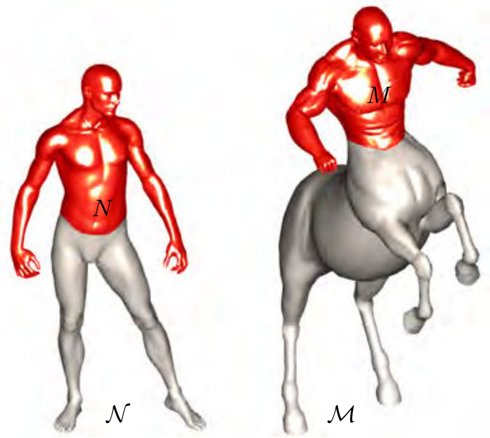


$$\mathbf{C}_{NM} = \mathbf{C}_{N\mathcal{N}}\mathbf{C}_{\mathcal{N}\mathcal{M}}\mathbf{C}_{\mathcal{M}M}$$

$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|}\frac{|\mathcal{M}|}{|M|}$$

Litani, Rodolà, BB, Cremers 2016

Key observation



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$$\text{slant} \propto \frac{|N|}{|\mathcal{N}|}\frac{|\mathcal{M}|}{|M|} = \frac{|\mathcal{N}|}{|\mathcal{M}|}$$

Litani, Rodolà, BB, Cremers 2016

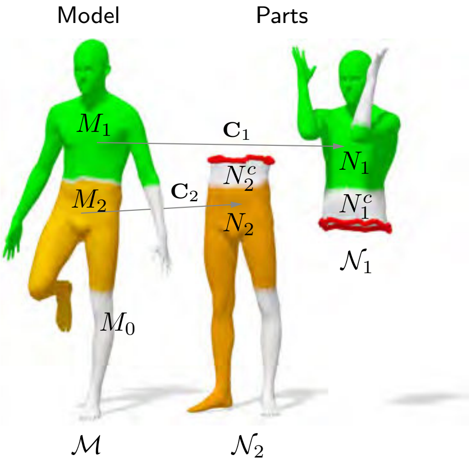
Non-rigid puzzles problem formulation

Input

- Model $\mathcal{M}$
- Parts $\mathcal{N}_1, \ldots, \mathcal{N}_p$

Output

- Segmentation $M_i \subseteq \mathcal{M}$
- Located parts $N_i \subseteq \mathcal{N}_i$
- Correspondences $\mathbf{C}_i$
- Clutter N_i^c
- Missing parts M_0

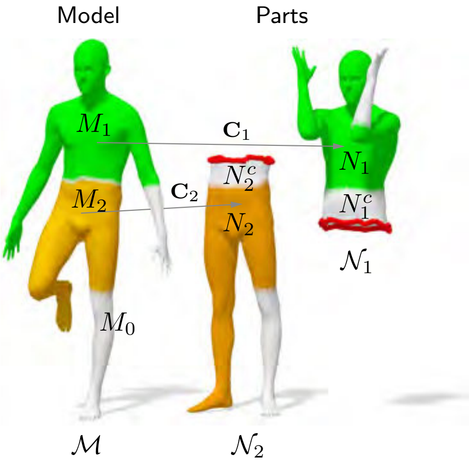


Litani, Rodolà, BB, Cremers 2016

Non-rigid puzzles problem formulation

- Data $\mathbf{F}_i$, $\mathbf{G}_i$
- Model basis $\mathbf{\Phi}$, $\mathbf{\Phi}(M_i)$
- Part bases $\mathbf{\Psi}_i$, $\mathbf{\Psi}_i(N_i)$
- Data term

$$\mathbf{F}_i^\top \mathbf{\Phi}(M_i) \approx \mathbf{G}_i^\top \mathbf{\Psi}_i(N_i)\mathbf{C}_i$$



Litani, Rodolà, BB, Cremers 2016

Non-rigid puzzles problem formulation

$$
\begin{aligned}
\min_{\substack{\mathbf{C}_i \\ M_i \subseteq \mathcal{M}, N_i \subseteq \mathcal{N}_i}} \quad & \sum_{i=1}^{p} \|\mathbf{G}_i^\top \mathbf{\Psi}_i(N_i)\mathbf{C}_i - \mathbf{F}_i^\top \mathbf{\Phi}(M_i)\|_{2,1} \\
& + \lambda_{\mathcal{M}} \sum_{i=0}^{p} \rho_{\text{part}}(M_i) + \lambda_{\mathcal{N}} \sum_{i=1}^{p} \rho_{\text{part}}(N_i) \\
& + \lambda_{\text{corr}} \sum_{i=1}^{p} \rho_{\text{corr}}(\mathbf{C}_i) \\
\text{s.t.} \quad & M_i \cap M_j = \emptyset \quad \forall i \neq j \\
& M_0 \cup M_1 \cup \cdots \cup M_p = \mathcal{M} \\
& |M_i| = |N_i| \geq \alpha |\mathcal{N}_i|,
\end{aligned}
$$

Litani, Rodolà, BB, Cremers 2016

Non-rigid puzzles problem formulation

$$
\begin{aligned}
\min_{\substack{\mathbf{C}_i \\ \mathbf{u}_i, \mathbf{v}_i}} \quad & \sum_{i=1}^{p} \|\mathbf{G}_i^\top \mathrm{diag}(\eta(\mathbf{u}_i)) \mathbf{\Psi}_i \mathbf{C}_i - \mathbf{F}_i^\top \mathrm{diag}(\eta(\mathbf{v}_i)) \mathbf{\Phi}\|_{2,1} \\
& + \lambda_{\mathcal{M}} \sum_{i=0}^{p} \rho_{\mathrm{part}}(\eta(\mathbf{v}_i)) + \lambda_{\mathcal{N}} \sum_{i=1}^{p} \rho_{\mathrm{part}}(\eta(\mathbf{u}_i)) \\
& + \lambda_{\mathrm{corr}} \sum_{i=1}^{p} \rho_{\mathrm{corr}}(\mathbf{C}_i) \\
\text{s.t.} \quad & \sum_{i=1}^{p} \eta(\mathbf{u}_i) = 1 \\
& \mathbf{a}_{\mathcal{M}}^\top \mathbf{u}_i = \mathbf{a}_{\mathcal{N}}^\top \mathbf{v}_i \geq \alpha \mathbf{a}_{\mathcal{N}_i}^\top \mathbf{1}
\end{aligned}
$$

Litani, Rodolà, BB, Cremers 2016

Convergence example

Outer iteration 1

Litani, Rodolà, BB, Cremers 2016

Convergence example

Outer iteration 2

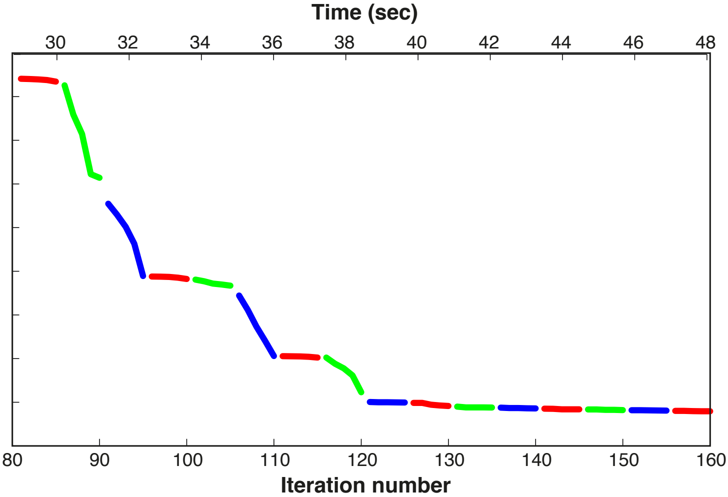
Litani, Rodolà, BB, Cremers 2016

Convergence example

Outer iteration 3

Litani, Rodolà, BB, Cremers 2016

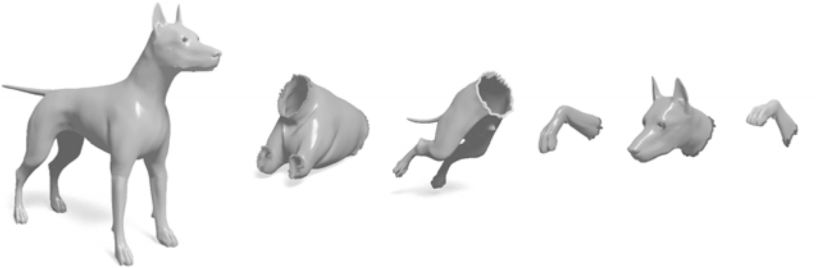
Convergence example



Litani, Rodolà, BB, Cremers 2016

"Perfect puzzle" example

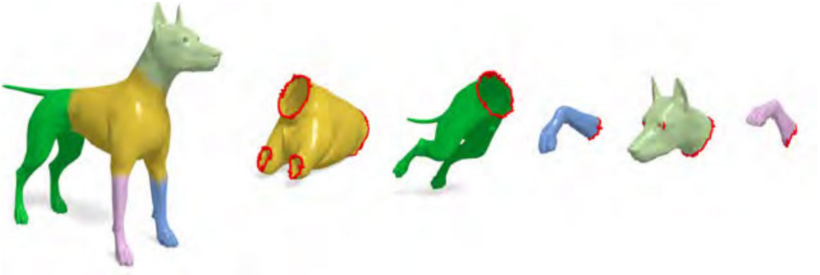
Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	No
Missing part	No
Data term	Dense (SHOT)



Litani, Rodolà, BB, Cremers 2016

“Perfect puzzle” example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	No
Missing part	No
Data term	Dense (SHOT)

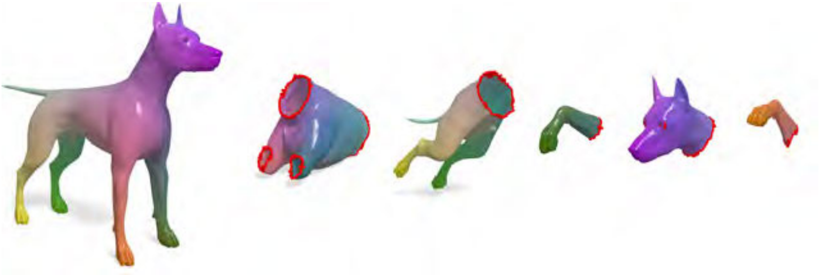


Segmentation

Litani, Rodolà, BB, Cremers 2016

“Perfect puzzle” example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	No
Missing part	No
Data term	Dense (SHOT)

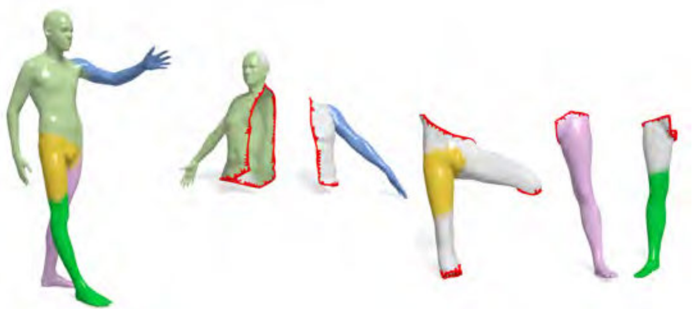


Correspondence

Litani, Rodolà, BB, Cremers 2016

Overlapping parts example

Model/Part	Synthetic (FAUST)
Transformation	Near-isometric
Clutter	**Yes (overlap)**
Missing part	No
Data term	Dense (SHOT)

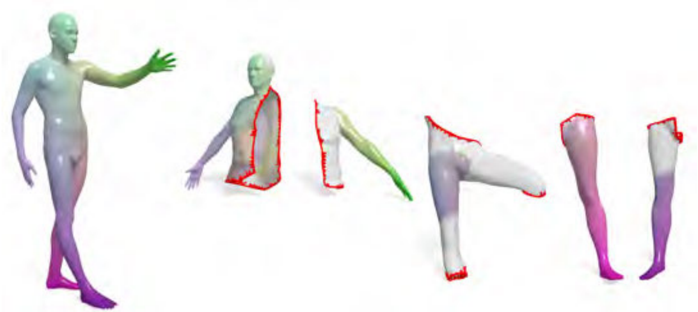


Segmentation

Litani, Rodolà, BB, Cremers 2016

Overlapping parts example

Model/Part	Synthetic (FAUST)
Transformation	Near-isometric
Clutter	**Yes (overlap)**
Missing part	No
Data term	Dense (SHOT)

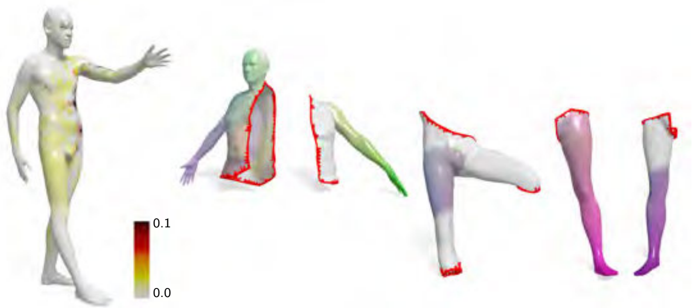


Correspondence

Litani, Rodolà, BB, Cremers 2016

Overlapping parts example

Model/Part	Synthetic (FAUST)
Transformation	Near-isometric
Clutter	**Yes (overlap)**
Missing part	No
Data term	Dense (SHOT)

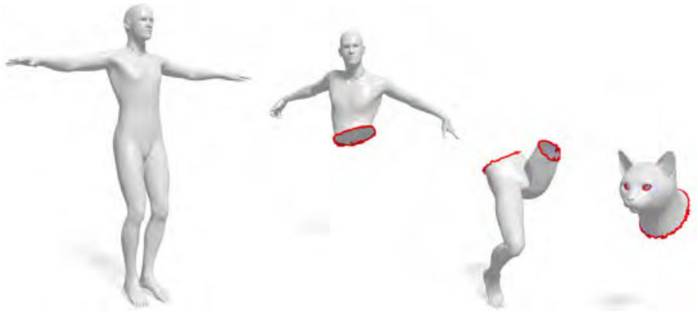


Correspondence error

Litani, Rodolà, BB, Cremers 2016

Missing parts example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	**Yes (extra part)**
Missing part	**Yes**
Data term	Dense (SHOT)



Litani, Rodolà, BB, Cremers 2016

Missing parts example

Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	**Yes (extra part)**
Missing part	**Yes**
Data term	Dense (SHOT)

Segmentation

Litani, Rodolà, BB, Cremers 2016

Missing parts example

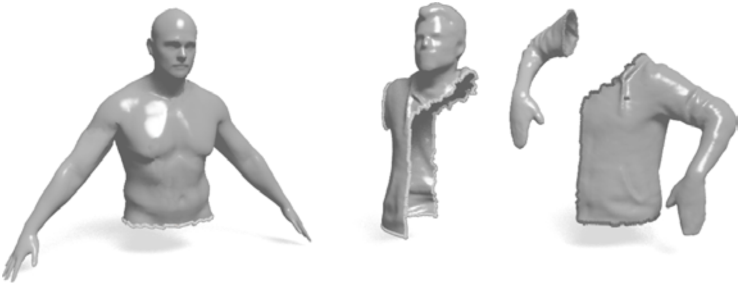
Model/Part	Synthetic (TOSCA)
Transformation	Isometric
Clutter	**Yes (extra part)**
Missing part	**Yes**
Data term	Dense (SHOT)

Correspondence

Litani, Rodolà, BB, Cremers 2016

Scanned data example

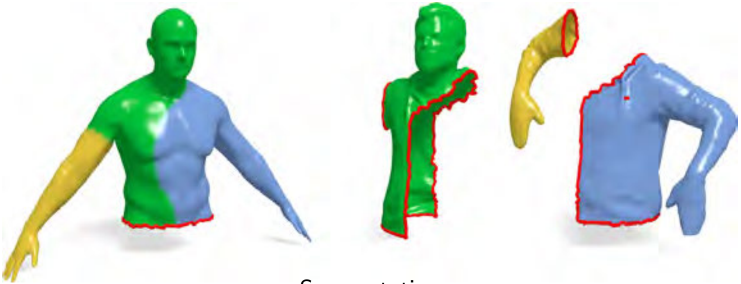
Model/Part	Synthetic (TOSCA) / Scan
Transformation	**Non-Isometric**
Clutter	No
Missing part	No
Data term	Sparse deltas



Litani, Rodolà, BB, Cremers 2016

Scanned data example

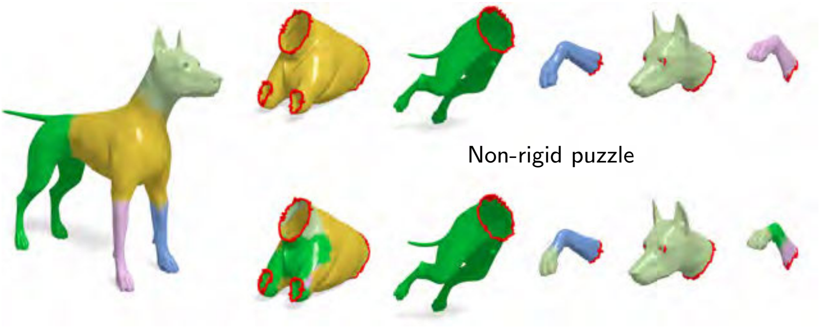
Model/Part	Synthetic (TOSCA) / Scan
Transformation	**Non-Isometric**
Clutter	No
Missing part	No
Data term	Sparse deltas



Segmentation

Litani, Rodolà, BB, Cremers 2016

Non-rigid puzzle vs Partial functional map



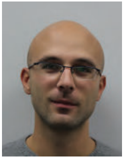
Non-rigid puzzle

Partial functional map (pair-wise)

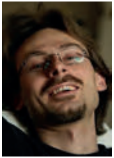
Rodolà, Cosmo, B, Torsello, Cremers 2016; Litani, Rodolà, BB, Cremers 2016

Summary

- New insights about spectral properties of Laplacians
- Extension of functional correspondence framework to the partial setting
- Practically working methods for challenging shape correspondence settings
- Code available (SGP Reproducibility Stamp)
- Some over-engineering - can be done simpler! (stay tuned...)



E. Rodolà



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Thank you!